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Nonparametric Identification of Incomplete Information Discrete Games with Non-equilibrium Behaviors

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Abstract

In the literature that estimates discrete games with incomplete information, researchers usually impose two assumptions. First, either the payoff function or the distribution of private information or both are restricted to follow some parametric functional forms. Second, players' behaviors are assumed to be consistent with the Bayesian Nash equilibrium. This paper jointly relaxes both assumptions. The framework non-parametrically specifies both the payoff function and the distribution of private information. In addition, each player's belief about other players' behaviors is also modeled as a nonparametric function. I allow this belief function to be any probability distribution over other players' action sets. This specification nests the equilibrium assumption when each player's belief corresponds to other players' actual choice probabilities. It also allows non-equilibrium behaviors when some players' beliefs are biased or incorrect. Under the above framework, this paper first derives a testable implication of the equilibrium condition. It then obtains the identification results for the payoff function, the belief function and the distribution of private information.

Topic: Econometric and statistical methods
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Résumé

Dans les travaux d'estimation des jeux discrets en information incomplète, les chercheurs formulent généralement deux hypothèses. D'abord, des contraintes sont imposées à la fonction de gains ou à la fonction de distribution des informations privées, ou aux deux à la fois, pour que celles-ci suivent des formes paramétriques. Ensuite, les comportements des joueurs sont considérés comme étant conformes au modèle d'équilibre de Nash bayésien. Dans cette étude, ces deux hypothèses sont écartées conjointement. Le cadre définit la fonction de gains et la fonction de distribution des informations privées de façon non paramétrique. De plus, la croyance de chaque joueur à l'égard des comportements des autres joueurs est modélisée sous la forme d'une fonction non paramétrique. La fonction des croyances peut correspondre à n'importe laquelle des distributions de probabilités qui représentent les ensembles d'actions des autres joueurs. Cette spécification intègre l'hypothèse d'équilibre lorsque la croyance de chaque joueur correspond aux probabilités réelles des choix des autres joueurs. Elle autorise aussi des comportements hors équilibre lorsque les croyances de certains joueurs sont biaisées ou erronées. De ce cadre, notre étude déduit d'abord l'implication vérifiable de la condition d'équilibre. Elle obtient ensuite les résultats d'identification pour la fonction de gains, la fonction des croyances et la fonction de distribution des informations privées.

Sujet : Méthodes économétriques et statistiques
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1 Introduction

Over the past two decades, econometric methods of incomplete information discrete games have been developed to estimate players' strategic interactions when they have asymmetric information.[1] In this literature, there are two common assumptions. First, the player's payoff function and/or the distribution of private information are assumed to be some parametric functions (henceforth, the parametric assumption). Second, players' observed choices are assumed to be consistent with the Bayesian Nash Equilibrium (henceforth, the equilibrium assumption). Under these two assumptions, researchers then estimate players' payoff functions and conduct counterfactual analysis.

Both the parametric assumption and the equilibrium assumption facilitate the identification and estimation of players' payoff functions. However, each assumption places strong restrictions on the econometric model and are potentially misspecified. For instance, the usual parametric restrictions include the linear payoff function and a Gumbelly (i.e., Logit model) or normally (i.e., Probit model) distributed private information. These restrictions are imposed for their statistical convenience but are difficult to justify by economic theories. In addition, the equilibrium assumption restricts each player to have an equilibrium/unbiased belief about other players' behaviors. However, games in the real world are often complicated. This complexity poses difficulties for a player to correctly predict others' strategies. Moreover, many games have multiple equilibria. This feature further complicates the construction of the unbiased belief. In these games, a player could be uncertain about which equilibrium strategy will be chosen by other players. Such a *strategic uncertainty* is first defined and studied by Van Huyck et al. (1990) and Crawford and Haller (1990) and could be prevalent in many applications of games. On the empirical side, economists have rejected the equilibrium condition in different types of games using both field and experimental data (Goeree and Holt, 2001; Goldfarb and Xiao, 2011; Aguirregabiria and Magesan, 2020).

Suppose that researchers incorrectly impose the parametric restriction and/or the equilibrium assumption: this could lead to incorrect estimates of players' payoff functions and counterfactual predictions. To address such an issue, this paper jointly relaxes both assumptions. In particular, the econometric model specifies both the payoff and the distribution of private information to be nonparametric functions. More-

[1] Examples include firm entry (Seim, 2006; Gowrisankaran and Krainer, 2011; Aradillas-López and Gandhi, 2016), product choice (Augereau et al., 2006; Sweeting, 2009), and social interaction (Brock and Durlauf, 2001; Bajari et al., 2010), among others. See a recent survey by Aradillas-López (2020).

over, each player's belief about other players' behaviors is also modeled as a nonparametric function. This belief function is allowed to be any probability distribution over other players' action sets. Intuitively, such a specification of belief nests the equilibrium assumption as a special case, when each player's belief corresponds to other players' actual conditional choice probabilities (CCPs). It also allows non-equilibrium behaviors, where some players have biased beliefs such that their beliefs differ from other players' true CCPs. Under the above-mentioned framework, I study the joint identification of the payoff function, the belief function, and the distribution of private information.

The identification results rely on an exclusion restriction that is commonly imposed in the existing literature. Specifically, I assume that there exists a player-specific payoff variable/shifter. This shifter affects only one player's payoff and has no impact on any other player's payoff. Importantly, almost every existing empirical application of incomplete information games exploits the identification power of the payoff shifter.[2] As shown by Aradillas-López (2010) and Bajari et al. (2010), such a shifter is usually necessary to identify the payoff function even under the parametric restriction and the equilibrium assumption. Given the exclusion restriction described above, this paper first considers a general binary choice game with $N \geq 2$ players. Suppose that the CCP of an arbitrary player, denoted by player i, remains constant across some realizations of the payoff shifters. For these realizations that satisfy the above condition of ***equal CCPs***, this paper derives a model restriction imposed on player i's belief. Under the equilibrium assumption, player i's belief equals other players' actual CCPs; therefore, the CCPs of other players must satisfy the same restriction. Since the CCPs can be consistently estimated, the above restriction turns out to be a testable implication of the equilibrium assumption. In particular, such an implication holds under nonparametric specifications of all model primitives. Importantly, when each player's payoff function and payoff shifter are continuous, there exist infinite realizations of the payoff shifters that satisfy the condition of equal CCPs. Consequently, the equilibrium assumption can be tested nonparametrically in a wide range of empirical applications. In the existing literature, this condition of equal CCPs has been exploited for both identification (Liu et al., 2017; Aguirregabiria, 2021) and estimation (Aradillas-López, 2012). Next, suppose that players' beliefs are unbiased/in equilibrium under a finite number of realizations of the payoff shifters, while they are allowed to be biased under all other

[2] A few exceptions exploit the identification power of multiple equilibria instead of the payoff shifter. See Sweeting (2009), de Paula and Tang (2012), and Aradillas-López and Gandhi (2016).

realizations. This restriction, referred to as the ***local unbiased belief assumption***, further achieves the nonparametric point identification of the payoff function and the distribution of private information. It also obtains an identified set for each player's belief function. When there are only $N = 2$ players, the identified set turns out to be a singleton so that each player's belief function is nonparametrically point identified.

The above local unbiased belief assumption may seem to be a little disappointing given the motivation to relax the equilibrium restriction. However, it is important to emphasize three points. First, the local unbiased belief assumption is imposed only on a finite number of realizations of the control variables. When some control variables are continuous, the region with the local unbiased belief assumption has a measure of zero. Therefore, it is a substantially weaker restriction than the equilibrium assumption in the existing literature which imposes a *global unbiased belief* restriction for every player. Specifically, each player's belief is assumed to be unbiased under every realization of the control variables. Second, in the real world, players could be familiar with some realizations of payoffs but are unfamiliar with others. Consider an example of firm competition. There could exist some values of the payoff shifter such that firms have experienced identical or similar realizations in the past or in other geographic markets. The learning process could lead firms to form unbiased beliefs under these familiar realizations; in contrast, firms may have biased beliefs under other, less familiar realizations. Such a phenomenon is also supported by experimental evidence (Goeree and Holt, 2001). Finally, this paper also derives a testable implication of the local unbiased belief assumption under an arbitrary realization of the payoff shifter. This test provides an empirical guidance for researchers on the choice of realizations to impose the condition of unbiased/correct beliefs.

This paper then extends the above results in binary choice games to a general game with $N \geq 2$ players, where each player has more than two actions. This multinomial choice game imposes two obstacles for the identification. Each obstacle is addressed by imposing an additional restriction on the econometric model. First, the condition of equal CCPs—required for the identification of model primitives—does not always hold in a multinomial choice game. To guarantee the equal CCPs condition, I impose an additional technical restriction. Second, in a binary choice game and under an appropriate location normalization, if two actions are chosen with equal probability, they must have the same deterministic expected payoff (i.e., the part of the expected payoff that excludes the private information). This relationship is the key to identifying the payoff function, the belief function, and the distribution of private information. However, it is not necessarily satisfied in a multinomial choice game. To address such an obstacle, I impose a *rank ordering property* on the distribution of private information. This property is first introduced by Manski (1975) and is subsequently applied by Goeree et al. (2005) and Fox (2007). It can be satisfied in a wide range of distributions of private information. Under the technical restriction and the rank ordering property, the identification results of all model primitives are generalized to multinomial choice games. At last, it is important to emphasize that the equilibrium condition and the local unbiased belief assumption can be tested under much weaker conditions and do not require the technical restriction and the rank ordering property.

Researchers have recognized the potential misspecifications of both the parametric restriction and the equilibrium assumption. However, recent literature usually relaxes one assumption but maintains the other one. For instance, under the equilibrium condition, Lewbel and Tang (2015) and Liu et al. (2017) relax the parametric assumption in a binary choice game. This paper extends their results in two major directions. First, the identification results are generalized to a game with more than two actions. Second, I show that the equilibrium assumption in their papers is more than sufficient for the identification results. Instead, a substantially weaker local unbiased belief assumption is enough to identify all model primitives. It further implies that the equilibrium assumption is testable, and this paper derives a testable implication. This second extension is closely related to another strand of literature that studies non-equilibrium behaviors in incomplete information games (Aradillas-López and Tamer, 2008; Aguirregabiria and Magesan, 2020; Aguirregabiria and Xie, 2021; Xie, forthcoming). Under the assumption that the distribution of private information is known by researchers, the above papers study the identification without the equilibrium assumption. Their results could be used to test the equilibrium restriction. However, when the distributional assumption on the private information is misspecified, the payoff function could be incorrectly estimated and the equilibrium behavior could be falsely rejected. In contrast, this paper avoids such issues of incorrect estimates and over-rejection. Specifically, I identify the payoff function and provide a test of the equilibrium assumption: both are robust to any distribution function.

To the best of my knowledge, Aguirregabiria (2021) is the only existing paper that also relaxes both assumptions. In the context of firm competition, he derives a testable implication of the equilibrium assumption that is robust to nonparametric specifications of both the payoff function and the distribution of private information (see his Proposition 5). This paper makes three major extensions. First, in addition to the data on firms' choices, Aguirregabiria assumes that researchers can also observe or estimate each firm's ex-post revenue. In contrast, the identification results obtained in this paper rely only on the choice data and do not require the revenue information. Consequently, this paper's results can be particularly useful when the data on revenue is limited. Second, the testable implication in Aguirregabiria (2021) is for binary choice games, while this paper extends it to multinomial choice games. Finally, Aguirregabiria shows only that the equilibrium assumption is testable, while he does not derive the identification of other model primitives. As a comparison, this paper further achieves the point identification of the payoff function, the distribution of private information, and the partial identification of the belief function.

In the context of discrete games with complete information, the equilibrium restriction and the parametric assumption have been relaxed by Kline (2015, 2016, 2018) and Kashaev and Salcedo (2021). Instead, this paper focuses on incomplete information games and complements the above studies on games with complete information.

The rest of this paper is organized as follows. Section 2 describes the empirical model of a static discrete game with incomplete information. This game consists of $N \geq 2$ players and each player has $(K+1)$ possible actions. Section 3 presents the identification conditions and model restrictions. I derive the identification results in a binary choice game (i.e., $K = 1$) in Section 4 and extend them to a multinomial choice game (i.e., $K > 1$) in Section 5. It also highlights the additional assumptions required for identification when the action space expands. Finally, I conclude in Section 6. All proofs are left to the Appendix.

2 A Simultaneous Discrete Game with Incomplete Information

This section describes the empirical framework of a discrete choice game with incomplete information. This framework has been extensively applied in the existing literature. There are $N \geq 2$ players. Letters i and j denote two arbitrary players. Letter $-i$ indexes all players other than i. Each player i simultaneously chooses an action, denoted by Y_i, from her action set $\{0, 1, \cdots, K\}$. This set consists of $(K+1)$ possible alternatives. Moreover, let the vector $\mathbf{Y}_{-i} = (Y_1, \cdots, Y_{i-1}, Y_{i+1}, \cdots, Y_N)'$ represent the action profile chosen by all players other than i. The utility/payoff function of player i is described by the following equation:

$$\tilde{U}_i(\mathbf{X},\mathbf{Z}_i,Y_i,\mathbf{Y}_{-i},\varepsilon_i) = U_i(\mathbf{X},\mathbf{Z}_i,Y_i,\mathbf{Y}_{-i}) + \varepsilon_i(Y_i), \tag{1}$$

where $\mathbf{X} \in \mathbb{R}^{L_\mathbf{x}}$ is a vector of control variables that could affect every player's payoff. Vector $\mathbf{Z}_i \in \mathbb{R}^{L_\mathbf{z}}$ is specific to player i. It affects only player i's payoff but has no impact on the payoffs of other players. In the existing literature, $\mathbf{Z}_i$ is referred to as the player-specific payoff shifter, and its existence is commonly assumed. As shown by Aradillas-López (2010) and Bajari et al. (2010), without such a shifter, the model is usually non-identified even under the parametric restriction and the equilibrium assumption. In addition, $\mathbf{X}$, $\mathbf{Z}_i$, and $\mathbf{Z}_{-i} = (\mathbf{Z}'_1,\cdots,\mathbf{Z}'_{i-1},\mathbf{Z}'_{i+1},\cdots,\mathbf{Z}'_N)'$ are common knowledge among players. As shown by Equation (1), player i's utility function $U_i(\cdot)$ depends on the common control variables $\mathbf{X}$, her payoff shifter $\mathbf{Z}_i$, her own action Y_i, and other players' choices $\mathbf{Y}_{-i}$. Importantly, this paper specifies $U_i(\cdot)$ as a ***nonparametric function***. In contrast to $(\mathbf{X},\mathbf{Z}_i,\mathbf{Z}_{-i})$, vector $\varepsilon_i = (\varepsilon_i(0),\varepsilon_i(1),\cdots,\varepsilon_i(K))'$ is player i's private information and is unknown by any of other players. An arbitrary variable $\varepsilon_i(k)$ in this vector affects player i's payoff of action $Y_i = k$. Finally, throughout this paper, bold letters (e.g., $\mathbf{X}$ and $\mathbf{x}$) denote vectors, italic letters (e.g., Y_i and y_i) denote scalars, capital letters (e.g., $\mathbf{X}$ and Y_i) denote random variables, and small letters (e.g., $\mathbf{x}$ and y_i) denote their realizations.

Without loss of generality, let us define $\pi_i(\mathbf{X},\mathbf{Z}_i,Y_i) = U_i(\mathbf{X},\mathbf{Z}_i,Y_i,\mathbf{Y}_{-i} = \mathbf{0})$ and $\delta_i(\mathbf{X},\mathbf{Z}_i,Y_i,\mathbf{Y}_{-i}) = U_i(\mathbf{X},\mathbf{Z}_i,Y_i,\mathbf{Y}_{-i}) - U_i(\mathbf{X},\mathbf{Z}_i,Y_i,\mathbf{Y}_{-i} = \mathbf{0})$, where $\mathbf{0}$ is an $(N-1)\times 1$ vector whose elements are all zeros. Function $\pi_i(\cdot)$ represents player i's payoff when all other players choose the base action, labelled as action 0. Consequently, $\pi_i(\cdot)$ is referred to as the *base return*. When some players deviate from their base action, they will have an impact on player i's payoff. This impact is captured by function $\delta_i(\cdot)$ and it is referred to as the *strategic effect*. By construction, $\delta_i(\mathbf{X},\mathbf{Z}_i,Y_i,\mathbf{Y}_{-i} = \mathbf{0}) = 0$. With the above definitions, player i's payoff function can be represented by the following Equation (2) without loss of generality:

$$\tilde{U}_i(\mathbf{X},\mathbf{Z}_i,Y_i,\mathbf{Y}_{-i},\varepsilon_i) = \underbrace{\pi_i(\mathbf{X},\mathbf{Z}_i,Y_i)}_{\textit{Base Return}} + \underbrace{\delta_i(\mathbf{X},\mathbf{Z}_i,Y_i,\mathbf{Y}_{-i})}_{\textit{Strategic Effect}} \cdot \mathbb{1}(\mathbf{Y}_{-i} \neq \mathbf{0}) + \varepsilon_i(Y_i). \tag{2}$$

Even though $\pi_i(\cdot)$ and $\delta_i(\cdot)$ are additively separable; by construction, Equation (2) actually specifies a nonparametric payoff function without additional restrictions and is equivalent to Equation (1). The rest of this paper then mainly considers the representation by Equation (2). Note that a game theoretic model implies that $\delta_i(\cdot, Y_i = y_i, \mathbf{Y}_{-i} = \mathbf{y}_{-i}) \neq 0$ for some $(y_i, \mathbf{y}_{-i})$; for instance, the strategic effect is non-zero for some action profile. Otherwise, the economic environment could be described by a single-agent model, not necessarily by a game theoretic model. This paper further focuses on the regular game such that any of the other players could impose a non-zero strategic effect on player i. Equivalently, for each $j \neq i$, there exists at least one profile (y_i, y_j) such that $\delta_i(\cdot, Y_i = y_i, Y_j = y_j, \mathbf{Y}_{-i,-j} = \mathbf{0}) \neq 0$. Suppose instead that player i's payoff is unaffected by player j's behaviors; then player j is redundant and could be excluded from the econometric model of player i's decision. Finally, given the identification conditions in this paper, the null hypothesis that player j has a non-zero strategic impact on player i is testable.

I study an incomplete information game where ε_i is player i's private information and all other model primitives are common knowledge among players. Assumption 1 states the restrictions imposed on the private information ε_i.

Assumption 1. *(a) For each player i, ε_i and $\varepsilon_{-i} = (\varepsilon'_1, \cdots, \varepsilon'_{i-1}, \varepsilon'_{i+1}, \cdots, \varepsilon'_N)'$ are independent conditional on $(\mathbf{X}, \mathbf{Z}_i, \mathbf{Z}_{-i})$.*

(b) For each player i, let $\Gamma_{i,\mathbf{X}}(\varepsilon_i)$ denote the cumulative distribution function (C.D.F.) of ε_i. Function $\Gamma_{i,\mathbf{X}}(\varepsilon_i)$ is absolutely continuous with respect to the Lebesgue measure. That is, $\Gamma_{i,\mathbf{X}}(\cdot)$ has a density with respect to the Lebesgue measure.

Assumption 1(a) restricts the private information to be independent across players conditional on common observables. This restriction has been imposed in many empirical applications of incomplete information games (Seim, 2006; Aradillas-López and Tamer, 2008; Sweeting, 2009; Bajari et al., 2010; Aradillas-López and Gandhi, 2016). Assumption 1(b) only restricts ε_i to have a well-defined density function over its support. As shown by Hotz and Miller (1993) and Norets and Takahashi (2013), it implies that the mapping between a player's conditional choice probabilities and her deterministic expected payoffs is bijective. This bijectivity plays a crucial role to establish the identification results. Importantly, $\Gamma_{i,\mathbf{X}}(\varepsilon_i)$ is ***nonparametrically*** specified. Moreover, I allow this C.D.F. to depend on the common knowledge $\mathbf{X}$, but restrict it to be independent of $\mathbf{Z}_i$. The support of ε_i could be either bounded or unbounded.

Let $Z_{i,l}$ denote an arbitrary variable in $\mathbf{Z}_i$. Moreover, $\mathbf{Z}_{i,-l} = (Z_{i,1}, \cdots, Z_{i,l-1}, Z_{i,l+1}, \cdots, Z_{i,L_\mathbf{z}})'$ represents all variables in player i's payoff shifters other than $Z_{i,l}$. Assumption 2 states the exclusion restriction that is commonly imposed in the existing literature.

Assumption 2. *For each player i and each $l \leq L_{\mathbf{Z}}$, $Z_{i,l}$ is a continuous variable. Moreover, $Z_{i,l}$ has exogenous variation over its support, conditional on $(\mathbf{X}, \mathbf{Z}_{i,-l}, \mathbf{Z}_{-i})$.*

Since the private information ε_i is continuous, the continuity of $\mathbf{Z}_i$ is required to trace out the distribution $\Gamma_{i,\mathbf{X}}(\cdot)$.[3] Most of the identification results hold when only a single payoff shifter exists (i.e., $L_{\mathbf{Z}} = 1$ and $\mathbf{Z}_i$ reduces to a scalar Z_i), while a few other results require the existence of more shifters (i.e., $L_{\mathbf{Z}} > 1$). The support of $\mathbf{Z}_i$ could be either bounded or unbounded.

Given the game structure described above, define $\sigma_i(\mathbf{X}, \mathbf{Z}_i, \mathbf{Z}_{-i}, \varepsilon_i) : \mathbb{R}^{L_{\mathbf{X}}+N\cdot L_{\mathbf{Z}}+K+1} \rightarrow \{0, 1, \cdots, K\}$ as a strategy function of player i. It maps from all information observed by player i to one of her actions in $\{0, \cdots, K\}$. Note that this paper focuses on the pure strategy. Since ε_i is continuously distributed, player i would have a unique optimal action with probability 1. Therefore, the focus on the pure strategy function is innocuous. Furthermore, define $B_i(\mathbf{Y}_{-i}|\mathbf{X}, \mathbf{Z}_i, \mathbf{Z}_{-i})$ as player i's belief function. In particular, $B_i(\mathbf{Y}_{-i} = \mathbf{y}_{-i}|\mathbf{X}, \mathbf{Z}_i, \mathbf{Z}_{-i})$ represents player i's believed probability that other players will choose the action profile $\mathbf{Y}_{-i} = \mathbf{y}_{-i}$, conditional on $(\mathbf{X}, \mathbf{Z}_i, \mathbf{Z}_{-i})$. This belief function depends on all variables that are common knowledge. Importantly, $B_i(\mathbf{Y}_{-i}|\mathbf{X}, \mathbf{Z}_i, \mathbf{Z}_{-i})$ is a ***nonparametric*** function with the restriction that it is a valid probability distribution (i.e., $0 \leq B_i(\mathbf{Y}_{-i}|\mathbf{X}, \mathbf{Z}_i, \mathbf{Z}_{-i}) \leq 1$, and $\sum_{\mathbf{y}_{-i}} B_i(\mathbf{Y}_{-i} = \mathbf{y}_{-i}|\mathbf{X}, \mathbf{Z}_i, \mathbf{Z}_{-i}) = 1$). More details about the micro-foundation of this belief function are left to the Appendix.

Given the strategy function $\sigma_i(\cdot)$ and Assumption 1(a), each player's behavior would be independent conditional on $(\mathbf{X}, \mathbf{Z}_i, \mathbf{Z}_{-i})$. If player i correctly figures out this conditional independence, there exists a dimension reduction of the belief function to facilitate the identification. Specifically, let $B_i^j(Y_j = y_j|\mathbf{X}, \mathbf{Z}_i, \mathbf{Z}_{-i})$ denote player i's belief about the probability that player j will choose $Y_j = y_j$. The conditional independence implies that $B_i(\mathbf{Y}_{-i}|\cdot)$ could be replaced by $B_i^j(Y_j|\cdot)$; for instance, $B_i(\mathbf{Y}_{-i}|\cdot) = \prod_{j \neq i}^N B_i^j(Y_j|\cdot)$. This replacement will reduce the dimension of player i's belief from $(K+1)^{N-1} - 1$ to $(N-1) \cdot K$ and could ease the identification. In this paper, I do not impose the conditional independence on player i's belief. Instead, I study the identification of the belief function $B_i(\mathbf{Y}_{-i}|\cdot)$ that permits arbitrary correlation among the actions of all players other than i. Such a specification allows two aspects of biased beliefs. First, player i could have incorrect expectation about any single player's CCP. Second, player i

[3] The identification results would also hold when some discrete variables are included in $\mathbf{Z}_i$. The key condition is that $\mathbf{Z}_i$ contains at least $L_{\mathbf{Z}}$ continuous variables. For notation simplicity, this paper suppresses discrete variables in $\mathbf{Z}_i$.

could also falsely believe that other players' behaviors are correlated, while they are actually independent.

With the payoff and belief functions defined above, player i's expected payoff of action Y_i is represented by the following equation:

$$E[U_i(\mathbf{X},\mathbf{Z}_i,\mathbf{Z}_{-i},Y_i,\varepsilon_i)] = \underbrace{\pi_i(\mathbf{X},\mathbf{Z}_i,Y_i) + \sum_{\mathbf{y}_{-i}\neq\mathbf{0}} \delta_i(\mathbf{X},\mathbf{Z}_i,Y_i,\mathbf{Y}_{-i}=\mathbf{y}_{-i}) \cdot B_i(\mathbf{Y}_{-i}=\mathbf{y}_{-i}|\mathbf{X},\mathbf{Z}_i,\mathbf{Z}_{-i})}_{=EU_i(\mathbf{X},\mathbf{Z}_i,\mathbf{Z}_{-i},Y_i)} + \varepsilon_i(Y_i). \tag{3}$$

To simplify the notation, Equation (3) defines a *deterministic expected payoff* function $EU_i(\cdot,Y_i) = \pi_i(\cdot,Y_i)$ $+\sum_{\mathbf{y}_{-i}\neq\mathbf{0}} \delta_i(\cdot,Y_i,\mathbf{Y}_{-i}=\mathbf{y}_{-i}) \cdot B_i(\mathbf{Y}_{-i}=\mathbf{y}_{-i}|\cdot)$. It represents the part of the expected payoff of action Y_i that depends on the common knowledge $(\mathbf{X},\mathbf{Z}_i,\mathbf{Z}_{-i})$ but is independent of the private information ε_i. Under the equilibrium assumption, $EU_i(\cdot)$ would be common knowledge and is deterministic. Specifically, it is a composite function of both the payoff and the belief. Note that player $-i$'s payoff shifter $\mathbf{Z}_{-i}$ will affect player i's $EU_i(\cdot)$ by indirectly affecting player i's belief. In addition, let $P_i(Y_i|\mathbf{X},\mathbf{Z}_i,\mathbf{Z}_{-i})$ denote the probability that player i will choose action Y_i conditional on $(\mathbf{X},\mathbf{Z}_i,\mathbf{Z}_{-i})$. This *conditional choice probability* (CCP) takes the following form:

$$\begin{aligned} P_i(Y_i=k|\cdot) &= \int \mathbb{1}\left[EU_i(\cdot,Y_i=k)+\varepsilon_i(k) \geq EU_i(\cdot,Y_i=k')+\varepsilon_i(k'),\ \forall k'\neq k\right] d\Gamma_{i,\mathbf{X}}(\varepsilon_i) \\ &= \tilde{\Gamma}^k_{i,\mathbf{X}}\left[EU_i(\cdot,Y_i=k)-EU_i(\cdot,Y_i=0),\cdots,EU_i(\cdot,Y_i=k)-EU_i(\cdot,Y_i=K)\right], \end{aligned} \tag{4}$$

where $\tilde{\Gamma}^k_{i,\mathbf{X}}(\cdot)$ denotes the C.D.F. of the differences of private information; i.e., $(\varepsilon_i(0)-\varepsilon_i(k),\cdots,\varepsilon_i(k-1)-\varepsilon_i(k),\varepsilon_i(k+1)-\varepsilon_i(k),\cdots,\varepsilon_i(K)-\varepsilon_i(k))'$. Note that $\tilde{\Gamma}^k_{i,\mathbf{X}}(\cdot)$ can be derived from $\Gamma_{i,\mathbf{X}}(\cdot)$.

In the literature that estimates discrete games with incomplete information, researchers usually assume that players' behaviors are consistent with the Bayesian Nash Equilibrium. This equilibrium assumption restricts each player to be perfectly rational in the sense that player i could correctly predict other players' CCPs given the available information. This corresponds to the restriction that $B_i(\cdot) = P_{-i}(\cdot)$ and is summarized by Remark 1.

Remark 1. *The behaviors of Bayesian Nash Equilibrium are described by the following restriction:* $B_i(\mathbf{Y}_{-i}|\mathbf{X},\mathbf{Z}_i,\mathbf{Z}_{-i}) = P_{-i}(\mathbf{Y}_{-i}|\mathbf{X},\mathbf{Z}_i,\mathbf{Z}_{-i}) = \prod_{j\neq i}^N P_j(Y_j|\mathbf{X},\mathbf{Z}_i,\mathbf{Z}_{-i}),\ \forall 1\leq i\leq N.$

Remark 1 emphasizes how this paper relaxes the Bayesian Nash Equilibrium. In particular, the framework nests the equilibrium assumption when each player has an unbiased belief (i.e., $B_i(\cdot) = P_{-i}(\cdot)$, $\forall i$). It also allows non-equilibrium behaviors if at least one player's belief is biased (i.e., $B_i(\cdot) \neq P_{-i}(\cdot)$ for some i). One of this paper's objectives is to identify $B_i(\cdot)$ and test the equilibrium assumption $B_i(\cdot) = P_{-i}(\cdot)$.

3 Identification Objectives, Conditions, and Model Restrictions

This paper considers the following identification problem. Given each player i's CCP $P_i(Y_i|\mathbf{X}, \mathbf{Z}_i, \mathbf{Z}_{-i})$,[4] can we identify this player's payoff functions $\pi_i(\cdot)$, $\delta_i(\cdot)$, belief function $B_i(\cdot)$, the C.D.F. of the differences of private information denoted by $\tilde{\Gamma}_{i,\mathbf{X}}(\cdot)$, and test the restriction of equilibrium beliefs $B_i(\cdot) = P_{-i}(\cdot)$?[5] This paper derives identification results using the variation of $(\mathbf{Z}_i, \mathbf{Z}_{-i})$. These results hold true for each realization of $\mathbf{X}$. Therefore, $\mathbf{X}$ is suppressed for the rest of this paper for notation simplicity.

First, consider the following two transformations of the payoff function and/or the private information: (1) $\hat{U}_i(\cdot) = U_i(\cdot) + c$ for some $c \neq 0$; (2) $\hat{\pi}_i(\cdot) = c \cdot \pi_i(\cdot)$, $\hat{\delta}_i(\cdot) = c \cdot \delta_i(\cdot)$, and $\hat{\varepsilon}_i = c \cdot \varepsilon_i$ for some $c > 0$ (Recall Equations (1) and (2) for the equivalency between $U_i(\cdot)$ and $\pi_i(\cdot)$, $\delta_i(\cdot)$). Equation (4) implies that any of these two transformations would preserve the same CCP for player i and are indistinguishable from each other. As in discrete choice models, a location and a scale normalization are required for the identification; they are summarized in Assumption 3.

Assumption 3. *(a)* $\pi_i(\mathbf{Z}_i, Y_i = 0) = 0$ *and* $\delta_i(\mathbf{Z}_i, Y_i = 0, \mathbf{Y}_{-i}) = 0$.

(b) For an arbitrary action profile $(y_i, \mathbf{y}_{-i})$ *where* $\mathbf{y}_{-i} \neq \mathbf{0}$ *and an arbitrary realization of* $\mathbf{Z}_i$ *denoted by* $\mathbf{z}_i$, $\left|\delta_i(\mathbf{Z}_i = \mathbf{z}_i, Y_i = y_i, \mathbf{Y}_{-i} = \mathbf{y}_{-i})\right| = 1$.

Assumption 3(a) normalizes player i's payoff of action 0 to be zero and is standard in discrete choice models. Assumption 3(b) normalizes the strategic effect of an arbitrary action profile under one arbitrary realization of $\mathbf{Z}_i$ to be 1. In contrast, the strategic effects of other action profiles and/or under other realizations of $\mathbf{Z}_i$ are unrestricted. This type of scale normalization is also imposed in Liu et al. (2017).

[4]Even though each player i's CCP $P_i(\cdot)$ is not directly observed by researchers, it could be consistently estimated. Consider a cross-sectional dataset of M independent games/observations. For each game m, researchers observe the common knowledge control variables $(\mathbf{x}_m, \mathbf{z}_{i,m}, \mathbf{z}_{-i,m})$ and each player's choice $(y_{i,m}, \mathbf{y}_{-i,m})$. With this dataset, the CCP $P_i(\cdot)$ can be consistently estimated when $M \to \infty$. Therefore, for the identification purpose, $P_i(\cdot)$ is assumed to be known by researchers.

[5]As in Train (2009), there are infinite distribution functions $\Gamma_{i,\mathbf{X}}(\cdot)$ that imply the same distribution of the difference of private information; i.e., $\tilde{\varepsilon}_i = (\varepsilon_i(1) - \varepsilon_i(0), \cdots, \varepsilon_i(K) - \varepsilon_i(0))'$. Since only the difference matters in discrete choice models, $\Gamma_{i,\mathbf{X}}(\cdot)$ cannot be identified and we can at most identify the distribution of $\tilde{\varepsilon}_i$. Such a distribution is denoted by $\tilde{\Gamma}_{i,\mathbf{X}}(\cdot)$.

In the literature that estimates discrete games, another practice is to normalize the marginal effect of a control variable on the payoff function; for instance, $\frac{\partial \pi_i(\mathbf{Z}_i=\mathbf{z}_i, Y_i=y_i)}{\partial Z_{i,l}} = 1$ for an arbitrary variable $Z_{i,l}$ and arbitrary realizations $\mathbf{z}_i$, $y_i \neq 0$ (Lewbel and Tang, 2015; Kline, 2015). As will be shown in the proof of my identification results, $\pi_i(\cdot)$ and $\frac{\partial \pi_i(\cdot)}{\partial Z_{i,l}}$ are identified as functions that are linear in $\delta_i(\cdot)$. Consequently, the scale normalization $\frac{\pi_i(\cdot)}{\partial Z_{i,l}} = 1$ is equivalent to Assumption 3(b). This paper chooses to normalize the scale of the strategic effect for notation convenience. In more detail, Lewbel and Tang (2015) and Kline (2015) specify $\mathbf{Z}_i$ to enter player i's payoff additively and linearly. Consequently, the marginal impact is a constant, and it is convenient to normalize such a constant to be 1. In contrast, I allow $\mathbf{Z}_i$ to enter nonparametrically and interactively into player i's payoff function. Therefore, more notations are required to emphasize that the marginal effect is evaluated at a particular realization of the control variables and action profiles. In addition, some of this paper's identification results exploit the variation of $\mathbf{Z}_{-i}$, but with a fixed value of $\mathbf{Z}_i$. Assumption 3(b) is convenient to prove these results. In contrast, with normalization $\frac{\pi_i(\cdot)}{\partial Z_{i,l}} = 1$, the proofs turn out to be cumbersome, as a notation intense transformation from $\frac{\pi_i(\cdot)}{\partial Z_{i,l}}$ to $\delta_i(\cdot)$ is required. See Liu et al. (2017) for a detailed discussion about the equivalent normalizations in discrete games.

Denote $\mathbf{P}_i(\mathbf{Z}_i, \mathbf{Z}_{-i}) = \big(P_i(Y_i = 1|\mathbf{Z}_i, \mathbf{Z}_{-i}), \cdots, P_i(Y_i = K|\mathbf{Z}_i, \mathbf{Z}_{-i})\big)'$ and $\mathbf{EU}_i(\mathbf{Z}_i, \mathbf{Z}_{-i}) = \big(EU_i(\mathbf{Z}_i, \mathbf{Z}_{-i}, Y_i = 1), \cdots, EU_i(\mathbf{Z}_i, \mathbf{Z}_{-i}, Y_i = K)\big)'$ as two $K \times 1$ vectors of player i's CCPs and deterministic expected payoffs, respectively. Note that since $\sum_{y_i} P_i(Y_i = y_i|\cdot) = 1$ and $\pi_i(\cdot, Y_i = 0) = 0$, $\delta_i(\cdot, Y_i = 0, Y_{-i}) = 0$ by Assumption 3(a), $P_i(Y_i = 0|\cdot)$ and $EU_i(\cdot, Y_i = 0)$ contain no additional information and they are suppressed from vectors $\mathbf{P}_i(\cdot)$ and $\mathbf{EU}_i(\cdot)$. Moreover, let $\mathbf{P}_{-i}(\mathbf{Z}_i, \mathbf{Z}_{-i})$ be a $\big((K+1)^{N-1} - 1\big) \times 1$ vector; each element in this vector represents the CCP of one particular action profile chosen by all players other than i. Again, the probability of the profile $\mathbf{Y}_{-i} = \mathbf{0}$ is excluded from the vector $\mathbf{P}_{-i}$. This paper studies the identification under the regular case where both the payoff function and the belief function are bounded and continuous in their arguments. Moreover, the elements in the random vector $\mathbf{P}_{-i}(\cdot)$ are linearly independent. Assumption 4 summarizes the above regularity conditions.

Assumption 4. *(a) Each player i's payoff functions $\pi_i(\cdot)$, $\delta_i(\cdot)$, and belief function $B_i(\cdot)$ are bounded and continuous in their arguments.*

(b) For a fixed realization $\mathbf{Z}_i = \mathbf{z}_i$, $P_{-i}(\mathbf{Y}_{-i}|\mathbf{Z}_i = \mathbf{z}_i, \mathbf{Z}_{-i})$ can be seen as a random variable with the

variation of $\mathbf{Z}_{-i}$*. Then, the elements in the random vector* $\mathbf{P}_{-i}(\mathbf{Z}_i = \mathbf{z}_i, \mathbf{Z}_{-i})$ *are linearly independent.*

Under Assumption 4(a), the value of the deterministic expected payoff function $EU_i(\mathbf{Z}_i, \mathbf{Z}_{-i}, Y_i)$ could remain constant across some realizations of $(\mathbf{Z}_i, \mathbf{Z}_{-i})$. Consequently, player i's CCP would hold fixed across these realizations. As described in the Introduction, this condition of equal CCPs is the key to my identification results. Assumption 4(b) states that $\mathbf{Z}_{-i}$ should have sufficient impact on other players' CCPs. This assumption is also imposed in the existing literature that establishes the identification results under the equilibrium restriction (Bajari et al., 2010; Liu et al., 2017). In more details, Assumption 4(b) implies that $\frac{\partial \mathbf{P}_{-i}(\mathbf{Z}_i,\mathbf{Z}_{-i})}{\partial \mathbf{Z}'_{-i}} \neq \mathbf{0}$. Under the equilibrium restriction, it further implies that $\frac{\partial \mathbf{P}_{i}(\mathbf{Z}_i,\mathbf{Z}_{-i})}{\partial \mathbf{Z}'_{-i}} \neq \mathbf{0}$. Intuitively, $\mathbf{Z}_{-i}$ affects player i's CCPs but does not affect her payoff function. Naturally, the variation of $\mathbf{Z}_{-i}$ would provide identification power. In particular, the equation to identify the payoff function mimics the structure of a linear regression. In this regression, other players' CCPs $P_{-i}(\mathbf{Y}_{-i}|\cdot)$ act as regressors, and the strategic effects $\delta_i(\cdot)$ represent the coefficients of these regressors. Analogously, the linear independence restriction by Assumption 4(b) could be interpreted as the non-multicollinearity condition in the linear regression model. Note that such a condition will be always satisfied in a two-player binary choice game, provided that $\mathbf{Z}_{-i}$ affects player $-i$'s CCPs. Finally, since $\mathbf{P}_{-i}(\cdot)$ can be consistently estimated, Assumption 4(b) is testable.

This paper's identification results focus on the range of function values such that each player's CCP satisfies $0 < P_i(Y_i|\cdot) < 1$. Given Assumption 4(a), the condition $0 < P_i(Y_i|\cdot) < 1$ always holds when ε_i has unbounded support. This unbounded support is satisfied in most of the existing literature, as researchers usually consider the Logit or Probit specifications. When ε_i has bounded support and consider an arbitrary action k, then any sufficiently negative payoffs of this action would imply $P_i(Y_i = k|\cdot) = 0$. Equivalently, if an action's choice probability is zero, it is impossible to identify this action's payoff. Therefore, when ε_i has bounded support, the identification results are obtained in the range of function values such that each action of player i is chosen with strictly positive probability.

I now describe the restrictions imposed by the model in Section 2. With the vector representations $\mathbf{P}_i(\mathbf{Z}_i, \mathbf{Z}_{-i})$ and $\mathbf{EU}_i(\mathbf{Z}_i, \mathbf{Z}_{-i})$, Equation (4) could be expressed in the following matrix form:

$$\mathbf{P}_i(\mathbf{Z}_i, \mathbf{Z}_{-i}) = \mathbf{G}_i\left[\mathbf{EU}_i(\mathbf{Z}_i, \mathbf{Z}_{-i})\right], \tag{5}$$

where $\mathbf{G}_i(\cdot)$ denotes the mapping from player i's deterministic expected payoffs to her CCPs. Under Assumption 1(b) and the condition $0 < P_i(Y_i|\cdot) < 1$, Hotz and Miller (1993) and Norets and Takahashi (2013) prove that this mapping is bijective. Therefore, function $\mathbf{G}_i(\cdot)$ is invertible, and its inversion is represented by the following equation:

$$\mathbf{EU}_i(\mathbf{Z}_i,\mathbf{Z}_{-i}) = \mathbf{F}_i\big[\mathbf{P}_i(\mathbf{Z}_i,\mathbf{Z}_{-i})\big]$$

or

$$\mathbf{\Pi}(\mathbf{Z}_i) + \mathbf{\Delta}_i(\mathbf{Z}_i)\cdot\mathbf{B}_i(\mathbf{Z}_i,\mathbf{Z}_{-i}) = \mathbf{F}_i\big[\mathbf{P}_i(\mathbf{Z}_i,\mathbf{Z}_{-i})\big], \tag{6}$$

where $\mathbf{F}_i(\cdot)$ represents the inverse of function $\mathbf{G}_i(\cdot)$. Letter $\mathbf{\Pi}_i(\cdot) = \big(\pi_i(\cdot,Y_i=1),\cdots,\pi_i(\cdot,Y_i=K)\big)'$ is a $K\times 1$ vector of player i's base returns. Term $\mathbf{\Delta}_i$ is a $K\times\big((K+1)^{N-1}-1\big)$ matrix. Its k^{th} row represents the strategic effect that each action profile $\mathbf{Y}_{-i}$ imposes on player i's payoff of action $Y_i = k$. Letter $\mathbf{B}_i(\cdot)$ is a $\big((K+1)^{N-1}-1\big)\times 1$ vector; each element in this vector represents player i's belief about the probability of one particular action profile chosen by other players. Note that the belief about the profile $\mathbf{Y}_{-i} = \mathbf{0}$ is excluded from the vector $\mathbf{B}_i(\cdot)$.

Equation (6) contains all model restrictions. It is the key equation for all identification results in this paper. Moreover, due to the bijectivity result by Hotz and Miller (1993) and Norets and Takahashi (2013), the identification of function $\mathbf{F}_i(\cdot)$ implies the identification of $\mathbf{G}_i(\cdot)$ and the C.D.F. of the difference of private information denoted by $\tilde{\Gamma}_i(\cdot)$. Therefore, the rest of this paper focuses on the identification of $\mathbf{F}_i(\cdot)$.

One implication of Equation (6) is that the payoff functions can be canceled out. It leaves a relationship between player i's belief and her inverted choice probability function $\mathbf{F}_i(\cdot)$. To see such a relationship, define $\tilde{\mathbf{B}}_i(\mathbf{Z}_i,\mathbf{z}_{-i}^{1,2}) = \mathbf{B}_i(\mathbf{Z}_i,\mathbf{Z}_{-i}=\mathbf{z}_{-i}^2) - \mathbf{B}_i(\mathbf{Z}_i,\mathbf{Z}_{-i}=\mathbf{z}_{-i}^1)$ and $\tilde{\mathbf{F}}_i(\mathbf{Z}_i,\mathbf{z}_{-i}^{1,2}) = \mathbf{F}_i[\mathbf{P}_i(\mathbf{Z}_i,\mathbf{Z}_{-i}=\mathbf{z}_{-i}^2)] - \mathbf{F}_i[\mathbf{P}_i(\mathbf{Z}_i,\mathbf{Z}_{-i}=\mathbf{z}_{-i}^1)]$ as the differences of player i's beliefs and inverted functions between two realizations of $\mathbf{Z}_{-i}$, denoted by $\mathbf{z}_{-i}^1$ and $\mathbf{z}_{-i}^2$. Moreover, let $\tilde{\mathbb{B}}_i(\mathbf{Z}_i,\mathbf{z}_{-i}^{1:(K+1)^{N-1}}) = \big(\tilde{\mathbf{B}}_i(\mathbf{Z}_i,\mathbf{z}_{-i}^{1,2}),\cdots,\tilde{\mathbf{B}}_i(\mathbf{Z}_i,\mathbf{z}_{-i}^{1,(K+1)^{N-1}})\big)$ and $\tilde{\mathbb{F}}_i(\mathbf{Z}_i,\mathbf{z}_{-i}^{1:(K+1)^{N-1}}) = \big(\tilde{\mathbf{F}}_i(\mathbf{Z}_i,\mathbf{z}_{-i}^{1,2}),\cdots,\tilde{\mathbf{F}}_i(\mathbf{Z}_i,\mathbf{z}_{-i}^{1,(K+1)^{N-1}})\big)$ be the corresponding matrices. Lemma 1 presents the relationship between player i's belief and her inverted choice probability function.

Lemma 1. *Under Assumptions 1–2 and consider any $(K+1)^{N-1}+1$ realizations of $\mathbf{Z}_{-i}$, denoted by $\mathbf{z}_{-i}^1$ to $\mathbf{z}_{-i}^{(K+1)^{N-1}+1}$. Suppose further that the matrices $\tilde{\mathbb{B}}_i(\mathbf{Z}_i,\mathbf{z}_{-i}^{1:(K+1)^{N-1}})$ and $\tilde{\mathbb{B}}_i(\mathbf{Z}_i,\mathbf{z}_{-i}^{2:(K+1)^{N-1}+1})$ are invertible, we then have the following relationship between player i's belief function and her inverted* *choice probability function:*

$$\tilde{\mathbb{F}}_i(\mathbf{Z}_i, \mathbf{z}_{-i}^{1:(K+1)^{N-1}}) \cdot \tilde{\mathbb{B}}_i^{-1}(\mathbf{Z}_i, \mathbf{z}_{-i}^{1:(K+1)^{N-1}}) = \tilde{\mathbb{F}}_i(\mathbf{Z}_i, \mathbf{z}_{-i}^{2:(K+1)^{N-1}+1}) \cdot \tilde{\mathbb{B}}_i^{-1}(\mathbf{Z}_i, \mathbf{z}_{-i}^{2:(K+1)^{N-1}+1}). \tag{7}$$

Proof. In the Appendix. □

Equation (7) obtains a relationship between $\mathbf{B}_i(\cdot)$ and $\mathbf{F}_i(\cdot)$ that is independent of player i's payoff function. Aguirregabiria and Magesan (2020) and Aguirregabiria and Xie (2021) derive a similar relationship in games with two players; Lemma 1 extends their results to games with $N \geq 2$ players. To better interpret this lemma, consider a simple 2×2 game. Equation (7) then turns to the following:

$$\frac{F_i\left[P_i(Y_i = 1|\mathbf{Z}_i, \mathbf{z}^2_{-i})\right] - F_i\left[P_i(Y_i = 1|\mathbf{Z}_i, \mathbf{z}^1_{-i})\right]}{B_i(Y_{-i}|\mathbf{Z}_i, \mathbf{z}^2_{-i}) - B_i(Y_{-i}|\mathbf{Z}_i, \mathbf{z}^1_{-i})} = \frac{F_i\left[P_i(Y_i = 1|\mathbf{Z}_i, \mathbf{z}^3_{-i})\right] - F_i\left[P_i(Y_i = 1|\mathbf{Z}_i, \mathbf{z}^2_{-i})\right]}{B_i(Y_{-i}|\mathbf{Z}_i, \mathbf{z}^3_{-i}) - B_i(Y_{-i}|\mathbf{Z}_i, \mathbf{z}^2_{-i})}. \tag{8}$$

In this two-player binary choice game, $F_i(\cdot)$ reduces to the inverse function of the C.D.F. of $(\varepsilon_i(0) - \varepsilon_i(1))$. Equation (8) would hold for any three realizations of $\mathbf{Z}_{-i}$, denoted by $\mathbf{z}^1_{-i}$ to $\mathbf{z}^3_{-i}$. Moreover, the invertibility condition of the matrix $\tilde{\mathbb{B}}_i(\cdot)$ in Lemma 1 turns to a non-zero condition of the denominator in Equation (8). This condition holds true as long as $\mathbf{Z}_{-i}$ could affect player i's belief $B_i(Y_{-i}|\cdot)$.

In Aguirregabiria and Magesan (2020) and Aguirregabiria and Xie (2021), the distribution of private information is assumed to be known by researchers. As a result, the inverted choice probability function $\mathbf{F}_i(\cdot)$ is also known. Naturally, Equation (7) characterizes an identified set of the belief function $B_i(\mathbf{Y}_{-i}|\cdot)$. This identified set could be used to test the equilibrium condition $B_i(\cdot) = P_{-i}(\cdot)$. However, the above argument is not applicable in this paper since I nonparametrically specify the distribution of private information. The first contribution of this paper is to extend Lemma 1. In particular, I derive a testable implication of the equilibrium assumption that is robust to the nonparametric specification of the distribution of private information. The next two sections present this testable implication and other identification results.

4 Identification in Binary Choice Games

This section establishes the identification results in binary choice games. Since each player has two actions, and the payoff of action 0 is normalized to zero, the argument Y_i could be removed from $\pi_i(\cdot)$ and $\delta_i(\cdot)$ in this section for notation simplicity. In particular, $\pi_i(\mathbf{Z}_i)$ and $\delta_i(\mathbf{Z}_i, \mathbf{Y}_{-i})$ represent player i's base return and strategic effect of action $Y_i = 1$. In this game with binary choice, Equation (6) then turns to:

$$\pi_i(\mathbf{Z}_i) + \sum_{\mathbf{y}_{-i} \neq \mathbf{0}} \delta_i(\mathbf{Z}_i, \mathbf{Y}_{-i} = \mathbf{y}_{-i}) \cdot B_i(\mathbf{Y}_{-i} = \mathbf{y}_{-i} | \mathbf{Z}_i, \mathbf{Z}_{-i}) = F_i\left[P_i(Y_i = 1 | \mathbf{Z}_i, \mathbf{Z}_{-i})\right]. \tag{9}$$

As described above, $F_i(\cdot)$ reduces to the inverse function of the C.D.F. of the difference of private information; i.e., $\tilde{\varepsilon}_i = \varepsilon_i(0) - \varepsilon_i(1)$. Assumption 5 normalizes the location of $\tilde{\varepsilon}_i$.

Assumption 5. *Let* $\tilde{\varepsilon}_i = \varepsilon_i(0) - \varepsilon_i(1)$*, then* $Median(\tilde{\varepsilon}_i) = 0$*.*

Consider the transformation such that $\hat{\pi}_i(\cdot) = \pi_i(\cdot) + c$ and $\hat{\tilde{\varepsilon}}_i = \tilde{\varepsilon}_i + c$ for some $c \neq 0$. Any of these transformations would preserve the same CCP of player i, given Equation (4). Therefore, the locations of $\pi_i(\cdot)$ and $\tilde{\varepsilon}_i$ are indistinguishable. Assumption 5 normalizes the location of $\tilde{\varepsilon}_i$ by imposing a zero median restriction.

To explain the intuition, Subsection 4.1 first sketches the identification results in a simple two-player binary choice game. Subsection 4.2 then formally establishes these results in a general binary choice game with $N \geq 2$ players.

4.1 Sketch of Identification Results

In a 2×2 game, $-i$ indexes the only player other than i. Therefore, in this subsection, other players' action profile $\mathbf{Y}_{-i}$ reduces to a scalar, denoted by Y_{-i}. In this binary choice game with two players, Equation (9) turns to the following:

$$\pi_i(\mathbf{Z}_i) + \delta_i(\mathbf{Z}_i, Y_{-i} = 1) \cdot B_i(Y_{-i} = 1 | \mathbf{Z}_i, \mathbf{Z}_{-i}) = F_i\left[P_i(Y_i = 1 | \mathbf{Z}_i, \mathbf{Z}_{-i})\right]. \tag{10}$$

I first describe a testable implication of the equilibrium assumption. Consider three pairs of realizations of $(\mathbf{Z}_i, \mathbf{Z}_{-i})$, denoted by $(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(l)})$ and $(\mathbf{z}_i^2, \mathbf{z}_{-i}^{2(l)})$ for $l = 1, 2, 3$. Each pair satisfies the condition $P_i(Y_i =$

$1|\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(l)}) = P_i(Y_i = 1|\mathbf{z}_i^2,\mathbf{z}_{-i}^{2(l)})$. Equivalently, player i's CCP is fixed across some variations of $(\mathbf{Z}_i,\mathbf{Z}_{-i})$. This condition, referred to as the condition of *equal CCPs*, is satisfied for infinite pairs of $(\mathbf{Z}_i,\mathbf{Z}_{-i})$ given Assumption 4(a). In particular, consider an entry game where the strategic effect $\delta_i(\cdot) < 0$. Moreover, suppose that $\mathbf{Z}_i$ is a single variable denoted by Z_i. It strictly reduces firm i's entry cost and therefore strictly increases its entry probability. Naturally, when Z_{-i} increases, player i would anticipate a higher entry probability $P_{-i}(Y_{-i} = 1|\cdot)$ by the other firm. Since $\delta_i(\cdot) < 0$, firm i would then lower its own $P_i(Y_i = 1|\cdot)$. Therefore, in this entry game, $P_i(Y_i = 1|Z_i,Z_{-i})$ is increasing in Z_i but decreasing in Z_{-i} and continuous in both arguments. Intuitively, for values $z_i^2 > z_i^1$, we can always find values $z_{-i}^2 > z_{-i}^1$ such that $P_i(Y_i = 1|z_i^1,z_{-i}^1) = P_i(Y_i = 1|z_i^2,z_{-i}^2)$, provided that $(z_i^2 - z_i^1)$ is not too large.[6] Given the continuity condition in Assumption 4(a), there exist infinite pairs of realizations that could hold $P_i(Y_i = 1|\cdot)$ constant. A similar argument applies to other types of games when $\delta_i(\cdot) > 0$. Under these three pairs of realizations, there is a testable implication of the equilibrium assumption described by the following equation:

$$\frac{P_{-i}(Y_{-i}|\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(3)}) - P_{-i}(Y_{-i}|\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(2)})}{P_{-i}(Y_{-i}|\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(2)}) - P_{-i}(Y_{-i}|\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(1)})} = \frac{P_{-i}(Y_{-i}|\mathbf{z}_i^2,\mathbf{z}_{-i}^{2(3)}) - P_{-i}(Y_{-i}|\mathbf{z}_i^2,\mathbf{z}_{-i}^{2(2)})}{P_{-i}(Y_{-i}|\mathbf{z}_i^2,\mathbf{z}_{-i}^{2(2)}) - P_{-i}(Y_{-i}|\mathbf{z}_i^2,\mathbf{z}_{-i}^{2(1)})}. \tag{11}$$

Equation (11) states that the equilibrium assumption imposes a restriction on the other player's CCP $P_{-i}(Y_{-i}|\cdot)$. The proof of this result follows a simple extension of Lemma 1 and is described below:

$$\begin{aligned}
\frac{P_{-i}(Y_{-i}|\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(3)}) - P_{-i}(Y_{-i}|\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(2)})}{P_{-i}(Y_{-i}|\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(2)}) - P_{-i}(Y_{-i}|\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(1)})} &= \frac{B_i(Y_{-i}|\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(3)}) - B_i(Y_{-i}|\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(2)})}{B_i(Y_{-i}|\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(2)}) - B_i(Y_{-i}|\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(1)})} \\
&= \frac{F_i\big[P_i(Y_i = 1|\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(3)})\big] - F_i\big[P_i(Y_i = 1|\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(2)})\big]}{F_i\big[P_i(Y_i = 1|\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(2)})\big] - F_i\big[P_i(Y_i = 1|\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(1)})\big]} \\
&= \frac{F_i\big[P_i(Y_i = 1|\mathbf{z}_i^2,\mathbf{z}_{-i}^{2(3)})\big] - F_i\big[P_i(Y_i = 1|\mathbf{z}_i^2,\mathbf{z}_{-i}^{2(2)})\big]}{F_i\big[P_i(Y_i = 1|\mathbf{z}_i^2,\mathbf{z}_{-i}^{2(2)})\big] - F_i\big[P_i(Y_i = 1|\mathbf{z}_i^2,\mathbf{z}_{-i}^{2(1)})\big]} \\
&= \frac{B_i(Y_{-i}|\mathbf{z}_i^2,\mathbf{z}_{-i}^{2(3)}) - B_i(Y_{-i}|\mathbf{z}_i^2,\mathbf{z}_{-i}^{2(2)})}{B_i(Y_{-i}|\mathbf{z}_i^2,\mathbf{z}_{-i}^{2(2)}) - B_i(Y_{-i}|\mathbf{z}_i^2,\mathbf{z}_{-i}^{2(1)})} \\
&= \frac{P_{-i}(Y_{-i}|\mathbf{z}_i^2,\mathbf{z}_{-i}^{2(3)}) - P_{-i}(Y_{-i}|\mathbf{z}_i^2,\mathbf{z}_{-i}^{2(2)})}{P_{-i}(Y_{-i}|\mathbf{z}_i^2,\mathbf{z}_{-i}^{2(2)}) - P_{-i}(Y_{-i}|\mathbf{z}_i^2,\mathbf{z}_{-i}^{2(1)})}.
\end{aligned}$$

[6]Note that this argument does not require player i to have unbiased beliefs. It just requires player i to understand that $P_{-i}(Y_{-i} = 1|\cdot)$ is increasing in Z_{-i}, but allows her to incorrectly predict the magnitude. Moreover, in the case that player i incorrectly predicts the direction so that she believes $P_{-i}(Y_{-i} = 1|\cdot)$ is decreasing in Z_{-i}. For values $z_i^2 > z_i^1$, we can still find $z_{-i}^2 < z_{-i}^1$, not $z_{-i}^2 > z_{-i}^1$, such that $P_i(Y_i = 1|z_i^1,z_{-i}^1) = P_i(Y_i = 1|z_i^2,z_{-i}^2)$.

The first and fifth lines follow the restriction of the equilibrium belief such that $B_i(\cdot) = P_{-i}(\cdot)$. The second and fourth lines follow a simple transformation of the results in Lemma 1 (see Equation (8)). The third line follows the conditions that $P_i(Y_i = 1|z_i^1, z_{-i}^{1(l)}) = P_i(Y_i = 1|z_i^2, z_{-i}^{2(l)})$ for $l = 1, 2, 3$.

Equation (11) derives a testable restriction of the equilibrium assumption under the condition of equal choice probability. This equal CCP condition has been exploited for both identification (Liu et al., 2017; Aguirregabiria, 2021) and estimation (Aradillas-López, 2012). When there are $N > 2$ players, the testable implication by Equation (11) turns to a rank restriction on a matrix of player $-i$'s CCPs.

In the context of firm competition, Proposition 5 in Aguirregabiria (2021) also provides a nonparametric test of the equilibrium assumption. However, in addition to the data on players' actual choices, Aguirregabiria (2021) also assumes that each firm's revenue can be observed by researchers. As a comparison, Equation (11) only requires observations of players' choices and is particularly useful when the information on revenue is unavailable or limited.

The second result establishes the point identification of the inverted choice probability function $F_i(\cdot)$. Note that the identification of $F_i(\cdot)$ implies that the C.D.F. of $\tilde{\varepsilon}_i$, denoted by $\tilde{\Gamma}_i(\cdot)$, is also identified. To identify $F_i(\cdot)$, it requires an additional assumption such that player i always has unbiased beliefs under one particular realization of $\mathbf{Z}_i$, say $\mathbf{z}_i^1$. For instance, $B_i(Y_{-i}|\mathbf{Z}_i = \mathbf{z}_i^1, \mathbf{Z}_{-i}) = P_{-i}(Y_{-i}|\mathbf{Z}_i = \mathbf{z}_i^1, \mathbf{Z}_{-i})$. In contrast, player i's belief is allowed to be biased under any other realizations of $\mathbf{Z}_i$. This assumption is referred to as the *local unbiased belief* restriction. It contrasts with the equilibrium condition in the existing literature that restricts each player to have unbiased belief at every realization of the control variables.

At first glance, the local unbiased belief assumption may seem to be against the motivation to relax the equilibrium restriction. However, a few notes should be emphasized. First, since variables in $\mathbf{Z}_i$ are continuous, the region with the unbiased belief restriction (i.e., only one realization $\mathbf{z}_i^1$) has a zero measure in the support of the payoff shifters. Therefore, the local unbiased belief assumption is a substantially weaker condition than the existing equilibrium restriction. Second, as described in the Introduction, realization $\mathbf{z}_i^1$ could represent the state that players are most familiar with. Therefore, the learning process could lead to a correct/unbiased belief. In contrast, beliefs could be potentially biased under other less familiar realizations. Third, Goeree and Holt (2001) conduct an experimental study with 10 different types of games, ranging from simple matching pennies, to coordination games, to complicated dynamic games with incomplete information. For each type of game, they find that the equilibrium condition holds for a particular realization of the payoffs. However, when they perturb the payoffs, players' behaviors substantially deviate from the equilibrium predictions. Their experimental evidence supports the local unbiased belief assumption in some states and justifies the potential biased beliefs under other states. Fourth, this paper also derives a testable implication of the unbiased belief restriction at any realization of $\mathbf{Z}_i$. This test could guide researchers to choose the realization that the local unbiased belief assumption holds. Note that such an assumption has also been exploited in the identification of incomplete information discrete games (Aguirregabiria and Magesan, 2020; Aguirregabiria and Xie, 2021; Xie, forthcoming).

Under the local unbiased belief assumption at $\mathbf{z}_i^1$, Equation (10) then turns to:

$$\pi_i(\mathbf{Z}_i=\mathbf{z}_i^1)+\delta_i(\mathbf{Z}_i=\mathbf{z}_i^1,Y_{-i}=1)\cdot P_{-i}(Y_{-i}=1|\mathbf{Z}_i=\mathbf{z}_i^1,\mathbf{Z}_{-i})=F_i\big[P_i(Y_i=1|\mathbf{Z}_i=\mathbf{z}_i^1,\mathbf{Z}_{-i})\big], \tag{12}$$

where $B_i(Y_{-i}|\mathbf{Z}_i=\mathbf{z}_i^1,\mathbf{Z}_{-i})$ is replaced by $P_{-i}(Y_{-i}|\mathbf{Z}_i=\mathbf{z}_i^1,\mathbf{Z}_{-i})$ due to the local unbiased belief assumption. Equation (12) first identifies the sign of the strategic effect. In particular, consider the variation of $\mathbf{Z}_{-i}$ while holding $\mathbf{Z}_i$ fixed at $\mathbf{z}_i^1$. Then $\delta_i(\mathbf{z}_i^1,Y_i=1)$ would be positive (negative) if an increase of $P_{-i}(Y_{-i}=1|\mathbf{z}_i^1,\mathbf{Z}_{-i})$ causes an increase (decrease) of $P_i(Y_i=1|\mathbf{z}_i^1,\mathbf{Z}_{-i})$. In addition, Assumption 3(b) normalizes the absolute value of the strategic effect to be one. Together with the identified sign, the value of $\delta_i(\mathbf{z}_i^1,Y_{-i}=1)$ is identified. Next, since $P_i(\cdot)$ is continuous in $\mathbf{Z}_{-i}$, there would exist a realization $\mathbf{Z}_{-i}=\mathbf{z}_{-i}$ such that $P_i(Y_i=1|\mathbf{z}_i^1,\mathbf{z}_{-i})=\frac{1}{2}$. Under this particular realization, Equation (12) becomes:

$$\pi_i(\mathbf{z}_i^1)+\delta_i(\mathbf{z}_i^1,Y_{-i}=1)\cdot P_{-i}(Y_{-i}=1|\mathbf{z}_i^1,\mathbf{z}_{-i})=F_i\big[P_i(Y_i=1|\mathbf{z}_i^1,\mathbf{z}_{-i})=\frac{1}{2}\big]=0.$$

Note that $F_i(\frac{1}{2})=0$ is due to the zero median condition by Assumption 5. The above equation would identify $\pi_i(\mathbf{z}_i^1)$ since $\delta_i(\mathbf{z}_i^1,Y_{-i}=1)$ has been identified and $P_{-i}(\cdot)$ is known by researchers. Given the identification of $\pi_i(\mathbf{z}_i^1)$ and $\delta_i(\mathbf{z}_i^1,Y_{-i}=1)$, Equation (12) then traces out the function $F_i(\cdot)$ with the variation of $\mathbf{Z}_{-i}$.

In single-agent static discrete choice models, Lewbel (2000) pioneers the approach of *special regressor* (i.e., an observed variable that enters additively and linearly into an agent's utility function). As shown by Lewbel (2000), this special regressor is required for the nonparametric identification of the distribution of the error term. Recently, Chen (2017) applies this approach to single-agent dynamic discrete choice models. In the context of discrete games with incomplete information, Lewbel and Tang (2015) introduce a player-specific special regressor. In particular, they assume that $\mathbf{Z}_i$ enters into player i's payoff function additively and linearly. Under this specification and the equilibrium assumption, Lewbel and Tang (2015) establish the nonparametric identification of all model primitives. As a comparison, this paper allows $\mathbf{Z}_i$ to enter nonparametrically and interactively into player i's payoff function. Consequently, my identification results do not require the special regressor. The reason is shown by Equation (12), where player $-i$'s CCP $P_{-i}(\cdot)$ can be seen as an observed variable. It enters additively and linearly into player i's expected payoff, with $\delta_i(\cdot)$ as its coefficient. Moreover, with a fixed $\mathbf{Z}_i$, $P_{-i}(\cdot)$ has exogenous variation provided by $\mathbf{Z}_{-i}$. This structure of incomplete information discrete games suggests that player $-i$'s CCP has already played the role of the special regressor. Therefore, introducing an additional special regressor into player i's payoff function is not required for the nonparametric identification of model primitives.

There exists a testable implication of the local unbiased belief assumption at realization $\mathbf{Z}_i = \mathbf{z}_i^1$. The test is simple based on the variation of $\mathbf{Z}_{-i}$. Specifically, suppose there are two realizations of $\mathbf{Z}_{-i}$, denoted by $\mathbf{z}_{-i}^1$ and $\mathbf{z}_{-i}^2$, such that player i has equal CCPs; for instance, $P_i(Y_i|\mathbf{z}_i^1, \mathbf{z}_{-i}^1) = P_i(Y_i|\mathbf{z}_i^1, \mathbf{z}_{-i}^2)$. This condition of equal CCPs implies that player i would have the same deterministic expected payoffs under these two realizations. Since player i has the same value of the payoff function (i.e., $\mathbf{Z}_i$ is fixed at $\mathbf{z}_i$), the equality of deterministic expected payoffs further implies that player i's beliefs at $\mathbf{z}_{-i}^1$ and $\mathbf{z}_{-i}^2$ must be equal. For instance, $B_i(Y_{-i} = 1|\mathbf{z}_i^1, \mathbf{z}_{-i}^1) = B_i(Y_{-i} = 1|\mathbf{z}_i^1, \mathbf{z}_{-i}^2)$. Under the local unbiased belief assumption such that $B_i(\cdot) = P_{-i}(\cdot)$ when $\mathbf{Z}_i = \mathbf{z}_i^1$, the above condition of equal beliefs turns to be a testable restriction imposed on player $-i$'s CCPs; for instance, $P_{-i}(Y_{-i} = 1|\mathbf{z}_i^1, \mathbf{z}_{-i}^1) = P_{-i}(Y_{-i} = 1|\mathbf{z}_i^1, \mathbf{z}_{-i}^2)$. To test such a restriction of the local unbiased belief assumption, there must exist two realizations, $\mathbf{z}_i^1$ and $\mathbf{z}_i^2$, such that the equal CCPs condition $P_i(Y_i|\mathbf{z}_i^1, \mathbf{z}_{-i}^1) = P_i(Y_i|\mathbf{z}_i^1, \mathbf{z}_{-i}^2)$ holds. These realizations always exist when $\mathbf{Z}_{-i}$ contains multiple variables (i.e., $L_\mathbf{z} > 1$) or when $\mathbf{Z}_{-i}$ is a single variable, denoted by Z_{-i} (i.e., $L_\mathbf{z} = 1$), but $P_i(Y_i = 1|z_i^1, Z_{-i})$ is non-monotone in Z_{-i}. The only case that the condition of equal CCPs cannot hold is when Z_{-i} is a single variable and $P_i(Y_i = 1|z_i^1, Z_{-i})$ is strictly monotone in Z_{-i}. Under such a case, $P_i(Y_i|z_i^1, z_{-i}^1) \neq P_i(Y_i|z_i^1, z_{-i}^2)$ for any $z_{-i}^1 \neq z_{-i}^2$, and the equal CCPs condition is never satisfied. Next, consider a game with $N > 2$ players: the testable restriction of the local unbiased belief assumption is a rank condition on a matrix of player $-i$'s CCPs. Unlike the case of two players, such a restriction in games with $N > 2$ players can be always tested, regardless of the dimension of $\mathbf{Z}_{-i}$ and the monotonicity of $P_i(\cdot)$. Subsection 4.2 formally establishes these results.

The last result of this paper deals with the identification of player i's payoff and belief functions. Rearrange Equation (8) and it yields:

$$B_i(Y_{-i}|\mathbf{Z}_i,\mathbf{Z}_{-i}) = B_i(Y_{-i}|\mathbf{Z}_i,\mathbf{z}^2_{-i}) + \frac{F_i\big[P_i(Y_i=1|\mathbf{Z}_i,\mathbf{Z}_{-i})\big]-F_i\big[P_i(Y_i=1|\mathbf{Z}_i,\mathbf{z}^2_{-i})\big]}{F_i\big[P_i(Y_i=1|\mathbf{Z}_i,\mathbf{z}^2_{-i})\big]-F_i\big[P_i(Y_i=1|\mathbf{Z}_i,\mathbf{z}^1_{-i})\big]}\cdot\big[B_i(Y_{-i}|\mathbf{Z}_i,\mathbf{z}^2_{-i})-B_i(Y_{-i}|\mathbf{Z}_i,\mathbf{z}^1_{-i})\big].$$

Since $F_i(\cdot)$ has been identified, the above equation suggests that the belief function $B_i(\cdot)$ is identified up to its values at only two realizations $\mathbf{Z}_{-i} = \mathbf{z}^1_{-i}$, $\mathbf{z}^2_{-i}$. Suppose that we further impose the local unbiased belief assumption at $\mathbf{z}^1_{-i}$, $\mathbf{z}^2_{-i}$, but allow player i to have biased beliefs under any other realizations of $\mathbf{Z}_{-i}$. It will identify $B_i(Y_{-i}|\mathbf{Z}_i,\mathbf{z}^1_{-i}) = P_{-i}(Y_{-i}|\mathbf{Z}_i,\mathbf{z}^1_{-i})$ and $B_i(Y_{-i}|\mathbf{Z}_i,\mathbf{z}^2_{-i}) = P_{-i}(Y_{-i}|\mathbf{Z}_i,\mathbf{z}^2_{-i})$. Consequently, $B_i(Y_i|\mathbf{Z}_i,\mathbf{Z}_{-i})$ is then identified under any other realizations of $\mathbf{Z}_{-i}$. Next, consider Equation (10), the exogenous variation of $\mathbf{Z}_{-i}$ affects $B_i(Y_{-i}=1|\cdot)$ but has no impact on player i's payoff function. Given the identification of $B_i(\cdot)$ and $F_i(\cdot)$, this exogenous variation of $\mathbf{Z}_{-i}$ implies that the base return $\pi_i(\cdot)$ and the strategic effect $\delta_i(\cdot)$ are identified (Aradillas-López, 2010; Bajari et al., 2010). In games with $N>2$ players, the payoff function is point identified while the belief function would be partially identified.

4.2 Identification Results

Based on the intuition described above, this subsection formally establishes the identification results in a general binary choice game with $N \geq 2$ players. First, I derive a testable implication of the equilibrium assumption. To see such an implication, similar to $\tilde{\mathbf{B}}_i(\cdot)$, define $\tilde{\mathbf{P}}_{-i}(\mathbf{Z}_i,\mathbf{z}^{1,2}_{-i}) = \mathbf{P}_{-i}(\mathbf{Z}_i,\mathbf{Z}_{-i}=\mathbf{z}^2_{-i}) - \mathbf{P}_{-i}(\mathbf{Z}_i,\mathbf{Z}_{-i}=\mathbf{z}^1_{-i})$ as the difference of other players' CCPs between two realizations $\mathbf{Z}_{-i}=\mathbf{z}^1_{-i}$, $\mathbf{z}^2_{-i}$, and let $\tilde{\mathbb{P}}_{-i}(\mathbf{Z}_i,\mathbf{z}^{1:l}_{-i}) = \big(\tilde{\mathbf{P}}_{-i}(\mathbf{Z}_i,\mathbf{z}^{1,2}_{-i}),\cdots,\tilde{\mathbf{P}}_{-i}(\mathbf{Z}_i,\mathbf{z}^{1,l}_{-i})\big)$ be the corresponding $(2^{N-1}-1)\times(l-1)$ matrix for any $l\geq 2$. Proposition 1 presents the testable implication of the equilibrium assumption.

Proposition 1. *Under Assumptions 1–4 and consider any* $2^{N-1}+1$ *pairs of realizations of* $(\mathbf{Z}_i,\mathbf{Z}_{-i})$*, denoted by* $(\mathbf{z}^1_i,\mathbf{z}^{1(l)}_{-i})$ *and* $(\mathbf{z}^2_i,\mathbf{z}^{2(l)}_{-i})$ $\forall l\leq 2^{N-1}+1$*, such that the following conditions hold:*

$$P_i(Y_i=1|\mathbf{Z}_i=\mathbf{z}^1_i,\mathbf{Z}_{-i}=\mathbf{z}^{1(l)}_{-i}) \neq P_i(Y_i=1|\mathbf{Z}_i=\mathbf{z}^1_i,\mathbf{Z}_{-i}=\mathbf{z}^{1(l')}_{-i}),\ \forall l\neq l',$$
$$P_i(Y_i=1|\mathbf{Z}_i=\mathbf{z}^1_i,\mathbf{Z}_{-i}=\mathbf{z}^{1(l)}_{-i}) = P_i(Y_i=1|\mathbf{Z}_i=\mathbf{z}^2_i,\mathbf{Z}_{-i}=\mathbf{z}^{2(l)}_{-i}),\ \forall l\leq 2^{N-1}+1.$$

Then the assumption that player i has unbiased/equilibrium beliefs implies the following testable restriction:

(a) If $N=2$, *then* $\frac{P_{-i}(Y_{-i}|\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(3)})-P_{-i}(Y_{-i}|\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(2)})}{P_{-i}(Y_{-i}|\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(2)})-P_{-i}(Y_{-i}|\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(1)})} = \frac{P_{-i}(Y_{-i}|\mathbf{z}_i^2,\mathbf{z}_{-i}^{2(3)})-P_{-i}(Y_{-i}|\mathbf{z}_i^2,\mathbf{z}_{-i}^{2(2)})}{P_{-i}(Y_{-i}|\mathbf{z}_i^2,\mathbf{z}_{-i}^{2(2)})-P_{-i}(Y_{-i}|\mathbf{z}_i^2,\mathbf{z}_{-i}^{2(1)})}$.

(b) If $N>2$, *then the rank of* $\left[\tilde{\mathbb{P}}_{-i}^{-1}(\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(1:2^{N-1})})\cdot\tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(2:2^{N-1}+1)})-\tilde{\mathbb{P}}_{-i}^{-1}(\mathbf{z}_i^2,\mathbf{z}_{-i}^{2(1:2^{N-1})})\cdot\tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^2,\mathbf{z}_{-i}^{2(2:2^{N-1}+1)})\right]$ *is at most* $2^{N-1}-2$, *note that this is a* $(2^{N-1}-1)\times(2^{N-1}-1)$ *matrix.*

Proof. In the Appendix. □

Proposition 1 considers $2^{N-1}+1$ pairs of realizations of $(\mathbf{Z}_i,\mathbf{Z}_{-i})$ that satisfy two conditions. These conditions require no further restrictions on the model and can be easily satisfied. Specifically, the first one, that $P_i(Y_i=1|\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(l)}) \neq P_i(Y_i=1|\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(l')})$, can easily hold because $\mathbf{Z}_{-i}$ has an impact on $P_i(Y_i|\cdot)$. Moreover, as described in Subsection 4.1, there exist infinite pairs of realizations such that the second condition of equal CCPs $P_i(Y_i=1|\mathbf{z}_i^1,\mathbf{z}_{-i}^{1(l)}) = P_i(Y_i=1|\mathbf{z}_i^2,\mathbf{z}_{-i}^{2(l)})$ holds. With any $2^{N-1}+1$ pairs of realizations that satisfy the above two conditions, Proposition 1 establishes a testable implication of the equilibrium assumption. Furthermore, the implication in Proposition 1(b) requires the matrix $\tilde{\mathbb{P}}_{-i}(\cdot)$ to be invertible. Such an invertibility condition would hold given the linear independence by Assumption 4(b). In addition, to test the above implication, one needs to estimate and test the rank of a matrix. This technique has been well developed in the econometrics literature: see Robin and Smith (2000), Kleibergen and Paap (2006), and Camba-Mendez and Kapetanios (2009). Finally, the testable implication in Proposition 1(b) suggests that a matrix of other players' CCPs is less than full rank and is therefore singular. Since the set of singular matrices has a zero measure in the space of random matrices, it suggests that the equilibrium condition imposes a strong restriction on the data, even under the nonparametric specification.

The second result of this paper establishes the point identification of the inverted choice probability function $F_i(\cdot)$. As described in Subsection 4.1, this result requires a local unbiased belief restriction as stated in Assumption 6. Note that this assumption is testable; the test will be described later.

Assumption 6. *There exists one realization of* $\mathbf{Z}_i$, *say* $\mathbf{z}_i^1$, *such that player i always has unbiased beliefs when* $\mathbf{Z}_i=\mathbf{z}_i^1$. *For instance,* $B_i(Y_{-i}|\mathbf{Z}_i=\mathbf{z}_i^1,\mathbf{Z}_{-i}) = P_{-i}(Y_{-i}|\mathbf{Z}_i=\mathbf{z}_i^1,\mathbf{Z}_{-i})$. *Moreover, player i's belief is allowed to be biased under any other realizations of* $\mathbf{Z}_i$.

Let $\mathcal{P}_i(\mathbf{z}_i)$ denote the image of player i's probability function $P_i(Y_i = 1|\mathbf{Z}_i = \mathbf{z}_i, \mathbf{Z}_{-i})$ when we only vary $\mathbf{Z}_{-i}$ but fix $\mathbf{Z}_i$ at $\mathbf{z}_i$. In particular, given the continuity condition by Assumption 4(a), $\mathcal{P}_i(\mathbf{z}_i) = \left[\min_{\mathbf{Z}_{-i}} P_i(Y_i = 1|\mathbf{z}_i, \mathbf{Z}_{-i}), \max_{\mathbf{Z}_{-i}} P_i(Y_i = 1|\mathbf{z}_i, \mathbf{Z}_{-i})\right]$. Furthermore, denote $int[\mathcal{P}_i(\mathbf{z}_i^1)]$ as the set of all interior points in $\mathcal{P}_i(\mathbf{z}_i^1)$. Proposition 2 establishes the point identification of the inverted choice probability function $F_i(\cdot)$.

Proposition 2. *Under Assumptions 1–6 and suppose that* $\frac{1}{2} \in int[\mathcal{P}_i(\mathbf{z}_i^1)]$, *then the inverted choice probability function* $F_i(p)$ *is point identified* $\forall p \in \mathcal{P}_i(\mathbf{z}_i^1)$.

Proof. In the Appendix. □

As described in Subsection 4.1, the identification of $F_i(\cdot)$ requires player i's CCP $P_i(Y_i = 1|\mathbf{z}_i^1, \mathbf{Z}_{-i})$ to go through the point of $\frac{1}{2}$. At this point, the deterministic expected payoff of action 1 equals the one of action 0, which is normalized to zero by Assumption 3(a). This equality is the key for the identification. Moreover, Proposition 2 identifies $F_i(p)$ in the region where $p \in \mathcal{P}_i(\mathbf{z}_i^1)$. When p lies outside of this region, some additional assumptions have to be imposed to identify $F_i(p)$. A simple one is to extend Assumption 6 to a finite number of other realizations of $\mathbf{Z}_i$. In particular, consider L realizations of $\mathbf{Z}_i$, denoted by $\mathbf{z}_i^1$ up to $\mathbf{z}_i^L$. With an appropriate choice of these L realizations, $\cup_{l=1}^L \mathcal{P}_i(\mathbf{z}_i^l)$ could well approximate the image of player i's CCP $P_i(Y_i = 1|\mathbf{Z}_i, \mathbf{Z}_{-i})$ when considering variations of both $\mathbf{Z}_i$ and $\mathbf{Z}_{-i}$. Therefore, if we assume that player i has unbiased beliefs when her payoff shifter equals any of these L realizations but allow her belief to be biased otherwise, the inverted choice probability function $F_i\left[P_i(Y_i = 1|\mathbf{Z}_i, \mathbf{Z}_{-i})\right]$ can be point identified for (almost) the entire image of player i's CCP $P_i(Y_i = 1|\cdot)$. The number of L required for identification depends on the impact of $\mathbf{Z}_{-i}$ on $P_i(\cdot)$ and is application specific. The important lesson is that we only need the local unbiased belief condition for a finite number of realizations. As described above, since variables in $\mathbf{Z}_i$ are all continuous, the region with the unbiased belief restriction has a zero measure in the space of control variables. Therefore, the local unbiased belief assumption is a substantially weaker condition than the equilibrium restriction in the existing literature. Importantly, the local unbiased belief assumption is testable, and Proposition 3 presents an implication of Assumption 6.

Proposition 3. *Suppose that Assumptions 1–4 hold. Consider any* 2^{N-1} *realizations of* $\mathbf{Z}_{-i}$, *denoted by* $\mathbf{z}_{-i}^l$ *for* $l \leq 2^{N-1}$, *such that* $P_i(Y_i|\mathbf{z}_i^1, \mathbf{z}_{-i}^l) = P_i(Y_i|\mathbf{z}_i^1, \mathbf{z}_{-i}^{l'})$ $\forall l \neq l'$. *Then, the assumption that player i has unbiased beliefs at realization* $\mathbf{Z}_i = \mathbf{z}_i^1$ *implies the following testable restriction:*

(a) If $N = 2$, then $P_{-i}(Y_{-i}|\mathbf{z}_i^1, \mathbf{z}_{-i}^1) = P_{-i}(Y_{-i}|\mathbf{z}_i^1, \mathbf{z}_{-i}^2)$.

(b) If $N > 2$, then the rank of the matrix $\tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1:2^{N-1}})$ is at most $2^{N-1} - 2$. Note that $\tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1:2^{N-1}})$ is a $(2^{N-1} - 1) \times (2^{N-1} - 1)$ matrix.

Proof. In the Appendix. □

Similar to Proposition 1, the testable implication of the local unbiased belief assumption in Proposition 3 also requires the condition of equal CCPs. However, it is important to note that the condition in Proposition 3 is slightly stronger than the one in Proposition 1. Specifically, a player's CCP holds fixed under some variations of both $\mathbf{Z}_i$ and $\mathbf{Z}_{-i}$ in Proposition 1. In contrast, the equal CCPs condition is required to hold with only the variation of $\mathbf{Z}_{-i}$ in Proposition 3. This stronger condition can be always satisfied in games with $N > 2$ players. To see this point, consider an entry game with 3 firms. For simplicity, suppose that $\mathbf{Z}_i$ is a single variable denoted by Z_i. Moreover, assume that a higher Z_i strictly reduces firm i's entry cost and strictly raises its entry probability. Due to strategic substitutability in the entry game, player i's entry probability $P_i(Y_i = 1|Z_i, Z_j, Z_{j'})$ is strictly increasing in its own Z_i but is strictly decreasing in the two other firms' Z_j, $Z_{j'}$. Due to continuity, we can find a pair of realizations $(z_j^1, z_{j'}^1)$ and $(z_j^2, z_{j'}^2)$, where $z_j^1 > z_j^2$ and $z_{j'}^1 < z_{j'}^2$, such that $P_i(Y_i = 1|Z_i, z_j^1, z_{j'}^1) = P_i(Y_i = 1|Z_i, z_j^2, z_{j'}^2)$. Essentially, we can find infinite pairs that hold firm i's entry probability fixed. In this game with $N > 2$ players, Assumption 6 implies a rank restriction on a matrix of other players' CCPs, as stated in Proposition 3(b). In contrast, in games with two players, the testable implication is an equality restriction as in Proposition 3(a). Cautiously, as described in Subsection 4.1, in this two-player game, it is not always possible to find a pair of realizations such that the equal CCPs condition holds. Such a condition could hold when $\mathbf{Z}_{-i}$ is a vector of multiple variables or when Z_{-i} is a single variable and $P_i(Y_i|\cdot)$ is non-monotone in Z_{-i}. However, the equal CCPs condition cannot hold when $P_i(\cdot)$ is strictly monotone in Z_{-i}.

Given the identification of $F_i(\cdot)$, the last result of this section establishes the identification of player i's payoff function and belief function. This identification result requires an additional assumption such that player i's belief is unbiased under 2^{N-1} realizations of other players' payoff shifters $\mathbf{Z}_{-i}$. This restriction, summarized by Assumption 7, is also imposed in Aguirregabiria and Magesan (2020), Aguirregabiria and Xie (2021), and Xie (forthcoming).

Assumption 7. *There exist 2^{N-1} realizations of $\mathbf{Z}_{-i}$, each denoted by $\mathbf{z}_{-i}^l$, such that player i always has* *unbiased beliefs when* $\mathbf{Z}_{-i} = \mathbf{z}^{l}_{-i}$ $\forall l \leq 2^{N-1}$. *Moreover,* $P_i(Y_i|\mathbf{Z}_i, \mathbf{Z}_{-i} = \mathbf{z}^{l}_{-i}) \neq P_i(Y_i|\mathbf{Z}_i, \mathbf{Z}_{-i} = \mathbf{z}^{l'}_{-i})$ $\forall l \neq l'$. *In contrast, player i's belief is allowed to be biased under any other realizations of* $\mathbf{Z}_{-i}$.

Proposition 4. *Suppose that the conditions met in Proposition 2 hold so that* $F_i(\cdot)$ *is point identified. Assumption 7 implies the following:*

(a) Player i's base return $\pi_i(\mathbf{Z}_i)$ *and strategic effect* $\delta_i(\mathbf{Z}_i, \mathbf{Y}_{-i})$ *are point identified.*

(b) When $N = 2$*, player i's belief function* $B_i(\mathbf{Y}_{-i}|\mathbf{Z}_i, \mathbf{Z}_{-i})$ *is point identified.*

(c) When $N > 2$*, Player i's belief function, represented by the* $(2^{N-1} - 1) \times 1$ *vector* $\mathbf{B}_i(\mathbf{Z}_i, \mathbf{Z}_{-i})$*, is partially identified on a hyperplane with a dimension of* $2^{N-1} - 2$.

Proof. In the Appendix. □

Under the assumption that the distribution of private information is known by researchers, Aguirregabiria and Magesan (2020) and Aguirregabiria and Xie (2021) obtain a special result of Proposition 4 in games with two players. This paper extends their results in two directions. First, the distribution of private information is nonparametrically specified and is first identified in Proposition 2. Second, Proposition 4 extends the result to games with $N > 2$ players. In these games, player i's behavior depends on her believed probabilities of multiple events (i.e., the action of each of the other players and their potential correlation). There exist infinite transformations of beliefs that could hold player i's deterministic expected payoff constant (i.e., mean-preserving transformation). These transformations are indistinguishable, and therefore the entire belief vector $\mathbf{B}_i(\mathbf{Z}_i, \mathbf{Z}_{-i})$ cannot be point identified. However, this vector could be partially identified on a hyperplane with a lower dimension, as in Proposition 4(c). In contrast, player i's belief in a two-player game is a scalar and can be point identified.

5 Identification in Multinomial Choice Games

This section extends the identification results in binary choice games as presented in Section 4 (i.e., $K = 1$) to games with more actions (i.e., $K > 1$). Specifically, player i's CCP is represented by a scalar $P_i(Y_i = 1|\mathbf{Z}_i, \mathbf{Z}_{-i})$ when $K = 1$, but expands to a $K \times 1$ vector $\mathbf{P}_i(\mathbf{Z}_i, \mathbf{Z}_{-i})$ when $K > 1$. Such a difference in the dimensions of CCPs introduces two obstacles in the extension to multinomial choice games. Each one requires an additional assumption to establish the identification results. The first obstacle is the construction of the equal CCPs condition. As described in Section 4, when $K = 1$, the continuity condition by Assumption 4 directly implies that $P_i(Y_i = 1|\mathbf{z}_i^1, \mathbf{z}_{-i}^1) = P_i(Y_i = 1|\mathbf{z}_i^2, \mathbf{z}_{-i}^2)$ for some pairs $(\mathbf{z}_i^1, \mathbf{z}_{-i}^1)$ and $(\mathbf{z}_i^2, \mathbf{z}_{-i}^2)$. This condition of equal CCPs is the key to identifying all model primitives. In more details, the local unbiased belief assumption at $\mathbf{z}_i^1$ would first identify the value of the inverted choice probability function $F_i[P_i(Y_i = 1|\mathbf{z}_i^1, \mathbf{z}_i^1)]$. In the second step, the equal CCPs condition implies that $F_i[P_i(Y_i = 1|\mathbf{z}_i^2, \mathbf{z}_i^2)]$ equals $F_i[P_i(Y_i = 1|\mathbf{z}_i^1, \mathbf{z}_i^1)]$ and is therefore identified. In the last step, the identification of $F_i[P_i(Y_i = 1|\mathbf{z}_i^2, \mathbf{z}_{-i}^2)]$ is exploited to identify player i's payoff function and belief function at realization $(\mathbf{z}_i^2, \mathbf{z}_{-i}^2)$. Unfortunately, when $K > 1$, Assumption 4 does not necessarily imply the equal CCPs condition. Specifically, there could exist no pairs of realizations such that the vector $\mathbf{P}_i(\mathbf{z}_i^1, \mathbf{z}_{-i}^1) = \mathbf{P}_i(\mathbf{z}_i^2, \mathbf{z}_{-i}^2)$. This obstacle is addressed by extending Assumption 4 and imposing an additional technical restriction on player i's CCP. This restriction is summarized by Assumption 4' and guarantees that the equal CCPs condition could also hold in multinomial choice games.

Assumption 4'. *(a) Assumption 4 holds.*
(b) For each player i, the rank of the matrix $\frac{\partial \mathbf{P}_i(\mathbf{Z}_i, \mathbf{Z}_{-i})}{\partial \mathbf{Z}'_{-i}}$ *is K. Recall that the dimension of* $\mathbf{Z}_i$ *is* $L_\mathbf{z}$ *and therefore,* $\frac{\partial \mathbf{P}_i(\mathbf{Z}_i, \mathbf{Z}_{-i})}{\partial \mathbf{Z}'_{-i}}$ *is a* $K \times (N-1)L_\mathbf{z}$ *matrix.*

Assumption 4'(b) requires the matrix $\frac{\partial \mathbf{P}_i(\mathbf{Z}_i, \mathbf{Z}_{-i})}{\partial \mathbf{Z}'_{-i}}$ to be full rank. First, recall that $\mathcal{P}_i$ represents the image of the function $\mathbf{P}_i(\mathbf{Z}_i, \mathbf{Z}_{-i})$, with variations of both $\mathbf{Z}_i$ and $\mathbf{Z}_{-i}$. Next, consider an arbitrary value of CCP $\mathbf{P}_i(\mathbf{z}_i^1, \mathbf{z}_{-i}^1)$ that lies in the interior of $\mathcal{P}_i$. Together with the continuity assumption, the full rank condition of $\frac{\partial \mathbf{P}_i(\mathbf{Z}_i, \mathbf{Z}_{-i})}{\partial \mathbf{Z}'_{-i}}$ implies that there exist infinite realizations, denoted by $(\mathbf{z}_i^2, \mathbf{z}_{-i}^2)$, such that $\mathbf{P}_i(\mathbf{z}_i^1, \mathbf{z}_{-i}^1) = \mathbf{P}_i(\mathbf{z}_i^2, \mathbf{z}_{-i}^2)$. Therefore, the condition of equal CCPs can be always satisfied. Importantly, since $\mathbf{P}_i(\cdot)$ can be consistently estimated, the full rank condition by Assumption 4'(b) is testable.

A necessary order condition for $\frac{\partial \mathbf{P}_i(\mathbf{Z}_i, \mathbf{Z}_{-i})}{\partial \mathbf{Z}'_{-i}}$ to have full rank is that $(N-1)L_\mathbf{z} \geq K \Rightarrow L_\mathbf{z} \geq \frac{K}{N-1}$. This relationship implies that, in a game with weakly more players than actions (i.e., $N \geq K+1$), the model can be identified with a single variable payoff shifter (i.e., $L_\mathbf{z} = 1$). In contrast, when a game has strictly more actions (i.e., $N < K+1$), the identification requires multiple variables as each player's payoff shifter (i.e., $L_\mathbf{z} > 1$). Importantly, when the matrix $\frac{\partial \mathbf{P}_i(\mathbf{Z}_i, \mathbf{Z}_{-i})}{\partial \mathbf{Z}'_{-i}}$ does not have full rank, the equilibrium assumption is still testable but with a different implication. The Appendix presents such a testable restriction. As a comparison, without Assumption 4'(b), I fail to obtain the identification of the payoff function, the belief function, and the distribution of private information.

Since $\frac{\partial \mathbf{P}_i(\mathbf{Z}_i,\mathbf{Z}_{-i})}{\partial \mathbf{Z}'_{-i}}$ can be seen as a random matrix given random variables $(\mathbf{Z}_i, \mathbf{Z}_{-i})$, the full rank condition by Assumption 4'(b) is likely to be satisfied in the data. Unfortunately, Assumption 4'(b) also excludes an important class of payoff structures in empirical games. To see this point, first note that by the chain rule, $\frac{\partial \mathbf{P}_i(\mathbf{Z}_i,\mathbf{Z}_{-i})}{\partial \mathbf{Z}'_{-i}} = \frac{\partial \mathbf{G}_i[\mathbf{EU}_i(\mathbf{Z}_i,\mathbf{Z}_{-i})]}{\partial \mathbf{EU}_i(\mathbf{Z}_i,\mathbf{Z}_{-i})'} \cdot \mathbf{\Delta}_i(\mathbf{Z}_i) \cdot \frac{\partial \mathbf{B}_i(\mathbf{Z}_i,\mathbf{Z}_{-i})}{\partial \mathbf{Z}'_{-i}}$. Consequently, a necessary condition for $\frac{\partial \mathbf{P}_i(\mathbf{Z}_i,\mathbf{Z}_{-i})}{\partial \mathbf{Z}'_{-i}}$ to have full rank is that the matrix of strategic effects $\mathbf{\Delta}_i(\mathbf{Z}_i)$ is also full rank. This full rank condition of $\mathbf{\Delta}_i$ contrasts the structure of *multiplicatively separable* strategic effects considered by Sweeting (2009), and Aradillas-López and Gandhi (2016), among others. In more details, these papers specify $\delta_i(\mathbf{Z}_i, Y_i, \mathbf{Y}_{-i}) = \hat{\delta}_i(\mathbf{Z}_i, Y_i) \cdot h_i(\mathbf{Z}_i, \mathbf{Y}_{-i})$ for some functions $\hat{\delta}_i(\cdot)$ and $h_i(\cdot)$. This specification implies that the rank of $\mathbf{\Delta}_i(\mathbf{Z}_i)$ is one and consequently violates Assumption 4'(b). However, it is important to emphasize two points. First, as described above, the equilibrium assumption is still testable even when Assumption 4'(b) fails. The Appendix presents such a testable restriction. Second, under the multiplicatively separable strategic effects, even though the model primitives are non-identified without the equilibrium restriction, this paper's identification arguments could still be useful to establish the identification results under the equilibrium condition.

Proposition 1'. *Suppose that Assumptions 1–3 and Assumption 4' hold. Consider any* $(K+1)^{N-1}+1$ *pairs of realizations of* $(\mathbf{Z}_i, \mathbf{Z}_{-i})$*, denoted by* $(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(l)})$ *and* $(\mathbf{z}_i^2, \mathbf{z}_{-i}^{2(l)})$ $\forall l \leq (K+1)^{N-1}+1$*, such that the following conditions hold:*

$$\mathbf{P}_i(\mathbf{Z}_i = \mathbf{z}_i^1, \mathbf{Z}_{-i} = \mathbf{z}_{-i}^{1(l)}) \neq \mathbf{P}_i(\mathbf{Z}_i = \mathbf{z}_i^1, \mathbf{Z}_{-i} = \mathbf{z}_{-i}^{1(l')}),\ \forall l \neq l'$$
$$\mathbf{P}_i(\mathbf{Z}_i = \mathbf{z}_i^1, \mathbf{Z}_{-i} = \mathbf{z}_{-i}^{1(l)}) = \mathbf{P}_i(\mathbf{Z}_i = \mathbf{z}_i^2, \mathbf{Z}_{-i} = \mathbf{z}_{-i}^{2(l)}),\ \forall l \leq (K+1)^{N-1}+1.$$

Then the assumption that player i has unbiased/equilibrium beliefs implies the following testable restriction:

(a) If $N=2$*, then* $\tilde{\mathbb{P}}_{-i}^{-1}(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(1:K+1)}) \cdot \tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(2:K+2)}) = \tilde{\mathbb{P}}_{-i}^{-1}(\mathbf{z}_i^2, \mathbf{z}_{-i}^{2(1:K+1)}) \cdot \tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^2, \mathbf{z}_{-i}^{2(2:K+2)})$.

(b) If $N>2$*, then* $\big[\tilde{\mathbb{P}}_{-i}^{-1}(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(1:(K+1)^{N-1})}) \cdot \tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(2:(K+1)^{N-1}+1)}) - \tilde{\mathbb{P}}_{-i}^{-1}(\mathbf{z}_i^2, \mathbf{z}_{-i}^{2(1:(K+1)^{N-1})}) \cdot \tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^2, \mathbf{z}_{-i}^{2(2:(K+1)^{N-1}+1)})\big]$ *has a rank of at most* $(K+1)^{N-1}-K-1$*. Note that the above is a* $\big((K+1)^{N-1}-1\big) \times \big((K+1)^{N-1}-1\big)$

matrix.

Proof. In the Appendix. □

Proposition 1' extends Proposition 1 to games with more than two actions. In particular, Proposition 1 can be seen as a special case of Proposition 1' when a game has binary choice (i.e., $K = 1$).

To see the second obstacle in multinomial choice games, consider a binary choice game first. The zero median condition of the private information (e.g., which is essentially a normalization) by Assumption 5 implies that $EU_i(\cdot, Y_i = 1) = EU_i(\cdot, Y_i = 0)$ if and only if $P_i(Y_i = 1|\cdot) = P_i(Y_i = 0|\cdot)$. This relationship is the key to identifying the inverted choice probability function $F_i(\cdot)$ and other model primitives thereafter. In contrast, in a multinomial choice game, $P_i(Y_i = k|\cdot) = P_i(Y_i = k'|\cdot)$ does not necessarily imply $EU_i(\cdot, Y_i = k) = EU_i(\cdot, Y_i = k')$ and vice versa. Therefore, it is substantially difficult to retrieve information from each player's CCP and establish the identification results. To address such an obstacle, this section imposes an additional restriction, as stated in Assumption 5'. Under this assumption, if two actions are chosen with equal probability, these two actions must have the same deterministic expected payoff.

Assumption 5'. *For any two actions, denoted by k and k', we have $EU_i(\mathbf{Z}_i, \mathbf{Z}_{-i}, Y_i = k) > EU_i(\mathbf{Z}_i, \mathbf{Z}_{-i}, Y_i = k')$ if and only if $P_i(Y_i = k|\mathbf{Z}_i, \mathbf{Z}_{-i}) > P_i(Y_i = k'|\mathbf{Z}_i, \mathbf{Z}_{-i})$.*

In discrete choice models, Assumption 5' is referred to as the rank ordering property. This property is first introduced by Manski (1975) and is subsequently applied by Goeree et al. (2005) and Fox (2007), among others. Specifically, the rank ordering property restricts an action to be chosen more frequently if and only if it has a higher deterministic expected payoff. This property can be satisfied for many distributions of private information. The simplest case is that $\varepsilon_i(Y_i)$ follows an identical nonparametric distribution over the real line and is independent across each action. In a slightly more complicated scenario, Assumption 5' could also be satisfied when the private information is correlated among actions. Consider that ε_i follows a multivariate normal distribution with the restrictions $Var(\varepsilon_i(k)) = \sigma^2 \; \forall k$ and $Cov(\varepsilon_i(k), \varepsilon_i(k')) = \rho\sigma^2 \; \forall k \neq k'$. This error structure with fixed variance and covariance across actions satisfies the rank ordering property. Furthermore, Goeree et al. (2005) provide a weaker sufficient condition, referred to as the exchangeability, for Assumption 5'. In more details, exchangeability requires the distribution function $\Gamma_i(\cdot)$ to be fixed for any permutation of ε_i. Unfortunately, not all distribution functions $\Gamma_i(\cdot)$ are consistent with the rank ordering property. Two important classes of discrete choice models that do not satisfy Assumption 5' are the nested Logit model and the random coefficient model (Berry et al., 1995).[7] It is also important to emphasize that the rank ordering property is always satisfied in binary choice games. Finally, Assumption 5' implies that $P_i(Y_i = k|\cdot) = P_i(Y_i = k'|\cdot)$ if and only if $EU_i(\cdot, Y_i = k) = EU_i(\cdot, Y_i = k')$. This condition is the key to establishing the identification of the model primitives.

Similar to the binary choice game in Section 4, the identification of the inverted choice probability function $\mathbf{F}_i(\cdot)$ requires Assumption 6 such that player i has unbiased beliefs under just one realization $\mathbf{Z}_i = \mathbf{z}_i^1$. Furthermore, in multinomial choice games, this assumption needs to be extended slightly such that two actions could be chosen with equal probability at $\mathbf{Z}_i = \mathbf{z}_i^1$. The following definition introduces two terminologies that facilitate the expression of this extension.

Definition 1. *(a) For each player i, a pair of actions k and k' is called **directly connected** at $\mathbf{Z}_i = \mathbf{z}_i$ if there exists a realization of $\mathbf{Z}_{-i}$, say $\mathbf{z}_{-i}$, such that $P_i(Y_i = k|\mathbf{Z}_i = \mathbf{z}_i, \mathbf{Z}_{-i} = \mathbf{z}_{-i}) = P_i(Y_i = k'|\mathbf{Z}_i = \mathbf{z}_i, \mathbf{Z}_i = \mathbf{z}_{-i})$.*
*(b) For each player i, a pair of actions k and k' is called **indirectly connected** at $\mathbf{Z}_i = \mathbf{z}_i$ if these two actions are not directly connected. However, there exists a sequence of actions $\{k, l, l', l'', \cdots, l''', k'\}$ where each pair of adjacent actions in this sequence is directly connected.*

Assumption 6'. *(a) Assumption 6 holds. For instance, player i always has unbiased beliefs when $\mathbf{Z}_i = \mathbf{z}_i^1$; while her belief is allowed to be biased under any other realizations of $\mathbf{Z}_i$.*
(b) For any two actions $k \neq k'$ of player i, they are either directly connected or indirectly connected at $\mathbf{Z}_i = \mathbf{z}_i^1$.

In binary choice games described in Section 4, the identification result in Proposition 2 requires player i's CCP $P_i(Y_i = 1|\cdot)$ to go through the point of $\frac{1}{2}$. At this point, $P_i(Y_i = 0|\cdot) = P_i(Y_i = 1|\cdot)$ so that actions 0 and 1 are directly connected. Assumption 6'(b) imposes a similar restriction in multinomial choice games. Since this restriction is imposed on player i's CCP, it is testable. Proposition 2' then establishes the point identification of the inverted choice probability function $\mathbf{F}_i(\cdot)$.

Proposition 2'. *Under Assumptions 1–3, 4'–6', the inverted choice probability function $\mathbf{F}_i(\mathbf{p})$ is point identified $\forall \mathbf{p} \in \mathcal{P}_i(\mathbf{z}_i^1)$.*

[7] Since the nested Logit is a parametric model, it faces less identification burden than the nonparametric problem considered in this paper. Therefore, even though the error structure of the nested Logit does not satisfy Assumption 5', it could be identified.

Proof. In the Appendix. □

Similar to Section 4, when the local unbiased belief assumption is imposed on sufficiently many but a finite number of realizations of $\mathbf{Z}_i$, the inverted choice probability function $\mathbf{F}_i\left[\mathbf{P}_i(\mathbf{Z}_i,\mathbf{Z}_{-i})\right]$ can be point identified for (almost) the entire image of player i's CCP $\mathbf{P}_i(\mathbf{Z}_i,\mathbf{Z}_{-i})$. Finally, the assumption that player i has unbiased belief when $\mathbf{Z}_i=\mathbf{z}_i^1$ is testable. Proposition 3' describes the testable implication.

Proposition 3'. *Suppose that Assumptions 1–3 and Assumption 4' hold. Consider any $(K+1)^{N-1}$ realizations of $\mathbf{Z}_{-i}$, denoted by $\mathbf{z}_{-i}^l$ $\forall l\le(K+1)^{N-1}$, such that $\mathbf{P}_i(\mathbf{z}_i^1,\mathbf{z}_{-i}^l)=\mathbf{P}_i(\mathbf{z}_i^1,\mathbf{z}_{-i}^{l'})$ $\forall l\neq l'$. Then, the assumption that player i has unbiased beliefs at realization $\mathbf{Z}_i=\mathbf{z}_i^1$ implies the following testable restriction:*

(a) If $N=2$, then the matrix $\tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^1,\mathbf{z}_{-i}^{1:K+1})=\mathbf{0}$.

(b) If $N>2$, then the rank of the matrix $\tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^1,\mathbf{z}_{-i}^{1:(K+1)^{N-1}})$ is at most $(K+1)^{N-1}-K-1$. Note that $\tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^1,\mathbf{z}_{-i}^{1:(K+1)^{N-1}})$ is a $\left((K+1)^{N-1}-1\right)\times\left((K+1)^{N-1}-1\right)$ matrix.

Proof. In the Appendix. □

In particular, Proposition 3 in Section 4 can be seen as a special case of Proposition 3' when each player has binary choice (i.e., $K=1$). Moreover, similar to the comparison in Section 4, the condition of equal CCPs required in Proposition 3' is slightly stronger than the one in Proposition 1'. Specifically, an order condition $(N-1)\cdot L_{\mathbf{z}}>K$ is sufficient for the equal CCPs condition in Proposition 3' to hold. Note that the order condition $(N-1)\cdot L_{\mathbf{z}}>K$ is slightly stronger than the one implied by Assumption 4'(b) (i.e., $(N-1)\cdot L_{\mathbf{z}}\ge K$).

Given the identification of $\mathbf{F}_i(\cdot)$, the last result of this paper establishes the identification of player i's payoff function and belief function. Compared with Assumption 7 in Section 4, the identification results require the local unbiased belief assumption under more realizations of $\mathbf{Z}_{-i}$. This is because multinomial choice games introduce additional action profiles and more unknown payoffs associated with these profiles. Consequently, more restrictions are needed for the identification. Assumption 7' summarizes these restrictions.

Assumption 7'. *There exist $(K+1)^{N-1}$ realizations of $\mathbf{Z}_{-i}$, denoted by $\mathbf{z}_{-i}^l$ $\forall l\le(K+1)^{N-1}$, such that player i always has unbiased beliefs when $\mathbf{Z}_{-i}=\mathbf{z}_{-i}^l$ $\forall l$. Moreover, $\mathbf{P}_i(\mathbf{Z}_i,\mathbf{Z}_{-i}=\mathbf{z}_{-i}^l)\neq\mathbf{P}_i(\mathbf{Z}_i,\mathbf{Z}_{-i}=\mathbf{z}_{-i}^{l'})$ $\forall l\neq l'$. In contrast, player i's belief is allowed to be biased under any other realizations of $\mathbf{Z}_{-i}$.*

Proposition 4'. *Suppose that the conditions met in Proposition 2' hold so that* $\mathbf{F}_i(\cdot)$ *is point identified. Assumption 7' implies the following:*

(a) Player i's base return $\pi_i(\mathbf{Z}_i, Y_i)$ *and strategic effect* $\delta_i(\mathbf{Z}_i, Y_i, \mathbf{Y}_{-i})$ *are point identified.*

(b) When $N = 2$*, player i's belief function* $B_i(\mathbf{Y}_{-i}|\mathbf{Z}_i, \mathbf{Z}_{-i})$ *is point identified.*

(c) When $N > 2$*, player i's belief function, represented by the* $\left((K+1)^{N-1} - 1\right) \times 1$ *vector* $\mathbf{B}_i(\mathbf{Z}_i, \mathbf{Z}_{-i})$*, is partially identified on a hyperplane with a dimension of* $(K+1)^{N-1} - K - 1$*.*

Proof. In the Appendix. □

In particular, Assumption 7 and Proposition 4 in Section 4 can be seen as special cases of Assumption 7' and Proposition 4' when each player has binary choice (i.e., $K = 1$).

6 Conclusion

This paper jointly relaxes two restrictions—the parametric assumption and the equilibrium assumption—that are commonly imposed in the literature of empirical discrete choice games with incomplete information. The model nonparametrically specifies both the payoff function and the distribution of private information. In addition, each player's belief function is allowed to be any probability distribution over other players' action sets. Under this framework, I first derive a testable implication of the equilibrium restriction. When players' beliefs are unbiased in a small subset of the space of the control variables, this paper further achieves the point identification of the payoff function and the distribution of private information. The belief function is partially identified in games with $N > 2$ players and is point identified in games with $N = 2$ players. Importantly, the subset with the unbiased belief restriction has a zero measure in the support of control variables. The null hypothesis of the unbiased belief assumption at any realization of the payoff shifter is also testable.

This paper derives a set of identification results but leaves the estimation method as a future topic. Deriving a consistent estimator based on this paper's identification results and testing the equilibrium restriction in a relevant empirical application are important areas for future research.

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A Appendix

Proof of Lemma 1: Consider the realizations $\mathbf{z}^1_{-i}$ and $\mathbf{z}^2_{-i}$ of $\mathbf{Z}_{-i}$ and plug them into Equation (6). We then obtain the following expressions:

$$\begin{aligned}\mathbf{\Pi}(\mathbf{Z}_i) + \mathbf{\Delta}_i(\mathbf{Z}_i)\cdot\mathbf{B}_i(\mathbf{Z}_i,\mathbf{z}^1_{-i}) &= \mathbf{F}_i\left[\mathbf{P}_i(\mathbf{Z}_i,\mathbf{z}^1_{-i})\right],\\ \mathbf{\Pi}(\mathbf{Z}_i) + \mathbf{\Delta}_i(\mathbf{Z}_i)\cdot\mathbf{B}_i(\mathbf{Z}_i,\mathbf{z}^2_{-i}) &= \mathbf{F}_i\left[\mathbf{P}_i(\mathbf{Z}_i,\mathbf{z}^2_{-i})\right].\end{aligned}$$

For notation simplicity, I exclude the random variable $\mathbf{Z}_{-i}$ as the argument of CCP and belief function but keep its realizations $\mathbf{z}^1_{-i}$, $\mathbf{z}^2_{-i}$ in the expression. This notation also applies to the rest of this Appendix. Subtracting the above two equations yields the following difference of deterministic expected payoffs:

$$\mathbf{\Delta}_i(\mathbf{Z}_i)\cdot\tilde{\mathbf{B}}_i(\mathbf{Z}_i,\mathbf{z}^{1,2}_{-i}) = \tilde{\mathbf{F}}_i(\mathbf{Z}_i,\mathbf{z}^{1,2}_{-i}). \tag{13}$$

Next, consider realizations $\mathbf{z}^1_{-i}$ to $\mathbf{z}^{(K+1)^{N-1}}_{-i}$ of $\mathbf{Z}_{-i}$. Equation (13) then turns to the following matrix form:

$$\mathbf{\Delta}_i(\mathbf{Z}_i)\cdot\tilde{\mathbb{B}}_i(\mathbf{Z}_i,\mathbf{z}^{1:(K+1)^{N-1}}_{-i}) = \tilde{\mathbb{F}}_i(\mathbf{Z}_i,\mathbf{z}^{1:(K+1)^{N-1}}_{-i}) \;\Rightarrow\; \mathbf{\Delta}_i(\mathbf{Z}_i) = \tilde{\mathbb{F}}_i(\mathbf{Z}_i,\mathbf{z}^{1:(K+1)^{N-1}}_{-i})\cdot\tilde{\mathbb{B}}_i^{-1}(\mathbf{Z}_i,\mathbf{z}^{1:(K+1)^{N-1}}_{-i}). \tag{14}$$

Equation (14) holds since $\tilde{\mathbb{B}}_i(\mathbf{Z}_i,\mathbf{z}^{1:(K+1)^{N-1}}_{-i})$ is invertible as required in Lemma 1. Similarly, for realizations $\mathbf{z}^2_{-i}$ to $\mathbf{z}^{(K+1)^{N-1}+1}_{-i}$ of $\mathbf{Z}_{-i}$, we have the following relationship:

$$\mathbf{\Delta}_i(\mathbf{Z}_i) = \tilde{\mathbb{F}}_i(\mathbf{Z}_i,\mathbf{z}^{2:(K+1)^{N-1}+1}_{-i})\cdot\tilde{\mathbb{B}}_i^{-1}(\mathbf{Z}_i,\mathbf{z}^{2:(K+1)^{N-1}+1}_{-i}). \tag{15}$$

Since $\mathbf{\Delta}_i(\mathbf{Z}_i)$ on the left-hand sides of Equations (14) and (15) is the same, we can equate these two equations and obtain the following:

$$\tilde{\mathbb{F}}_i(\mathbf{Z}_i,\mathbf{z}^{1:(K+1)^{N-1}}_{-i})\cdot\tilde{\mathbb{B}}_i^{-1}(\mathbf{Z}_i,\mathbf{z}^{1:(K+1)^{N-1}}_{-i}) = \tilde{\mathbb{F}}_i(\mathbf{Z}_i,\mathbf{z}^{2:(K+1)^{N-1}+1}_{-i})\cdot\tilde{\mathbb{B}}_i^{-1}(\mathbf{Z}_i,\mathbf{z}^{2:(K+1)^{N-1}+1}_{-i}).$$

This corresponds to Equation (7) in Lemma 1 and completes the proof. □

Proof of Proposition 1: Under the condition that player i has unbiased beliefs, we have $B_i(\cdot) = P_{-i}(\cdot)$

when $\mathbf{Z}_i = \mathbf{z}_i^1$. Plug this relationship into Equation (7) in Lemma 1 and obtain the following relationship:

$$\begin{aligned}
&\tilde{\mathbb{F}}_i(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(1:2^{N-1})}) \cdot \tilde{\mathbb{P}}_{-i}^{-1}(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(1:2^{N-1})}) = \tilde{\mathbb{F}}_i(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(2:2^{N-1}+1)}) \cdot \tilde{\mathbb{P}}_{-i}^{-1}(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(2:2^{N-1}+1)}) \\
\Rightarrow\ &\tilde{\mathbb{F}}_i(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(1:2^{N-1})}) \cdot \tilde{\mathbb{P}}_{-i}^{-1}(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(1:2^{N-1})}) \cdot \tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(2:2^{N-1}+1)}) = \tilde{\mathbb{F}}_i(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(2:2^{N-1}+1)}). \qquad (16)
\end{aligned}$$

Similarly, for realizations $(\mathbf{z}_i^2, \mathbf{z}_{-i}^{2(1)})$ to $(\mathbf{z}_i^2, \mathbf{z}_{-i}^{2(2^{N-1}+1)})$, we can get the following equation:

$$\tilde{\mathbb{F}}_i(\mathbf{z}_i^2, \mathbf{z}_{-i}^{2(1:2^{N-1})}) \cdot \tilde{\mathbb{P}}_{-i}^{-1}(\mathbf{z}_i^2, \mathbf{z}_{-i}^{2(1:2^{N-1})}) \cdot \tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^2, \mathbf{z}_{-i}^{2(2:2^{N-1}+1)}) = \tilde{\mathbb{F}}_i(\mathbf{z}_i^2, \mathbf{z}_{-i}^{2(2:2^{N-1}+1)}). \qquad (17)$$

Due to the condition of equal CCPs, we have $\tilde{\mathbb{F}}_i(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(1:2^{N-1})}) = \tilde{\mathbb{F}}_i(\mathbf{z}_i^2, \mathbf{z}_{-i}^{2(1:2^{N-1})})$ and $\tilde{\mathbb{F}}_i(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(2:2^{N-1}+1)}) = \tilde{\mathbb{F}}_i(\mathbf{z}_i^2, \mathbf{z}_{-i}^{2(2:2^{N-1}+1)})$. Plug these conditions into Equations (16) and (17) and equate these two equations. It will yield the following relationship:

$$\tilde{\mathbb{F}}_i(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(1:2^{N-1})}) \cdot \big[\tilde{\mathbb{P}}_{-i}^{-1}(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(1:2^{N-1})}) \cdot \tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(2:2^{N-1}+1)}) - \tilde{\mathbb{P}}_{-i}^{-1}(\mathbf{z}_i^2, \mathbf{z}_{-i}^{2(1:2^{N-1})}) \cdot \tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^2, \mathbf{z}_{-i}^{2(2:2^{N-1}+1)})\big] = \mathbf{0}. \qquad (18)$$

In binary choice games, $\tilde{\mathbb{F}}_i(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(1:2^{N-1})})$ is a $1 \times (2^{N-1} - 1)$ vector. Moreover, the condition $P_i(Y_i = 1 | \mathbf{z}_i^1, \mathbf{z}_{-i}^{1(l)}) \neq P_i(Y_i = 1 | \mathbf{z}_i^1, \mathbf{z}_{-i}^{1(l')})\ \forall l \neq l'$ implies that all elements in $\tilde{\mathbb{F}}_i(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(1:2^{N-1})})$ are non-zeros. Therefore, when $N > 2$, Equation (18) implies that any one row of the matrix $\big[\tilde{\mathbb{P}}_{-i}^{-1}(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(1:2^{N-1})}) \cdot \tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(2:2^{N-1}+1)}) - \tilde{\mathbb{P}}_{-i}^{-1}(\mathbf{z}_i^2, \mathbf{z}_{-i}^{2(1:2^{N-1})}) \cdot \tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^2, \mathbf{z}_{-i}^{2(2:2^{N-1}+1)})\big]$ is a linear combination of other rows. Consequently, the rank of such a $(2^{N-1} - 1) \times (2^{N-1} - 1)$ matrix is at most $2^{N-1} - 2$. The testable implication of the equilibrium belief when $N = 2$ has been proved in Subsection 4.1. This completes the proof. $\square$

Proof of Proposition 2: This part proves the result in games with $N > 2$ players. The result when $N = 2$ is described in Subsection 4.1. Consider $(2^{N-1} - 1)$ realizations of $\mathbf{Z}_{-i}$, denoted by $\mathbf{z}_{-i}^1$ to $\mathbf{z}_{-i}^{2^{N-1}-1}$, such that $P_i(Y_i = 1 | \mathbf{z}_i^1, \mathbf{z}_{-i}^l) = \frac{1}{2}$ for all $1 \leq l \leq 2^{N-1} - 1$. As discussed in the main text (see the discussion after Proposition 3), there are infinite pairs of realizations that satisfy the above condition. Under Assumptions 5 and 6, the inverted choice probability Equation (9) turns to the following equation system:

$$\pi_i(\mathbf{z}_i^1) + \sum_{\mathbf{y}_{-i} \neq \mathbf{0}} \delta_i(\mathbf{z}_i^1, \mathbf{y}_{-i}) \cdot P_{-i}(\mathbf{y}_{-i} | \mathbf{z}_i^1, \mathbf{z}_{-i}^l) = F_i\big[P_i(1 | \mathbf{z}_i^1, \mathbf{z}_{-i}^l) = \frac{1}{2}\big] = 0,\ \forall l \leq 2^{N-1} - 1. \qquad (19)$$

Note that $B_i(\cdot)$ is replaced by $P_{-i}(\cdot)$ due to the local unbiased belief assumption at $\mathbf{Z}_i = \mathbf{z}_i^1$. Consider an arbitrary action profile $\mathbf{y}_{-i}^1 \neq \mathbf{0}$ and move this term to the right-hand side of the above equation. It would yield the following one:

$$\pi_i(\mathbf{z}_i^1) + \sum_{\mathbf{y}_{-i} \neq \mathbf{0}, \mathbf{y}_{-i}^1} \delta_i(\mathbf{z}_i^1, \mathbf{y}_{-i}) \cdot P_{-i}(\mathbf{y}_{-i}|\mathbf{z}_i^1, \mathbf{z}_{-i}^l) = -\delta_i(\mathbf{z}_i^1, \mathbf{y}_{-i}^1) \cdot P_{-i}(\mathbf{y}_{-i}^1|\mathbf{z}_i^1, \mathbf{z}_{-i}^l), \ \forall l \leq 2^{N-1} - 1. \tag{20}$$

Suppose the terms on the right-hand side are known. Equation system (20) then consists of $(2^{N-1} - 1)$ equations and $(2^{N-1} - 1)$ unknowns (i.e., $\pi_i(\mathbf{z}_i^1)$ and $\delta_i(\mathbf{z}_i^1, \mathbf{y}_{-i})\ \forall \mathbf{y}_{-i} \neq \mathbf{0}, \mathbf{y}_{-i}^1$). Given the linear independence condition by Assumption 4(b), the rank condition is satisfied. Therefore, all payoffs $\pi_i(\mathbf{z}_i^1)$ and $\delta_i(\mathbf{z}_i, \mathbf{Y}_{-i} \neq \mathbf{y}_{-i}^1)$ could be written as linear transformations of the strategic effect of the action profile $\mathbf{y}_{-i}^1$, denoted by $\delta_i(\mathbf{z}_i^1, \mathbf{y}_{-i}^1)$.

Next, consider the inverted choice probability with the random variables $\mathbf{Z}_{-i}$ as the arguments:

$$\pi_i(\mathbf{z}_i^1) + \sum_{\mathbf{y}_{-i} \neq \mathbf{0}} \delta_i(\mathbf{z}_i^1, \mathbf{y}_{-i}) P_{-i}(\mathbf{y}_{-i}|\mathbf{z}_i^1, \mathbf{Z}_{-i}) = F_i\left[P_i(1|\mathbf{z}_i^1, \mathbf{Z}_{-i})\right]. \tag{21}$$

Since all payoffs are linear transformations of $\delta_i(\mathbf{z}_i^1, \mathbf{y}_{-i}^1)$, Equation (21) is equivalent to:

$$T(\mathbf{Z}_{-i}) \cdot \delta_i(\mathbf{z}_i^1, \mathbf{y}_{-i}^1) = F_i\left[P_i(1|\mathbf{z}_i^1, \mathbf{Z}_{-i})\right], \tag{22}$$

where $T(\mathbf{Z}_{-i})$ represents the linear transformation that has been identified. Equation (22) would identify the sign of $\delta_i(\mathbf{z}_i^1, \mathbf{y}_{-i}^1)$. In addition, Assumption 3(b) normalizes the scale of this term to be 1; it consequently identifies the value of $\delta_i(\mathbf{z}_i^1, \mathbf{y}_{-i}^1)$. As a consequence, all payoffs $\pi_i(\mathbf{z}_i^1)$ and $\delta_i(\mathbf{z}_i^1, \mathbf{Y}_{-i})$ are identified. Given the identification of these payoffs, Equation (21) implies that the variation of $\mathbf{Z}_{-i}$ would identify the value of $F_i(p)\ \forall p \in \mathcal{P}_i(\mathbf{z}_i)$. This completes the proof. □

Proof of Proposition 3: This part proves the result in games with $N > 2$. The result when $N = 2$ is described in Subsection 4.1. Consider any 2^{N-1} realizations of $\mathbf{Z}_{-i}$ such that $P_i(Y_i|\mathbf{z}_i^1, \mathbf{z}_{-i}^l) = P_i(Y_i|\mathbf{z}_i^1, \mathbf{z}_{-i}^{l'})\ \forall l \neq l'$. For these realizations, Equation (13) turns to

$$\mathbf{\Delta}_i(\mathbf{z}_i^1) \cdot \tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1:2^{N-1}}) = \tilde{\mathbb{F}}_i(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1:2^{N-1}}) = \mathbf{0}. \tag{23}$$

Note that $B_i(\cdot)$ is replaced by $P_{-i}(\cdot)$ when $\mathbf{Z}_i = \mathbf{z}_i^1$, due to the local unbiased belief condition by Assumption 6. $\tilde{\mathbb{F}}_i(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1:2^{N-1}}) = \mathbf{0}$ is due to the condition of equal CCPs. In this binary choice game, $\boldsymbol{\Delta}_i(\mathbf{z}_i^1)$ is a $1 \times (2^{N-1} - 1)$ vector who has at least two non-zero elements (i.e., each of other players will have an impact on player i's payoff). Therefore, Equation (23) implies that the matrix $\tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1:2^{N-1}})$ has a rank at most $2^{N-1} - 2$. It corresponds to the result in Proposition 3(b) and completes the proof. □

Proof of Proposition 4: Under Assumption 7, for a fixed $\mathbf{Z}_i$, Equation (14) implies that the strategic effect is point identified as:

$$\boldsymbol{\Delta}_i(\mathbf{Z}_i) = \tilde{\mathbb{F}}_i(\mathbf{Z}_i, \mathbf{z}_{-i}^{1:2^{N-1}}) \cdot \tilde{\mathbb{P}}_{-i}^{-1}(\mathbf{Z}_i, \mathbf{z}_{-i}^{1:2^{N-1}}).$$

On the right-hand side of this equation, player i's belief $B_i(\cdot)$ is replaced by other players' CCPs $P_{-i}(\cdot)$ when $\mathbf{Z}_{-i} = \mathbf{z}_i^l \; \forall l \leq 2^{N-1}$, due to Assumption 7. The matrix $\tilde{\mathbb{P}}_{-i}(\cdot)$ is invertible given the linear independence condition by Assumption 4(b). Moreover, the inverted choice probability function $F_i(\cdot)$ is identified in Proposition 2. Consequently, all terms on the right-hand side are known and the strategic effect $\boldsymbol{\Delta}_i(\mathbf{Z}_i)$ is identified.

Given the identification of $\delta_i(\cdot)$, the base return $\pi_i(\mathbf{Z}_i)$ for any value of $\mathbf{Z}_i$ can be identified as:

$$\pi_i(\mathbf{Z}_i) = F_i\big[P_i(Y_i = 1|\mathbf{Z}_i, \mathbf{z}_{-i}^1)\big] - \sum_{\mathbf{y}_{-i} \neq \mathbf{0}} \delta_i(\mathbf{Z}_i, \mathbf{y}_{-i}) \cdot P_{-i}(\mathbf{y}_{-i}|\mathbf{Z}_i, \mathbf{z}_{-i}^1).$$

For values of $(\mathbf{Z}_i, \mathbf{Z}_{-i})$ where player i is allowed to have biased beliefs, her belief function must satisfy the following inverted choice probability equation:

$$\pi_i(\mathbf{Z}_i) + \sum_{\mathbf{y}_{-i} \neq \mathbf{0}} \delta_i(\mathbf{Z}_i, \mathbf{y}_{-i}) \cdot B_i(\mathbf{y}_{-i}|\mathbf{Z}_i, \mathbf{Z}_{-i}) = F_i\big[P_i(1|\mathbf{Z}_i, \mathbf{Z}_{-i})\big].$$

Given the identification of $\pi_i(\cdot)$, $\delta_i(\cdot)$, and $F_i(\cdot)$, the above equation implies that the $(2^{N-1} - 1) \times 1$ vector of player i's belief $\mathbf{B}_i(\mathbf{Z}_i, \mathbf{Z}_{-i})$ is identified on a hyperplane with a dimension of $2^{N-1} - 2$. In games with $N = 2$ players, the belief vector turns to be a scalar and is point identified. This completes the proof. □

Proof of Proposition 1': Consider Equation (18) in the proof of Proposition 1. In games with $K > 1$, $\tilde{\mathbb{F}}_i(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(1:(K+1)^{N-1})})$ is a $K \times \big((K+1)^{N-1} - 1\big)$ matrix. This matrix has a rank of K as implied by Assumption 4'(b). Therefore, when $N = 2$, Equation (18) implies that $\tilde{\mathbb{P}}_{-i}^{-1}(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(1:K+1)}) \cdot \tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(2:K+2)}) -$

$\tilde{\mathbb{P}}_{-i}^{-1}(\mathbf{z}_i^2, \mathbf{z}_{-i}^{2(1:K+1)}) \cdot \tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^2, \mathbf{z}_{-i}^{2(2:K+2)}) = \mathbf{0}$. It yields Proposition 1'(a).

In games with $N > 2$ players, Equation (18) implies that any K rows of the matrix $\tilde{\mathbb{P}}_{-i}^{-1}(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(1:(K+1)^{N-1})}) \cdot \tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1(2:(K+1)^{N-1}+1)}) - \tilde{\mathbb{P}}_{-i}^{-1}(\mathbf{z}_i^2, \mathbf{z}_{-i}^{2(1:(K+1)^{N-1})}) \cdot \tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^2, \mathbf{z}_{-i}^{2(2:(K+1)^{N-1}+1)})$ can be written as linear combinations of other $(K+1)^{N-1} - 1 - K$ rows. Therefore, the above matrix has a rank of at most $(K+1)^{N-1} - 1 - K$. This is the result in Proposition 1'(b) and completes the proof. □

Proof of Proposition 2': Without loss of generality, consider that action 0 is directly connected with action 1 at $\mathbf{Z}_i = \mathbf{z}_i^1$. Therefore, $P_i(Y_i = 0|\mathbf{z}_i^1, \mathbf{z}_{-i}) = P_i(Y_i = 1|\mathbf{z}_i^1, \mathbf{z}_{-i})$ for some realization $\mathbf{Z}_{-i} = \mathbf{z}_{-i}$. Moreover, given Assumption 4', there exist infinite realizations of $\mathbf{Z}_{-i}$ that equate these two actions' CCPs. Consider $(K+1)^{N-1} - 1$ such realizations, denoted by $\mathbf{z}_{-i}^1$ to $\mathbf{z}_{-i}^{(K+1)^{N-1}-1}$. Under Assumption 6'(a) such that player i has unbiased beliefs when $\mathbf{Z}_i = \mathbf{z}_i^1$, the above realizations imply the following equation system:

$$\pi_i(\mathbf{z}_i^1, Y_i = 1) + \sum_{\mathbf{y}_{-i} \neq \mathbf{0}} \delta_i(\mathbf{z}_i^1, Y_i = 1, \mathbf{y}_{-i}) P_{-i}(\mathbf{y}_{-i}|\mathbf{z}_i^1, \mathbf{z}_{-i}^l) = EU_i(\mathbf{z}_i^1, \mathbf{z}_{-i}^l, Y_i = 0) = 0, \ \forall l \leq (K+1)^{N-1} - 1. \tag{24}$$

The above system has the same mathematical structure as Equation (19) in the proof of Proposition 2. Therefore, the same argument in binary choice games trivially applies, and $\pi_i(\mathbf{z}_i, Y_i = 1)$ and $\delta_i(\mathbf{z}_i^1, Y_i = 1, \mathbf{y}_{-i})$ are identified $\forall \mathbf{y}_{-i} \neq \mathbf{0}$. Given Assumption 6'(b), each action k is either directly or indirectly connected to action 1 at $\mathbf{Z}_i = \mathbf{z}_i^1$. Through these connections and by the same argument, $\pi_i(\mathbf{z}_i, Y_i = k)$ and $\delta_i(\mathbf{z}_i^1, Y_i = k, \mathbf{y}_{-i})$ are identified for any k.

Finally, note that Assumption 6'(a) implies the following:

$$\mathbf{\Pi}_i(\mathbf{z}_i^1) + \mathbf{\Delta}_i(\mathbf{z}_i^1) \cdot \mathbf{P}_{-i}(\mathbf{z}_i^1, \mathbf{Z}_{-i}) = \mathbf{F}_i\left[\mathbf{P}_i(\mathbf{z}_i^1, \mathbf{Z}_{-i})\right].$$

Given the identification of $\mathbf{\Pi}_i(\mathbf{z}_i^1)$ and $\mathbf{\Delta}_i(\mathbf{z}_i^1)$ and with the variation of $\mathbf{Z}_{-i}$, the above equation identifies the inverted choice probability $\mathbf{F}_i(\mathbf{p})$ $\forall \mathbf{p} \in \mathcal{P}_i(\mathbf{z}_i^1)$. It completes the proof. □

Proof of Proposition 3': Consider any $(K+1)^{N-1}$ realizations of $\mathbf{Z}_{-i}$ such that $\mathbf{P}_i(\mathbf{z}_i^1, \mathbf{z}_{-i}^l) = \mathbf{P}_i(\mathbf{z}_i^1, \mathbf{z}_{-i}^{l'})$ $\forall l \neq l'$. For these realizations, Equation (13) turns to

$$\mathbf{\Delta}_i(\mathbf{z}_i^1) \cdot \tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1:(K+1)^{N-1}}) = \tilde{\mathbb{F}}_i(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1:(K+1)^{N-1}}) = \mathbf{0}. \tag{25}$$

Note that $B_i(\cdot)$ is replaced by $P_{-i}(\cdot)$ when $\mathbf{Z}_i = \mathbf{z}_i^1$ due to Assumption 6'(a). $\tilde{\mathbb{F}}_i(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1:(K+1)^{N-1}}) = \mathbf{0}$ is due to the condition of equal CCPs such that $\mathbf{P}_i(\mathbf{z}_i^1, \mathbf{z}_{-i}^l) = \mathbf{P}_i(\mathbf{z}_i^1, \mathbf{z}_{-i}^{l'})\ \forall l \neq l'$. In this multinomial choice game, $\mathbf{\Delta}_i(\mathbf{z}_i^1)$ is a $K \times \big((K+1)^{N-1} - 1\big)$ matrix with rank K (see the main text for the proof of full rank of $\mathbf{\Delta}_i(\cdot)$). Therefore, when $N = 2$, Equation (25) implies that $\tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1:K+1}) = \mathbf{0}$, which corresponds to Proposition 3'(a). When $N > 2$, Equation (25) implies that the matrix $\tilde{\mathbb{P}}_{-i}(\mathbf{z}_i^1, \mathbf{z}_{-i}^{1:(K+1)^{N-1}})$ has a rank of at most $(K+1)^{N-1} - 1 - K$, which corresponds to Proposition 3'(b). It completes the proof. □

Proof of Proposition 4': Under Assumption 7', for a fixed $\mathbf{Z}_i$, Equation (14) implies that the strategic effect is point identified as:

$$\mathbf{\Delta}_i(\mathbf{Z}_i) = \tilde{\mathbb{F}}_i(\mathbf{Z}_i, \mathbf{z}_{-i}^{1:(K+1)^{N-1}}) \cdot \tilde{\mathbb{P}}_{-i}^{-1}(\mathbf{Z}_i, \mathbf{z}_{-i}^{1:(K+1)^{N-1}}).$$

On the right-hand side of this equation, player i's belief $B_i(\cdot)$ is replaced by other players' CCPs $P_{-i}(\cdot)$ when $\mathbf{Z}_{-i} = \mathbf{z}_{-i}^l\ \forall l \leq (K+1)^{N-1}$ due to Assumption 7'. Moreover, the inverted choice probability function $\mathbf{F}_i(\cdot)$ is identified in Proposition 2'. Consequently, all terms on the right-hand side are known and the strategic effect $\mathbf{\Delta}_i(\mathbf{Z}_i)$ is identified.

Given the identification of $\mathbf{\Delta}_i(\cdot)$, the vector of base returns $\mathbf{\Pi}_i(\mathbf{Z}_i)$ is identified as:

$$\mathbf{\Pi}_i(\mathbf{Z}_i) = \mathbf{F}_i\big[\mathbf{P}_i(\mathbf{Z}_i, \mathbf{z}_{-i}^1)\big] - \mathbf{\Delta}_i(\mathbf{Z}_i) \cdot \mathbf{P}_{-i}(\mathbf{Z}_i, \mathbf{z}_{-i}^1).$$

For values of $(\mathbf{Z}_i, \mathbf{Z}_{-i})$ where player i is allowed to have biased beliefs, her belief function must satisfy the following inverted choice probability equation:

$$\mathbf{\Pi}_i(\mathbf{Z}_i) + \mathbf{\Delta}_i(\mathbf{Z}_i) \cdot \mathbf{B}_i(\mathbf{Z}_i, \mathbf{Z}_{-i}) = \mathbf{F}_i\big[\mathbf{P}_i(\mathbf{Z}_i, \mathbf{Z}_{-i})\big].$$

The above equation contains K restrictions. Note that $\mathbf{\Pi}_i(\cdot)$, $\mathbf{\Delta}_i(\cdot)$, and $\mathbf{F}_i(\cdot)$ have been identified. Moreover, the rank of $\mathbf{\Delta}_i(\cdot)$ is K. Therefore, the above equation system implies that the $\big((K+1)^{N-1} - 1\big) \times 1$ vector of player i's belief $\mathbf{B}_i(\mathbf{Z}_i, \mathbf{Z}_{-i})$ is identified on a hyperplane with a dimension of $(K+1)^{N-1} - K - 1$. In games with $N = 2$ players, the belief vector has a dimension of K and is therefore point identified. This completes the proof. □

Micro-Foundation of the Belief Function: First, consider the case that player i correctly anticipates that all other players' behaviors are independent conditional on $(\mathbf{X}, \mathbf{Z}_i, \mathbf{Z}_{-i})$. Let $\sigma_j^{B_i}(\mathbf{X}, \mathbf{Z}_i, \mathbf{Z}_{-i}, \varepsilon_j) : \mathbb{R}^{L_{\mathbf{x}}+N\cdot L_{\mathbf{z}}+K+1} \to \{0, 1, \cdots, K\}$ denote player i's belief about the strategy function that will be used by player j. In particular, given $(\mathbf{X}, \mathbf{Z}_i, \mathbf{Z}_{-i}, \varepsilon_j)$, function $\sigma_j^{B_i}(\cdot)$ generates the realization of Y_j that player i believes player j will choose. Due to simultaneity and incomplete information, player i's actual strategy function $\sigma_i(\cdot)$, actual choice Y_i, and private information ε_i, are unobserved by player j. Consequently, none of these variables has an impact on player j's behaviors and they are excluded from player i's believed strategy function $\sigma_j^{B_i}(\cdot)$. In addition, since player i does not observe ε_j, she needs to integrate her believed strategy function over player j's private information to obtain an estimate of player j's CCP. Specifically, $B_i^j(Y_j = y_j|\mathbf{X}, \mathbf{Z}_i, \mathbf{Z}_{-i}) = \int \mathbb{1}\left[\sigma_j^{B_i}(\mathbf{X}, \mathbf{Z}_i, \mathbf{Z}_{-i}, \varepsilon_j) = y_j\right] d\Gamma_{j,\mathbf{X}}(\varepsilon_j)$. Moreover, because player i correctly anticipates that all other players' behaviors are conditionally independent, her belief about other players' CCPs is represented by $B_i(\mathbf{Y}_{-i}|\mathbf{X}, \mathbf{Z}_i, \mathbf{Z}_{-i}) = \prod_{j\neq i}^N B_i^j(Y_j|\mathbf{X}, \mathbf{Z}_i, \mathbf{Z}_{-i})$.

A slight modification of the above structure could allow player i to incorrectly believe that other players' behaviors are conditionally correlated. In particular, let $\sigma_{-i}^{B_i}(\mathbf{X}, \mathbf{Z}_i, \mathbf{Z}_{-i}, \varepsilon_{-i}) : \mathbb{R}^{L_{\mathbf{x}}+N\cdot L_{\mathbf{z}}+(N-1)\cdot(K+1)} \to \{0, 1\cdots, K\}^{N-1}$ denote player i's believed strategy function that will be used by all other players. Specifically, given $(\mathbf{X}, \mathbf{Z}_i, \mathbf{Z}_{-i}, \varepsilon_{-i})$, function $\sigma_{-i}^{B_i}(\cdot)$ generates the realization of $\mathbf{Y}_{-i}$ that player i believes other players will choose. Importantly, this believed strategy function allows arbitrary correlation among the actions chosen by all players other than i. Again, $\sigma_{-i}^{B_i}(\cdot)$ has to be integrated to form player i's belief function $B_i(\cdot)$. For instance, $B_i(\mathbf{Y}_{-i} = \mathbf{y}_{-i}|\mathbf{X}, \mathbf{Z}_i, \mathbf{Z}_{-i}) = \int \mathbb{1}[\sigma_{-i}^{B_i}(\mathbf{X}, \mathbf{Z}_i, \mathbf{Z}_{-i}, \varepsilon_{-i}) = \mathbf{y}_{-i}] d\Gamma_{-i,\mathbf{X}}(\varepsilon_{-i})$. Note that since I impose no restrictions on $\sigma_{-i}^{B_i}(\cdot)$, the above micro-foundation is consistent with and nests many concepts other than the Bayesian Nash Equilibrium, for instance, the rationalizability proposed by Bernheim (1984) and Pearce (1984). Finally, by construction, $B_i(\cdot)$ is a nonparametric function without any restrictions except that it is a valid probability distribution.

Testable Restriction of the Equilibrium Assumption when $\frac{\partial \mathbf{P}_i(\mathbf{Z}_i, \mathbf{Z}_{-i})}{\partial \mathbf{Z}'_{-i}}$ ***Is Rank-Deficient***: The rest of this Appendix derives an testable implication of the equilibrium assumption in multinomial choice games, but without Assumption 4'(b). Specifically, it is a restriction of the equilibrium condition when the matrix $\frac{\partial \mathbf{P}_i(\mathbf{Z}_i, \mathbf{Z}_{-i})}{\partial \mathbf{Z}'_{-i}}$ has less than full rank. In the rest of this Appendix, Assumption 4' is replaced by the following Assumption 4''.

Assumption 4''. *(a) Assumption 4 holds.*

(b) For each player i, the $K \times (N-1)L_{\mathbf{Z}}$ *matrix* $\frac{\partial \mathbf{P}_i(\mathbf{Z}_i,\mathbf{Z}_{-i})}{\partial \mathbf{Z}'_{-i}}$ *has less than full rank. Moreover,* $(N-1)L_{\mathbf{Z}} \geq K$.

Assumption 4''(b) considers the same order condition (i.e., $(N-1)L_{\mathbf{Z}} \geq K$) as in Assumption 4'(b) but focuses on the case when $\frac{\partial \mathbf{P}_i(\mathbf{Z}_i,\mathbf{Z}_{-i})}{\partial \mathbf{Z}'_{-i}}$ has less than full rank. Together with Proposition 1', the results in this Appendix indicate that the equilibrium restriction can be tested in any games as long as researchers observe sufficiently many variables as each player's specific payoff shifter.

Since the rank of $\frac{\partial \mathbf{P}_i(\mathbf{Z}_i,\mathbf{Z}_{-i})}{\partial \mathbf{Z}'_{-i}}$ is less than K, there are infinite non-zero vectors of change of $\mathbf{Z}_{-i}$—denoted by $d\mathbf{Z}_{-i}$—such that $\frac{\partial \mathbf{P}_i(\mathbf{Z}_i,\mathbf{Z}_{-i})}{\partial \mathbf{Z}'_{-i}} \cdot d\mathbf{Z}_{-i} = \mathbf{0}$. Equivalently, there are infinite pairs of $\mathbf{z}^l_{-i}$ and $\mathbf{z}^{l'}_{-i}$ such that $\mathbf{P}_i(\mathbf{Z}_i,\mathbf{z}^l_{-i}) = \mathbf{P}_i(\mathbf{Z}_i,\mathbf{z}^{l'}_{-i})$. Note that this condition of equal CCPs considers the variation of $\mathbf{Z}_{-i}$ but with a fixed value of $\mathbf{Z}_i$. Similar to the proof in Proposition 3', we can consider any $(K+1)^{N-1}$ realizations of $\mathbf{Z}_{-i}$ such that the above equal CCPs condition holds. It then yields the following equation, similar as Equation (25).

$$\mathbf{\Delta}_i(\mathbf{Z}_i) \cdot \tilde{\mathbb{P}}_i(\mathbf{Z}_i, \mathbf{z}_{-i}^{1:(K+1)^{N-1}}) = \mathbf{0}. \tag{26}$$

Note that $\tilde{\mathbb{P}}_i(\mathbf{Z}_i, \mathbf{z}_{-i}^{1:(K+1)^{N-1}})$ is a $\left((K+1)^{N-1}-1\right) \times \left((K+1)^{N-1}-1\right)$ matrix. Therefore, Equation (26) implies that $rank[\tilde{\mathbb{P}}_i(\mathbf{Z}_i, \mathbf{z}_{-i}^{1:(K+1)^{N-1}})] \leq (K+1)^{N-1} - 1 - rank[\mathbf{\Delta}_i(\mathbf{Z}_i)]$.

The next step is to determine the rank of $\mathbf{\Delta}_i(\cdot)$. Note that in the main text, $\mathbf{\Delta}_i(\cdot)$ has full rank given Assumption 4'(b). In contrast, it is likely to be rank deficient given Assumption 4''(b). By the chain rule, we have the following relationship:

$$\frac{\partial \mathbf{P}_i(\mathbf{Z}_i,\mathbf{Z}_{-i})}{\partial \mathbf{Z}'_{-i}} = \frac{\partial \mathbf{G}_i[\mathbf{EU}_i(\mathbf{Z}_i,\mathbf{Z}_{-i})]}{\partial \mathbf{EU}_i(\mathbf{Z}_i,\mathbf{Z}_{-i})'} \cdot \mathbf{\Delta}_i(\mathbf{Z}_i) \cdot \frac{\partial \mathbf{B}_i(\mathbf{Z}_i,\mathbf{Z}_{-i})}{\partial \mathbf{Z}'_{-i}}. \tag{27}$$

Equation (27) identifies a lower bound of the rank of $\mathbf{\Delta}_i(\cdot)$. For instance, $rank[\Delta_i(\mathbf{Z}_i)] \geq rank[\frac{\partial \mathbf{P}_i(\mathbf{Z}_i,\mathbf{Z}_{-i})}{\partial \mathbf{Z}'_{-i}}]$. Together with the result in Equation (26), it implies the following Proposition 1''.

Proposition 1''. *Suppose that Assumptions 1–3 and Assumption 4'' hold. Consider any* $(K+1)^{N-1}$ *realizations of* $\mathbf{Z}_{-i}$*, denoted by* $\mathbf{z}^l_{-i}$ $\forall l \leq (K+1)^{N-1}$*, such that* $\mathbf{P}_i(\mathbf{Z}_i,\mathbf{z}^l_{-i}) = \mathbf{P}_i(\mathbf{Z}_i,\mathbf{z}^{l'}_{-i})$ $\forall l \neq l'$*. Then the assumption that player i has unbiased/equilibrium beliefs implies the following testable restriction for* *each* $\mathbf{Z}_i$*:*

$$rank[\tilde{\mathbb{P}}_i(\mathbf{Z}_i, \mathbf{z}_{-i}^{1:(K+1)^{N-1}})] \leq (K+1)^{N-1} - 1 - rank[\frac{\partial \mathbf{P}_i(\mathbf{Z}_i, \mathbf{Z}_{-i})}{\partial \mathbf{Z}'_{-i}}].$$

Note that since $\frac{\partial \mathbf{P}_i(\mathbf{Z}_i, \mathbf{Z}_{-i})}{\partial \mathbf{Z}'_{-i}}$ contains non-zero elements, its rank is at least 1. Therefore, the restriction by Proposition 1'' is non-trivial. In fact, the region that satisfies such a restriction has a measure of zero.