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The Dynamic Canadian Debt Strategy Model

Nicolas Audet
Financial Markets Department
Bank of Canada
naudet@bankofcanada.ca

Joe Ning
Financial Markets Department
Bank of Canada
zning@bankofcanada.ca

Adam Epp
Financial Markets Department
Bank of Canada
adamepp@bankofcanada.ca

Jeffrey Gao
Financial Markets Department
Bank of Canada
jgao@bankofcanada.ca

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Abstract

We present a dynamic debt strategy model framework designed to assist sovereign debt portfolio managers in choosing an optimal debt issuance strategy. The model consists of two parts: a simulation engine and a debt issuance optimization engine. The main innovation of this framework is the introduction of dynamic issuance strategies, which allow issuance decisions to vary over time based on the model's simulated state variables. We apply this framework to Canada's specific debt management setting and show that these dynamic strategies, when compared with the deterministic issuance strategies of the original Canadian debt strategy model, bring considerable improvements to the costs and risks of available debt portfolios.

Topics: Debt management; Econometric and statistical methods; Financial markets; Fiscal policy
JEL codes: H63, H68, G11, G17, C61

Résumé

Nous présentons un cadre détaillant un modèle dynamique de gestion de la dette, conçu pour guider les gestionnaires de dette souveraine dans le choix d'une stratégie d'émission optimale. Le modèle comporte deux parties : un moteur de simulation et un moteur d'optimisation des émissions. L'aspect le plus novateur de notre cadre est l'introduction de stratégies d'émission dynamiques, qui permettent une variation des décisions au fil du temps en fonction des variables d'état simulées du modèle. Nous appliquons notre cadre au contexte spécifique de la gestion de la dette au Canada. Nous démontrons ainsi qu'en comparaison avec la stratégie déterministe du modèle canadien original de gestion de la dette, ces nouvelles stratégies dynamiques réduisent considérablement les coûts et les risques des portefeuilles de dette disponibles.

Sujets : Gestion de la dette ; Méthodes économétriques et statistiques ; Marchés financiers ; Politique budgétaire
Codes JEL : H63; H68; G11; G17; C61

1. Introduction

Sovereign debt managers face the task of structuring issuances in a way that minimizes debt costs—subject to limitations on risk—while considering qualitative objectives such as preserving the liquidity of secondary markets and maintaining predictable issuance plans. Moreover, with economic conditions constantly changing, debt managers need to know how they should alter their issuance decisions over time.

In this paper, we introduce a modelling framework in which debt managers can adapt their issuances to different changes in the macro-financial state. A dynamic debt strategy model is a simulation-based model in which debt issuances are a function of the prevailing macroeconomic and financial conditions of each simulated path and point in time. This contrasts with the more classical approach where issuance weights are deterministic; see Bolder and Deeley (2011), Bergstrom, Holmlund and Lindberg (2002), and Pick and Anthony (2006).

This model solves for the same general problem faced by debt managers as in Bolder and Deeley (2011): to find a financing strategy that best achieves the policy objectives given an uncertain future while capturing the appropriate debt and fiscal mechanics and economic interactions. Our innovation is in how we define a financing strategy—instead of being a fixed set of issuance weights in each instrument, it is a function of issuance weights on the state variables. This new definition of a financing strategy offers several key improvements compared with the deterministic definition.

Models with deterministic issuance weights suffer from two main drawbacks. First, these models provide little guidance on deciding issuance in the short term because issuance amounts are fixed and not adaptable to the current environment. Second, using such a model for deciding on current issuances leads to a time inconsistency between the current decision and the issuance decisions that will be made in subsequent years. The model's mapping between issuance strategies and the cost-risk space makes sense only if the debt manager commits to the multi-year strategy proposed by the model. With fixed weights, this would mean that, at all time steps, the debt manager should issue with the same fixed weights until the model horizon, which is generally not what they do. In practice, cost-minimizing debt managers do adjust issuances based on current and expected future economic conditions. This inconsistency is precisely what breaks the mapping between issuance strategies and the cost-risk space that the strategies entail.

The discussion above highlights that debt strategy modelling is about not only replicating the dynamics of the economic environment but also the decisions of the debt managers, which points to making issuance decisions contingent on economic conditions. Not only will that model be more realistic, but it will also stand a better chance of not suffering from the time inconsistency problem mentioned above.

Moreover, such models can provide information on the short-term issuance strategy as well as insights on how the debt manager should react to a changing environment.

Our paper therefore aims at addressing that shortcoming. We follow from Belton et al. (2018) in the sense that our framework allows issuances to depend on the economic, financial or fiscal environment. Our work differs from Belton et al. (2018) by the functional forms mapping state variables to issuances: in Belton et al.'s framework, issuances deviate from a "base" allocation using a linear combination of issuance kernels, which are intended to induce a bias toward either short-, middle- or long-term debt issuances. In our model, we use a more generic functional form of "softmax," which is often used as the activation function of a neural network to normalize the output. We believe that our specification is more flexible and avoids issues related to negative issuances,[1] which are a possibility with Belton et al.'s model.

While this dynamic debt strategy model framework is intended to be applied by any sovereign debt manager, the specifications of the model will vary based on each debt manager's unique policy objectives, data sources, and economic and market realities. In this paper, we apply this framework to the case of Canada, building on the existing Canadian debt strategy model's core elements (described in Bolder and Deeley 2011). The precise settings and results are thus illustrative and are intended to provide guidance to others who wish to adapt this framework for their own application.

We take the approach of dynamically optimizing government borrowing costs for a given level of risk under uncertainty. Our micro portfolio approach, which treats debt management as distinct from the government's broader fiscal policy, is consistent with the practice of most sovereign debt managers worldwide (Blommestein and Hubig 2012), including Canada.

The remainder of this paper is structured as follows. Section 2 provides an overview of the workflow of the debt optimization problem. Section 3 contains an in-depth description of the components of the model. Section 4 presents preliminary results of a specific debt optimization, which includes conditioning issuances on various macro-financial variables, based on the model specifications for Canada. Section 5 concludes. Descriptions of the simulation process, the issuance strategy optimization and the dynamic issuance strategy itself can be found in the appendices.

[1] Negative issuances do occur in reality as debt buy-back. However, these operations are generally either for cash management purpose or for improving secondary market liquidity. They therefore fall outside the scope of this paper.

2. Overview of the issuance strategy optimization problem

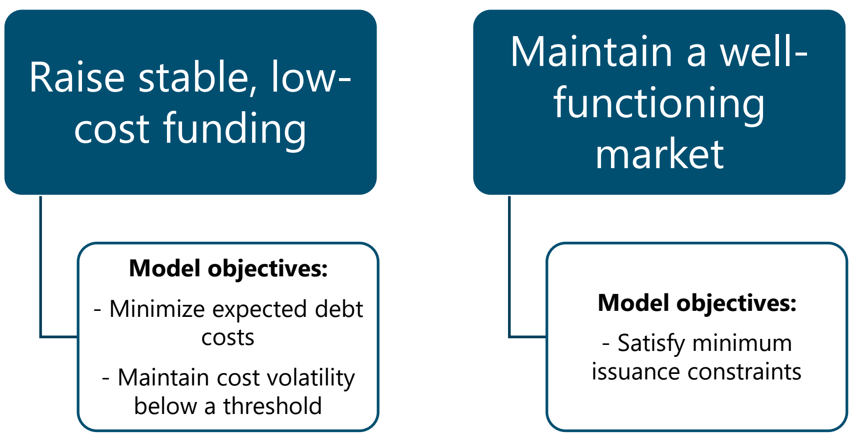
2.1. Translating policy objectives into model objectives

In general, a debt manager aims to achieve two primary objectives:[2]

- to provide the government with stable, low-cost funding
- to maintain a well-functioning market for government securities

From the point of view of the debt strategy modeller, these policy objectives must be translated into the model's objectives (**Figure 1**).

Figure 1: Policy and model objectives



The first objective can be understood as choosing the issuance strategy in a way that minimizes the expected debt costs,[3] subject to maintaining the volatility of costs below a certain threshold. Spanning a range of values for the threshold for cost volatility generates the efficient frontier, which maps the minimum expected cost for a range of

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[2] In this document, the objective of the debt manager is understood to be to reduce expected debt costs and their volatility by choosing only amounts to issue for each bond sector, with the fiscal, monetary and macroeconomic environment being essentially independent of the debt manager's decisions. This is a narrower definition of the manager's objectives than what is found in a sizable share of the literature on optimal debt management. That literature uses a canonical framework of a Ramsey planner, where a household's intertemporal utility is optimized over consumption and leisure, and where the government chooses tax and bond policy. See Faraglia et al. (2019) for an illustration of this framework. Section 3.4 in our paper briefly discusses how the related concepts of tax smoothing can be accommodated by our framework.

[3] Debt costs are understood to be the amounts paid as coupon on bonds.

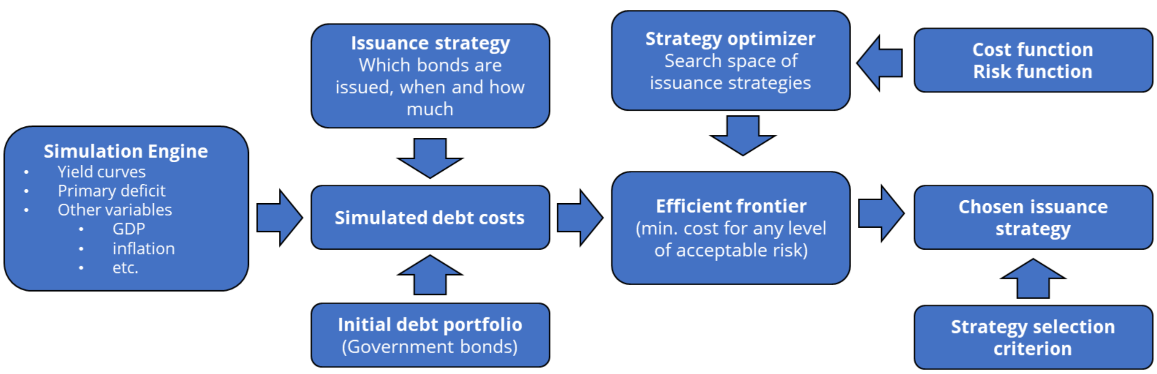
risk levels. This part of the issuance strategy optimization is perfectly analogous to common mean-variance portfolio selection as in Markowitz (1952).

The second objective aims at maintaining a well-functioning market for government securities, which reflects the external benefit of having sufficient levels of liquidity in secondary markets. In terms of issuance strategy optimization, this objective can be implemented as a constraint on minimum levels of issuances for each bond type. The determination of these levels is outside the scope of this paper. We therefore treat the minimum issuance constraints as given. See **Appendix C** for the technical implementation of minimum issuance constraints for a dynamic issuance strategy.

2.2. Overview of the strategy optimization process

This section provides an overview of the workflow used for the optimization of the issuance strategy. The process is illustrated in **Figure 2**.

Figure 2: Issuance strategy optimization workflow



The first component in the optimization of the issuance strategy is the **simulation engine**, which simulates the financial, fiscal and macroeconomic environment. The financial variables consist of the term structure of interest rates at which government bonds are issued. The fiscal variable is intended to describe the evolution of the government's funding needs, which are modelled through the primary government deficit. Macroeconomic variables complete the description of the environment.[4]

The second component is the definition of the **issuance strategy**. In a simple *static* model, the strategy would be defined as a set of weights (one per term issued), and issuances in each sector would equal the total funding requirements multiplied by the

[4] Note that our framework implies that the simulation engine produces data that are independent of the issuance strategy. This means that we exclude the possibility that the issuance strategy can influence, for example, the dynamics of macroeconomic variables. This is a simplification that is made to ease the numerical burden associated with optimizing the debt issuance strategy. Estimating the cost and risk associated with any given strategy would otherwise require re-running the simulation engine.

weight assigned to that sector. A slightly more sophisticated *deterministic* model would assign weights with a deterministic function of time. In this *dynamic* model, issuance weights are assigned with a function of both time and the state variables generated by the simulation engine. Introducing this association of state variables to issuance is the key innovation of our framework.

The next element is the calculation of the **simulated debt costs** associated with a particular issuance strategy. This calculation simply relies on an accounting-style description of the portfolio dynamics.

This is then followed by the definition of both the **cost and risk functions**. These are typically defined as a function of the debt costs but can also include other variables, such as the government's primary deficit,[5] gross domestic product (GDP) or consumer price index (CPI),[6] debt stock, etc. The cost function is typically a measure of central tendency, such as the expected total debt cost, while the risk function is typically a measure of dispersion, such as the variance of the cost or a tail risk measure.

The next step is to pass the objective function to a **strategy optimizer**, which finds the issuance strategies that minimize the cost measure while respecting an upper bound constraint on the risk measure. Spanning different risk values generates the efficient frontier, from which debt managers can pick a debt issuance strategy. The selection of a specific point on the efficient frontier depends on the debt manager's aversion to risk and therefore falls outside the scope of this paper.

3. Model components

This section provides a detailed description of the model components described in section 2. In particular, section 3.3 describes the issuance strategies that form the core of the dynamic model's added value. As noted, the specifications can be adjusted based on the issues and environment facing each individual debt manager.[7]

We use the following notational conventions throughout the paper:

- time—$t \in (1 \ldots T)$
    - a variable indexed by t is $\mathcal{F}_t$-measurable
    - today is $t = 0$; the first simulated date is $t = 1$

[5] This can be included to rewrite the optimization in terms of the total deficit, which is the sum of the primary deficit (generated by the simulation engine) and the debt costs.

[6] This could be used to deflate the debt costs using GDP or CPI growth.

[7] Note that both for simplicity and because real return bonds are no longer issued by the Canadian government, we assume throughout this paper that only nominal bonds are issued by the government. Adding real return bonds should not be particularly complex, although care must be taken to ensure the debt costs include the effect of the indexation of the principal amount to the price index. See Bolder and Deeley (2011) for a discussion on how this can be done.

- in general, the simulation horizon will be relatively long (for example: $T = 60$, or 15 years)

- bond maturity—$m \in (1 \dots M)$
  - a bond issued at t with maturity m matures at $t + m$[8]
- time and maturities are measured (and indexed) in quarters
  - bond year fraction: $\theta = 0.25$
- simulated path—$p \in (1 \dots P)$
- state variable index[9]—$n \in (1 \dots N)$

The order of the indexes is always (t, m, p) or (t, n, p).

3.1. The simulation engine

3.1.1. General requirements

The simulation engine is intended to produce:

- Bond yields—$y_{p,m,t}$ observed at time t for a bond reaching maturity at time $t + m$. The coupon paid by this bond will be $\theta y_{p,m,t}$ for each unit of the bond issued, which is to be paid at times $t + 1, \dots, t + m$. The principal will be repaid at time $t + m$.

  Note that we assume all bonds pay a coupon, which is not the case for bills (for $m \in \{1,2,4\}$). This approximation is done for simplicity.

- Primary deficit—$D_{p,t}$, which represents the government's funding needs, excluding debt costs and rollover of bonds that have reached their maturity.

The simulation engine can also generate other variables, such as inflation, output or other macroeconomic variables. These variables can be added if they are part of the calculation of the objective function, the universe of state variables that inform issuance weights, or if they improve the properties of the simulated distribution of yields or the primary deficit (such as adding to the predictive power).

3.1.2. Our simulation engine

The debt issuance strategy optimization in this paper is based on a vector autoregression (VAR) similar to Diebold and Li (2006) and augmented with fiscal (primary deficit) and macroeconomic variables (inflation and output).

Let x_t represent the model state variables, which are:

- L_t—yield curve level factor from the Nelson-Siegel decomposition[10]
- S_t—yield curve slope factor from the Nelson-Siegel decomposition
- C_t—yield curve curvature factor from the Nelson-Siegel decomposition

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[8] In practice, bond maturities are restricted to the set of available financing instruments (for example: T-bills, 2-year, 5-year, 10-year, 30-year).

[9] These variables form a basis for the dynamic issuance strategy, which will be described in section 3.2.3.

[10] See Nelson and Siegel (1987) for details on the yield curve decomposition.

- $D_t = \log(1 + PD_t/GDP_t)$, where PD_t is the quarterly primary deficit[11]
- $G_t = \Delta \log(GDP_t)$
- $\pi_t = \Delta \log(CPI_t)$

Details on the data sources (all macro variables seasonally adjusted):

- inflation—core CPI (CANSIM vector number v41690924) [12]
- GDP—nominal GDP (v62305784)[13]
- deficit—primary deficit, which is the Government of Canada's negative fiscal balance minus debt costs:
    - fiscal balance: net lending or net borrowing (CANSIM vector number v62425704)[14]
    - debt costs: interest on debt (CANSIM vector number v62425704)[15]
- yields—yield curves for zero-coupon Government of Canada bonds published by the Bank of Canada, with the three Nelson-Siegel factors estimated from rates at the relevant tenors (3-month, 6-month, 12-month, 2-year, 3=year, 5-year, 10-year and 30-year)

Note that all interest rates and associated costs and risks are in nominal terms.

The simulation engine is based on this VAR specification:[16]

$$x_t = c + \sum_{l=1}^{L} \Phi_l \, x_{t-l} + C \, \varepsilon_t. \qquad (1)$$

Here, x_t is stationary, $\varepsilon_t \sim iid \; N(0, I_K)$ and $\Omega = CC'$ is the covariance matrix of the VAR innovation. Typically, CC' is the Cholesky decomposition of Ω, with C being a lower triangular with real and positive diagonal entries. In this paper we set $L = 1$.

The model parameters can be estimated via ordinary least squares (OLS).

3.1.3. Simulation output conditioning

In general, the average output of the simulation engine will not match the expectations of expert forecasters. Also, debt managers will often want to align the simulation output with those forecasts or with various other scenarios. The main challenge with this requirement is to "discipline" the simulation output in a way that is meaningful from a probabilistic point of view.

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[11] The transformation used for D_t is intended to ensure its stationarity and improve its similarity to a Gaussian distribution.

[12] The data are first subsampled to quarterly frequency. Inflation is then computed as the first difference of the log-CPI.

[13] Note that "quarterly" GDP is annualized and requires multiplying the data by 0.25. The growth of GDP is the first difference of the log of quarterly GDP.

[14] Note that the "quarterly" fiscal balance is again obtained by multiplying the data by 0.25.

[15] Again, the data needs to be multiplied by 0.25 to get quarterly costs.

[16] VAR is used here because it is transparent, analytically tractable, easily estimated, as has the property of a "best linear predictor." Though it is not a fundamental feature, so other specifications could be used.

Appendix A describes such a method. The forecast data for the various model state variables are available from the following sources:

- data on rates, inflation and GDP growth—Department of Finance Survey of Private Sector Economic Forecasters
- deficit data (primary deficit/GDP)—various Department of Finance Canada Federal Budget web pages (for example, 2023 Budget, Table A1.5)

3.2. Portfolio dynamics

The debt portfolio dynamics are defined from the following variables, using the same notation as in Belton et al. (2018):

- $G_{p,m,t}$—the issued amounts at time t of bonds reaching maturity at time $t+m$. These bonds are assumed to be issued at par and pay a coupon rate $y_{p,m,t}$ until maturity.

- $C_{p,t}$—the total debt costs, which are the coupons paid at time t

- $R_{p,t}$—the total redemption amounts, which are the payments on bonds that have reached maturity at time t

In general, $G_{p,m,t}$ can take any positive value[17] that respects the relevant constraints. Also, the issuance amounts can be a function of only those variables known to the debt manager, standing on path p at time t.

The redemption amounts and debt costs are calculated as:

$$R_{p,t} = \sum_{m=1}^{M} G_{p,m,t-m} \qquad (2)$$

$$C_{p,t} = \theta \sum_{m=1}^{M} \sum_{i=1}^{m} y_{p,m,t-i}\, G_{p,m,t-i}. \qquad (3)$$

Issuances must be chosen in a way that respects the budget constraint, which specifies that the amounts raised through issuance must cover the primary deficit, bond redemptions and debt costs:

$$\sum_{m=1}^{M} G_{p,m,t} = D_{p,t} + R_{p,t} + C_{p,t}. \qquad (4)$$

Equations (2), (3) and (4) describe the portfolio dynamics. Note that equations (2) and (3) could refer to past issuances and coupon rates (with time indexes $t \leq 0$) and would thus take the composition of the initial debt stock as an input. Through equations (2) and (3), equation (4) connects past and current issuance amounts.

[17] We ignore the possibility of bond buybacks.

3.3. Issuance strategies

This section describes three ways issuance strategies can be characterized. This model uses a dynamic issuance strategy, which allows issuance to respond to different states.

All issuance strategies described below are based on issuance weights, which are defined as follows:[18]

$$w_m = \frac{G_{p,m,t}}{\sum_{l=1}^{M} G_{p,l,t}}. \tag{5}$$

This implies that $\sum_m w_m = 1$. Furthermore, we impose that $w_m > 0$ for all m, which makes it impossible for the debt manager to invest in some sectors while issuing in others.

3.3.1. Fixed issuance strategy

The simplest way to parametrize the issuance strategy is to assume that each sector forms a fixed percentage of the total amount issued:

$$G_{p,m,t} = w_m\left(D_{p,t} + R_{p,t} + C_{p,t}\right). \tag{6}$$

This strategy is the simplest, but obviously it has the drawback of not using information from the model state variables, and it also implies a discontinuity with the issuances at time $t = 0$.

3.3.2. Deterministic issuance strategy

A simple way to improve on the discontinuity of the static issuance strategy is to assume that the proportion of issuances for each sector is given by a deterministic function of time:

$$G_{p,m,t} = \left(\left(1 - \phi(t)\right)w_{m,0} + \phi(t)w_{m,\infty}\right)\left(D_{p,t} + R_{p,t} + C_{p,t}\right). \tag{7}$$

Here, $D_{p,t} + R_{p,t} + C_{p,t}$ is total issuances, via the budget constraint from equation (4). Therefore, we have

$$w_{m,t} = \left(1 - \phi(t)\right)w_{m,0} + \phi(t)w_{m,\infty}. \tag{8}$$

[18] Note that in order to enforce minimum issuance amounts $\left(G_{p,m,t}^{Min}\right)_{m=1}^{M}$, one can apply the issuance weights on the funding needs net of the sum of minimum issuance amounts, and then add back the minimum issuances amounts.

Therefore, the term $w_{m,t}$ corresponds to the percentage, or weight, of the issuances in bonds with term m. They are not indexed with the path and are therefore deterministic.

The term $w_{m,0}$ is the initial issuance weights at time $t = 0$ and corresponds to

$$w_{m,0} = \frac{G_{p,m,0}}{\sum_{l=1}^{M} G_{p,l,0}}. \tag{9}$$

The function $\phi(t)$ is defined such that $\phi(0) = 0$, $\phi(\infty) \to 1$ and $\phi(x) > 0$. It is therefore used as an interpolation coefficient between current issuances $w_{m,0}$ and long-term issuances $w_{m,\infty}$. Introducing this interpolation guarantees that issuance weights will not deviate too quickly from current issuance weights to ensure the consistency of government's issuance strategy near the initial time.

The parameters $w_{m,\infty}$ represent the long-term issuance weights. They are the choice variable for the portfolio manager. As we describe in the next sections, the values for $w_{m,\infty}$ can be found through the optimization of the cost and risk trade-off, as defined by the specific objective function guiding the portfolio manager.

Despite this improvement, this strategy is still deterministic and has the same drawback of not using information from the model state variables—and hence the issues of time inconsistency and short-term ineffectiveness described in the introduction.

Note that a static strategy is simply a restricted form of a deterministic strategy where $\phi(t) = 1$ for all t (i.e., all the weight is on long-term issuance, none is on current).

3.3.3. Dynamic issuance strategy

The dynamic issuance strategy parametrization is a generalization of the deterministic issuance strategy. It allows issuance weight to be a function of the state variables of the model. Issuances are defined similarly as before, with the exception that weights can now vary across paths:

$$G_{p,m,t} = \Big(\big(1 - \phi(t)\big)w_{m,0} + \phi(t)w_{p,m,t}\Big)\left(D_{p,t} + R_{p,t} + C_{p,t}\right). \tag{10}$$

With the dynamic issuance strategy, the issuance weights are defined as a function of the state variables. Let $\left(V_{p,n,t}\right)_{n=1}^{N}$ be a set of N state variables generated by the model. We call the set of those variables the "issuance basis." These can be any transformation of any variable that is observable at time t or earlier, along path p. These can include simulated variables such as those in x_t, variables that are part of the portfolio dynamics (past issuances, debt stock composition), lagged variables, deterministic functions of time, or combinations of all of these. By convention, we assume that a constant is included, $V_{p,1,t} = 1$.

The mapping relationship between issuance weights and the chosen state variables is flexible; however, it should satisfy two basic criteria in practice:

- $w_{p,m,t} \geq 0$ to ensure the non-negative issuance weights
- $\sum_m w_{p,m,t} = 1$ to ensure the total sum of issuance weights equals 1

In theory, any mapping function that satisfies the above criteria can be a candidate for the issuance reaction in our modelling framework. The refinement of the issuance reaction function is out of the scope of this paper, though one could consider using more advanced techniques such as machine learning.

For this paper, we choose a linear-logistic functional form, which satisfies the two criteria above:[19]

$$w_{p,m,t} = \frac{\exp\left(\sum_n V_{p,n,t}\beta_{n,m}\right)}{\sum_l \exp\left(\sum_n V_{p,n,t}\beta_{n,l}\right)}. \tag{11}$$

The above reaction function considers only the first-order linear basis functions of state variables; however, it can be easily generalized to incorporate the nonlinear basis functions when necessary. **Appendix C** offers a more detailed discussion.

The set of parameters $\beta_{n,m}$ is the choice variable for the debt portfolio manager. By choosing a fixed set of β values, the debt manager will be able to vary the composition of issuances according to the state of the financial, fiscal or macroeconomic environment. For example, the portfolio manager can adjust the issuances of short- versus long-term bonds, depending on the steepness of the yield curve, expectations of increasing or decreasing rate levels, etc.

This β choice variable here contrasts with the direct weights ($w_{m,\infty}$) choice variable in the deterministic model.

Note that the deterministic issuance strategy is simply a restricted form of the dynamic issuance strategy where $N = 1$ (i.e., no state variables besides constant term). Thus, a dynamic strategy will never be worse than a deterministic strategy in terms of the defined objective function.

[19] It is possible to use other mappings. We find this form works with our data and objectives.

3.4. Cost and risk functions

Both the cost and risk functions here are functions of the simulated debt costs $C_{p,t}$, which implicitly are functions of the parameters of the issuance strategy. To make the notation more concise, we remove that dependence from the notation.

Here the cost function is simply defined as the average debt cost over all paths and timesteps without discounting:

$$Cost(\theta) = \frac{1}{TP}\sum_{t,p} C_{p,t}. \tag{12}$$

Conditional cost volatility is the risk function used in our examples. It represents the volatility of the conditional surprise in debt costs. It is defined by the variance of the residual of a regression of $C_{p,t}$ on $C_{p,t-1}$ and a constant. Technically, this can be written as:

$$Risk(\theta) = \min_{a,b} \frac{1}{(T-1)P}\sum_{t,p}\left(C_{p,t} - \left(a - b\, C_{p,t-1}\right)\right)^2. \tag{13}$$

The estimators for (a, b) can be obtained through OLS.

3.4.1. Alternative formulations of the cost and risk functions

The framework we present in this paper can accommodate other definitions of the objective function. For example, the alternative definitions of costs and risks used in Belton et al. (2018) can be used in our framework, such as defining costs in terms of the cost-to-GDP ratio:

$$Cost_{GDP}(\theta) = \frac{1}{TP}\sum_{t,p} \frac{C_{p,t}}{GDP_{p,t}}. \tag{14}$$

The definition of risk can also be modified in a similar way to obtain risks in terms of costs over GDP:

$$Risk_{GDP}(\theta) = \min_{a,b} \frac{1}{(T-1)P}\sum_{t,p}\left(\frac{C_{p,t}}{GDP_{p,t}} - \left(a - b\, \frac{C_{p,t-1}}{GDP_{p,t-1}}\right)\right)^2. \tag{15}$$

The value of $GDP_{p,t}$ can be obtained by integrating the GDP growth produced by the simulation engine. A similar specification would be to consider debt costs in real terms by discounting them using the simulated price level index, which can be obtained by integrating the simulated inflation along each path.

One specific variation for the objective function could be to measure risks not in terms of the volatility of debt costs but instead in terms of the volatility of the total deficit, which is $C_{p,t} + D_{p,t}$. This definition is interesting for two reasons:

- First, the risk measure will be dependent on the correlation between costs and the primary deficit. Because states with high growth are generally associated with both higher yields and a lower primary deficit, we should expect a negative correlation between $C_{p,t}$ and $D_{p,t}$, which will be stronger when more short-term debt is issued, as noted in Belton et al. (2018). Consequently, this formulation will highlight the importance of short-term issuances when the debt manager is concerned with the total fiscal volatility.
- Second, since this risk measure focuses on minimizing volatility, it is more consistent with the idea of government tax smoothing. See Blommestein and Hubig (2012) for details.

3.5. Issuance strategy optimization

The issuance strategy optimization is assumed to be of the mean-variance type, where expected costs are minimized over the choice variables of the issuance strategy, subject to a given risk measure being lower than some maximum threshold. The set of all optimum expected costs for any level of maximum threshold risk form the efficient frontier.

The choice variables of the strategy are denoted ρ, which are:

- $w_{m,\infty}$ when optimizing over deterministic issuance strategies
- $\beta_{n,m}$ when optimizing over dynamic strategies

The debt manager optimization problem can be written as:

$$\rho^*(R) = \underset{\rho}{\operatorname{argmin}}\, Cost(\rho) \;\; s.t.\, Risk(\rho) \le R. \tag{16}$$

The efficient frontier is simply the parametric curve $\left(Cost\left(\rho^*(R)\right), Risk\left(\rho^*(R)\right)\right)$, with R spanning a broad set of possible risk levels.

Note that the issuance strategy optimization is numerically complicated due to having multiple degrees of freedom. In practice, the robustness of the optimization results strongly relies on the soundness of the initial guess. As part of this model, we propose a novel idea of iterative random search around non-dominated data points to generate the initial guess for the gradient-descent-based algorithm of optimization. **Appendix C** presents the details of our implementation.

3.6. Choosing the issuance basis for dynamic strategies

The choice of issuance basis affects the performance of dynamic strategies. In theory, all variables forming a Markovian state-space representation of the system describing economic and portfolio-related variables should be part of the set $\left(V_{p,n,t}\right)_{n=1}^{N}$ of state variables informing the issuance decisions.[20] This would include the variables generated by the simulation engine as well as the structure of the bond portfolio.

We choose to use a small number of variables produced by the simulation engine. To remain parsimonious in the specification of the model and to improve the tractability of the issuance function, we normally try several combinations of those variables, and for each of those combinations we compute the efficient frontier. We continue adding new variables until the efficient frontier is no longer materially improved by the addition of new variables.

4. Results

This section presents preliminary results obtained from the issuance strategy optimization procedure presented above to showcase how this dynamic framework can be applied.

We use the following set-up, based on some Canada-specific settings and inputs, for this test:

- The simulation engine is estimated over data from 1996Q1 to 2024Q1. The private sector forecasts used for conditioning the simulation output are from 2024Q1.
- The initial debt portfolio is the actual Canadian domestic debt portfolio from 2024Q1.[21]
- The simulation engine generates 4,000 paths and uses a 15-year horizon.
- The issuance basis variables that we test are *level* and *slope*.[22] All issuance strategies that we use are described in sections 3.3.2 and 3.3.3, with the deterministic strategy having a constant as its only issuance basis variable. The possible combinations are therefore deterministic, level, slope, and level and slope.

[20] In our case, it can be shown that this would be the last L values of the state variables from the simulation engine, plus all the future cashflows from the already-issued debt.

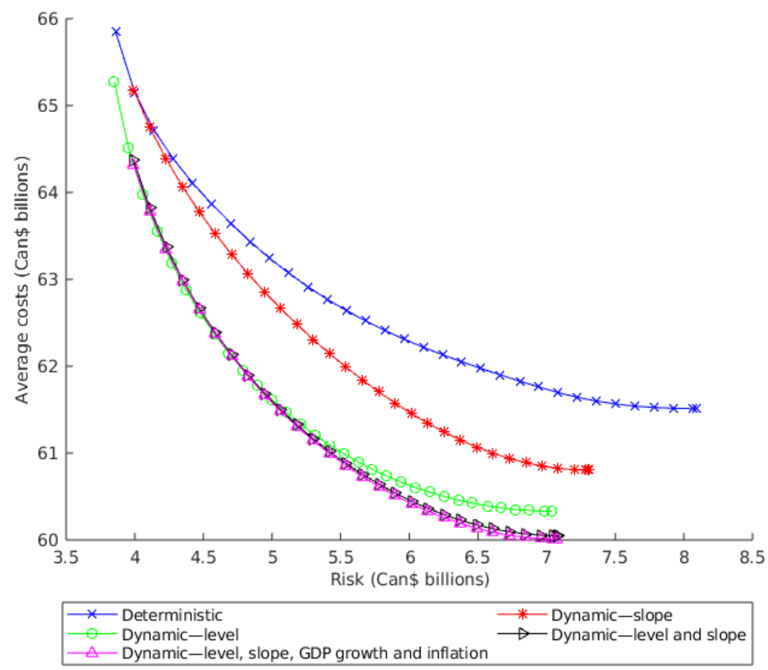
[21] Note that it does include real return bonds, which are not part of this issuance strategy optimization analysis but are nevertheless included with their CPI-adjusted costs properly taken into account, as in Bolder and Deeley (2011).

[22] Several other variables were also tested. It appears that in all tests, the *level* variable carries the most importance in terms of its capacity to improve the efficient frontier. *Inflation* also has some importance, albeit to a much lesser extent.

4.1. Comparing deterministic and dynamic efficient frontiers

The first step in analyzing the impact of the choice of issuance basis variables is to compute the associated efficient frontiers (**Chart 1**).

Chart 1: Efficient frontiers for deterministic and dynamic issuance strategies



Note: Dollar amounts are per year. Conditional cost volatility is used for the risk function.

Because the various combinations of issuance basis form a set of embedded models, we find that adding variables to the set of issuance basis variables reduces the expected costs for all levels of risk. For example, the deterministic issuance strategy is dominated by all dynamic strategies, while the *level* and *slope* strategies are dominated by the *level and slope* strategy.

4.2. Interpretation of the results

In this section, we offer some insights on the results from the previous section. Namely, we explore why the *level* variable provides most of the improvement in the efficient frontier.

We find that the *level* correlates with the realized term premium and therefore informs the trade-off between the various terms of issuance. This follows from the assumption of a stationary and mean-reverting process for rates: high rates today imply decreasing rates in the future, making it advantageous to issue short-term debt.

We also investigate an alternative set-up for the simulation engine where rates are non-stationary. We find an increased importance of the *slope* variable for predicting the realized term premium. This coincides with our finding that *slope* brings the most improvement to the issuance strategy in terms of efficient frontier.

4.2.1. Interpretation under stationary rates

Under the standard assumption of stationary, mean-reverting interest rates (an I(0) process), the *level* factor in a dynamic issuance strategy provides the single largest improvement. This indicates that of all state variables investigated so far (*level*, *slope*, *GDP growth* and *inflation*), *level* has the greatest predictive power for how well a decision will turn out.

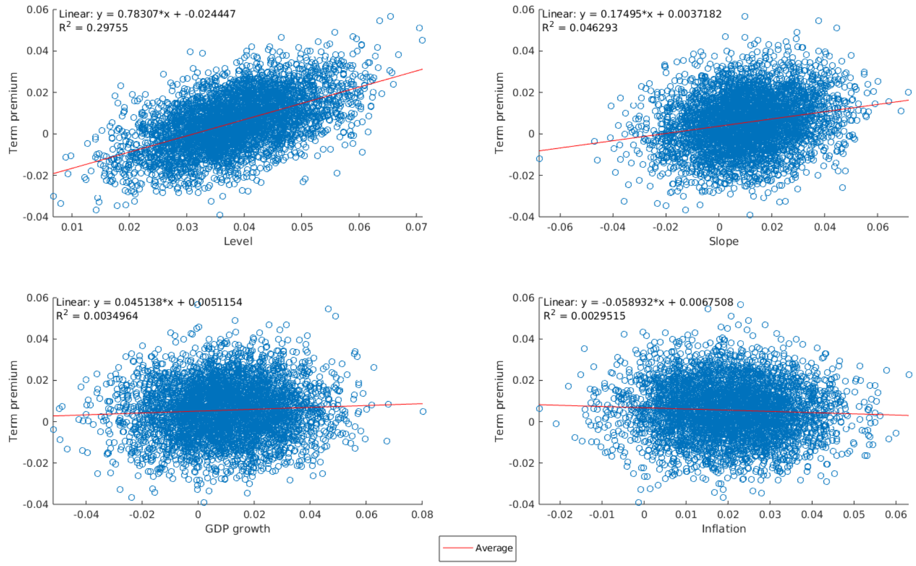
Since the debt manager's decision is basically about issuing short-term versus long-term debt, a good proxy for success (or failure) of each decision is the realized term premium—how much more costly it is to issue a long bond versus issuing a series of short bonds for that same period.

Thus, to test if *level* is the most significant factor, we look at the relationship between each state variable $V(t)$ at t and the realized term premium (RTP) for 10-year bonds at $t + n$, which is defined as:

$$RTP_{p,n,t} = y_{p,n,t} - \frac{1}{n}\sum_{i=0}^{n-1} y_{p,1,t+i},$$

where $n = 40$ quarters (or 10 years). **Chart 2** shows the RTP for $t = 20$. The RTP broadly reflects the additional costs of issuing a 10-year bond versus issuing 3-month bills (and continuously refinancing) for those 10 years. In general, a higher RTP means that issuing short was a good decision; lower means it was bad. A state variable $V(t_0)$ with good predictive power should thus have a higher absolute correlation—positive or negative—with the RTP.

Chart 2: The relationship between state variables and the average term premium, with stationary interest rates



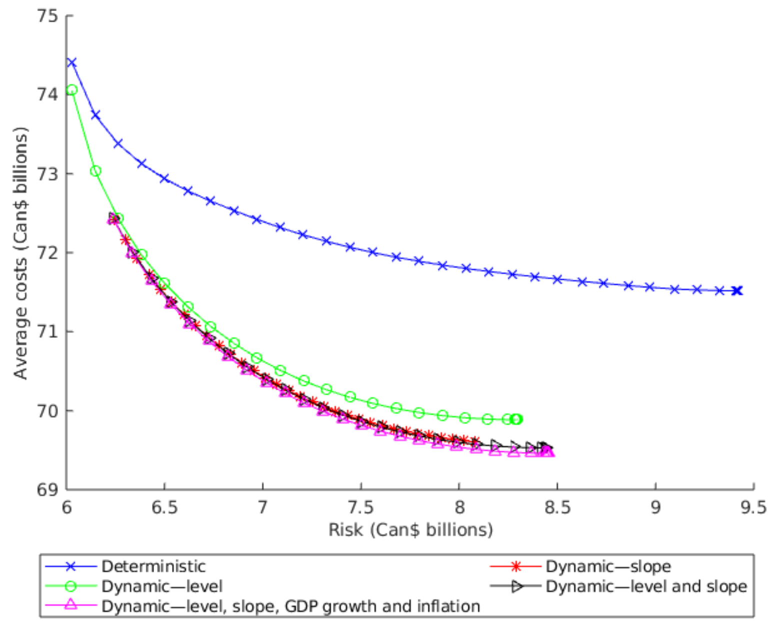
Note: Panels show the relationship between each variable and the 10-year average term premium. The state variable scenarios use an I(0) process and assume interest rates are stationary and mean-reverting.

As **Chart 2** shows, the *level* variable tends to have higher correlation with RTP, which is consistent with it having the largest cost-risk performance gain in the efficient frontier (**Chart 1**). Conversely, *GDP growth* and *inflation* are less correlated and do not provide much improvement.

4.2.2. Interpretation under non-stationary rates

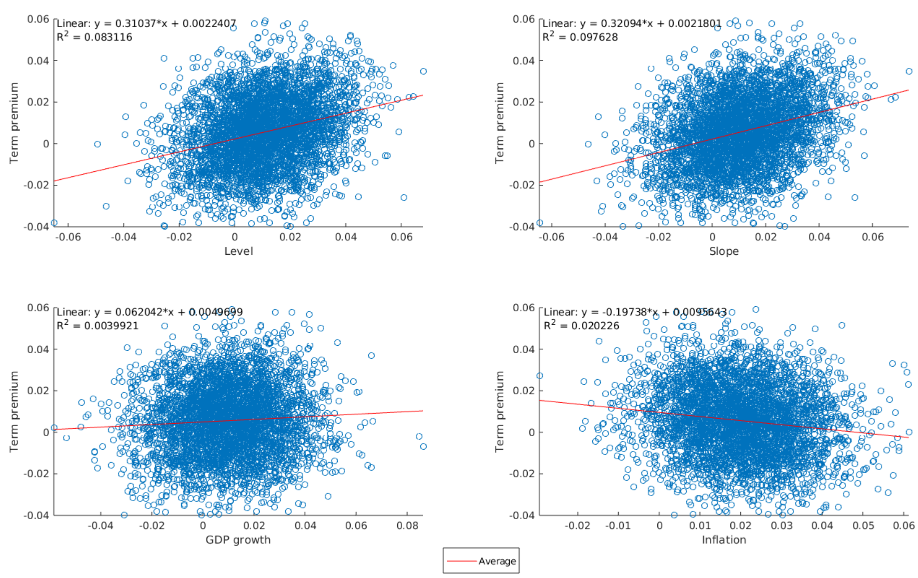
Under a different simulation engine assumption, *level* may not necessarily be the most informative state variable. In an alternative formulation for the simulation model, we assume that the first difference of *level* is stationary, instead of the variable itself being stationary. With this assumption, our experimental results seem to indicate that the *slope* variable brings the most important improvement to the efficient frontier (**Chart 3**). This is consistent with the slightly higher correlation to RTP that we have observed, which is shown in **Chart 4**.

Chart 3: Efficient frontiers for deterministic and dynamic strategies with non-stationary interest rates



Note: Dollar amounts are per year. Conditional cost volatility is used for the risk function. We use an I(1) process for the level variable.

Chart 4: The relationship between state variables and the average term premium, with non-stationary interest rates



Note: Panels show the relationship between each variable and the 10-year average term premium. The state variable scenarios use an I(1) process for the *level* variable and assume interest rates are non-stationary.

Since the issuance allocations from a dynamic model are directly a product of simulation engine assumptions, it is important that the engine be carefully set up in a way that reflects the policy-makers’ view of market and economic dynamics.

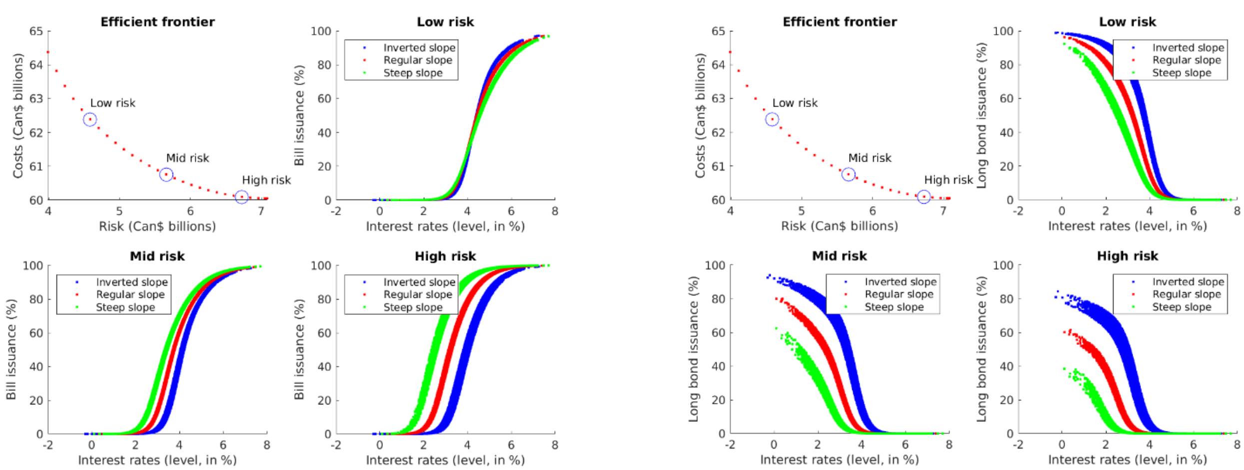
4.3. Analyzing dynamic issuance strategies

Dynamic issuance strategies are more complex to visualize than those that are static, which have a single weight for each sector. A dynamic issuance strategy can be visually represented in two ways. One is through a cross-section view of the issuance strategy, which shows how issuance allocation changes across a range of state variable values for each risk level. Another is through a time series view of the issuance strategy, which shows that at each time point, the average issuance of a dynamic strategy differs from the corresponding deterministic issuance for each risk level.

4.3.1. Cross-section view of the issuance strategies

By studying how issuance in short (bill) and long varies based on the prevailing values of our two main state variables (*level* and *slope*) at different points on the efficient frontier in **Chart 5**, several observations can be made.

Chart 5: Issuances of short and long bonds when level and slope vary



Note: Short bonds are bills with a term of up to one year; long bonds are those with 10- and 30-year terms. Percentiles of distribution are as follows: inverted slope, 10%–20%; regular slope, 45%–55%; steep slope, 80%–90%. Dollar amounts are per year. Conditional cost volatility is used for the risk function.

First, regardless of the steepness (*slope*) of the yield curve, bill issuance increases monotonically with the increase of interest rate levels, while long bond issuance decreases in a similar manner. This is consistent with our finding noted in section 4.1 that higher *level* tends to predict a higher RTP, which favours shorter issuance.

Thus, the debt manager might consider dynamically adjusting issuance based on the change of yield environments. For example, they might consider long-biased issuance in low-yield environments and issuing more bills in high-yield environments.

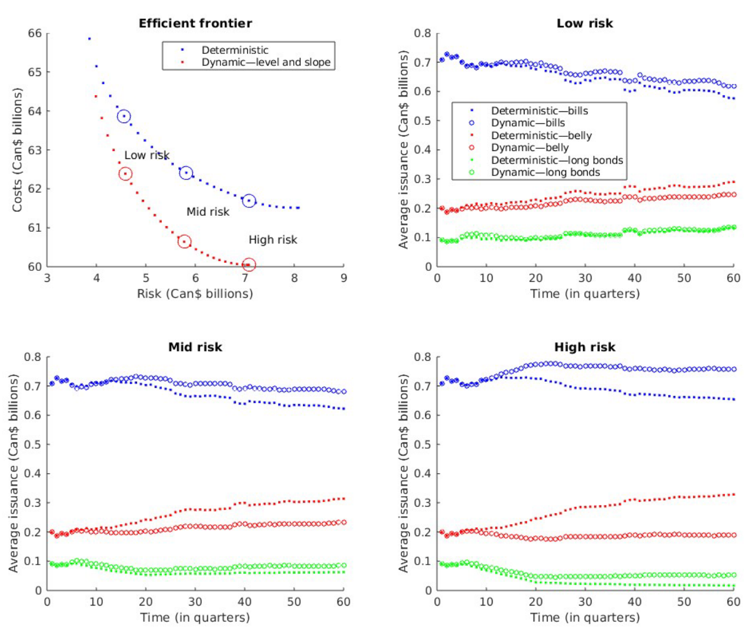
Second, a steeper curve is better for short issuance and worse for long issuance, and vice-versa. However, in line with the results discussed in section 4.1, in our standard I(0) simulation, *slope* appears to have a smaller overall impact than *level* in determining the issuance.

These results are robust to different risk levels. All else being equal, a higher risk level means more bill issuance (more cost volatility from refinancing, but a lower point on the yield curve), while a lower risk level means more long bond issuance (the opposite).

4.3.2. Time series view of the issuance strategies

Chart 6 shows expected issuances over time for both the deterministic strategy and the dynamic strategy with *level* and *slope*. The low, mid- and high-risk strategies of the deterministic and dynamic strategies are for the same level of risk.

Chart 6: Expected issuance weights versus time



Note: Short bonds are bills with a term of up to one year; long bonds are those with 10- and 30-year terms. Dollar amounts are per year. Conditional cost volatility is used for the risk function.

At each risk level, the average issuance from the dynamic strategy differs from the corresponding deterministic issuance, and this difference is greater the further out in the simulation. This shows that deterministic strategies cannot be thought of as a first-order approximation of dynamic strategies (see **Appendix C** for details). This also suggests that only the ability to react to changing state variables in the future affects the relative favourability between sectors.

5. Conclusion

We propose an enhanced debt strategy modelling framework that gives the debt portfolio manager flexibility to dynamically adjust their issuance strategy according to the changing macroeconomic and interest rate environments. Analyzing the optimal dynamic reaction functions for issuance strategy on an issuance basis can provide insightful guidance for the decision-making process of debt issuance. Applying this framework to Canada's case, we find that the level of interest rates currently provides the most relevant information for optimal dynamic issuance strategies; however, incorporating the slope of the yield curve adds further but diminishing improvement.

6. References

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Appendix A: Simulation conditioning

This section broadly follows Jarociński (2010) with the addition of a method for applying "soft conditioning."

Starting from a vector autoregression (VAR) model with "L" lags:

$$y_t = c + \sum_{l=1}^{L} \Phi_l \, y_{t-l} + C \, \varepsilon_t. \tag{17}$$

Here, y_t has size N and is stationary, $\varepsilon_t \sim iid\ N(0, I_K)$ and $\Omega = CC'$ is the covariance matrix of the VAR innovation. In general, CC' is the Cholesky decomposition of Ω, with C being a lower triangular with real and positive diagonal entries.

Centring the model and rewriting it in terms of $z_t = y_t - \mu$, with $\mu = E[y_t] = (I_K - \sum_l \Phi_l)^{-1} c$, we have:

$$z_t = \sum_{l=1}^{L} \Phi_l \, z_{t-l} + C \, \varepsilon_t. \tag{18}$$

We can write the companion form of the VAR model as:

$$\tilde{z}_t = F \tilde{z}_{t-1} + G \tilde{\varepsilon}_t, \tag{19}$$

where $\tilde{z}_t = (z_t' \quad \dots \quad z_{t-L+1}')'$, $F = \begin{pmatrix} \Phi_1 & \dots & \Phi_L \\ I_{(L-1)N} & & 0_{(L-1)N,N} \end{pmatrix}$, $G = \begin{pmatrix} C & 0_{N,(L-1)N} \\ 0_{(L-1)N,N} & 0_{N,N} \end{pmatrix}$ and $\tilde{\varepsilon}_t = (\varepsilon_t' \quad 0_{N,(L-1)N})'$.

A1.1 Conditioning

Assume that we know the parameters of the VAR. We want to simulate its state variables over $[1 \dots T]$, conditional on some starting point z_0. First, write the VAR in "impulse response" for z_t:

$$z_t = F^t_{(1:N,:)} \tilde{z}_0 + C\varepsilon_t + \Psi_1 C \varepsilon_{t-1} + \dots + \Psi_{t-1} C \varepsilon_1. \tag{20}$$

Here, we have $\Psi_i = F^i_{(1:N,1:N)}$, which is the $N \times N$ left-upper block of F raised to its i^{th} power, while $F^t_{(1:N,:)}$ is the first N rows of F^t.

The previous equation shows that all z_t are linear functions of $(\varepsilon_1 \dots \varepsilon_t)$ and are therefore jointly Gaussian. We can write all (future) timesteps (and variables) as:

$$\begin{pmatrix} z_1 \\ \dots \\ z_T \end{pmatrix} = \begin{pmatrix} F^1_{(1:N,:)} \\ \dots \\ F^T_{(1:N,:)} \end{pmatrix} \tilde{z}_0 + \begin{pmatrix} C & 0 & 0 \\ \vdots & \ddots & \vdots \\ \Psi_{T-1} C & \dots \Psi_1 C & \dots C \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \dots \\ \varepsilon_T \end{pmatrix}, \tag{21}$$

or in short form:

$$z = H \tilde{z}_0 + R\varepsilon. \tag{22}$$

We want to simulate P paths for z:

$$z^{(p)} = H\tilde{z}_0 + R\varepsilon^{(p)}. \tag{23}$$

We need to choose the correct distribution for $\varepsilon^{(p)}$, which is conditional on

$$Az^{(p)} = b^{(p)}, \tag{24}$$

where A is a $q \times TK$ array, $b^{(p)}$ is a $q \times 1$ random vector drawn from some random distribution. Using equation (23), the condition in equation (24) can be written as $AR\varepsilon^{(p)} = b^{(p)} - AH\tilde{z}_0$ or $\tilde{R}\varepsilon^{(p)} = r^{(p)}$, where $\tilde{R} = AR$ and $r^{(p)} = b^{(p)} - AH\tilde{z}_S$. Given that the joint distribution of $\varepsilon^{(p)}$ and $\tilde{R}\varepsilon^{(p)}$ is:

$$\begin{pmatrix} \varepsilon^{(p)} \\ \tilde{R}\varepsilon^{(p)} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} I & \tilde{R}' \\ \tilde{R} & \tilde{R}\tilde{R}' \end{pmatrix}\right), \tag{25}$$

we therefore have:

$$\left(\varepsilon^{(p)} \middle| \tilde{R}\varepsilon^{(p)} = r^{(p)}\right) \sim N\left(\tilde{R}'\left(\tilde{R}\tilde{R}'\right)^{-1} r^{(p)}, I - \tilde{R}'\left(\tilde{R}\tilde{R}'\right)^{-1}\tilde{R}\right). \tag{26}$$

We can therefore simulate with equation (23) while imposing equation (24) by drawing from the conditional distribution for $\varepsilon^{(p)}$ from equation (26).

A1.2 Numerical considerations

It is numerically more efficient to simulate $\varepsilon^{(p)}$ by first applying the singular value decomposition[23] to $\tilde{R} = U\left(D\ 0_{q\times(T-q)}\right)(V_1\ V_2)'$ where D is a $q \times q$ diagonal array of singular values of $\tilde{R}$, $U'U = UU' = I_{q\times q}$, $(V_1V_2)'(V_1V_2) = (V_1V_2)(V_1V_2)' = I_{NT\times NT}$, where V_1 is the first q right singular vectors, and V_2 is the $(NT - q)$ orthogonal vectors. It is possible to show that we can generate the conditional distribution of $\left(\varepsilon^{(p)} \middle| \tilde{R}\varepsilon^{(p)} = r^{(p)}\right)$ by setting:

$$\left(\varepsilon^{(p)} \middle| \tilde{R}\varepsilon^{(p)} = r^{(p)}\right) = V_1D^{-1}U'r + V_2\eta. \tag{27}$$

A1.3 Soft conditioning

If only the expected value of $b^{(p)}$ is known, i.e., $E\left[b^{(p)}\right] = \mu_b$ (for example: when imposing that the expected rates, GDP growth, inflation, etc., as simulated by the conditioned

[23] See, for example, Press et al. (1992).

model must match an exogenously provided expected value, such as one coming from a survey), one can impose that the covariance structure from the conditioned model be the same as the covariance structure of the conditional model.

In the unconditioned model:

$$Cov\left(b^{(p)}\right) = Cov\left(Az^{(p)}\right) = Cov\left(A(H\tilde{z}_S + R\varepsilon)\right) = ARR'A' = \tilde{R}\tilde{R}'. \quad (28)$$

One can therefore draw $b^{(p)} \sim N\left(\mu_b, \tilde{R}\tilde{R}'\right)$.

Note that by preserving the estimated covariance structure of $b^{(p)} = Az^{(p)}$ we are also preserving the covariance structure of $z^{(p)}$:

$$\begin{aligned} Cov\left(z^{(p)}\right) &= E\left[Cov\left(z^{(p)} \middle| Az^{(p)} = b^{(p)}\right)\right] + Cov\left[E\left(z^{(p)} \middle| Az^{(p)} = b^{(p)}\right)\right] \\ &= \left(RR' - R\tilde{R}'\left(\tilde{R}\tilde{R}'\right)^{-1}\tilde{R}R'\right) + R\tilde{R}'\left(\tilde{R}\tilde{R}'\right)^{-1}\tilde{R}\tilde{R}'\left(\tilde{R}\tilde{R}'\right)^{-1}\tilde{R}R' \quad (29) \\ &= RR'. \end{aligned}$$

In simple terms, this means that assuming that $Cov\left(b^{(p)}\right) = \tilde{R}\tilde{R}'$ will preserve the covariance structure of $z^{(p)}$, which means that the co-movements of the modelled variables are not affected by this assumption.

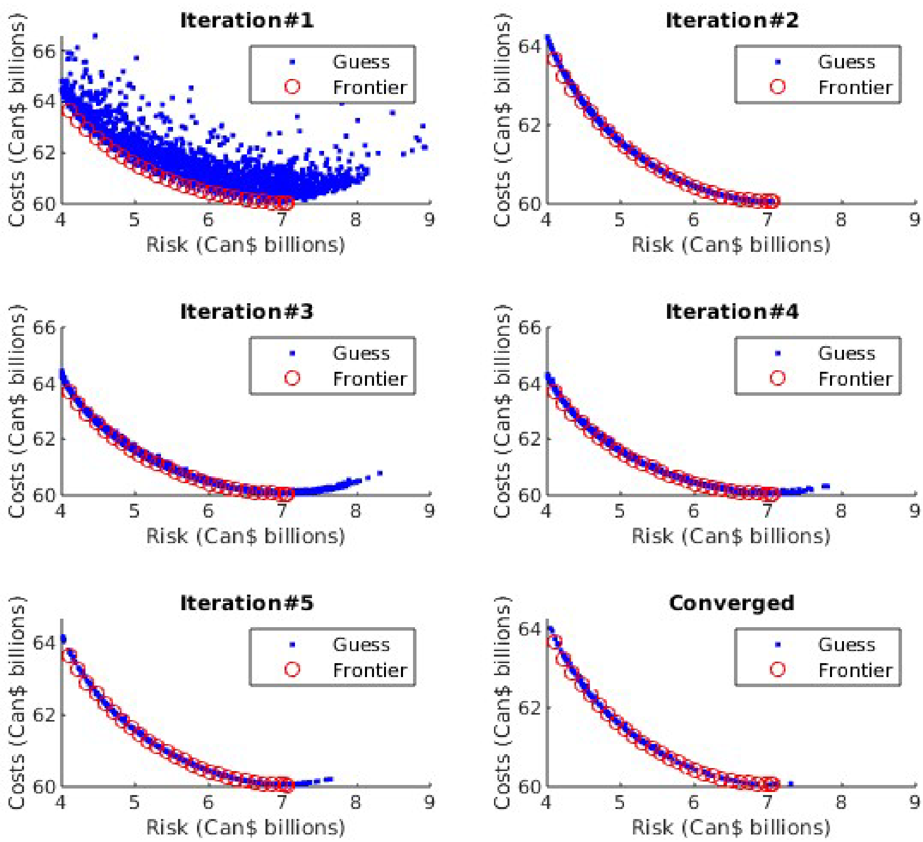
For example, assume that in the sample data we observe that high inflation is positively correlated with higher rates. If we impose the view that inflation should be higher in the conditioned model than in the unconditional model, then we should also find that rates will be higher in the conditioned model than in the unconditioned model. In other words, views on one variable will affect other variables in a way that is consistent with the data.

Appendix B: Iterative random search optimization

We develop a novel approach of generating good-quality initial guesses for the optimization problem. We find this approach is useful in practice to avoid being trapped by local minima, which is commonly encountered for multivariate optimization problems. The computational efficiency is also satisfactory compared with the global optimization approaches, e.g., genetic algorithm. The detailed approach is described as follows:

- Step 1: Generate a set of initial guesses of issuance strategy and calculate the pairs of (costs, risks). Given the risk steps, find the corresponding non-/least-dominated data points.
- Step 2: Keep all the non-/least-dominated data points, and redo the random sampling around these data points.
    - Create the Delaunay mesh of the non-/least-dominated data points.
    - For each triangle in the Delaunay mesh, a user-specified number (e.g., 50) of random samplings are generated as follows:
        - generate random weights $(\alpha_1, \alpha_2, \alpha_3)$ such that $0 \leq \alpha_i \leq 1$ and $\sum_i \alpha_i = 1$, thus the new point is $\alpha_1\beta_1^i + \alpha_2\beta_2^i + \alpha_3\beta_3^i$
        - apply the random noise adjustment to the new points; the covariance matrix of the noise is calculated based on the regression of β ($Y_{n\times n_\beta}$) & (costs, risks) ($X_{n\times 2}$) of the non/least-dominated data points (total number n):
            - $\epsilon = Y - X \times B,\ while\ B = (X'X)^{-1}(X'Y)$
            - $Cov_\epsilon = \frac{\epsilon'\epsilon}{n-2}$
            - $Noise\ = N(0, Cov_\epsilon)$
- Step 3: Repeat Step 1 and Step 2 until the cost difference of all non-dominated data points of the given risk steps converges to the preset tolerance. The progress of the iterative random searches is shown in **Chart B-1** as an example.

Chart B-1: The progress of iterative random searches around non-dominated data points



Note: Dollar amounts are per year. Conditional cost volatility is used for the risk function.

Appendix C: Extending to a nonlinear issuance strategy

Recall the definition of issuance weights from equation (5):

$$w_{p,m,t} = \frac{G_{p,m,t}}{\sum_{l=1}^{M} G_{p,l,t}}.$$

We can perform the following transformation on the weights and define:

$$x_{p,m,t} = \ln\left(\frac{w_{p,m,t}}{w_{p,M,t}}\right).$$

This transformation can be inverted as follows:

$$w_{p,m,t} = \frac{\exp(x_{p,m,t})}{\sum_{l=1}^{M} \exp(x_{p,l,t})}.$$

Since the transformation between $\left(x_{p,m,t}\right)_{m=1}^{M}$ and $\left(w_{p,m,t}\right)_{m=1}^{M}$ is bijective, one can think of an issuance strategy as a mapping from the model's state variables toward $\left(w_{p,m,t}\right)_{m=1}^{M}$, or equivalently as a mapping from the model's state variables toward $\left(x_{p,m,t}\right)_{m=1}^{M}$. In this section, we follow the latter formalism.

Let's assume that an optimal mapping $f^*: \mathcal{V} \to \mathcal{X}$ exists from the set of all possible values for state variables $\mathcal{V}$ and the set of all possible values for our issuance weights $\mathcal{X}$. We write the m[th] element of f^* as $x_{p,m,t} = f_m^*(V_{p,t})$ for $V_{p,t} = \left(V_{p,n,t}\right)_{n=1}^{N} \in \mathcal{V}$.

C1.1 The deterministic issuance strategy

Let's take a zero-degree Taylor expansion of f^* around 0, leading to the approximated reaction function $f^{(0)}$:

$$x_{p,m,t}^{(0)} = f_m^{(0)}(V_{p,t}) = f_m^*(0).$$

This means that we have $w_{p,m,t}^{(0)} = \frac{\exp(f_m^*(0))}{\sum_{l=1}^{M} \exp\left(f_l^*(0)\right)}$. If we write $\beta_m^{(0)} = f_m^*(0)$ we have:

$$w_{p,m,t}^{0} = \frac{\exp\left(\beta_m^{(0)}\right)}{\sum_{l=1}^{M} \exp\left(\beta_l^{(0)}\right)}.$$

This corresponds to the weights used in the deterministic model from section 3.3.

C1.2 The linear dynamic issuance strategy

Taking a first-degree Taylor expansion of f^*:

$$x_{p,m,t}^{(1)} = f_m^{(1)}(V_{p,t}) = f_m^*(0) + \sum_{n=1}^{N} V_n \frac{\partial f_m^*}{\partial V_n}(0).$$

Let's write $\beta_m^{(1)} = f_m^*(0)$ as before. We add the parameters $\beta_{n,m}^{(1)} = \frac{\partial f_m^*}{\partial V_n}(0)$. This means that we have:

$$w_{p,m,t}^{(1)} = \frac{\exp\left(\beta_m^{(1)} + \sum_{n=1}^{N} V_{p,n,t}\, \beta_{n,m}^{(1)}\right)}{\sum_{l=1}^{M} \exp\left(\beta_l^{(1)} + \sum_{n=1}^{N} V_{p,n,t}\, \beta_{n,l}^{(1)}\right)},$$

which corresponds to the reaction function defined in section 3.3.

C1.3 Higher-order issuance strategies

We can simply extend the technique above and take a second-degree Taylor expansion of f^*:

$$x_{p,m,t}^{(2)} = f_m^{(2)}\left(V_{p,t}\right) = f_m^*(0) + \sum_{n=1}^{N} V_{p,n,t} \frac{\partial f_m^*}{\partial V_n}(0) + \frac{1}{2}\sum_{n=1}^{N}\sum_{l=1}^{N} V_n V_l \frac{\partial^2 f_m^*}{\partial V_n \partial V_l}(0).$$

Defining $\beta_m^{(2)} = f_m^*(0)$, $\beta_{n,m}^{(2)} = \frac{\partial f_m^*}{\partial V_n}(0)$ and $\beta_{n,l,m}^{(2)} = \frac{1}{2}\frac{\partial^2 f_m^*}{\partial V_n \partial V_l}(0)$ we have:

$$w_{p,m,t}^{(2)} = \frac{\exp\left(\beta_m^{(2)} + \sum_{n=1}^{N} V_n\, \beta_{n,m}^{(2)} + \sum_{n=1}^{N}\sum_{l=1}^{N} V_n V_l\, \beta_{n,l,m}^{(2)}\right)}{\sum_{j=1}^{M} \exp\left(\beta_j^{(2)} + \sum_{n=1}^{N} V_n\, \beta_{n,j}^{(2)} + \sum_{n=1}^{N}\sum_{l=1}^{N} V_n V_l\, \beta_{n,l,j}^{(2)}\right)},$$

which is a quadratic extension of the previous reaction function.

Note that in the linear dynamic issuance strategy, the issuance basis variables are:

$$V_{p,t} = (1 \quad V_{p,1,t} \quad \cdots \quad V_{p,N,t}).$$

The quadratic issuance strategy can be obtained simply by adding new issuance basis variables:

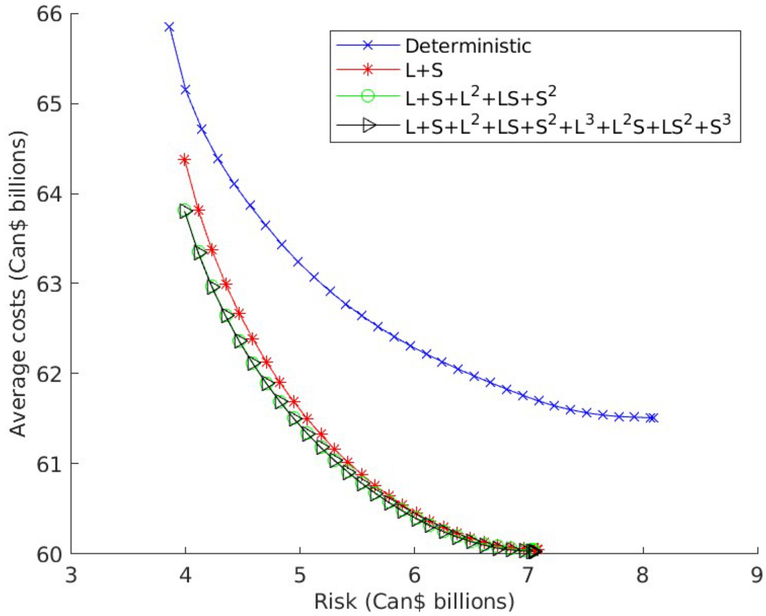
$$V_{p,t} = \left(1 \quad V_{p,1,t} \quad \cdots \quad V_{p,N,t} \quad V_{p,1,t}^2 \quad V_{p,1,t}V_{p,2,t} \quad \cdots \quad V_{p,N,t}^2\right).$$

C1.4 Setting the order of expansion for the issuance strategy

Assuming that f^* respects the conditions for convergence for the Taylor series, higher orders to the expansion should take successive approximations closer to f^*. One could determine an appropriate expansion order by computing the associated efficient frontier. Since the optimization problem associated with a lower order of expansion is constrained versions of the optimization problem associated with a higher order of expansion, it is guaranteed that the efficient frontier for higher orders of expansion will dominate. One can therefore increase the order of expansion up to the point where the improvement of the efficient frontier becomes negligible.

In **Chart C-1** we show the efficient frontiers associated with orders of expansion ranging from 0 (deterministic) to 3 for the state variables *level* and *slope*. One can clearly observe that successive orders of expansion improve the cost and risk trade-offs, as each successive order of expansion dominates the previous one.

Chart C-1: Efficient frontiers with and without the expansion of state variables



Note: L is level; S is slope. Dollar amounts are per year. Conditional cost volatility is used for the risk function.

Also, adding an extra order of expansion seems to have a diminishing effect on the improvement of the efficient frontier. The main improvement appears to be between the orders 1 and 0. Moreover, we can see that the difference between linear and quadratic is fairly small, while the difference between quadratic and cubic is practically null. This suggests that an issuance strategy based on a quadratic expansion of the state variables would constitute a good approximation of the optimal issuance strategy.

Note that the convergence of the efficient frontier tends to show that using a more complex functional form for the issuance strategy (for example, based on an artificial neural network) or using a functional form based on the Taylor expansion other than the one used in this section is unlikely to bring any significant improvement when compared with the one from the quadratic issuance strategy.