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Estimating the Portfolio-Balance Effects of the Bank of Canada's Government of Canada Bond Purchase Program

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Abstract

I propose a novel dynamic portfolio-balance model of the yield curve for Government of Canada bonds to evaluate the portfolio-balance effects of the Bank of Canada's Government of Canada Bond Purchase Program. My results suggest that this program, launched on March 27, 2020, in response to the COVID-19 pandemic, lowered the weighted average maturity of the Government of Canada's debt by approximately 1.4 years. This in turn reduced Canadian 10-year and 5-year zero-coupon yields by 84 and 52 basis points, respectively.

Topics: Asset pricing; Central bank research; Coronavirus disease (COVID-19); Interest rates; Monetary policy
JEL codes: E43, E52, G12, H63

Résumé

Je propose un nouveau modèle de portefeuille dynamique de la courbe de rendement des obligations du gouvernement du Canada pour évaluer les effets de portefeuille du Programme d'achat d'obligations du gouvernement du Canada de la Banque du Canada. Les résultats obtenus indiquent que ce programme, lancé le 27 mars 2020 en réponse à la pandémie de COVID-19, a permis d'abaisser l'échéance moyenne pondérée de la dette du gouvernement canadien d'environ 1,4 an. Cela a entraîné du même coup une diminution du rendement des obligations coupon zéro à dix ans et à cinq ans de 84 et 52 points de base, respectivement.

Sujets : Évaluation des actifs; Recherches menées par les banques centrales; Maladie à coronavirus (COVID-19); Taux d'intérêt; Politique monétaire
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1 Introduction

In response to the economic fallout from the COVID-19 pandemic, the Bank of Canada undertook extensive measures to stabilize financial markets and support the Canadian economy. As the pandemic-induced lockdowns in March 2020 caused a sharp increase in the demand for liquidity and led to severe market dysfunction, the Bank of Canada lowered their policy rate to the effective lower bound and launched several asset purchase programs. Such programs targeted various segments of the financial market, including government, mortgage, corporate bonds, commercial paper, and bankers' acceptances.[1] These asset purchase programs, unprecedented in scale, represented 23.6% of Canada's 2019 nominal GDP.[2]

Of these programs, the Government of Canada Bond Purchase Program (GBPP), launched on March 27, 2020, played a pivotal part in the Bank of Canada's response to the pandemic in that it accounted for almost 80% of purchased assets. Initially, the GBPP aimed to address the severe market dysfunction and restore liquidity in the government bond market by purchasing at least $5 billion of Government of Canada (GoC) bonds per week. However, by July 2020, as market conditions normalized, the GBPP transitioned to a quantitative easing (QE) program to provide additional monetary stimulus while the policy rate was constrained by the effective lower bound. This shift was designed to lower long-term borrowing costs in the economy, given that purchases of GoC bonds of a given maturity tend to bid up their price, thus lowering the interest rate that the bond pays to its holders. This lower interest rate, in turn, transmits to mortgages and corporate loans, which stimulates more borrowing and spending to support economic recovery and help the Bank of Canada achieve its inflation target.[3]

[1] In addition to these asset purchase programs, the Bank of Canada introduced an extended repo facility and a contingent term repo facility to support commercial bank funding conditions and counter any severe market-wide liquidity stresses. See Johnson (2023) for a review of the Bank of Canada's market operations related to COVID-19.

[2] See Appendix A in CGFS (2023).

[3] See Beaudry (2020) and Kozicki (2024).

Modern versions of the portfolio-balance theories of the yield curve support the use of these large-scale asset purchase (LSAP) programs to put downward pressure on long-term interest rates (see Greenwood and Vayanos, 2014; Vayanos and Vila, 2021). In such models, risk-averse investors do not view all assets as perfect substitutes due to their differences in liquidity, risk characteristics, tax treatment, regulatory constraints, or institutional preferences.

In this paper, we focus on the imperfect substitutability among government bonds that naturally arises from their differing sensitivities to interest rate movements. Specifically, investors are concerned about the risk of capital losses on their bond holdings when interest rates increase. Therefore, long-term and short-term bonds are seen as imperfect substitutes because long-term bond prices are more sensitive to interest rate changes (i.e., when interest rates rise, the prices of long-term bonds fall more than the price of short-term bonds, and vice versa). As a result, investors demand higher yields on long-term bonds to compensate for the greater interest-rate risk compared to short-term bonds. Consequently, by swapping private holdings of longer-maturity assets (such as long-term government bonds) for short-term assets (such as settlement balances), a program like the GBPP can reduce private investors' exposure to interest-rate risk and thus put downward pressure on long-term interest rates.[4]

Given the unprecedented scale of the Bank of Canada's GBPP, this paper attempts to quantify the portfolio-balance effects of this large-scale asset purchase program on the Canadian yield curve. Specifically, our contribution is twofold. First, we quantify the reduction in interest-rate risk implied by the Bank of Canada's GBPP by computing the reduction in the average maturity of the outstanding GoC marketable debt due to the GBPP purchases. Our calculations suggest that, at its peak in November 2021, the

[4]While in this paper we focus on the imperfect substitutability caused by interest-rate risk differences across long-term and short-term government bonds, these bonds can also differ in their degree of pledgeability as collateral (see, i.e., Williamson, 2016, for a theoretical model exploring these implications on the degree of pledgeability as collateral on the impact of central bank purchases of government bonds).

GBPP had lowered the weighted average maturity of the portfolio of the GoC investor by about 1.4 years. For comparison, the pre-pandemic Federal Reserve Board's large-scale asset purchase programs, implemented in the U.S. as part of the Fed's quantitative easing programs, lowered the weighted average maturity of the portfolio of U.S. Treasury marketable debt by approximately 1.7 years (see Greenwood et al., 2016).

Second, in a similar exercise to that in Hamilton and Wu (2012), we exploit an equivalence between modern portfolio-balance models and Gaussian dynamic term structure models (GDTSMs) to quantify the importance of the portfolio-balance effects of the Bank of Canada's GBPP on nominal bond yields. This equivalence allows us to calibrate a portfolio-balance model based on those of Greenwood and Vayanos (2014) and Vayanos and Vila (2021) by estimating a GDTSM by maximum likelihood using the approach outlined in Joslin et al. (2011). Our estimates suggest that lowering the average maturity of the GoC marketable debt by approximately 1.4 years translates to a reduction of the Canadian 10-year (5-year) zero-coupon yield by 84 (52) basis points.

In this paper, we focus on the portfolio-balance effects of the Bank of Canada's GBPP on the GoC bond yield curve. Two additional potential channels exist for central bank asset purchases to affect long-term interest rates. For one, Gagnon et al. (2011), Krishnamurthy and Vissing-Jorgensen (2011), and Joyce et al. (2011), among others, have suggested a market functioning/liquidity channel through which central bank asset purchases could affect bond risk premia. In this case, central bank purchases can enhance market functioning by reducing the liquidity premia on bonds demanded by market participants, thus making it easier to sell bonds. This channel was likely particularly important during the initial phase of the pandemic. On the other hand, as noted by Bauer and Rudebusch (2014), among others, the signalling channel recognizes that asset purchases contain news about future monetary policy. Therefore, the announcement of an asset purchase program can lead market participants to revise their expectations of future short-term rates, thus affecting long-term interest rates. Both channels should put additional downward pressure on long-term interest rates. Therefore, our estimates of the portfolio-balance effects of the Bank of Canada's GBPP could be viewed as a lower bound on the overall effects of this program on the Canadian yield curve.

Related literature. Our work complements a growing literature on evaluating the Bank of Canada's response to the COVID-19 pandemic. Arora et al. (2021) analyze intraday movements in GoC bond yields in the hour after the Bank of Canada first announced the GBPP on March 27, 2020, at 9 a.m. They find that the announcement of the GBPP had a strong and immediate impact, with 10-year benchmark GoC bond yields declining by about 10 basis points (bps) immediately after the announcement. Azizova et al. (2024), using an event study to evaluate the impact of the GBPP program, find that the GBPP had an announcement effect on long-term government bond yields of a decline of 10 and 20 bps. Further, since these announcements do not capture that there was some expectation that the Bank of Canada would purchase GoC bonds even before the GBPP was announced, they perform a back-of-the-envelope counterfactual and estimate that it may have had an impact of almost 80 bps on 10-year bond yields, an estimate that is in line with our results. Finally, using a macrofinance model based on Zhang's (2021), they map the effect of a decline in 10-year bond yields by almost 80 bps into impacts on GDP and inflation to find a peak impact of about 3% on real GDP and 1.8 annualized percentage points on inflation (although they report a lot of uncertainty around the size of this impact).

Tombe (2023) presents a historical perspective of the Bank of Canada's balance sheet, revenue streams, and expenditures, including the effects of the Bank of Canada's response to the COVID-19 pandemic. In particular, he finds that the overall effect of the Bank of Canada's purchases of GoC bonds during the pandemic lowered the average time to maturity of federal government debt by approximately 1.5 years. Our results are complementary to Tombe (2023). First, we can isolate the effect of the GBPP by differentiating between the Bank of Canada's holdings of GoC bonds due to standard operations (i.e., bond purchases to offset government deposits and the issuance of banknotes) and the Bank of Canada's holdings of bond purchases under the GBPP. Second, we focus on the weighted average maturity reduction implied by the GBPP rather than the reduction in the average time to maturity of the federal government since focusing on weighted average maturity provides a better measure of the interest-rate risk removed by the Bank of Canada from a theoretical perspective.

Paper outline. In Section 2, we quantify the reduction in interest-rate risk implied by the Bank of Canada's GBPP. Section 3 presents our portfolio-balance model of the term structure of interest rates. In Section 4, we quantitatively assess the importance of the portfolio-balance effects of the Bank of Canada's GBPP on the Canadian yield curve. Section 5 concludes.

2 The Bank of Canada's Government Bond Purchase Program

The GBPP, introduced by the Bank of Canada on March 27, 2020, aimed to mitigate severe dislocations in the GoC bond market precipitated by the COVID-19 pandemic. The program committed to purchasing at least $5 billion in GoC bonds weekly, encompassing all maturities across the yield curve, and pledged to continue these purchases until economic recovery was well underway. This intervention was initially designed to restore liquidity and ensure the proper functioning of the government bond market, which a widespread surge in the demand for cash had significantly disrupted.

As market conditions stabilized by mid-2020, the GBPP transitioned from focusing on restoring market functioning to serving as a QE tool to provide additional monetary stimulus. This shift redirected the program's objective toward exerting downward pressure on long-term interest rates in the economy to bolster economic growth. By absorbing a significant volume of government bonds, the Bank of Canada effectively reduced the supply of GoC bonds available to private investors, lowering their exposure to interest-rate risk and thus putting downward pressure on long-term interest rates.

In this section, we quantify the reduction in interest-rate risk implied by the Bank of Canada's GBPP by calculating the weighted average maturity of the GoC's marketable debt, with and without considering the consolidation of the Bank of Canada's holdings of GoC debt purchased under the GBPP.

2.1 Data

To quantify the reduction in the amount of interest rate risk borne by private investors due to the Bank of Canada's GBPP, we collect data on bond characteristics (issue date, coupon rate, maturity date, and face value outstanding) on every GoC bond outstanding between January 2018 and March 2024.

We also collect data on the Bank of Canada holdings of each of these bonds. Importantly, in addition to the bond purchases under the GBPP, the Bank of Canada also acquires GoC bonds "under normal course," primarily through non-competitive bids at government securities auctions, to offset government deposits and the issuance of banknotes, which represent liabilities on its balance sheet.[5] For this reason, we distinguish in our data set between holdings of GoC bonds acquired under normal course in the primary market and holdings due to the GBPP acquired in the secondary market.

2.2 The maturity structure of GoC debt

Following Doepke and Schneider (2006) and Greenwood and Vayanos (2014), we construct the maturity structure of government debt at a given date by (i) breaking the stream of

[5]As laid out in the Bank of Canada's "Statement of Policy Governing the Acquisition and Management of Financial Assets" (https://www.bankofcanada.ca/2021/09/statement-policy-governing-acquisition-management-financial-assets-bank-canada-balance-sheet/), asset purchases conducted under normal course are governed by three key principles: prudence, transparency, and neutrality.

each bond's cash flows into coupon and principal payments and (ii) aggregating cash flows across individual bonds. In this way, total payments due n quarters from date t are

$$D_t^{(n)} = C_t^{(n)} + PR_t^{(n)} = \sum_i C_{i,t}^{(n)} + \sum_i PR_{i,t}^{(n)}, \tag{2.1}$$

where $C_t^{(n)}$ is the total coupon payment, derived by summing over the cash flows related to the coupon bond payment $C_{i,t}^{(n)}$ that each bond i is due to make n quarters from date t. In contrast, $PR_t^{(n)}$ is the total principal payment, derived by summing over bonds the principal payments $PR_{i,t}^{(n)}$ that each bond i is due to make n quarters from date t. Specifically, we compute the maturity structure of the GoC debt up to 120 quarters (30 years).[6]

Similarly, we construct an equivalent maturity structure for the GoC debt where we consolidate the Bank of Canada holdings of GoC bonds due to its purchases under normal course, and a maturity structure of the GoC debt where we consolidate the Bank of Canada total holdings of GoC bonds. In both cases, we include the liabilities issued by the Bank of Canada (i.e., cash and settlement balances) to finance the purchase of these GoC bonds when consolidating the Bank of Canada holdings into the maturity structure of the GoC debt. We denote the total payments due n quarters from date t from the bonds in the Bank of Canada holdings due to its purchases under normal course by $S_t^{(n)}$, while the payments related to all the Bank of Canada bond holdings is denoted by $H_t^{(n)}$. Note that the difference between $H_t^{(n)}$ and $S_t^{(n)}$ captures the maturity structure of the Bank of Canada holdings of GoC bonds due to the GBPP purchases.

Motivated by the work of Greenwood and Vayanos (2014) and Vayanos and Vila (2021), who show that bond supply shocks can affect the compensation demanded by fixed-income investors if they change the amount of interest rate risk that they are exposed to, we summarize the maturity structure of GoC debt at a given date by computing the

[6] As in Greenwood and Vayanos (2014), we aggregate any payments beyond 120 quarters (30 years) in the 120-quarter bucket.

weighted average maturity of the outstanding GoC debt:

$$WAM_t^{(GoC\ total)} = \frac{\sum_{n=0}^{N=120} D_t^{(n)} e^{-ny_t^{(n)}} n}{\sum_{n=0}^{N=120} D_t^{(n)} e^{-ny_t^{(n)}}}, \tag{2.2}$$

where $y_t^{(n)}$ is the yield on n-quarter nominal zero-coupon bonds. Note that equation (2.2) can be expressed as

$$WAM_t^{(GoC\ total)} = \sum_{n=0}^{N=120} \omega_t^{(n)} n, \tag{2.3}$$

where the weights $\omega_t^{(n)}$ satisfy

$$\omega_t^{(n)} = \frac{D_t^{(n)} e^{-ny_t^{(n)}}}{\sum_{n=0}^{N=120} D_t^{(n)} e^{-ny_t^{(n)}}}. \tag{2.4}$$

Note that, by weighting the maturity structure of the stream of future payments by the GoC by the present value of each cash flow, the weighted average maturity coincides with the Macaulay duration of the outstanding GoC debt.

Similarly, we compute the weighted average maturity of the GoC bond outstanding with the consolidation of the Bank of Canada's holdings of GoC bonds acquired under normal course,[7]

$$WAM_t^{(Consolidated\ BoC\ normal)} = \frac{\sum_{n=0}^{N=120} \left[D_t^{(n)} - S_t^{(n)}\right] e^{-ny_t^{(n)}} n}{\sum_{n=0}^{N=120} D_t^{(n)} e^{-ny_t^{(n)}}}, \tag{2.5}$$

and the weighted average maturity of the GoC bond outstanding with the consolidation of the Bank of Canada's total holdings of GoC bonds,

$$WAM_t^{(Consolidated\ BoC\ holdings)} = \frac{\sum_{n=0}^{N=120} \left[D_t^{(n)} - H_t^{(n)}\right] e^{-ny_t^{(n)}} n}{\sum_{n=0}^{N=120} D_t^{(n)} e^{-ny_t^{(n)}}}. \tag{2.6}$$

[7]Note that the denominator in the definitions of the weighted average maturity of the GoC bond outstanding with the consolidation of the Bank of Canada's holdings of GoC bonds acquired under normal course and the total Bank of Canada holdings of GoC bonds (equations 2.5 and 2.6, respectively) remain unchanged with respect to the definition of the weighted average maturity of the GoC total bond outstanding (equation 2.2). This is because, under the GBPP, the Bank of Canada is swapping long-term GoC bonds with liabilities with (essentially) zero duration (either currency or settlement balances). For this reason, the present value of the Bank of Canada's total holdings of GoC bonds, which is given by $\sum_{n=0}^{N=120} H_t^{(n)} e^{-ny_t^{(n)}}$ (which represents the zero-duration liabilities that the Bank of Canada has issued to the public and that, therefore, does not appear in the numerator of the consolidated weighted average maturity), needs to be added to the present value of the GoC bond outstanding net of the Bank of Canada's total holdings of GoC bonds, which is $\sum_{n=0}^{N=120} \left[D_t^{(n)} - H_t^{(n)}\right] e^{-ny_t^{(n)}}$, resulting in a denominator that is equal to $\sum_{n=0}^{N=120} D_t^{(n)} e^{-ny_t^{(n)}}$.

These measures of the weighted average maturity of the outstanding GoC debt can move around due to changes in the yield curve (even if the maturity structure of the outstanding debt does not change). For this reason, we follow Greenwood et al. (2016) and compute our weighted average maturity measures based on a constant term structure of interest rates (in this case, the daily average term structure of interest rates from January 2018 to March 2024) to isolate the changes in the weighted average maturity of the outstanding GoC debt due to changes in the Bank of Canada holdings of GoC bonds from those due to changes in the yield curve.

In Figure 1, we display these three measures of the average maturity of the outstanding GoC marketable debt. The thick solid line represents the weighted average maturity of the outstanding GoC marketable debt. The thin solid line represents the average maturity of the outstanding GoC marketable debt that consolidates the Bank of Canada holdings of GoC debt due to the Bank of Canada's purchases of GoC bonds under normal course. Finally, the thin dashed line displays the average maturity of the outstanding GoC marketable debt that consolidates the Bank's total holdings of GoC debt. We note, again, that the difference between the weighted average maturity of the outstanding GoC debt consolidated with the Bank of Canada's holdings due to its normal course operations (thin solid line) and the total holdings isolates the effect of the GBPP (thin dashed line).

From January 2018 to early 2020, the weighted average maturity of the GoC outstanding debt remained stable at around 5.5 years. This period of stability is consistent with the GoC maintaining a consistent mix of short-term and long-term debt, balancing its debt maturity profile to effectively manage refinancing risks and low-cost funding needs. The Bank of Canada purchases of GoC bonds conducted under normal course lowered the average maturity of the GoC investor portfolio by approximately 0.6 years. The consistent gap between both lines underscores the regular and predictable impact of the BoC's normal course operations on the duration of GoC debt during the pre-pandemic period.

With the onset of the pandemic in March 2020, the weighted average maturity of the outstanding GoC debt fell by approximately 1 year from the pre-pandemic average of approximately 5 years, primarily due to the GoC's increased issuance of short-term Treasury Bills to rapidly raise liquidity and address immediate fiscal needs during the crisis. At this time, the Bank of Canada also increased the purchases of GoC Treasury Bills at auctions in the primary market to support a liquid and well-functioning short-term GoC borrowing.[8] As a result, the maturity removal due to the Bank of Canada's purchases of GoC bonds conducted under normal course remained close to their pre-pandemic levels (0.5 vs. 0.6 years) compared to the pre-pandemic average maturity of the outstanding GoC debt of 5.5 years.

As the pandemic progressed, the weighted average maturity of outstanding GoC debt gradually began to return to its pre-pandemic levels of around 5 years, reflecting the government's gradual shift toward issuing longer-term bonds once market conditions stabilized and the initial liquidity crunch eased. Similarly, the Bank of Canada's extraordinary actions pivoted from focusing on restoring market functioning to serving as a QE tool to provide additional monetary stimulus while the policy rate was constrained by the effective lower bound. Effectively, as the Bank of Canada ramped up its GBPP purchases starting in March 2020, the program significantly impacted the interest rate risk that private investors were exposed to.[9] By November 2021, coinciding with the peak of GoC bond holdings under the GBPP, the GBPP had lowered the weighted average maturity of the portfolio of the GoC investor by about 1.4 years. For comparison, the pre-pandemic LSAP programs implemented by the Fed had lowered the average maturity of the U.S. Treasury debt by 1.7 years by the end of their QE3 program (see Greenwood et al., 2016).

As the Bank of Canada started the process to normalize its balance sheet by first

[8] We note that these incremental purchases of GoC Treasury Bills from April 21, 2020, to November 24, 2020, are included in the Bank of Canada's holdings of GoC bonds acquired under normal course.

[9] Similarly, the Bank of Canada started to return to a more normal participation rate in the auction purchases of GoC bonds for its normal course in July 2020 and halted the incremental purchases of GoC Treasury Bills.

stopping net new purchases of GoC bonds in October 2021, when it moved from QE to the reinvestment phase, and then halting all purchases in April 2022, the GBPP maturity removal started diminishing from the peak of 1.4 years. By the end of March 2024, the impact of the GBPP on the average maturity of the GoC outstanding debt was still 0.87 years.[10]

We now present the details of a portfolio-balance model in the spirit of Greenwood and Vayanos (2014) and Vayanos and Vila (2021), which we use to calibrate the effects of lowering the weighted average maturity of the GoC marketable debt on the Canadian yield curve.

3 A Portfolio-Balance Model

In this section, we introduce the portfolio-balance model that we will use to quantify the portfolio-balance effects of the Bank of Canada GBPP. This model is a simplified version of Diez de los Rios (2024), where further details can be found.

3.1 State variables

We assume that the state of the economy is described by a $(M \times 1)$ vector of latent state variables (or pricing factors), $\mathbf{x}_t$. The dynamic evolution of the state variables under the physical measure, $\mathbb{P}$, is given by a Gaussian VAR(1) process:

$$\mathbf{x}_{t+1} = \mathbf{\Phi}_{x0} + \mathbf{\Phi}_{xx}\mathbf{x}_t + \boldsymbol{\varepsilon}_{x,t+1}, \tag{3.1}$$

where, to guarantee the pricing factors are stationary, the eigenvalues of $\mathbf{\Phi}_{xx}$ lie inside the unit circle in the complex plane. Further, let $\boldsymbol{\varepsilon}_{x,t+1} \sim iid\, N(0, \mathbf{\Sigma}_{xx})$, where $\mathbf{\Sigma}_{xx}$ is a symmetric positive definite matrix with unique Cholesky decomposition given by $\mathbf{\Sigma}_{xx}^{1/2}$.

[10] In addition, the maturity removal due to the GoC purchases under normal course has been steadily declining from 0.6 years in November 2021 to 0.4 years by March 2024, reflecting that the Bank of Canada also halted purchases in the primary market (i.e., under normal course) in April 2022.

3.2 Assets

Short-term nominal bond. Let $B_t^{(1)}$ be the nominal price (i.e., in dollars) at time t of a one-period zero-coupon nominal bond, $b_t^{(1)} \equiv \log[B_t^{(1)}]$ be its log price, and $r_t = -b_t^{(1)}$ be the short-term nominal interest rate in this economy, which we assume is affine in the set of state variables:

$$r_t = \delta_{x0} + \boldsymbol{\delta}_x' \mathbf{x}_t. \tag{3.2}$$

Long-term nominal bonds. In addition to the one-period nominal bond, we further assume that there is a set of n-period (default-free) nominal zero-coupon bonds with maturities $n = 2, \ldots, N$. Consistent with the notation for the one-period nominal bond, we denote the nominal price (i.e., in dollars) at time t of an n-period nominal zero-coupon bond that pays 1 dollar at date $t+n$ by $B_t^{(n)}$, its log price by $b_t^{(n)} \equiv \log B_t^{(n)}$, and its (log) yield by $y_t^{(n)} \equiv -b_t^{(n)}/n$. Specifically, let the gross (nominal) return on the n-period nominal zero-coupon bond be $R_{t+1}^{(n)} = \frac{B_{t+1}^{(n-1)}}{B_t^{(n)}}$ for $n = 1, \ldots, N$, where $B_{t+1}^{(0)} = 1$ given that a nominal one-period zero-coupon bonds pays 1 dollar at date $t+1$. Importantly, nominal long-term bonds are subject to interest rate risk because these assets suffer a capital loss if short-term nominal interest rates rise unexpectedly.

The log return on the n-period nominal zero-coupon bond is given by $r_{t+1}^{(n)} \equiv \log\left[R_{t+1}^{(n)}\right] = b_{t+1}^{(n-1)} - b_t^{(n)}$, for $n = 1, \ldots, N$, while its log excess return over the short-term nominal interest rate from time t to $t+1$ is

$$rx_{t+1}^{(n)} = b_{t+1}^{(n-1)} - b_t^{(n)} - r_t, \qquad \text{for } n = 2, \ldots, N. \tag{3.3}$$

Taking expectations on equation 3.3 yields a difference equation that can be iterated forward to obtain the following expression for the yield of the n-period nominal zero-coupon bond:

$$y_t^{(n)} = \underbrace{\left[\sum_{k=0}^{n-1} E_t\left(r_{t+k}\right)\right]}_{\text{Expectations component}} + \underbrace{\left[\sum_{k=0}^{n-1} E_t\left(rx_{t+k+1}^{(n-k)}\right)\right]}_{\text{Term premium}}. \tag{3.4}$$

The first term in equation 3.4 is the average path of expected future nominal short rates. The second term is, on the other hand, a (nominal) term premium. Since $rx_{t+1}^{(n)}$ is the log return from holding an n-period nominal zero-coupon bond in excess of investing in the nominal one-period bond, we have that the term premium component captures the (expected) additional return required by investors to hold the long-term nominal bond (which is exposed to interest-rate risk) as opposed to holding the short-term nominal bond (which is risk free). We advance that changes in the amount of interest-rate risk that financial market participants are exposed to, and consequently the compensation that they demand for bearing this risk, will affect bond yields by affecting the second term in equation 3.4.

3.3 Arbitrageurs

As in Greenwood and Vayanos (2014), we assume that nominal bonds are issued by a government and are traded by both (identical) arbitrageurs and other investors. We assume that arbitrageurs choose a portfolio of nominal bonds that maximize their expected utility over their nominal wealth. Specifically, we assume that arbitrageurs have power utility with a (constant) coefficient of relative risk aversion γ. Therefore, the arbitrageurs' portfolio choice problem can be expressed as

$$\max_{\left\{d_t^{(n)}\right\}_{n=1}^{N}} \frac{E_t W_{t+1}^{1-\gamma} - 1}{1-\gamma}, \tag{3.5}$$

where $d_t^{(n)}$ is the portfolio weight in the nominal n-period zero-coupon bond and W_{t+1} denotes the arbitrageurs' wealth at time $t+1$.

The arbitrageurs' wealth evolves across time according to the following budget constraint:

$$\frac{W_{t+1}}{W_t} = R_{t+1}^{(p)} = \sum_{n=2}^{N} d_t^{(n)} R_{t+1}^{(n)} + \left[1 - \sum_{n=2}^{N} d_t^{(n)}\right] R_{t+1}^{(r)}, \tag{3.6}$$

where $R_{t+1}^{(p)}$ is the gross return on the arbitrageurs' portfolio, and $R_{t+1}^{(r)} \equiv \exp(r_t)$ is the gross return from investing in the short-term nominal bond. Note that the portfolio weight invested in the nominal one-period bond is given by $d_t^{(1)} = 1 - \sum_{n=2}^{N} d_t^{(n)}$.

Following Campbell and Viceira (2001), we solve the arbitrageurs' portfolio choice problem by assuming that the gross return on the arbitrageurs' portfolio, $R_{t+1}^{(p)}$, is conditionally lognormal (an assumption that we verify below), which implies that the arbitrageurs' wealth at time $t+1$ is conditionally lognormal as well. Specifically, taking logs on both equations 3.5 and 3.6 and using the properties of a lognormal variable, we can rewrite the arbitrageurs' portfolio choice problem as

$$\max \underbrace{E_t r_{t+1}^{(p)} + \frac{1}{2}\sigma_{pt}^2}_{\log E_t\left[R_{t+1}^{(p)}\right]} - \gamma \sigma_{pt}^2, \tag{3.7}$$

where $r_{t+1}^{(p)} \equiv \log\left[R_{t+1}^{(p)}\right]$, and $\sigma_{pt}^2 = Var_t\left[r_{t+1}^{(p)}\right]$ is the conditional variance of the log portfolio return.

To link the log returns on the underlying assets to the log return on the portfolio, we follow Campbell and Viceira (2001) again. Specifically, using a second-order Taylor approximation of the portfolio return in equation 3.6, we obtain that the log return of the arbitrageurs' portfolio in excess of the log return from investing in the nominal short-term rate bond is

$$rx_{t+1}^{(p)} \equiv \log\left[\frac{R_{t+1}^{(p)}}{R_{t+1}^{(r)}}\right] \approx \mathbf{d}_t'\mathbf{rx}_{t+1} + \frac{1}{2}\mathbf{d}_t'\boldsymbol{\sigma}_t^2 - \frac{1}{2}\mathbf{d}_t'\boldsymbol{\Sigma}_t\mathbf{d}_t, \tag{3.8}$$

with $\mathbf{d}_t = \left[d_t^{(2)}, \ldots, d_t^{(N)}\right]$, $\mathbf{rx}_{t+1} = \left[rx_t^{(1)}, \ldots, rx_t^{(N)}\right]'$, $\boldsymbol{\Sigma}_t = Var_t(\mathbf{rx}_{t+1})$, and $\boldsymbol{\sigma}_t^2 = diag(\boldsymbol{\Sigma}_t) = \left[\sigma_{2,t}^2, \ldots, \sigma_{n,t}^2\right]$. Campbell and Viceira (2001) noted that this Taylor approximation is exact in continuous time, given that higher-order terms converge to zero over shorter and shorter time intervals.

Substituting equation 3.8 into equation 3.7 and taking derivatives with respect to the portfolio weights, $\mathbf{d}_t$, we arrive at the following first-order condition for the arbitrageurs' portfolio choice problem:

$$E_t\mathbf{rx}_{t+1} + \frac{1}{2}\boldsymbol{\sigma}_t^2 = \gamma\boldsymbol{\Sigma}_t\mathbf{d}_t \tag{3.9}$$

if expressed in matrix form and

$$E_t rx_{t+1}^{(n)} + \frac{1}{2} Var_t \left[rx_{t+1}^{(n)} \right] = \gamma \sum_{j=2}^{N} Cov_t \left[rx_{t+1}^{(n)}, rx_{t+1}^{(j)} \right] d_t^{(j)} \qquad \text{for } n = 2, \ldots, N \tag{3.10}$$

when specialized to the case of the excess return on the n-period zero-coupon bond. Note that this solution is equivalent to the multiple-asset mean-variance solution once we convert from log returns to simple returns. As in the case of the portfolio-balance models of Greenwood and Vayanos (2014), Greenwood et al. (2018), and Vayanos and Vila (2021), the arbitrageurs trade off mean against variance in the portfolio return. However, in our setup, we have that the relevant mean return for arbitrageurs with power utility is the mean simple return (similar to the setup in Campbell and Viceira, 2001).

3.4 Bond supply

We model the net supply of nominal long-term bonds available to the arbitrageurs as an affine function of the state variables, $\mathbf{x}_t$. Specifically, we assume that the value of the net supply at time t of a nominal bond with maturity n available to the arbitrageur is given by $s_t^{(n)} W_t$, where

$$s_t^{(n)} = s_{x0}^{(n)} + \mathbf{s}_x^{(n)\prime} \mathbf{x}_t, \qquad \text{for } n = 2, \ldots, N. \tag{3.11}$$

Importantly, our specification nests different cases, such as (i) a constant supply of nominal bonds; (ii) the case that the net supply of nominal bonds available to the arbitrageurs is exogenous, price-inelastic, and described by a one-factor model (see Greenwood and Vayanos, 2014), and (iii) the case that there is a preferred-habitat sector with demand functions that are linear and decreasing in the (log) price of the nominal bond (see Vayanos and Vila, 2021).

3.5 Equilibrium

As in Greenwood and Vayanos (2014), Greenwood et al. (2018), and Vayanos and Vila (2021), we solve for a rational expectations solution of the model. Specifically, we solve for the endogenous nominal bond price equilibrium process that is consistent with (i) the nominal short rate given by equation 3.2 and (ii) equilibrium in the nominal bond market (i.e., $d_t^{(n)} = s_t^{(n)}$ for all $n = 2, \ldots, N$).

Specifically, we conjecture that equilibrium log bond prices are affine functions of the state variables, $\mathbf{x}_t$:

$$b_t^{(n)} = -b_{x0}^{(n)} - \mathbf{b}_x^{(n)\prime}\mathbf{x}_t \qquad \text{for } n = 1, \ldots, N. \tag{3.12}$$

Consequently, we have that the zero-coupon bond yields will also be affine in the set state variables:

$$y_t^{(n)} = a_{x0}^{(n)} + \mathbf{a}_x^{(n)\prime}\mathbf{x}_t, \tag{3.13}$$

with $a_{x0}^{(n)} = b_{x0}^{(n)}/n$ and $\mathbf{a}_x^{(n)} = \mathbf{b}_x^{(n)}/n$.

3.5.1 Excess returns on nominal zero-coupon bonds

Substituting the guesses for the (log) bond prices in equation 3.12 into the expression for the excess returns equations for the nominal zero-coupon bonds in equation 3.3, we have that the log return from investing in the nominal n-period zero-coupon bond, for $n = 2, \ldots, N$, in excess of the nominal short-term rate satisfies:

$$rx_{t+1}^{(n)} = \left[b_{x0}^{(n)} - b_{x0}^{(n-1)} - \mathbf{b}_x^{(n-1)\prime}\boldsymbol{\Phi}_{x0} - \delta_{x0}\right] + \left[\mathbf{b}_x^{(n)\prime} - \mathbf{b}_x^{(n-1)\prime}\boldsymbol{\Phi}_{xx} - \boldsymbol{\delta}_x'\right]\mathbf{x}_t - \mathbf{b}_x^{(n-1)\prime}\boldsymbol{\varepsilon}_{x,t+1}. \tag{3.14}$$

Importantly, note that since $\boldsymbol{\varepsilon}_{x,t+1}$ is conditionally normally distributed, the log returns on the nominal bonds in excess of the real short-term interest rates are conditionally normal as well (cf. equation 3.14). Since, conditional on the information available at time t, the arbitrageurs' log portfolio return is a linear combination of the log excess returns on the nominal bonds (cf. equation 3.7), we have that the log portfolio return is also conditionally normally distributed.

3.5.2 Arbitrageurs' first-order condition

We now use the expression for the bond excess return in equation 3.14 to compute the (co)variance terms in equation 3.10 and thus solve the arbitrageurs' optimization problem:

Lemma 1. *The arbitrageurs' first-order condition implies that*

$$E_t rx_{t+1}^{(n)} + \frac{1}{2}\sigma_{n,t}^2 = -\mathbf{b}_x^{(n-1)\prime}\boldsymbol{\lambda}_{xt} \qquad \text{for } n = 2, \ldots, N, \tag{3.15}$$

where

$$\boldsymbol{\lambda}_{x,t} = \gamma\boldsymbol{\Sigma}_{xx}\sum_{j=2}^{N}\left[-\mathbf{b}_x^{(j-1)}d_t^{(j)}\right]. \tag{3.16}$$

Equation 3.15 implies that the expected excess return from investing in the nominal bonds, corrected by a convexity term, is equal to the (inner) product of the sensitivity of the excess returns on the nominal bonds, $\mathbf{b}_x^{(n-1)}$, and a price of risk term, $\boldsymbol{\lambda}_{x,t}$ that captures the expected excess return per unit of sensitivity demanded by arbitrageurs as compensation for being exposed to the shocks, $\boldsymbol{\varepsilon}_{x,t+1}$. Consistent with the absence of arbitrage, this compensation per unit of factor sensitivity is the same for all nominal bonds. Finally, we note that the price of risk $\boldsymbol{\lambda}_{x,t}$ depends on the overall sensitivity of the arbitrageurs' portfolio to that factor: $\sum_{j=2}^{N}\left[-\mathbf{b}_x^{(j-1)}d_t^{(j)}\right]$. Consequently, the riskier the arbitrageurs' portfolio is, the higher the compensation (per unit of factor sensitivity) they demand for holding such a portfolio.

3.5.3 Solution of the model

Similar to the models in Greenwood and Vayanos (2014) and Vayanos and Vila (2021), the assumption of the absence of arbitrage does not impose restrictions on the prices of risk. Once more, we will follow these authors and determine the prices of risk that are consistent with market clearing in the bond markets. Specifically, we have that, in equilibrium,

$$d_t^{(n)} = s_t^{(n)} \qquad \text{for } n = 2, \ldots, N. \tag{3.17}$$

Substituting $rx_{t+1}^{(n)}$ from equation 3.14 and the market-clearing conditions in equation 3.17 into the first-order condition of the arbitrageurs' portfolio choice problem in equation 3.15, we find a set of N affine equations in $\mathbf{x}_t$. Further, by setting the constant terms and linear terms in $\mathbf{x}_t$ to zero, we can find a set of difference equations defining the equilibrium bond loadings $\left\{b_{x0}^{(n)}, \mathbf{b}_x^{(n)\prime}\right\}_{n=1}^{N}$. We will collect the results in the next theorem.

Theorem 1. *The equilibrium bond price loadings,* $\left\{b_{x0}^{(n)}, \mathbf{b}_x^{(n)\prime}\right\}_{n=1}^{N}$, *satisfy the following set of difference equations:*

$$\mathbf{b}_x^{(n)\prime} = \mathbf{b}_x^{(n-1)\prime}\mathbf{\Phi}_{xx}^{\mathbb{Q}} + \boldsymbol{\delta}_x^{\prime}, \tag{3.18}$$

$$b_{x0}^{(n)} = b_{x0}^{(n-1)} + \mathbf{b}_x^{(n-1)\prime}\mathbf{\Phi}_{x0}^{\mathbb{Q}} - \frac{1}{2}\mathbf{b}_x^{(n-1)\prime}\mathbf{\Sigma}_{xx}\mathbf{b}_x^{(n-1)} + \delta_{x0}, \tag{3.19}$$

with initial conditions given by $b_{x0}^{(1)} = \delta_{x0}$, *and* $\mathbf{b}_x^{(1)} = \boldsymbol{\delta}_x$, *and where* $\mathbf{\Phi}_{x0}^{\mathbb{Q}}$ *and* $\mathbf{\Phi}_{xx}^{\mathbb{Q}}$ *satisfy*

$$\mathbf{\Phi}_{x0}^{\mathbb{Q}} = \mathbf{\Phi}_{x0} + \gamma\mathbf{\Sigma}_{xx}\sum_{j=2}^{N}\mathbf{b}_x^{(j-1)}s_{x0}^{(j)}, \tag{3.20}$$

$$\mathbf{\Phi}_{xx}^{\mathbb{Q}} = \mathbf{\Phi}_{xx} + \gamma\mathbf{\Sigma}_{xx}\sum_{j=2}^{N}\mathbf{b}_x^{(j-1)}\mathbf{s}_x^{(j)\prime}, \tag{3.21}$$

Equation 3.21 has a solution if γ *is sufficiently close to zero.*

Notice that the solution to the equilibrium bond loadings in equation 3.18 requires solving a fixed-point problem. On one hand, we have that the nominal bond loadings $\left\{\mathbf{b}_x^{(n)}\right\}_{n=1}^{N}$ in equation 3.18 depend on $\mathbf{\Phi}_{xx}^{\mathbb{Q}}$. However, the matrix $\mathbf{\Phi}_{xx}^{\mathbb{Q}}$ in equation 3.21 depends on the nominal bond loadings through the term $\left\{\mathbf{b}_x^{(n)}\right\}_{n=1}^{N}$. This happens because, when arbitrageurs are risk averse ($\gamma \neq 0$), the equilibrium price of risk, $\boldsymbol{\lambda}_{x,t}$, depends on the overall sensitivity of the arbitrageurs' portfolio to the state variables, $\sum_{j=2}^{N}\left[-\mathbf{b}_x^{(j-1)}s_t^{(j)}\right]$, which depends on the bond factor loadings themselves.

To discuss the existence of a solution to this fixed-point problem in equation 3.21, we note that when γ approaches zero, a continuity argument suggests that the mapping for the fixed-point problem becomes a contraction as both sides of this equation converge to $\mathbf{\Phi}_{xx}$ when $\gamma \to 0$. Therefore, applying the Banach contraction mapping theorem, we have that equation 3.21 admits a unique solution when γ is sufficiently close to zero.

Similar to Hamilton and Wu (2012), who demonstrated that portfolio-balance models might serve as a foundation for the affine price of risk specification within GDTSMs, we have that both equation 3.18 and equation 3.19 coincide with the standard zero-coupon bond pricing recursions in a standard GDTSM in which the dynamic evolution of the state variables under the risk-neutral measure, $\mathbb{Q}$, is given by the following Gaussian VAR(1) process:

$$\mathbf{x}_{t+1} = \mathbf{\Phi}^{\mathbb{Q}}_{x0} + \mathbf{\Phi}^{\mathbb{Q}}_{xx}\mathbf{x}_t + \boldsymbol{\varepsilon}^{\mathbb{Q}}_{x,t+1}, \tag{3.22}$$

where $\boldsymbol{\varepsilon}^{\mathbb{Q}}_{x,t+1} \sim iid\, N(0, \mathbf{\Sigma}_{xx})$.

In the next section, we exploit this equivalence between portfolio-balance models and GDTSMs to calibrate our model and quantify the importance of the portfolio-balance effects of the Bank of Canada's GBPP on the Canadian yield curve.

4 Policy Analysis and Calibration

We now show how to analyze the effects of central bank bond purchase policies within our portfolio-balance model. Specifically, we follow the approach outlined in Vayanos and Vila (2021) and, consistent with the Bank of Canada purchases under the GBPP, assume that central bank purchases only concern government bonds.

As in Vayanos and Vila (2021), we model central bank purchases of nominal bonds as a decrease $\Delta s^{(n)}_{x0}$ in the intercept of the (relative) amount of bonds with maturity n supplied to the arbitrageurs at time t (see equation 3.11). Furthermore, we assume that this increase (i) is unanticipated, (ii) takes place at time zero, (iii) can be well approximated by a one-factor model, and (iv) reverts deterministically to zero at a rate of ϕ_β:

$$\Delta s^{(n)}_{x0} = s^{(n)}_{\beta}\Delta\overline{\beta}\phi^{t}_{\beta}, \qquad \text{for } n = 1, \ldots, N, \tag{4.1}$$

where the coefficient $s^{(n)}_{\beta}$ measures the sensitivity of the supply to changes in the (deterministic) factor $\overline{\beta}$.

The following proposition collects the impact of such an increase on bond prices.

Proposition 1. *The impact on the nominal bond prices due to the change in the supply* *of government bonds given by equation 4.1 satisfies*

$$\Delta b_t^{(n)} = -b_\beta^{(n)} \Delta\overline{\beta}\phi_\beta^t, \tag{4.2}$$

where

$$b_\beta^{(n)} = b_\beta^{(n-1)}\phi_\beta + \gamma \mathbf{b}_x^{(n-1)\prime} \mathbf{\Sigma}_{xx} \sum_{j=1}^{N} \mathbf{b}_x^{(j-1)} s_\beta^{(j)}, \tag{4.3}$$

where the recursion is initialized with $b_\beta^{(1)} = 0$.

Equation 4.3 allows us to quantitatively assess the importance of the portfolio-balance effects of the Bank of Canada's GBPP on nominal yields by proceeding as follows.

First, we exploit the relationship between portfolio-balance models and GDTSMs established above to obtain estimates of $\mathbf{\Sigma}_{xx}$ and $\mathbf{b}_x^{(n)}$ for $n = 1, \ldots, N$, by estimating a three-factor GDTSM by maximum likelihood using the approach outlined in Joslin et al. (2011).

Second, we assume that the persistence of the Bank of Canada's government bond purchases is similar to that of the Federal Reserve Board's large-scale asset purchase programs. This assumption implicitly captures that, at the time, the economic effects of the pandemic were expected to last (at least) as long as the effects of the financial crisis of 2007–08. In particular, we follow King (2019) and set $\phi_\beta = 0.9608$ to match a half-life of the quantitative easing shocks of approximately 4.5 years on average, as in Carpenter et al. (2015). To check the robustness of our results, we also consider the following values, $\phi_\beta = 0.9753$ (half-life of 6.9 years) and $\phi_\beta = 0.9512$ (half-life of 3.5 years), which coincide with the persistence of the quantitative easing shocks used in Vayanos and Vila (2021).

Third, we assume that the sequence $\left\{ s_\beta^{(n)} \right\}_{n=1}^{N}$ satisfies

$$s_\beta^{(n)} = 1 - \frac{2n}{N+1}. \tag{4.4}$$

Since $\sum_{n=1}^{N} s_\beta^{(n)} = 0$, we have that this specification for $s_\beta^{(n)}$ implies that central bank bond purchases reduce the amount of long-term bonds and increase the amount of short-term bonds in equal measures. That is, changes in $\overline{\beta}$ do not alter the total value of bonds available to the arbitrageurs but affect only the weighted average maturity of their portfolio.[11] Importantly, since only the weighted sum $\sum_{j=1}^{N} \mathbf{b}_x^{(j-1)} s_\beta^{(j)}$ enters in equation 4.3, the exact specification of the individual supply factor loadings, $s_\beta^{(n)}$, is not of first-order importance for the pricing of bonds as long as it can capture the variation in the weighted average maturity of the portfolio that the arbitrageur needs to hold (see King, 2019, for further discussion).

Specifically, equation 4.4 implies that the impact on the weighted average maturity, ΔWAM_t, due to the change in the supply of government bonds given by equation 4.1 is

$$\Delta\text{WAM}_t = -\Delta\overline{\beta}\phi_\beta^t \sum_{j=1}^{N} j s_\beta^{(j)}. \tag{4.5}$$

We thus calibrate $\Delta\overline{\beta}$ to ensure that the effect at time zero of the supply shock in equation 4.1 decreases the weighted average maturity of the portfolio of bonds that arbitrageurs are required to hold by 1.4 years to match the decline in the weighted average maturity of the GoC debt outstanding resulting from the Bank of Canada's GBPP, as described in Section 2 above.

Finally, we set the arbitrageurs' relative risk aversion coefficient to $\gamma = 20$, which, as shown in Diez de los Rios (2024), is needed for portfolio-balance models to match the estimated impacts of the Federal Reserve's pre-pandemic LSAP announcements.[12]

We now discuss in more detail the estimation of the GDSTM underlying the calibration of our portfolio-balance model.

[11] This specification for $s_\beta^{(n)}$ can be viewed as the discrete-time version of the functional forms for the sensitivity of the bond supply to the supply factor in Greenwood et al. (2015) and King (2019).

[12] Specifically, Diez de los Rios (2024) follows Greenwood et al. (2015) in assuming that the total price impact on 10-year yields for all the Federal Reserve's pre-pandemic LSAP announcements is 1.50%, an assumption that can be justified by (i) the fact that the cumulative reduction of 10-year bond equivalents available to investors due to the LSAP programs implemented between late 2008 and mid-2014 was roughly $3 trillion (see Greenwood et al., 2016) and (ii) the estimate that an LSAP announcement of a $500 billion purchase of 10-year bond equivalents reduces 10-year yields by 25 bps (Williams, 2014). Therefore, the author calibrates the value of γ such that the effect of a 1.7-year decrease in the weighted average maturity of the portfolio of U.S. Treasury government bonds (i.e., the duration removed by the Fed's pre-pandemic LSAP programs) is 1.50%, which yields $\gamma = 20$.

4.1 Estimation of GDTSM

To calibrate our model, we start by estimating an $M = 3$ factor GDTSM by maximum likelihood (ML) using the approach of Joslin et al. (2011). Importantly, we follow these authors in adopting their canonical specification to guarantee the identification of the model by assuming that (i) $\delta_{x0} = 0$; (ii) $\boldsymbol{\delta}_x = \mathbf{1}_M$, where $\mathbf{1}_M$ is $(M \times 1)$ vector of ones; (iii) $\boldsymbol{\Phi}^{\mathbb{Q}}_{x0,1} = k^{\mathbb{Q}}_{\infty}$ and $\boldsymbol{\Phi}^{\mathbb{Q}}_{x0,j} = 0$ for $j = 2, \ldots, M$; and (iv) $\boldsymbol{\Phi}^{\mathbb{Q}}_{xx}$ is a diagonal matrix whose non-zero elements are collected in the vector $\boldsymbol{\phi}^{\mathbb{Q}}_{x}$. Specific details on the estimation approach can be found in the appendix.

Our data set consists of end-of-quarter observations from March 1986 (1986Q1) to December 2023 (2023Q4) of Canadian zero-coupon bond yields for maturities of three and six months and one, two, three, five, seven, and ten years, obtained from the Bank of Canada website.

Table 1 summarizes the results from the ML estimation of the GDTSM. In panel a, we present the results of estimating the VAR dynamics for the state variables in equation 3.1. We note that most of the parameters describing the physical dynamics of the factors are statistically different from zero, and the fact that this is the case for most of the off-diagonal elements of $\boldsymbol{\Phi}_{xx}$ and $\boldsymbol{\Sigma}^{1/2}_{xx}$ suggests the importance of modelling the interdependencies and feedback mechanisms between different factors that influence the term structure of interest rates.

Panel b presents estimates of the parameters driving the dynamics of the state variables under the risk-neutral measure and inflation. In line with previous results in the literature, the dynamics of the state variables under the risk-neutral measure are more persistent than under the physical measure, in that the eigenvalues of $\boldsymbol{\Phi}^{\mathbb{Q}}_{xx}$, captured by the diagonal elements of this matrix, are larger than the eigenvalues of $\boldsymbol{\Phi}_{xx}$. The largest eigenvalue of $\boldsymbol{\Phi}^{\mathbb{Q}}_{xx}$ is very close to 1 (0.9946), a feature needed to replicate the level factor that characterizes the term structure of GoC bond yields. The diagonal elements of $\boldsymbol{\Phi}^{\mathbb{Q}}_{xx}$ are all very precisely estimated due to the cross-equation restrictions imposed by the no-arbitrage conditions on the cross-section of interest rates. Finally, we note that we fit the cross-section of bond yields with a root mean square pricing error (RMSPE) of less than 10 bps.

4.2 Policy analysis

Figure 2 shows the effect of a decrease in the weighted average maturity of GoC debt by 1.4 years on the nominal yield curve at time zero for three different values of the persistence of the QE shock. Under the baseline case (QE shock with a half-life of 4.5 years), our model suggests that the portfolio-balance effects of the Bank of Canada's GBPP on the 10-year yield were 84 basis points.

We find that the portfolio-balance effects of a QE shock increase in the bond's maturity in that these effects are more pronounced on longer-term yields: Longer-term bond prices are more sensitive to changes in interest rates, so investors require greater compensation for holding these bonds. For example, the portfolio-balance effects of the Bank of Canada's GBPP on the two-year yield were 20 basis points. In comparison, the effects on the five-year and 10-year yields were 52 and 84 basis points, respectively.

Interestingly, this effect is nonlinear because investors are forward-looking and anticipate that since the QE shock is temporary, its effects will wane over the life of the bond. Consequently, they require less additional compensation for holding bonds with increasing maturities.

As mentioned above, we also check the robustness of our results to different values of the persistence of the QE shock by also analyzing the effects of the GBPP on Canadian yields for the two values of the persistence of the QE program used in the calibration exercise in Vayanos and Vila (2021). In this case, we find a more persistent program (half-life of 6.9 years) increases the impact on the 10-year (five-year) yield to 98 (56) basis points, while a less persistent program (half-life of 3.5 years) reduces the impact of the GBPP on the 10-year (five-year) yield to 77 (50) basis points. Importantly, as the persistence of the QE shock increases, the relationship between the portfolio-balance effect on yields and the bond's maturity becomes more linear.

Finally, to provide a sense of the persistence of the effects of the Bank of Canada's GBPP on Canadian yields, we also compute the effect of the 0.85-year reduction in the average maturity on the outstanding GoC debt, which corresponds to the average maturity reduction due to the Bank of Canada's holdings of GoC debt purchased under the GBPP as of the end of March 2024. In this case, our model suggests that the portfolio-balance impact of a 0.85-year reduction in the average maturity of the outstanding GoC debt is 51 (32) basis points on the 10-year (five-year) yield (approximately 60% of the peak impact).

5 Final remarks

We propose a novel dynamic portfolio-balance model of the term structure of interest for Canada to evaluate the portfolio-balance effects of the Bank of Canada's Government Bond Purchase Program.

Specifically, we quantify the reduction in interest-rate risk implied by the Bank of Canada's GBPP by calculating the weighted average maturity of the GoC's marketable debt, with and without considering the consolidation of the Bank of Canada's holdings of GoC debt purchased under the GBPP. Our calculations suggest that, at its peak in November 2021, the GBPP had lowered the weighted average maturity of the GoC investor's portfolio by about 1.4 years (or 5.8 quarters). Further, by mapping the effect of removing 1.4 years of maturity from the GoC debt market, our model suggests that the Bank of Canada's GBPP successfully put downward pressure on Canadian longer-term bond yields by reducing the term premium component of Canadian 10-year (five-year) zero-coupon yields by 84 (52) basis points.

Our estimates of the GBPP's portfolio-balance effects are based on a closed-economy model. However, as noted by Kabaca (2016) and Diez de los Rios and Shamloo (2017), the transmission of QE shock may be different in small open economies such as Canada. For this reason, it would be interesting to extend our framework to let arbitrageurs invest in bonds denominated in different currencies. Along these lines, Greenwood et al. (2023) and Gourinchas et al. (2024) present two-country portfolio-balance models of the term structure of interest rates that could be used to explore the portfolio-balance effects of QE shocks in small open economies. We leave such an exercise for further research.

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Appendix

A Proofs

A.1 Proof of Theorem 1

Note that equation 3.14 implies that the conditional expectation of the log return from investing in the nominal n-period zero-coupon bond in excess of the nominal short-term rate satisfies

$$E_t rx_{t+1}^{(n)} = \left[b_{x0}^{(n)} - b_{x0}^{(n-1)} - \mathbf{b}_x^{(n-1)\prime}\mathbf{\Phi}_{x0} - \delta_{x0}\right] + \left[\mathbf{b}_x^{(n)\prime} - \mathbf{b}_x^{(n-1)\prime}\mathbf{\Phi}_{xx} - \boldsymbol{\delta}_x^{\prime}\right]\mathbf{x}_t, \tag{A.1}$$

while the conditional covariance between the excess returns from investing in two nominal bonds with different maturities n and j is given by

$$Cov_t\left[rx_{t+1}^{(n)}, rx_{t+1}^{(j)}\right] = \mathbf{b}_x^{(n-1)\prime}\mathbf{\Sigma}_{xx}\mathbf{b}_x^{(j-1)}. \tag{A.2}$$

Consequently, we have that the left-hand side (LHS) of the arbitrageurs' FOC in equation 3.10 is

$$\begin{aligned} E_t rx_{t+1}^{(n)} + \frac{1}{2}Var_t\left[rx_{t+1}^{(n)}\right] = & \left[b_{x0}^{(n)} - b_{x0}^{(n-1)} - \mathbf{b}_x^{(n-1)\prime}\mathbf{\Phi}_{x0} - \delta_{x0}\right] + \frac{1}{2}\mathbf{b}_x^{(n-1)\prime}\mathbf{\Sigma}_{xx}\mathbf{b}_x^{(n-1)} \\ & + \left[\mathbf{b}_x^{(n)\prime} - \mathbf{b}_x^{(n-1)\prime}\mathbf{\Phi}_{xx} - \boldsymbol{\delta}_x^{\prime}\right]\mathbf{x}_t. \end{aligned} \tag{A.3}$$

On the other hand, the right-hand side (RHS) of equation 3.10 evaluated at the bond market-clearing conditions in equation 3.17 is equal to

$$\gamma\sum_{j=2}^{N} Cov_t\left[rx_{t+1}^{(n)}, rx_{t+1}^{(j)}\right]s_t^{(j)} = \gamma\mathbf{b}_x^{(n-1)\prime}\mathbf{\Sigma}_{xx}\sum_{j=2}^{N}\mathbf{b}_x^{(j-1)}\left[s_{x0}^{(j)} + \mathbf{s}_x^{(j)\prime}\mathbf{x}_t\right]. \tag{A.4}$$

Collecting terms for $\mathbf{x}_t$ in equation A.3 and matching coefficients with equation A.4, we arrive at the following recursions for the loadings of the equilibrium (log) prices of the nominal bonds:

$$\mathbf{b}_x^{(n)\prime} = \mathbf{b}_x^{(n-1)\prime}\mathbf{\Phi}_{xx}^{\mathbb{Q}} + \boldsymbol{\delta}_x^{\prime}, \tag{A.5}$$

where $\mathbf{\Phi}_{xx}^{\mathbb{Q}} = \mathbf{\Phi}_{xx} + \gamma\mathbf{\Sigma}_{xx}\sum_{j=2}^{N}\mathbf{b}_x^{(j-1)}\mathbf{s}_x^{(j)\prime}$, as defined in equation 3.21 in the main text.

Similarly, by collecting and matching terms for the constant on both equation A.3 and equation A.4, we arrive at the following recursion for the constant of the equilibrium (log) prices of the nominal bonds:

$$b_{x0}^{(n)} = b_{x0}^{(n-1)} + \mathbf{b}_x^{(n-1)\prime} \mathbf{\Phi}_{x0}^{\mathbb{Q}} - \frac{1}{2} \mathbf{b}_x^{(n-1)\prime} \mathbf{\Sigma}_{xx} \mathbf{b}_x^{(n-1)} + \delta_{x0}, \tag{A.6}$$

where $\mathbf{\Phi}_{x0}^{\mathbb{Q}} = \mathbf{\Phi}_{x0} + \gamma \mathbf{\Sigma}_{xx} \sum_{j=2}^{N} \mathbf{b}_x^{(j-1)} s_{x0}^{(j)}$, as defined in equation 3.20 in the main text.

A.2 Proof of Proposition 1

To consider the effects of an unanticipated change $\Delta s_{x0}^{(n)}$ in the intercept of the supply of the nominal n-period bond available to the arbitrageur, as given in equation 4.1, we start by conjecturing that nominal bond prices are affine in the pricing factors $\mathbf{x}_t$ and the term $\Delta\overline{\beta}\phi_{\beta}^{t}$:

$$b_t^{(n)} = -b_{x0}^{(n)} - \mathbf{b}_x^{(n)\prime} \mathbf{x}_t - b_{\beta}^{(n)} \Delta\overline{\beta}\phi_{\beta}^{t} \qquad \text{for } n = 1, \ldots, N. \tag{A.7}$$

Following the same steps as in the proof of Theorem 1, we find that the coefficients for the nominal bond price coefficients for the constant and $\mathbf{x}_t$ remain unchanged with respect to the general case. However, the nominal bond price coefficients for the QE factor satisfy

$$b_{\beta}^{(n)} = b_{\beta}^{(n-1)} \phi_{\beta} + \gamma \mathbf{b}_x^{(n-1)\prime} \mathbf{\Sigma}_{xx} \sum_{j=2}^{N} \mathbf{b}_x^{(j-1)} s_{\beta}^{(n)}. \tag{A.8}$$

B Estimation details

We now provide additional details on the estimation of the GDTSM underlying our analysis of the portfolio-balance effects of the Bank of Canada's GBPP.

Specifically, it is particularly useful to resort to its state-space representation of the observed variables (i.e., the nominal bond yields). In a general state-space representation, there is a transition equation that describes the dynamic evolution of the state factors over time and a measurement equation that relates the observed data to the state factor. In our case, the VAR dynamics in equation 3.1 can be interpreted as the transition equation, while the conjectures for the bond yields in equation 3.13, respectively, are the measurement equations linking the observed data to the underlying state factor.

Let us denote $\widetilde{y}_t^{(n)}$ as the observed yields, which we assume are subject to measurement error. The vector $\mathbf{y}_t = \left[y_t^{(1)}, ...y_t^{(N)}\right]'$, on the other hand, comprises the model-implied yields that stack the affine mapping in equation 3.13 for the maturities used in the estimation of the model. Correspondingly, $\widetilde{\mathbf{y}}_t$ represents the vector of observed yields. Let $\boldsymbol{\eta}_{y,t}$ be a zero-mean bond yield measurement error that is iid across time, independent of the innovations to the pricing factor, $\boldsymbol{\varepsilon}_{x,t}$, and that has a covariance matrix $\boldsymbol{\Sigma}_{\eta_y\eta_y}$.

Given this notation, the measurement equations of the GDTSM state-space representation satisfy

$$\widetilde{\mathbf{y}}_t = \mathbf{a}_{x0} + \mathbf{a}_x\mathbf{x}_t + \boldsymbol{\eta}_{y,t}, \tag{B.1}$$

where, given the canonical representation of our GDTSM adopted in the paper, $\mathbf{a}_x$ is a nonlinear function of $\boldsymbol{\phi}_x^{\mathbb{Q}}$ (the eigenvalues of $\boldsymbol{\Phi}_{xx}^{\mathbb{Q}}$), and $\mathbf{a}_{x0}$ and is a nonlinear function of $k_\infty^{\mathbb{Q}}$, $\boldsymbol{\phi}_x^{\mathbb{Q}}$, and $\boldsymbol{\Sigma}_{xx}$. The transition equation is, on the other hand, given by

$$\mathbf{x}_{t+1} = \boldsymbol{\Phi}_{x0} + \boldsymbol{\Phi}_{xx}\mathbf{x}_t + \boldsymbol{\varepsilon}_{x,t+1}. \tag{B.2}$$

Given the latent nature of the pricing factors, estimation could, in principle, be achieved via Kalman filtering. Instead, we follow Joslin et al. (2011) in (i) working with bond state variables that are linear combinations (i.e., portfolios) of the yields themselves, $\mathbf{f}_t = \mathbf{P}'\widetilde{\mathbf{y}}_t$, where $\mathbf{P}$ is a full-rank matrix of weights, and in (ii) further assuming that $\mathbf{f}_t$ is observed perfectly, i.e., $\mathbf{P}'\boldsymbol{\eta}_{y,t} = \mathbf{0}$.

These assumptions allow us to rotate our set of latent factors, $\mathbf{x}_t$, into a set of observed factors and, in turn, factorize the joint likelihood function of the observed variables into the marginal component of the (observed) linear combination of yields, $\mathbf{f}_t$, and the conditional components corresponding to all the individual yields and inflation, given $\mathbf{f}_t$. Specifically, we have that the vector of observable pricing factors, $\mathbf{f}_t$, is simply an affine (invariant) transformation of the original latent factors, $\mathbf{x}_t$:

$$\mathbf{f}_t = \mathbf{P}'\widetilde{\mathbf{y}}_t = \mathbf{P}'\mathbf{a}_{x0} + (\mathbf{P}'\mathbf{a}_x)\mathbf{x}_t + \mathbf{P}'\boldsymbol{\eta}_{y,t} = \mathbf{c} + \mathbf{D}\mathbf{x}_t, \tag{B.3}$$

where $\mathbf{c} = \mathbf{P}'\mathbf{a}_{x0}$ and $\mathbf{D} = \mathbf{P}'\mathbf{a}_x$.

Consequently, by applying the properties of affine transformations for GDTSMs,[13] we have that our model is observationally equivalent to an alternative state-space model with measurement equations given by

$$\mathbf{f}_t^{\perp} = \mathbf{a}_{f0}^{\perp} + \mathbf{a}_f^{\perp}\mathbf{f}_t + \boldsymbol{\eta}_{y,t}, \tag{B.4}$$

and transition equation given by

$$\mathbf{f}_{t+1} = \boldsymbol{\Phi}_{f0} + \boldsymbol{\Phi}_{ff}\mathbf{f}_t + \boldsymbol{\varepsilon}_{f,t+1}, \tag{B.5}$$

where $\mathbf{f}_t^{\perp} = \mathbf{P}'_{\perp}\widetilde{\mathbf{y}}_t$, $\mathbf{P}'_{\perp}$ is a basis for the orthogonal component of the row span of $\mathbf{P}'$, $\boldsymbol{\varepsilon}_{f,t+1} \sim iid\, N(0, \boldsymbol{\Sigma}_{ff})$, and:

$$\boldsymbol{\Phi}_{ff} = \mathbf{D}\boldsymbol{\Phi}_{xx}\mathbf{D}^{-1}, \tag{B.6}$$

$$\boldsymbol{\Phi}_{f0} = (\mathbf{I} - \mathbf{D}\boldsymbol{\Phi}_{xx}\mathbf{D}^{-1})\mathbf{c} + \mathbf{D}\boldsymbol{\Phi}_{x0}, \tag{B.7}$$

$$\boldsymbol{\Sigma}_{ff} = \mathbf{D}\boldsymbol{\Sigma}_{xx}\mathbf{D}', \tag{B.8}$$

$$\mathbf{a}_f^{\perp} = \mathbf{P}'_{\perp}\mathbf{a}_x\mathbf{D}^{-1}, \tag{B.9}$$

$$\mathbf{a}_{f0}^{\perp} = \mathbf{P}'_{\perp}\left(\mathbf{a}_{x0} - \mathbf{a}_x\mathbf{D}^{-1}\mathbf{c}\right). \tag{B.10}$$

Note that focusing on $\mathbf{f}_t^{\perp} = \mathbf{P}'_{\perp}\widetilde{\mathbf{y}}_t$ in the measurement equation of this (rotated) model eliminates the redundant measurement equations for $\mathbf{f}_t$, given that these are perfectly observed.

Importantly, and despite of this rotation into observable pricing factors, $\mathbf{a}_f^{\perp}$ remains a nonlinear function of $\boldsymbol{\phi}_x^{\mathbb{Q}}$ only, while $\mathbf{a}_{f0}^{\perp}$ is a nonlinear function of $k_{\infty}^{\mathbb{Q}}$, $\boldsymbol{\phi}_x^{\mathbb{Q}}$, and $\boldsymbol{\Sigma}_{xx}$. On the other hand, neither of the coefficients of the conditional mean of the (observable)

[13] See Dai and Singleton (2000) for further details.

factors, $\mathbf{\Phi}_{f0}$ and $\mathbf{\Phi}_{ff}$, enter into the functional forms of the cross-sectional bond yield parameters. Given this separation between cross-sectional and physical dynamics, and given that the VAR dynamics remain unrestricted, Joslin et al. (2011) propose a two-step maximum likelihood estimator of the parameters of the GDTSM.

In the first step, Joslin et al. (2011) propose to estimate $\mathbf{\Phi}_{f0}$ and $\mathbf{\Phi}_{ff}$ by OLS given that, since the VAR dynamics are unrestricted, OLS recovers the ML estimates of the conditional mean parameters (Zellner, 1962).

Then, in the second step, Joslin et al. (2011) suggest estimating the remaining parameters of the model ($k_{\infty}^{\mathbb{Q}}$, $\boldsymbol{\phi}_{x}^{\mathbb{Q}}$, and $\mathbf{\Sigma}_{xx}$) via numerical maximization of the likelihood function, taking as given the estimates obtained in the first step.[14]

Finally, we recover the parameters driving the conditional mean of latent factors, $\mathbf{\Phi}_{x0}$ and $\mathbf{\Phi}_{xx}$, by undoing the affine transformations above.

[14] We further assume that $\mathbf{\Sigma}_{\eta_y \eta_y} = \sigma_{\eta_y}^2 \times \mathbf{P}_{\perp} \mathbf{P}'_{\perp}$. This guarantees that $\mathbf{P}'_{\perp} \mathbf{\Sigma}_{\eta_y \eta_y} \mathbf{P}_{\perp} = \mathbf{0}$ and allows us to concentrate $\sigma_{\eta_y}^2$ from the likelihood function through $\hat{\sigma}_{\eta_y}^2 = \sum_{t=1}^{T} \sum_{n} \left[\widetilde{y}_t^{(n)} - y_t^{(n)} \right]^2 / (T \times (N - M))$ where T is the length of the sample, N is the number of bonds used for the estimation, and M is the number of factors.

Table 1
ML estimates of GDTSM

Panel a: Physical dynamics of the state variables

$100 \times \mathbf{\Phi}_{x0}$	$\mathbf{\Phi}_{xx}$			$100 \times \mathbf{\Sigma}_{xx}^{1/2}$		
0.0170	0.9882***	0.0556**	0.0655	0.1357***	0	0
(0.0187)	(0.0144)	(0.0345)	(0.0411)	(0.0079)		
0.0691	-0.0735*	0.8758***	0.1689	-0.0418	0.3759***	0
(0.0531)	(0.0409)	(0.0750)	(0.1154)	(0.0314)	(0.0427)	
-0.0751	0.0710**	0.0184	0.6015***	-0.0474*	-0.2926***	0.1312***
(0.0458)	(0.0354)	(0.0654)	(0.0983)	(0.0265)	(0.0450)	(0.0077)

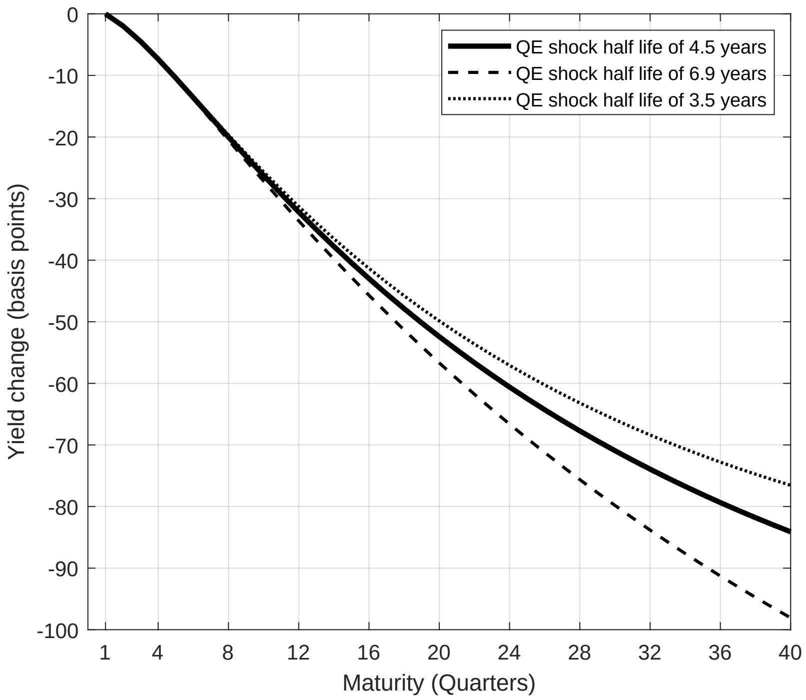
Panel b: Risk-neutral dynamics of the state variables

$100 \times \mathbf{\Phi}_{x0}^{\mathbb{Q}}$	$\mathbf{\Phi}_{xx}^{\mathbb{Q}}$		
0.0189***	0.9946***	0	0
(0.0009)	(0.0005)		
0	0	0.8482***	0
		(0.0081)	
0	0	0	0.6475***
			(0.0202)

Maximum likelihood estimates for a sample period of 1986Q1 to 2023Q4. The dynamics are given by $\mathbf{x}_{t+1} = \mathbf{\Phi}_{x0} + \mathbf{\Phi}_{xx}\mathbf{x}_t + \boldsymbol{\varepsilon}_{x,t+1}$, where $\boldsymbol{\varepsilon}_{x,t+1} \sim iidN(0, \mathbf{\Sigma}_{xx})$. Bond (log) prices satisfy $b_t^{(n)} = -b_{x0}^{(n)} - \mathbf{b}_x^{(n)'}\mathbf{x}_t$ where $\mathbf{b}_x^{(n)'} = \mathbf{b}_x^{(n-1)'}\mathbf{\Phi}_{xx}^{\mathbb{Q}} + \mathbf{b}_x^{(1)'}$, $b_{x0}^{(n)} = b_{x0}^{(n-1)} + b_{x0}^{(1)} + \mathbf{b}_x^{(n-1)'}\mathbf{\Phi}_{x0}^{\mathbb{Q}} - \frac{1}{2}\mathbf{b}_x^{(n-1)'}\mathbf{\Sigma}_{xx}\mathbf{b}_x^{(n-1)}$, and initial conditions given by $\mathbf{b}_x^{(1)} = \mathbf{1}_M$ and $b_0^{(1)} = 0$. The eigenvalues of $\mathbf{\Phi}_{xx}$ are 0.9735, 0.9072, and 0.5848. Asymptotic standard errors are in parentheses. ***, **, and * indicate a parameter estimate that is statistically different from zero at the 1%, 5%, and 10% levels, respectively.

Figure 1
Weighted average maturity of the Government of Canada marketable debt

Figure 2
Effect of the Bank of Canada's GBPP on Canadian nominal yields



Time-zero impact on the term structure of Canadian zero-coupon bond yields of an unanticipated quantitative easing shock that (i) decreases the WAM of the Government of Canada debt market by 1.4 years, (ii) reverts deterministically to zero, (iii) when the relative risk aversion is $\gamma = 20$, and (iv) for different values of the persistence of the QE shock, using the estimated parameters of the GDTSM from Table 1.