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The Anatomy of Sentiment-Driven Fluctuations

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Abstract

We show that sentiments—self-fulfilling changes in beliefs that are orthogonal to fundamentals—can drive persistent aggregate fluctuations under rational expectations in a beauty contest game. Such fluctuations can occur even in the absence of exogenous aggregate fundamental shocks. Moreover, sentiments alter the volatility and persistence of aggregate outcomes in response to fundamental shocks. We provide (i) necessary conditions under which sentiments can affect aggregate outcomes in equilibrium and (ii) conditions under which sentiments drive persistent fluctuations and when they only affect aggregate outcomes contemporaneously. We also show that sentiment equilibria are stable under least-squares learning while the fundamental equilibrium is not.

Topics: Business fluctuations and cycles, Economic models
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1 Introduction

Can a change in sentiments induce persistent macroeconomic fluctuations? Even though this is a very attractive proposition and has captured the minds of economists at least since Keynes and Pigou, this idea has been very hard to formalize under rational expectations. We revisit this question.

We explore this question in the context of a beauty contest model that has a unique equilibrium under full information. A key feature of the information structure is that agents receive noisy *endogenous* signals about the aggregate action in the economy which are potentially confounded by noises. Agents must parse this information in order to determine their own appropriate action. This reliance of each agent's action on the aggregate action can induce complementarities even if the primitives of the model do not feature any coordination motive. This induced strategic complementarity allows for persistent fluctuations driven by self-fulfilling changes in beliefs. We refer to these self-confirming changes in beliefs as *sentiments*, and aggregate fluctuations driven by these changes as *sentiment-driven fluctuations*. These sentiment-driven fluctuations are independent of changes in fundamentals such as technology, preferences or government policies. In fact, they can even exist in an economy without any change in these aggregate fundamentals and this is common knowledge.

Importantly, our definition of sentiments is fundamentally *different* from the way the term "sentiments" is used in the fast-growing theoretical and empirical literature which studies expectations-driven fluctuations.[1] This literature has largely modeled sentiments as an *exogenous* stochastic process which alters the agents' first-order beliefs or higher-order beliefs about fundamentals. As a result, these exogenous changes in *sentiments* can affect aggregate outcomes. In contrast, sentiment shocks are not exogenously imposed in our environment. Instead, sentiments are self-fulfilling changes in beliefs of agents. Thus, whether these changes in beliefs are self-confirming and whether they can affect aggregate outcomes is disciplined by equilibrium. In this sense, sentiments are *endogenous* in our environment. The aforementioned observability of the aggregate action is a necessary condition for our notion of endogenous sentiments to exist, while it is not crucial for the exogenous sentiments to play a role. One may wonder, why does this distinction between exogenous and endogenous sentiments matter? Since sentiments in our setting are endogenous and disciplined by equilibrium, our setup has very different normative implications regarding how policy can and should respond to sentiments. Policy cannot affect sentiments if they are governed by an exogenously specified process. However, in our setting, a fundamental equilibrium always exists when a sentiment equilibrium exists; this multiplicity of equilibria opens the door for policy intervention which targets certain equilibria over others.

We show that these endogenous sentiments can drive fluctuations in the economy even absent any fluctuations in aggregate fundamentals, i.e., sentiment-driven fluctuations can manifest even if aggregate fundamentals are common knowledge. This is another important distinction relative to the large literature mentioned above which models sentiments as an exogenous stochastic process. In these models, agents are unable to perfectly separate the noise or changes in sentiments from the

[1] See for example Angeletos and La'O (2010, 2013), Lorenzoni (2009), Barsky and Sims (2012), Acharya (2013), Nimark (2014), Rondina and Walker (2020), Angeletos et al. (2018), Huo and Takayama (2015) among many others.

changes in aggregate fundamentals. Thus, agents misattribute changes in sentiments to changes in fundamentals and thus behave as if fundamentals had changed. In our model, this is not the case; sentiments can drive persistent fluctuations even if the aggregate fundamental is common knowledge.

While our notion of sentiments is the same as in Benhabib et al. (2015), the focus of our paper is very different. Benhabib et al. (2015) provide an illustration of how sentiments can generate stochastic self-fulfilling rational expectations equilibrium, but they only consider a static environment and do not study whether sentiments can generate persistent fluctuations. In contrast, we study a general environment and present necessary conditions under which sentiment equilibria can exist and also conditions under which sentiments can drive persistent fluctuations, providing a theoretical foundation for a common finding in the empirical literature.[2] Next, we lay out in detail the contribution of this paper relative to the existing literature.

As mentioned above, the first contribution of this paper is to provide necessary conditions under which sentiment equilibria can exist. In doing so, we provide a practical way to check whether sentiment-driven fluctuations can arise in equilibrium. Importantly, these conditions are framed in terms of exogenously specified primitives of the environment and thus allow one to establish whether sentiment equilibria exist or not *without* solving the model. To the best of our knowledge, this is the first such characterization in the literature studying endogenous sentiments.

Next, the paper provides general conditions under which sentiments can have prolonged effects on aggregate outcomes, and when they can only have short-lived effects. Again, we provide conditions in terms of the primitives of the model; in particular, the information set of agents, under which sentiments can drive persistent fluctuations. Importantly, this characterization does not depend on the private information that agents may possess. Our analysis shows that if agents observe both (i) the history of realizations of past aggregate actions with a one period lag and (ii) the history of realizations of past aggregate fundamentals with a one period lag, then sentiments *cannot* drive persistent fluctuations. In this case, sentiments can at most affect contemporaneous aggregate outcomes. More generally, our analysis shows that if agents observe both (i) the history of realizations of past aggregate actions with a k-period lag and (ii) the history of realizations of past aggregate fundamentals with a k-period lag, then sentiment-driven fluctuations can be described by a $\mathrm{MA}(k-1)$ process. This result uncovers a key property of standard models with information frictions. A commonly made assumption in this literature to ensure tractability is to assume that agents observe past aggregate variables without any noise, either immediately or with a finite lag.[3] While this literature has not focused on endogenous sentiments as a driver of business cycle fluctuations, our results highlight that this commonly made assumption on the information set of agents greatly reduces the possibility of persistent sentiment-driven fluctuations to begin with.

We also present a practical way to construct sentiment equilibria in which sentiments can drive persistent aggregate fluctuations. In Section 4.2, we present some simple economic environments and show that these models can generate sentiment-driven fluctuations which resemble the identified

[2]See, for example, Benhabib and Spiegel (2017), Lagerborg et al. (n.d.) among others.

[3]The standard practice in this literature has been to either (1) assume that agents observe past realizations after a k-period lag (see, for example, Hellwig (2002)), or (2) use a model solution algorithm which involves truncating the history of realizations after a finite number of periods (see for example Lorenzoni (2009)).

response in the empirical literature. We also show that the presence of sentiments can imbue additional persistence to fluctuations driven by changes in aggregate fundamentals. In particular, we show that in an environment where fundamental shocks drive transitory aggregate responses in the fundamental equilibrium, the same change in aggregate fundamentals can generate a *hump-shaped* response in aggregate outcomes in a sentiment equilibrium. This shows that sentiments alter the economy's response to aggregate fundamentals because they play the role of *endogenous* noise shocks, hampering agents' ability to infer fundamentals from endogenous signals. Thus, sentiments also serve as amplification and propagation mechanisms with regard to fundamental shocks. Finally, we also show that sentiment equilibria are stable under least-squares learning in both the static and dynamic cases, while the fundamental equilibrium is not.

Related Literature As previously mentioned, our notion of sentiments is closely related to Benhabib et al. (2015). However, while they focus on a static environment, our paper focuses on the dynamic response of the economy to changes in sentiments. Another closely related paper is Chahrour and Gaballo (2016), where the sentiment is interpreted as the limit of a fundamental equilibrium where the variance of the fundamental shock goes to zero. Their work is also in the context of a static setting. Unlike them, we argue that sentiments can drive persistent fluctuations even when aggregate fundamentals are known to be fixed. In particular, we show that the existence of sentiment-driven fluctuations does not hinge on the existence of aggregate fundamental shocks in the first place.

In the dispersed information literature, sentiments, confidence, or animal spirits are often modeled as exogenous shocks to agents' expectations. For example, common noise in signals observed by agents serves as an exogenous shock to agents' first-order beliefs about the fundamental, such as in Angeletos and La'O (2010) and Barsky and Sims (2012), among many others. The sentiment shock in Angeletos and La'O (2013) instead alters agents' higher-order beliefs about the fundamental. Different from previous studies, our sentiments are not imposed onto the model by adding noise to the information set in an ad hoc way. Rather, they are generated endogenously, i.e., they are disciplined by the rational expectations equilibrium. However, as previously mentioned, our endogenous sentiments can play a role similar to an exogenous common noise. Hébert and La'O (2020) study conditions under which rationally inattentive agents would choose signals that would result in non-fundamental aggregate volatility.[4] They study how the properties of agents' information costs relate to the properties of equilibria in beauty contest games, and provide the necessary and sufficient conditions that information costs for signal acquisition must satisfy in order to rule out or allow non-fundamental volatility in equilibrium. In our paper, we take the signals available to agents as given and provide conditions on the structure of signals under which sentiment equilibria exist.

The sentiment equilibria that we obtain are also closely related to correlated equilibria of Aumann (1974), as further developed by Maskin and Tirole (1987).[5,6] Maskin and Tirole (1987) study an

[4]The non-fundamental fluctuations in their paper include those driven by exogenous common noises.

[5]See also Peck and Shell (1991).

[6]Aumann et al. (1988) provide an excellent overview of the relation between correlated and sunspot equilibria under asymmetric information with a set of examples in market games that in the limit converge to a competitive equilibrium, and also illustrate that under asymmetric information there can be correlated equilibria even though the fundamental

economy with a unique fundamental equilibrium, and correlated equilibria exist only if there are Giffen goods. In our model, all goods can be normal and demand functions downward sloping. In a linear Gaussian economy, Bergemann and Morris (2013) and Chahrour and Ulbricht (2017) characterize the set of correlated equilibria and construct the corresponding information process (without sentiment shocks) that supports a particular allocation in the set. Our exercise instead starts from a particular information structure, and explores the set of equilibria that can be supported by the given primitives.

Since sentiment-driven fluctuations in our paper take the form of self-confirming beliefs about aggregate outcomes, one could interpret these as sunspots. However, it is important to realize that the sentiment equilibria that we characterize are not simple sunspot randomizations over multiple fundamental equilibria, as in many macroeconomic models. There exists a significant literature showing that sunspot equilibria can occur in models where the fundamental equilibrium is unique. The seminal paper of Cass and Shell (1983) demonstrates this in a two period model with a unique fundamental equilibrium by introducing securities traded in the first period, with returns that are sunspot contingent and can induce wealth effects. Spear (1989) studies an overlapping generations model with two islands where prices in one island act as sunspots for the other. Peck and Shell (1991) obtain a similar result by postulating imperfect competition and non-Walrasian trades in the post-sunspot market that also gives rise to wealth effects. In contrast, Mas-Colell (1992) and Gottardi and Kajii (1999) explicitly rule out securities with payoffs contingent on sunspot realizations, but trading is possible due to heterogeneous endowments and preferences in the first period. Thus according to Gottardi and Kajii (1999), what accounts for the existence of sunspot equilibria is "potential multiplicity" in future spot markets that results from trades that take place in the first period. It is clear that these are not the forces generating multiplicity of equilibria in our economy, as agents do not trade assets and do not make any inter-temporal decisions. Instead, the multiple equilibria in our model arise due to signal extraction problems in a setting with endogenous information sources. The specific environments and contexts of these papers are very different, and involve assumptions that are arguably not suitable from an applied-macroeconomics perspective. In contrast, we show that a very commonly used environment in the macroeconomics literature also permits the existence of correlated equilibria, and that in these settings sentiments can account for persistent aggregate fluctuations.

2 Environment and equilibrium concept

2.1 Best response

We consider a standard beauty contest game such as in Morris and Shin (2002). Our economy consists of a continuum of agents indexed by $i \in [0, 1]$. Agent i wants to choose an action $a_{i,t}$ every period which depends on their idiosyncratic fundamental shock $z_{i,t}$, an aggregate fundamental shock

equilibrium is unique. For an excellent discussion, see also Forges and Peck (1995).

θ_t and the economy-wide aggregate action a_t. Assume that the optimal action by agent i is given by

$$a_{i,t} = \delta \mathbb{E}_{i,t}\theta_t + \gamma \mathbb{E}_{i,t} a_t + \alpha \mathbb{E}_{i,t} z_{i,t}, \tag{1}$$

where $a_t = \int a_{i,t} di$ is defined as the aggregate action and $\mathbb{E}_{i,t} x_t$ denotes the expectation of a random variable x_t conditional on agent i's information set at date t. The parameters α and δ can take any value on the real line but we impose that $\gamma < 1$. This assumption ensures that the strength of strategic complementarity is not strong enough to generate multiple equilibria as in Cooper and John (1988) – our environment features a unique full-information fundamental equilibrium. The processes for idiosyncratic and aggregate fundamental are given by

$$z_{i,t} = \mathbf{h}(L)\mathbf{u}_{i,t} = \sum_{k=0}^{\infty} \mathbf{h}_k \mathbf{u}_{i,t-k}, \tag{2}$$

$$\theta_t = \mathbf{g}(L)\mathbf{v}_t = \sum_{k=0}^{\infty} \mathbf{g}_k \mathbf{v}_{t-k}, \tag{3}$$

where $\mathbf{u}_{i,t}$ and $\mathbf{v}_t$ are sequences of Gaussian white noise innovations to the idiosyncratic and aggregate fundamental respectively. Even though the idiosyncratic and aggregate fundamentals are univariate stochastic processes, we allow them to be driven by a vector of innovations. In equation (2), $\mathbf{u}_{i,t}$ is a n_u-vector of idiosyncratic shocks to agents' fundamental and satisfies an adding-up constraint $\int_i \mathbf{u}_{i,t} di = \mathbf{0}$ at each date t. In contrast, $\mathbf{v}_t$ is a n_v-vector and is common across all agents. Furthermore, we assume that $\mathbf{h}(L)$ and $\mathbf{g}(L)$ are potentially infinite-order one-sided polynomials in positive powers of the lag operator L.[7] We do not impose any restrictions on $\mathbf{h}(L)$ and $\mathbf{g}(L)$ except square-summability which implies that $z_{i,t}$ and θ_t are linear stationary processes. Bold-face letters indicate vectors and matrices while non-bold variables indicate scalars.

2.2 Information structure

Agents have access to both *exogenous* and *endogenous* sources of information. The distinction between exogenous and endogenous sources is whether signals depend on interactions among agents or not. In other words, the informativeness of *endogenous* signals is determined in equilibrium. We model the information as the set of $n_s \geq 1$ signals $\mathbf{x}_{i,t}$:

$$\mathbf{x}_{i,t} \quad = \quad \mathbf{A}(L)a_t + \mathbf{B}(L)\boldsymbol{\nu}_t + \mathbf{C}(L)\boldsymbol{\zeta}_{i,t}, \tag{4}$$

where $\boldsymbol{\nu}_t = [\mathbf{v}_t, \boldsymbol{\eta}_t]$ and $\boldsymbol{\zeta}_{i,t} = [\mathbf{u}_{i,t}, \boldsymbol{\varsigma}_{i,t}]$. Here $\boldsymbol{\eta}_t$ is a n_η-vector which denotes signal noise common across all agents, while $\boldsymbol{\varsigma}_{i,t}$ is a n_ς-vector which denotes signal noise particular to one agent. $\mathbf{A}(L)$ is a $n_s \times 1$ vector, while $\mathbf{B}(L)$ is a $n_s \times n_\nu$ matrix (where $n_\nu = n_v + n_\eta$) and $\mathbf{C}(L)$ is a $n_s \times n_\zeta$ matrix (where $n_\zeta = n_u + n_\varsigma$). Furthermore, $\mathbf{A}(L)$, $\mathbf{B}(L)$, and $\mathbf{C}(L)$ are square summable, one-sided polynomials in the lag operator L. In other words, the signals can only depend on past and current

[7] As is convention, we define the lag operator L as $Lx_t := x_{t-1}$, $L^{-1}x_t := x_{t+1}$ and $L^n x_t = x_{t-n}$.

changes (not future), not only in aggregate and idiosyncratic fundamentals but also in aggregate and idiosyncratic noise. Importantly, the signal specification (4) is flexible enough to accommodate both exogenous and endogenous information. $\mathbf{A}(L)a_t$ captures the dependence of signals on endogenous equilibrium outcomes; setting $\mathbf{A}(L) = 0$ implies an information set which contains only exogenous information. In what follows, we refer to $\boldsymbol{\nu}_t$ and $\boldsymbol{\zeta}_{i,t}$ as *primitive* shocks in order to distinguish them from *sentiment* shocks. In what follows, it will be convenient to label all the signals which have $\mathbf{A}(L) = 0$ as $\mathbf{y}_{i,t}$. Finally, the information set of agent i can then be represented as

$$\mathcal{I}_{i,t} = \mathbb{V}(\mathbf{y}_i^t) \vee \mathbb{V}(\mathbf{x}_i^t / \ \mathbf{y}_i^t),$$

where $\mathbb{V}(\mathbf{y}_i^t)$ denotes the smallest sub-space spanned (at date t) by the past and current realizations of exogenous information $\mathbf{y}_i^t$. The space $\mathbb{V}(\mathbf{x}_i^t / \ \mathbf{y}_i^t)$ is defined analogously but for *endogenous* sources of information, where $\mathbf{x}_i^t / \ \mathbf{y}_i^t$ denotes the subset of the signals $\mathbf{x}_i^t$ for which $\mathbf{A}(L) \neq 0$. Finally, since we impose rational expectations, all agents have knowledge of the cross-equation restrictions, implying that agents know that the dynamics of the economy are determined by (1) $-$ (4).

Some economic examples While we do not provide explicit micro-foundations for the best response (1), it is easy to do so. In fact, many commonly studied economic settings admit best responses of this form. For example, Angeletos and La'O (2010) consider a real-business cycle model where a continuum of firms are subject to dispersed productivity shocks and private information about the economy's aggregate conditions. The optimal quantity $q_{i,t}$ of firm i at date t satisfies

$$q_{i,t} = \delta \mathbb{E}_{i,t}\theta_t + \gamma \mathbb{E}_{it} q_t + \alpha \mathbb{E}_{i,t} z_{i,t},$$

where θ_t denotes the level of aggregate productivity, $z_{i,t}$ denotes the level of firm i's idiosyncratic productivity and q_t denotes aggregate output. Here, α, γ and δ are functions of deep parameters such as trade linkages, Frisch elasticity, curvature of the production function, etc. Angeletos and La'O (2010) allow for both private and public signals, which can be represented in terms of our notation as

$$\mathbf{x}_{i,t} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\mathbf{A}(L)} q_t + \underbrace{\begin{bmatrix} g(L) & 0 \\ 1 & 1 \end{bmatrix}}_{\mathbf{B}(L)} \begin{bmatrix} v_t \\ \eta_t \end{bmatrix} + \underbrace{\begin{bmatrix} h(L) & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{C}(L)} \begin{bmatrix} u_{i,t} \\ \varsigma_{i,t} \end{bmatrix}, \tag{5}$$

where $\theta_t = g(L)v_t$ denotes the aggregate fundamental and $z_{i,t} = h(L)u_{i,t}$ denotes the idiosyncratic fundamental; the first signal is simply $x_{i,t}^1 = \theta_t + z_{i,t}$. Since η_t denotes exogenous common noise, the second signal, $x_{i,t}^2 = v_t + \eta_t$, is common across all agents, i.e., it is a noisy signal of the current innovation to θ. As Angeletos and La'O (2010) discuss, η_t also contributes to aggregate fluctuations. The signals described above are exogenous; they have $\mathbf{A}(L) = 0$. Now, suppose agents also receive a noisy private signal about the realization of aggregate output: $x_{i,t}^3 = q_t + \varsigma_{i,t}$. This third signal is an endogenous signal and can be incorporated into our setup as follows:

$$\mathbf{x}_{i,t} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{A}(L)} q_t + \underbrace{\begin{bmatrix} g(L) & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}}_{\mathbf{B}(L)} \begin{bmatrix} v_t \\ \eta_t \end{bmatrix} + \underbrace{\begin{bmatrix} h(L) & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{C}(L)} \begin{bmatrix} u_{i,t} \\ \varsigma_{i,t} \end{bmatrix}. \tag{6}$$

As another example, consider a variant of the monetary model studied in Woodford (2002). A continuum of monopolistically competitive firms $i \in [0,1]$ must set their individual prices $p_{i,t}$ in accordance with fundamentals (nominal aggregate demand θ_t and idiosyncratic productivity $z_{i,t}$) but also close to the average price p_t others set

$$p_{i,t} = \delta \mathbb{E}_{i,t}\theta_t + (1-\delta)\mathbb{E}_{i,t}p_t + \alpha \mathbb{E}_{i,t} z_{i,t},$$

where θ_t denotes nominal demand, which is controlled by the monetary authority (aggregate fundamental), and $z_{i,t}$ denotes firm i's level of idiosyncratic productivity (the idiosyncratic fundamental). Again, the equation above is a special case of (1) with $\gamma = 1-\delta < 1$. In the full model, $1-\delta$ measures the degree of strategic complementarity in pricing and is a function of various deep parameters, such as the elasticity of substitution between varieties, the curvature of the production function, etc. How responsive a firm is to idiosyncratic shocks $z_{i,t}$ depends again on deep parameters such as curvature of the production function but also on policy. Appendix D shows that a policy which subsidizes firms with low realizations of $z_{i,t}$ and taxes firms with high realizations of $z_{i,t}$ can lower α.

2.3 Equilibria

We focus on linear rational expectations equilibria, which we classify into two classes: *fundamental* equilibrium and *sentiment* equilibrium.

Definition 1 (Fundamental Equilibrium)**.** *In a fundamental equilibrium, the aggregate action is driven purely by changes in aggregate primitive shocks* $\boldsymbol{\nu}_t = [\mathbf{v}_t, \boldsymbol{\eta}_t]$

$$a_t = \psi(L)\boldsymbol{\nu}_t, \tag{7}$$

and a_t *is consistent with the agents' optimal choice*

$$a_t = \int \Big\{ \delta \mathbb{E}_{i,t}\theta_t + \gamma \mathbb{E}_{i,t} a_t + \alpha \mathbb{E}_{i,t} z_{i,t} \Big\} di. \tag{8}$$

In a fundamental equilibrium, aggregate fluctuations are driven solely by changes in primitive shocks. For example, these shocks can be aggregate TFP or preference shocks. Furthermore, our definition allows fundamental equilibria to include situations in which agents may not directly observe the fundamental θ_t. In fact, this definition is broad enough to include the equilibria studied in the large literature, where sentiments are modeled as *shocks* to agents' beliefs. In such settings, aggregate noise in signals can also result in aggregate fluctuations. Thus, fundamental equilibria encompass the standard full-information equilibrium as well as those in economies with information frictions. In the latter, the definition is general enough to include equilibria with both exogenous information and endogenous information.

Definition 2 (Sentiment Equilibrium)**.** *Consider any payoff irrelevant white noise process $\epsilon_t \sim N(0,1)$, which is orthogonal to primitive shocks. In a sentiment equilibrium, the aggregate action responds to both primitive shocks and payoff irrelevant ϵ_t:*

$$a_t = \psi(L)\boldsymbol{\nu}_t + \phi(L)\epsilon_t, \tag{9}$$

where $\phi(L)$ has no roots for $|L| < 1$. Moreover, a_t is consistent with the agents' optimal choice

$$a_t = \psi(L)\boldsymbol{\nu}_t + \phi(L)\epsilon_t = \int \Big\{\delta\mathbb{E}_{i,t}\theta_t + \gamma\mathbb{E}_{i,t}a_t + \alpha\mathbb{E}_{i,t}z_{i,t}\Big\}di. \tag{10}$$

The key difference between the two classes is that in addition to fluctuations driven by forces in a fundamental equilibrium, the sentiment equilibrium allows for aggregate fluctuations to also arise due to changes in a random variable ϵ_t.[8] By construction, ϵ_t is orthogonal to changes in fundamentals $\boldsymbol{\nu}_t$ and exogenous noise $\boldsymbol{\eta}_t$. Importantly, ϵ_t does not directly enter the best-response (1), implying that it is not directly payoff relevant and also does not directly appear in the signals (4) as $\{\epsilon_t\} \perp \{\boldsymbol{\nu}_t, \boldsymbol{\zeta}_{i,t}\}$.

In what follows, we will refer to the random variable $\phi(L)\epsilon_t$ as the sentiment and ϵ_t as the sentiment shock. Clearly, the dynamic variance-covariance matrix of sentiments is given by $\phi(L)\phi(L^{-1})$ and is determined as part of the equilibrium and thus, sentiments and their properties are *endogenous* in our environment. In particular, in the fundamental equilibrium, both the variance and autocorrelation functions are 0, but these are endogenously determined in the sentiment equilibrium.

Finally, notice that the definition of sentiment equilibria restricts attention to a setting where $\phi(L)$ is invertible, i.e., there is no $z \in \mathbb{C}$ such that $|z| < 1$ and $\phi(z) = 0$. Appendix A.1 shows that this is without loss of generality. To see this, notice that if we consider an equilibrium in which $\phi(L)$ is not invertible, then we can always construct another *observationally equivalent* equilibrium in which $a_t = \psi(L)\boldsymbol{\nu}_t + \widetilde{\phi}(L)\epsilon_t$, where $\widetilde{\phi}(L)$ is invertible and satisfies $\phi(L)\phi(L^{-1}) = \widetilde{\phi}(L)\widetilde{\phi}(L^{-1})$. This shows that a sentiment equilibrium is unique up to the autocovariance-generating function and hence we can focus on the invertible case.

How might a sentiment equilibrium come to exist? Since ϵ_t does not feature directly in the best response (1) or in signals (4), the only way in which ϵ_t can affect aggregate outcomes is through the endogenous part of the signal $\mathbf{A}(L)a_t$. If agents believe that the aggregate action a_t is affected by sentiment shocks ϵ_t, i.e., $a_t = \psi(L)\boldsymbol{\nu}_t + \phi(L)\epsilon_t$, with $\phi(L) \neq 0$, then the endogenous signals $\mathbf{x}_{i,t}$ load on ϵ_t. Consequently, each agent's inference about the aggregate fundamental θ_t, idiosyncratic fundamental $z_{i,t}$ and the aggregate action a_t is affected by the realization of ϵ_t. Thus, sentiments can only affect aggregate fluctuations if agents believe that sentiment shocks ϵ_t affect

[8] Sentiment equilibria lie in the Hilbert space $\mathcal{H}(\boldsymbol{\nu}, \epsilon)$, while fundamental equilibria lie in the Hilbert space $\mathcal{H}(\boldsymbol{\nu})$.

aggregate outcomes and these beliefs are confirmed in equilibrium. Whether this is the case or not depends on the specifics of the economic environment, as we discuss next.

2.4 Existence of sentiment equilibria

We now present some necessary conditions under which sentiment equilibria can exist. Since we are concentrating on linear rational expectations equilibria, in any equilibria the optimal decision of agent i at date t is given by a linear function of the signals that agent i observes:

$$a_{i,t} = \mathbf{\Pi}(L)\mathbf{x}_{i,t}.$$

Here, $\mathbf{\Pi}(L)$ is a $1 \times n_s$ polynomial matrix and denotes the weights that the agent chooses to put on current and past signals in equilibrium. The following Lemma establishes a necessary condition for existence of sentiment equilibrium:

Lemma 1. *Let $\mathbf{\Pi}(L)$ denote an individual agent's equilibrium action, i.e., $a_{i,t} = \mathbf{\Pi}(L)\mathbf{x}_{i,t}$. Then in any sentiment equilibrium with $\phi(L) \neq 0$, the following must be true:*

$$\begin{aligned} \mathbf{\Pi}(L)\mathbf{A}(L) &= 1, &(11)\\ \mathbf{\Pi}(L)\mathbf{B}(L) &= \mathbf{0}. &(12) \end{aligned}$$

Proof. See Appendix A.2. □

To understand the content of Lemma 1, note that in equilibrium, each agent's action at date t is a linear function of the past and current realizations of signals $\{\mathbf{x}_{i,s}\}_{s=-\infty}^{t}$, i.e., $a_{i,t} = \mathbf{\Pi}(L)\mathbf{x}_{i,t}$:

$$a_{i,t} \quad = \quad \mathbf{\Pi}(L)\mathbf{x}_{i,t} = \mathbf{\Pi}(L)\Big(\mathbf{A}(L)a_t + \mathbf{B}(L)\boldsymbol{\nu}_t + \mathbf{C}(L)\boldsymbol{\zeta}_{i,t}\Big). \qquad (13)$$

Aggregating across all agents and using the *perceived* law of motion for $a_t = \psi(L)\boldsymbol{\nu}_t + \phi(L)\epsilon_t$, the *actual* law of motion can be written as

$$a_t = \mathbf{\Pi}(L)\Big(\mathbf{A}(L)\psi(L) + \mathbf{B}(L)\Big)\boldsymbol{\nu}_t + \mathbf{\Pi}(L)\mathbf{A}(L)\phi(L)\epsilon_t.$$

In equilibrium, the actual and perceived laws of motion must be the same:

$$a_t = \psi(L)\boldsymbol{\nu}_t + \phi(L)\epsilon_t = \mathbf{\Pi}(L)\Big(\mathbf{A}(L)\boldsymbol{\psi}(L) + \mathbf{B}(L)\Big)\boldsymbol{\nu}_t + \mathbf{\Pi}(L)\mathbf{A}(L)\phi(L)\epsilon_t.$$

For this expression to hold for any realizations of $\boldsymbol{\nu}_t$ and $epsilon_t$, the following must be true:

$$\phi(L)\Big[1 - \mathbf{\Pi}(L)\mathbf{A}(L)\Big] = 0 \qquad \text{and} \qquad \psi(L)\Big[1 - \mathbf{\Pi}(L)\mathbf{A}(L)\Big] = \mathbf{\Pi}(L)\mathbf{B}(L).$$

For a sentiment equilibrium to exist, i.e., $\phi(L) \neq 0$, it must be that $\mathbf{\Pi}(L)\mathbf{A}(L) = 1$. Similarly, for the equality to hold for any realization of $\boldsymbol{\nu}_t$, it must be that $\mathbf{\Pi}(L)\mathbf{B}(L) = \mathbf{0}$.

It may seem odd that a necessary condition for existence of sentiment equilibria is framed in terms of the endogenous object $\mathbf{\Pi}(L)$. Even though the explicit properties of the policy rule $\mathbf{\Pi}(L)$ are determined as part of equilibrium, and depend on the details of the information process and the economic environment, (11) provides some elementary insights on the existence of sentiment equilibria. Conditions (11) and (12) effectively impose requirements for the equilibrium process $\phi(L)$ and $\psi(L)$. These conditions essentially boil down to solving a system of nonlinear equations. As we show in Sections 3 and 4, the practical requirement imposed by these conditions is that the optimal action $a_{i,t}$ described in (1) responds strongly to changes in aggregate and idiosyncratic fundamentals, i.e., either $\alpha = \frac{\partial a_{i,t}}{\partial z_{i,t}}$ or $\delta = \frac{\partial a_{i,t}}{\partial \theta_t}$ are large enough. Importantly, this is only a necessary condition for existence of sentiment equilibria; whether sentiment equilibria exist is finally determined by whether the non-linear set of equations defining $\phi(L)$ and $\psi(L)$ admit a solution with $\phi(L) \neq 0$.[9]

The role of endogenous information in generating sentiment equilibrium The law of motion $\phi(L)$ having multiple solutions implies that multiple equilibria exist. In the environment we study, a fundamental equilibrium with $\phi(L) = 0$ always exists. What can give rise to additional equilibria? An obvious candidate is the strength of the strategic complementarity. However, we restrict attention to the case with $\gamma < 1$, which implies a unique fundamental equilibrium. Nonetheless, as we show in Section 3.1, endogenous information induces complementarities in actions even when the primitives of the environment do not feature any strategic complementarities. A large enough α and δ in the conditions described above ensure that the induced complementarities are strong enough to generate additional equilibria beyond the fundamental one. These additional equilibria are the sentiment equilibria. In the examples that we study in this paper, there is only one additional sentiment equilibrium, but there could potentially be multiple sentiment equilibria.[10]

Appendix A.2 further uses the information in Lemma 1 to construct a practical way of checking whether an economic environment admits sentiment equilibria. The Theorem below formally presents this additional necessary condition for the existence of sentiment equilibria.

Theorem 1. *For a sentiment equilibrium with $\phi(L) \neq 0$ to exist, a necessary condition is*

$$rank \underbrace{\begin{bmatrix} \mathbf{A}(L) & \mathbf{B}(L) \end{bmatrix}}_{n_s \times (n_\nu + 1)} = rank \underbrace{\begin{bmatrix} \mathbf{A}(L) & \mathbf{B}(L) \\ 1 & \mathbf{0} \end{bmatrix}}_{(n_s+1)\times(n_\nu+1)}, \tag{14}$$

where the rank of a matrix $\mathbf{X}(L)$ is defined as $\max_{z \in \mathbb{C}} rank\Big(\mathbf{X}(z)\Big)$.

Proof. See Appendix A.2. □

[9]Notice that conditions (11)-(12) imply that each agent's action (13) can be written as $a_{i,t} = a_t + \mathbf{C}(L)\boldsymbol{\zeta}_{i,t}$. This does *not* mean that agents do not react to aggregate fundamentals. Condition (12) implies that in any sentiment equilibrium, agents optimally put weights on their signals such that they track the aggregate action. Since the aggregate action $a_t = \psi(L)\boldsymbol{\nu}_t + \phi(L)\epsilon_t$, it responds to aggregate fundamentals. Consequently, $a_{i,t}$ also responds to aggregate fundamentals in the same way.

[10]In each of the examples we study, there are technically two sentiment equilibria, so the sentiment equilibrium is unique to a sign. However, both of these have the same variance and autocovariances and thus we focus on the positive one. This is without loss of generality.

Condition (14) is a necessary condition for sentiment equilibria to exist, given an economic environment. Since (14) only depends on exogenous objects $\mathbf{A}(L)$ and $\mathbf{B}(L)$, which are known once the signal structure has been specified, Theorem 1 provides a practical way to rule out the existence of sentiment equilibrium without solving the model. To see why it is necessary for this condition to be satisfied for a sentiment equilibrium to exist, notice that conditions (11)-(12) can be written as

$$\underbrace{\begin{bmatrix} \mathbf{A}'(L) \\ \mathbf{B}'(L) \end{bmatrix}}_{(n_\nu+1)\times n_s} \underbrace{\mathbf{\Pi}'(L)}_{n_s \times 1} = \underbrace{\begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}}_{(n_\nu+1)\times 1} .$$

The expression above can be viewed as a linear system of $n_\nu + 1$ equations in n_s unknowns and may not have any solution. The rank condition (14) is simply a necessary condition for a solution to exist.

Endogenous vs Exogenous Sentiments Since the definition of sentiment equilibria in our paper and Benhabib et al. (2015) is identical, the distinction between the endogenous sentiments in this paper and the literature on exogenous sentiments is also the same as the distinction set forth in Benhabib et al. (2015). We discuss this distinction next. In our environment, whether sentiments can affect aggregate outcomes or not is determined as part of equilibrium. Risking abuse of terminology, one could thus claim that our sentiments are *endogenous*, as whether they can exist and how they can affect aggregate outcomes are disciplined by equilibrium. In using the term *endogenous*, we mean that whenever a sentiment equilibrium exists, there also exists a fundamental equilibrium, and therefore one can think of sentiments not being imposed from the outset. In contrast, the large literature following Angeletos and La'O (2013), Barsky and Sims (2012) and others models sentiments as *exogenous* shocks to either first-order or higher-order beliefs of agents. There is no sense in which the definition of equilibrium can disallow these shocks from affecting aggregate outcomes. In fact, given our broad definition of *primitive* shocks (which includes noise shocks), these exogenous sentiments would get classified under our definition of fundamental equilibrium.

Finally, the analysis in this paper raises an important question as to whether one can distinguish between endogenous sentiment-driven or exogenous sentiments/noise-driven fluctuations. From a positive perspective, since our sentiment equilibria are correlated equilibria, it follows directly from the results in Bergemann and Morris (2013) that one can find an exogenous information structure which can rationalize the strategies as a Bayesian-Nash equilibrium. In this sense, it is possible to find some specification of exogenous noise shocks/exogenous sentiments which would have identical aggregate dynamics as a sentiment equilibrium. But the opposite need not be true, since equilibrium disciplines sentiment-driven fluctuations, but noise shocks are exogenously specified.

Even though it may be difficult to distinguish between exogenous and endogenous sentiments from a positive point of view, there is a clear distinction from a normative perspective. As aforementioned, whenever sentiment equilibria exist, there also exists a fundamental equilibrium. In particular, as we discuss in Section 3.5, modeling sentiments as endogenous opens the door to policy intervention, where a policy maker can use policy to pick between one of the many sentiment equilibria (if there are multiple sentiment equilibria) and the fundamental equilibrium. This allows the planner to eliminate sentiment-driven fluctuations altogether. The models with exogenous sentiments typically feature a unique equilibrium, leaving no such scope for policy intervention.

3 Forces at play

Now that we have defined the environment, we begin by presenting a simple example to uncover which ingredients are essential in generating sentiment-driven fluctuations and which ingredients are not. Even though our final goal is to explore the possibility of persistent sentiment-driven fluctuations, we start with some static examples to build intuition regarding the existence of sentiment equilibria. These static examples also make it easier to see how the existence results laid out in the previous section can be used in a practical sense. Furthermore, when we do study the dynamic case in Section 4, we will again make use of these same economic environments. Doing so helps clearly delineate the features which are required to generate persistent sentiment-driven fluctuations.

3.1 A simple example with no fundamental shocks

We start by presenting a simple example which will nevertheless prove to contain an element of universality. This example is based on the environment studied in Benhabib et al. (2015). Starting with this example is instructive, as many readers may not be familiar with the concept of a sentiment equilibrium, and thus we start with some static examples before proceeding to our ultimate goal of studying persistent sentiment-driven fluctuations. While this example features a particular information structure, the broad idea continues to hold for more general information structures, as we show in various examples throughout the paper.

We assume that the aggregate fundamental θ is constant at $\theta = 0$ and that this fact is common knowledge. Further, assume that the idiosyncratic fundamental z_i is i.i.d. $N(0, \sigma_z^2)$ across agents. In this case, the best response function (1) simplifies to

$$a_i = \gamma \mathbb{E}_i a + \alpha \mathbb{E}_i z_i, \tag{15}$$

where $a = \int a_i di$ denotes the aggregate action. Agent i observes a noisy endogenous signal:

$$x_i = a + z_i.$$

The signal x_i provides agent i with information about the aggregate action a but is contaminated by the idiosyncratic fundamental z_i. The important thing about the signal x_i is not the exact form it takes, but rather the fact that it provides agents with information about the endogenous average action a. It is straightforward to see that the optimal action of agent i must take the form:

$$a_i = \pi x_i = \pi a + \pi z_i, \tag{16}$$

where the second equality follows from the definition of x_i. Importantly, π is determined as part of equilibrium. Aggregating the decisions of all agents,

$$a = \int a_i di = \pi a + \pi \int z_i di = \pi a. \tag{17}$$

Equation (17) implies that $a(1-\pi) = 0$ in equilibrium. Notice that if $\pi \neq 1$, then the only equilibrium is one in which $a = 0$. This is the unique *fundamental equilibrium*. But if $\pi = 1$, then (17) is satisfied for any level of a. Of course, π is determined as part of equilibrium and thus, it remains to show whether in equilibrium π equals 1. In order to determine the equilibrium value of π, we conjecture that in equilibrium, the aggregate action can be described by

$$a = \phi\epsilon, \qquad \epsilon \sim N(0,1), \tag{18}$$

where ϕ is also determined as part of equilibrium. As previously defined, we refer to ϵ as the sentiment shock, which is independent of idiosyncratic and aggregate fundamentals. Plugging (18) into the expression for x_i yields

$$x_i = \phi\epsilon + z_i. \tag{19}$$

Equation (19) shows that the signal x_i provides agent i information about their idiosyncratic fundamental z_i but is potentially contaminated by the sentiment shock ϵ. Importantly, the precision of this signal depends on ϕ: with $\phi = 0$, the signal is fully informative about z_i; if $\phi \neq 0$, x_i is a noisy signal of z_i.

Verifying the Fundamental Equilibrium In a *fundamental equilibrium* $\phi = 0$, i.e., the aggregate outcome is $a = 0$ and is unaffected by the sentiment shock ϵ. With $\phi = 0$, the signal x_i in (19) perfectly informs agent i about the actual realization of her idiosyncratic fundamental, i.e., $\mathbb{E}_i z_i = z_i$ and so $\mathbb{E}_i a = 0$. Consequently, using (15) agent i's optimal action is given by

$$a_i = \alpha \mathbb{E}_i z_i = \alpha x_i \qquad \Rightarrow \qquad a = \int a_i di = \alpha \int z_i di = 0.$$

Also, $\pi = \alpha \neq 1$ (except in the knife-edge case) confirming that $\phi = 0$ is indeed an equilibrium.

Can $\pi = 1$ in equilibrium? Since we just saw that $\phi = 0$ implies $\pi \neq 1$, then if there exist equilibria with $\pi = 1$, it must be with $\phi \neq 0$. In this case, and unlike the fundamental equilibrium, (19) does not have infinite precision, and the signal does not allow an agent to perfectly infer the realization of z_i:

$$\mathbb{E}_i z_i = \frac{\sigma_z^2}{\phi^2 + \sigma_z^2} x_i \qquad \text{and} \qquad \mathbb{E}_i a = \frac{\phi^2}{\phi^2 + \sigma_z^2} x_i,$$

where σ_z^2 denotes the variance of z_i. Using these in (15) and comparing with (16), it follows that:

$$\pi = \frac{\phi^2}{\phi^2 + \sigma_z^2}\gamma + \left(1 - \frac{\phi^2}{\phi^2 + \sigma_z^2}\right)\alpha. \tag{20}$$

Then, for $\pi = 1$, it must be the case that:

$$|\phi| = \sigma_z\sqrt{\frac{\alpha - 1}{1 - \gamma}}. \tag{21}$$

Thus, in addition to the fundamental equilibrium, the endogenous signal also supports an additional *sentiment* equilibrium in which sentiments can affect aggregate outcomes a even though it is common knowledge that the aggregate fundamental $\theta = 0$. Technically, there are two sentiment equilibria. One in which $a = \sigma_z\sqrt{\frac{\alpha-1}{1-\gamma}}\epsilon$ and the other with the sign flipped $a = -\sigma_z\sqrt{\frac{\alpha-1}{1-\gamma}}\epsilon$. Throughout the rest of the paper, we restrict attention to the positive solution for ϕ. This is without loss of generality since both cases result in the same variance of the aggregate outcome.

This example shows that sentiments can provide an amplification mechanism – the unconditional variance of the aggregate outcome is given by $\sigma_z^2\left(\frac{\alpha-1}{1-\gamma}\right) > 0$, even though it is common knowledge that aggregate fundamentals are fixed. Such an equilibrium exists as long as $\alpha > 1$. In this sense, endogenous information is necessary but not sufficient for sentiment equilibria to exist.

In this particular example, the optimal action a_i must be very responsive to idiosyncratic fundamentals (α large enough) for a sentiment equilibrium to exist. However, the fact that α needs to be large is not a general requirement. In fact, we present another example later in this section where $\alpha = 0$ and a sentiment equilibrium exists. As we show, in that example the relevant condition for existence of sentiment equilibrium is that the best response be very responsive to aggregate fundamentals. Either condition ensures that in equilibrium, agents respond strongly to the realization of the endogenous signal x_i.

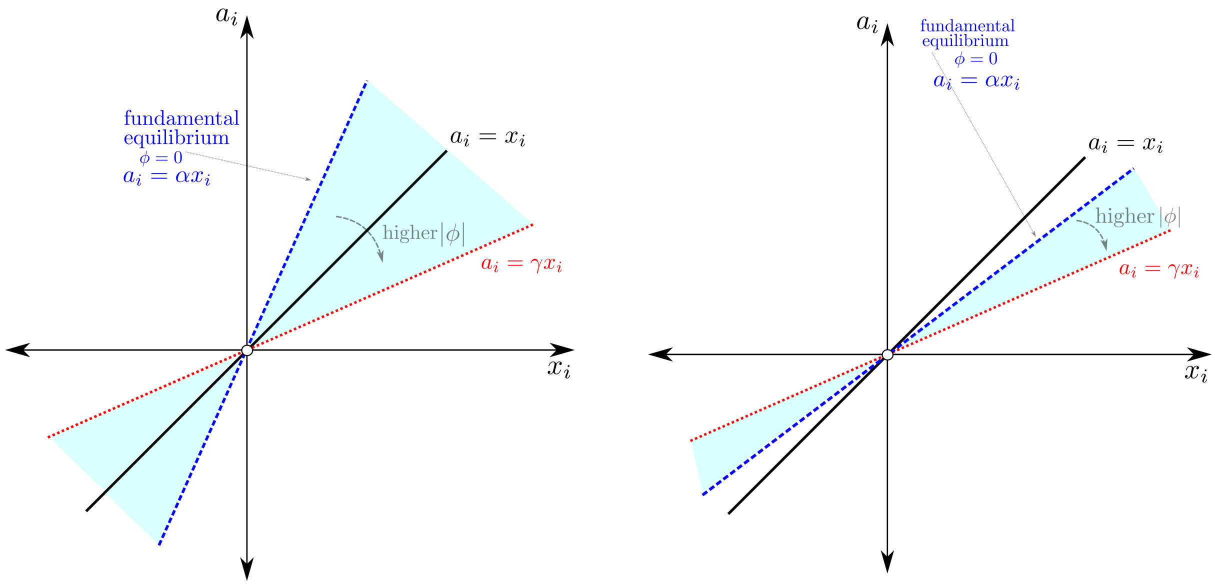
In order to understand this condition better, it is useful to cast this example in terms of the nomenclature of Section 2. Since this example is static, conditions (11) and (12) simplify to $\mathbf{A}(0)\mathbf{\Pi}(0) = 1$ and $\mathbf{B}(0)\mathbf{\Pi}(0) = 0$.[11] In this simple example, $\mathbf{A}(0) = 1$ is a scalar and $\mathbf{\Pi}(0) = \pi$,[12] so condition (11) simplifies to the requirement $\pi = 1$. This requirement is in turn satisfied when $\alpha > 1$.[13] Figure 1 shows graphically why we need $\alpha > 1$ for a sentiment equilibrium to exist in this case. All the lines in the figure are of the form $a_i = \pi x_i$ for different values of π. In both panels, the blue-dashed line denotes the equilibrium response of agents in the fundamental equilibrium $a = \alpha x_i$, in which case $\phi = 0$, while the red-dotted line denotes $a_i = \gamma x_i$. Since $\gamma < 1$, the red line is flatter than the black line $a_i = x_i$ (which is the condition needed for a sentiment equilibrium to exist). The black line ($a_i = x_i$) represents the condition which needs to be satisfied for a sentiment equilibrium to exist ($\pi = 1$). Equation (20) shows that π can be written as a convex combination of α and γ,

[11] Since agents observe only one signal, $n_s = 1$ and so the dimensionality of $\mathbf{A}$ is 1×1. Since the aggregate shock is known to be zero at all times, $\mathbf{B}$ is a matrix of zeros. Without loss of generality, we can set $n_\nu = 1$ in this case and so the dimensionality of $\mathbf{B}$ is also 1×1.

[12] Since there is only one signal $n_s = 1$, the dimensionality of $\mathbf{\Pi} = 1 \times n_s = 1 \times 1$.

[13] Condition (12) is trivially satisfied in this case as $\mathbf{B}(0) = 0$.

with the weights depending on $|\phi|$. Clearly, $\phi = 0$ picks the fundamental equilibrium ($\pi = \alpha$), while $\phi \to \infty$ would set $\pi = \gamma$. The shaded area in the figure denotes the range of π that can be obtained depending on the value of α. When $\alpha > 1$ (Figure 1a), the blue line is steeper than the 45-degree black line, and hence the the black line ($\pi = 1$) is part of the shaded area, i.e., one can find a $|\phi|$ between 0 and ∞ for which $\pi = 1$, and so the sentiment equilibrium exists. In contrast, when $\alpha < 1$ (Figure 1b), the blue line which represents the fundamental equilibrium lies below the 45-degree line. The black line $\pi = 1$ then lies outside the shaded region, and hence there is no $|\phi| \in (0, \infty)$ for which $\pi = 1$, and hence the sentiment equilibrium doesn't exist. Thus, Figure 1 shows that for a sentiment equilibrium to exist, the individual action must respond strongly enough to the signal x_i in the fundamental equilibrium, i.e., the blue line must be steeper than the 45-degree line.



(a) $\alpha > 1$

(b) $\alpha < 1$

Figure 1: **Existence of sentiment equilibrium**

Strategic complementarity The existence of sentiment equilibria does not require strong strategic complementarities, i.e., $\gamma > 1$. In fact, (21) shows that sentiment equilibria can exist even if $\gamma \leq 0$ (no complementarity or strategic substitutability). Endogenous information induces complementarities even when the primitive economy may not feature any. In our environment, agents' actions a_i are correlated with the aggregate action a as they all observe an endogenous signal which loads on a. The equilibrium covariance between a_i and a is $cov(a_i, a) = \sigma_z^2 \left(\frac{\alpha-1}{1-\gamma}\right) > 0$ even if $\gamma \leq 0$. This induced complementarity is bigger, the larger the value of α, and thus, a large enough α facilitates existence of additional sentiment equilibrium. This covariance is 0 in the fundamental equilibrium.

3.2 Adding aggregate fundamentals to the mix

The example above showed that sentiments can affect outcomes independently of aggregate fundamentals. More generally, not only can sentiments affect aggregate outcomes directly, they can also affect how the economy responds to changes in aggregate fundamentals, as the following example shows. To draw contrast with the previous example, we assume that it is common knowledge to all agents that the idiosyncratic fundamental $z_i = 0$ for all i. The best response (1) simplifies to

$$a_i = \delta \mathbb{E}_i \theta + \gamma \mathbb{E}_i a. \tag{22}$$

We assume that agent i observes the following signals:

$$x_i^1 = \theta + \varsigma_i \qquad \text{and} \qquad x_i^2 = a - \theta. \tag{23}$$

The first signal x_i^1 is a private noisy *exogenous* signal where $\theta \sim N(0,1)$ is the aggregate fundamental and $\varsigma_i \sim N(0, \sigma_\varsigma^2)$ is private noise. The second signal x_i^2 is an *endogenous* signal which loads on both the aggregate action a and the aggregate fundamental θ. Giving the model a business cycle interpretation, a_i denotes the output decision of firm i, the average action a is simply the aggregate output, and θ denotes the level of aggregate productivity. Then the endogenous signal $x_i^2 = a - \theta$ reveals the level of aggregate employment to each firm. If we interpret the model as a pricing problem of firms in a monetary context, a_i denotes the price set by firm i, p denotes the aggregate price level, θ can be interpreted as nominal aggregate demand, and x_i^2 is a signal that reveals the realization of real output.

Agent i's optimal decision must take the form $a_i = \pi_1 x_i^1 + \pi_2 x_i^2$. In a fundamental equilibrium, the aggregate action is driven by the fundamental θ only, i.e., $a = \psi\theta$. In this case, $x_i^2 = (\psi - 1)\theta$ will perfectly reveal the realization of θ, rendering the other signal x_i^1 worthless. It is then easy to see that the unique fundamental equilibrium is the same as the full-information equilibrium, in which

$$a = \frac{\delta}{1-\gamma}\theta. \tag{24}$$

However, in addition to this fundamental equilibrium, there also exists a sentiment equilibrium. In a sentiment equilibrium, a depends on both the fundamental θ and the sentiment shock ϵ, i.e., $a = \psi\theta + \phi\epsilon$, where ψ and ϕ are determined as part of equilibrium. Consequently, the signal x_i^2 can be rewritten as $x_i^2 = (\psi - 1)\theta + \phi\epsilon$ and the average action a can then be written as

$$a = \psi\theta + \phi\epsilon = \int a_i di = [\pi_1 + \pi_2(\psi - 1)]\,\theta + \pi_2\phi\epsilon.$$

Consistency requires that $\pi_2 = 1$ and $\pi_1 - \pi_2 = 0$. Appendix B.2 shows that when the sentiment equilibrium exists, we have[14]

$$|\phi| = \frac{\sigma_\varsigma\sqrt{\delta + \gamma - (1 + \sigma_\varsigma^2)}}{1-\gamma} \qquad \text{and} \qquad \psi = \frac{\delta - \sigma_\varsigma^2}{1-\gamma}. \tag{25}$$

In order for the sentiment equilibrium to exist, we need $\delta > 1 + \sigma_\varsigma^2 - \gamma$ and $\sigma_\varsigma > 0$. In other

[14]As in the previous example, the sentiment equilibrium is unique up to the sign of ϕ. We concentrate on $\phi > 0$ without loss of generality since the variance is identical in both cases.

words, the optimal action must be responsive enough to changes in the aggregate fundamental (large enough δ). This condition is analogous to the requirement that α be large enough for the sentiment equilibrium to exist in the previous example. Again, it is useful to interpret this requirement for existence of sentiment equilibria through the lens of Theorem 1. In this example, we have $\mathbf{A}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}'$, $\mathbf{B}(0) = \begin{bmatrix} 1 & -1 \end{bmatrix}'$ and $\mathbf{\Pi}(0) = \begin{bmatrix} \pi_1 & \pi_2 \end{bmatrix}$.[15] So condition (11) is the same as $\pi_2 = 1$ and (12) is the same as $\pi_1 - \pi_2 = 0$. These conditions ensure that the individual action a_i is responsive enough to changes in the endogenous signal (x_i^2 in this case) and are satisfied as long as δ is large enough.

Comparing the fundamental equilibrium (24) with the sentiment equilibrium (25), the average action now not only responds to changes in sentiments ϵ, but at the same time the presence of sentiments dampens the response of the average action to changes in the aggregate fundamental: $\frac{\delta - \sigma_\zeta^2}{1-\gamma} < \frac{\delta}{1-\gamma}$. While in the fundamental equilibrium, the endogenous signal x_i^2 enabled each agent to infer θ perfectly, in the sentiment equilibrium, agents can no longer infer the true realization of θ as the sentiment shock ϵ acts as *endogenous* noise. Returning to the interpretation of this example as the monetary economy, this would mean that in the fundamental equilibrium, the aggregate price adjusts to changes in nominal demand θ, keeping real output constant. However, in the sentiment equilibrium, agents can no longer infer the level of nominal demand perfectly. Consequently, individual prices and hence the aggregate price do not change enough, leading to changes in real output.

3.3 Does a sentiment equilibrium exist given any information structure?

Before proceeding to the analysis of the dynamic case, it is useful to discuss when sentiment equilibria exist. The two examples above suggested that given a signal structure, it may always be possible to find conditions under which sentiment equilibria exist. However, Theorem 1 states that this is *not* the case. Next, we present some examples to demonstrate the usefulness of Theorem 1 in helping ascertain whether, given a signal structure, a sentiment equilibrium exists.

Starting with the example in Section 3.1, it is easy to see that condition (14) is satisfied,

$$\text{rank}\begin{bmatrix} \mathbf{A}(L) & \mathbf{B}(L) \end{bmatrix} = \text{rank}\begin{bmatrix} 1 & 0 \end{bmatrix} = 1 = \text{rank}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \text{rank}\begin{bmatrix} \mathbf{A}(L) & \mathbf{B}(L) \\ 1 & 0 \end{bmatrix},$$

and that this is consistent with the existence of the sentiment equilibrium. Next, suppose we modify the signal in the example in Section 3.1 to

$$x_i = a + z_i + \eta,$$

where η denotes exogenous public noise. This signal corresponds to $\mathbf{A}(L) = 1$ and $\mathbf{B}(L) = 1$.[16]

[15] In this example, agents observe $n_s = 2$ signals. So the dimensionality of $\mathbf{A}$ is 2×1 and since there is only one aggregate fundamental $n_\nu = 1$, the dimension of $\mathbf{B}$ is $n_s \times n_\nu = 2 \times 1$. Finally, since $n_s = 2$, the dimension of $\mathbf{\Pi} = 1 \times n_s = 1 \times 2$.

[16] Since there is only one signal $n_s = 1$ and there is one source of aggregate noise, $n_\nu = 1$. So, $\mathbf{A}$ is $n_s \times 1 = 1 \times 1$ and $\mathbf{B}$ is $n_s \times n_\nu = 1 \times 1$.

Since rank $\begin{bmatrix}1 & 1\end{bmatrix} <$ rank $\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$, violating (14), no sentiment equilibrium exists. However, if we had changed the signal structure so that the common noise enters through a second signal,

$$x_i^1 = z_i + a \qquad \text{and} \qquad x_i^2 = a + \eta.$$

In this case, the number of signals is $n_s = 2$ and so the dimensionality of $\mathbf{A} = \begin{bmatrix}1 & 1\end{bmatrix}'$ is 2×1. Also, since there is one source of aggregate noise (η), we have $n_\nu = 1$ and the dimensionality of $\mathbf{B} = \begin{bmatrix}0 & 1\end{bmatrix}'$ is 2×1. In this case the rank condition (14) is satisfied, implying that sentiment equilibria can exist.

$$\text{rank}\begin{bmatrix}\mathbf{A}(L) & \mathbf{B}(L)\end{bmatrix} = \text{rank}\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} = 2 = \text{rank}\begin{bmatrix}1 & 0\\ 1 & 1\\ 1 & 1\end{bmatrix} = \text{rank}\begin{bmatrix}\mathbf{A}(L) & \mathbf{B}(L)\\ 1 & 0\end{bmatrix}$$

In fact, Benhabib et al. (2015) construct a sentiment equilibrium with this signal structure. Thus, Theorem 1 provides a practical way to check whether a model permits sentiment equilibria.

3.4 Learnability

The above examples show that both sentiment and fundamental equilibria can exist for the same parameter values in the same economic environment. Is there any reason to select one over the other? Often, equilibria which are stable under learning are selected over those that are not. Next, in the context of the two examples above, we show that the sentiment equilibria are stable under *least-squares learning* while the fundamental equilibrium is not.

Let's start with the example in Section 3.1. Recall that in this case, the fundamental equilibrium featured $a = 0$ while the sentiment equilibrium featured $a = \phi\epsilon$, where ϕ is defined in (21). Suppose that agents do not know which equilibrium is being played and they perceive that a is given by $a = \widehat{\phi}\epsilon$. Given the perceived response of a, the actual response of a can be written as

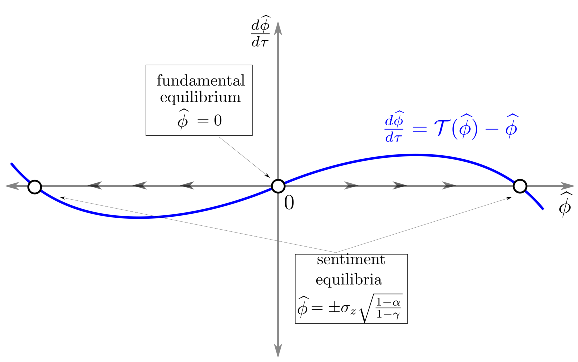
$$a = \mathcal{T}\left(\widehat{\phi}\right)\epsilon \qquad \text{where} \qquad \mathcal{T}\left(\widehat{\phi}\right) = \widehat{\phi}\left[\frac{\gamma\widehat{\phi}^2 + \alpha\sigma_z^2}{\widehat{\phi}^2 + \sigma_z^2}\right].$$

The equation $\widehat{\phi} = \mathcal{T}\left(\widehat{\phi}\right)$ has three zeros or stationary points: one at $\widehat{\phi} = 0$ (the fundamental equilibrium) and two others at $\widehat{\phi} = \pm\phi$ (as long as $\alpha > 1$). As is standard, the expectational stability of the stationary point is determined by the differential equation

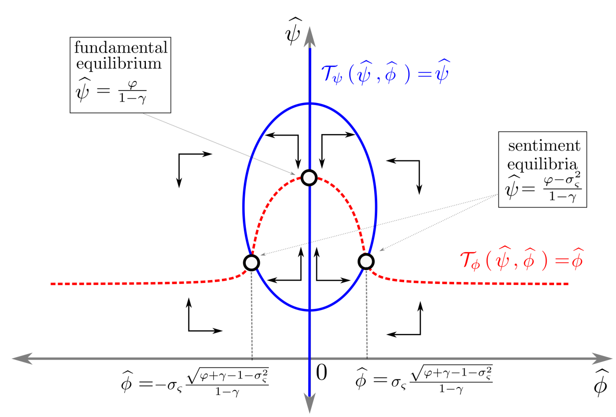
$$\frac{d\widehat{\phi}}{d\tau} = \mathcal{T}\left(\widehat{\phi}\right) - \widehat{\phi}.$$

If the differential equation is asymptotically stable at the stationary point $\widehat{\phi}$, then the system is said to be E-stable (Evans and Honkapohja, 2012). Evans and Honkapohja (2012) further show that E-stability guarantees that an equilibrium is stable under least-squares learning. Appendix C.1.1 shows that $\mathcal{T}'(0) - 1 > 0$, while $\mathcal{T}'(\phi) - 1 < 0$, implying that by the *E-stability principle*, the sentiment equilibrium is stable while the fundamental equilibrium is not. Figure 2a depicts this graphically by plotting a phase diagram for the differential equation above and shows that the fundamental equilibrium $\widehat{\phi} = 0$ is not stable under least-squares learning, while the two sentiment equilibria are.

It is easy to see why this might be the case. Suppose agents start with the belief that the variance of sentiments $\widehat{\phi}^2 > 0$. Given this belief, agents believe that the aggregate action responds to sentiment shocks and hence so does the signal x_i. Given a large α, agents individually respond strongly to sentiment shocks. As a result, the actual realization of a also responds strongly, and this feeds positively into the updated forecast of the variance. Eventually, this process converges to one of the sentiment equilibria rather than the fundamental equilibrium. The only way agents can converge to the fundamental equilibrium is if they start with the belief that the variance $\widehat{\phi} = 0$. In this case, agents believe that the aggregate action is unresponsive to sentiment shocks and thus, so is the signal x_i. Given these beliefs, an agent's optimal action does not respond to sentiments, causing the actual aggregate action to also not respond to sentiments. Starting from any non-zero $\widehat{\phi}$, agents' beliefs cannot converge to the fundamental equilibrium as long as $\alpha > 1$. The condition $\alpha > 1$ plays two separate although related roles. Recall from the discussion in Section 3.1 that in order for the sentiment equilibrium to exist, we required that agents respond strongly to the endogenous signal, i.e., $\alpha > 1$. It is precisely this strong reaction which also makes the sentiment equilibrium learnable. For the sentiment equilibrium to be stable under learning, we need the current forecast of the variance to positively feedback into future forecasts strongly enough, which requires that agents respond strongly to signals, given their current forecasts of the variance.



(a) no fundamental shock



(b) fundamental and sentiment shock

Figure 2: **Stability under learning.** In panel (a), the blue curve represents the differential equation $\frac{d\widehat{\phi}}{d\tau} = \mathcal{T}(\widehat{\phi}) - \widehat{\phi}$. In panel (b), the red-dashed curve represents $\mathcal{T}_\psi\left(\widehat{\psi}, \widehat{\phi}\right) - \widehat{\psi} = 0$ and the blue curve represents $\mathcal{T}_\phi\left(\widehat{\psi}, \widehat{\phi}\right) - \widehat{\phi} = 0$.

The example in Section 3.2 also shares the same properties. Suppose agents perceive the aggregate action to take the form $a = \widehat{\psi}\theta + \widehat{\phi}\epsilon$. Appendix C.1.2 shows that the actual form of a in terms of $\widehat{\psi}$

and $\widehat{\phi}$ satisfies:

$$a = \mathcal{T}_{\psi}\left(\widehat{\psi}, \widehat{\phi}\right)\theta + \mathcal{T}_{\phi}\left(\widehat{\psi}, \widehat{\phi}\right)\epsilon,$$

where the exact expressions of $\mathcal{T}_{\psi}\left(\widehat{\psi}, \widehat{\phi}\right)$ and $\mathcal{T}_{\phi}\left(\widehat{\psi}, \widehat{\phi}\right)$ are presented in the appendix. The appendix shows that the system of equations $\widehat{\psi} = \mathcal{T}_{\psi}\left(\widehat{\psi}, \widehat{\phi}\right)$ and $\widehat{\phi} = \mathcal{T}_{\phi}\left(\widehat{\psi}, \widehat{\phi}\right)$ has three roots or stationary points : the fundamental equilibrium $\left(\widehat{\psi}, \widehat{\phi}\right) = \left(\frac{\delta}{1-\gamma}, 0\right)$ and the two sentiment equilibria $\left(\widehat{\psi}, \widehat{\phi}\right) = \left(\frac{\delta-\sigma_{\varsigma}^2}{1-\gamma}, \pm\frac{\sigma_{\varsigma}\sqrt{\delta+\gamma-(1+\sigma_{\varsigma}^2)}}{1-\gamma}\right)$.[17] The expectational stability of each stationary point is determined by the matrix differential equation system:

$$\frac{d}{d\tau}\begin{bmatrix}\widehat{\psi}\\ \widehat{\phi}\end{bmatrix} = \begin{bmatrix}\mathcal{T}_{\psi}\left(\widehat{\psi}, \widehat{\phi}\right) - \widehat{\psi}\\ \mathcal{T}_{\phi}\left(\widehat{\psi}, \widehat{\phi}\right) - \widehat{\phi}\end{bmatrix}.$$

Figure 2b depicts the phase diagram associated with this differential equation system and shows that any initial condition with $\widehat{\phi}_0 \neq 0$ converges to one of the two sentiment equilibria, implying that the sentiment equilibria are stable under least-squares learning while the fundamental equilibrium is not.[18] Unless the agents' starting value for $(\widehat{\psi}, \widehat{\phi})$ satisfies $\widehat{\phi} = 0$, the system converges to the sentiment equilibrium. Again, if agents attribute part of the variance of the aggregate outcome to sentiment shocks, then given a large δ, their actions respond strongly to the endogenous signal which causes their forecast of the contribution of sentiments to aggregate volatility to converge towards the sentiment equilibrium. Only if agents initially attribute all the volatility in the aggregate action to the aggregate fundamental does their forecast of aggregate volatility eventually converge to the fundamental equilibrium.

This finding highlights a key difference between exogenous sentiments and our notion of endogenous sentiments which has not been previously identified in the literature. Recall that exogenous sentiments are classified as part of primitive shocks and hence generate fundamental equilibria under Definition 1: it follows from our findings above that *exogenous sentiments* generate fundamental equilibria which are not stable under least-squares learning.

3.5 Normative implications of exogenous and endogenous sentiments

Since sentiment-driven fluctuations are disciplined by equilibrium in our environment, this opens the door to policy intervention where a policy maker can use policy to pick between one of the many sentiment equilibria (if there are multiple sentiment equilibria) and the fundamental equilibrium. Thus, the planner can choose to eliminate sentiment equilibria altogether, if she so desires. To see this, recall that sentiment equilibria in the examples above existed only if the individual action was responsive enough to idiosyncratic and aggregate fundamentals, i.e., α and/or δ were large enough. While we take the best response function (1) as a primitive, as aforementioned, such a best response function can be derived in many commonly used economic models. Furthermore, in these models, α, δ

[17] Again, the last two stationary points exist as long as δ is large enough for sentiment equilibria to exist.

[18] See Appendix C.1.2 for a proof.

and γ are themselves functions of deep parameters and policy responses. Thus, by affecting the values of α and δ, policy can eliminate sentiment equilibria altogether. We illustrate this in Appendix D in the context of a monetary model in which firms have to solve their pricing problem. The appendix shows that a policy which subsidizes firms who draw a low idiosyncratic productivity z_i and taxes those who draw high idiosyncratic productivity z_i lowers the sensitivity of individual pricing decisions to realizations of z_i (lowering α). Such a policy can then prevent sentiment equilibrium from existing altogether. This is a key distinction from the literature on exogenous sentiments. In these models, the equilibrium is typically unique and hence policy cannot pick between different equilibria.

4 Beyond static sentiment equilibria

We can now proceed to the main question this paper strives to answer. What are the conditions under which sentiments can have persistent aggregate effects? In other words, we study equilibria in which $\phi(L) = \sum_{\tau=0}^{\infty} \phi_\tau L^\tau$, where $\phi_\tau \neq 0$ at least for some $\tau > 0$. While Theorem 1 established necessary conditions for the existence of sentiment equilibria, it did not guarantee that sentiments lead to persistent fluctuations in such equilibria. In order for persistent sentiment-driven fluctuations to arise in equilibrium, additional conditions must be satisfied, as we show next.

4.1 Necessary conditions for persistent sentiments

To proceed, it is useful to define the following assumptions on the information set.

Assumption 1. *Past realizations of the aggregate action are observable with a k-period lag, i.e.,* $a^{t-k} \in \mathcal{I}_{i,t}$

Assumption 2. *The k-period ago realization of primitive shocks can be perfectly inferred from exogenous information, i.e.,* $\mathbb{V}(\boldsymbol{\nu}^{t-k}) \subseteq \mathbb{V}(\mathbf{y}_i^t)$.

If Assumption 1 is satisfied, at date t, each agent i observes the aggregate outcomes up till date $t-k$, i.e., each agent knows the sequence a^{t-k}. Similarly, Assumption 2 ensures that at any date t, each agent can perfectly infer the realization of primitive shocks up till date $t-k$ by observing the set of exogenous signals $\mathbf{y}_{i,t}$.[19] Theorem 2 formally defines the conditions under which sentiments can have persistent aggregate effects.

Theorem 2. *If Assumptions 1 and 2 are satisfied, then in any sentiment equilibria* $\phi(L) = \sum_{\tau=0}^{\infty} \phi_\tau L^\tau$, *it must be the case that* $\phi_\tau = 0$ *for all* $\tau \geq k$. *In other words, a sentiment shock* ϵ_t *at date* t *cannot affect outcomes after date* $t + k$.

Proof. See Appendix A.3.[20] □

[19] For Assumption 2 to be satisfied, it is not necessary for agents to observe $\boldsymbol{\nu}_{t-k}$ directly. For example, if the aggregate fundamental follows an AR(1) process, $\theta_t = \rho\theta_{t-1} + v_t$, observing past fundamentals $\{\theta^{t-1}\}$ allows agents to infer past shocks $\{v^{t-1}\}$ perfectly.

[20] The proof relies on the result from Definition 2 and Appendix A.1 that we can focus on invertible $\phi(L)$.

Theorem 2 states that if agents observe past aggregate actions and fundamentals perfectly with a finite lag k, the effects of a sentiment shock dies out after a finite number of periods. In particular, dynamics of the aggregate outcome driven by a sentiment shock can be described by a moving average process where the maximum lag length is $k-1$.

A few remarks are in order. First, while this theorem might seem trivial at first glance, it is by no means obvious that observing information about past realizations will be fully revealing. For example, Rondina and Walker (2020) show that even in a setting where agents can observe past realizations of the aggregate variable, they may not be able to infer the true innovations if the underlying fundamental process is non-invertible. In our case, the allowable response to sentiments can also be non-invertible in principle. However, Theorem 3 in Appendix A.1 proves that we can restrict attention to invertible processes without loss of generality. Second, the statement of Theorem 2 holds under very general conditions, as we do not impose any restrictions on the number of shocks or the number of signals. Importantly, the theorem holds for *any* private information that each agent might possess. Finally, even though Assumptions 1-2 imply that agents can observe past aggregate actions and fundamentals, there is no supposition that they observe idiosyncratic fundamentals perfectly. In fact, agents can still have persistent forecast errors about their individual fundamentals. However, Theorem 2 makes clear that these forecast errors about the idiosyncratic fundamental *cannot* translate into persistent aggregate fluctuations.

Corollary 1. *If Assumptions 1 and 2 hold for $k = 1$, then in any sentiment equilibria $\phi(L) = \phi(0)$.*

A direct corollary of Theorem 2 is that if at date t, each agent observes the realization of the aggregate fundamental θ_{t-1} and a_{t-1}, then the *only* sentiment equilibrium is the one in which changes in sentiments at date t can only affect aggregate outcomes contemporaneously, i.e., $\partial a_{t+s}/\partial \epsilon_t = 0$ for all $s > 0$. This result is independent of any private information that agents may possess or other signals that they might observe. Corollary 1 also implies that the static examples in Section 3 can equally be interpreted in terms of a dynamic environment in which each agent observes (or can infer) θ_{t-1} and a_{t-1} at date t. The upshot of Theorem 2 and in particular Corollary 1 is that in order for sentiment-driven fluctuations to display persistence, Assumption 1 and/or Assumption 2 do not hold for $k = 1$, i.e., $a^{t-1} \notin \mathcal{I}_{i,t}$ and/or $\mathbb{V}(\mathbf{y}_i^t) \not\supseteq \mathbb{V}(\boldsymbol{\nu}^{t-1})$. This powerful characterization provides a helpful insight to the large literature which studies sentiment-driven equilibria such as Benhabib et al. (2013, 2015) and Chahrour and Gaballo (2016), among others. While this literature has largely concentrated on studying i.i.d. fluctuations driven by sentiments, Theorem 2 and Corollary 1 serve as a guide by uncovering the minimum ingredients required to construct and study equilibria in which sentiments can drive persistent fluctuations.

These results also provide additional insight about the large literature studying economies with information frictions. While the models used in this literature are very similar to the setting studied in this paper, their focus has largely been on fundamental equilibrium. In order to avoid the complexity of dealing with the problem referred to as *forecasting the forecasts of others* (Townsend, 1983), this literature has commonly made the assumption that the realizations of aggregate fundamentals and aggregate outcomes in the past become common knowledge after a short lag. Theorem 2 shows that this assumption on the information set greatly reduces the extent to which sentiments can have persistent effects.

Finally, notice that Theorem 2 is *not* about the existence of sentiment equilibrium; the statement of Theorem 2 is conditional on a sentiment equilibrium existing. This raises the question whether there exist any equilibria in which sentiments can drive persistent and predictable aggregate fluctuations even if we relax these assumptions. Next, we extend the example studied in Section 3.1 and show that persistent sentiment-driven fluctuations can emerge when Assumptions 1 and 2 only hold for $k > 1$. This example shows that the set of such sentiment equilibria is not empty.

Example: a_{t-1} is not observed at date t. We start with the dynamic counterpart to the example studied in Section 3.1 where the aggregate fundamental is fixed at $\theta_t = 0$ for all time and this is common knowledge. Recall that the best response of agent i at any date t can be written as $a_{i,t} = \gamma \mathbb{E}_{i,t} a_t + \alpha \mathbb{E}_{i,t} z_{i,t}$, which is identical to equation (15) except that we have appended time-subscripts. It follows from Theorem 2 that if agents observed a_{t-1} at date t, then any sentiment equilibrium must take the form $a_t = \phi \epsilon_t$, where ϕ is defined in equation (21). Now relax this assumption and assume that at date t, agents cannot observe the realization of a_{t-1} but can observe the aggregate outcome with two lags. This can be formalized as each agent i receiving two signals at each date:

$$x^1_{i,t} = a_t + z_{i,t} \qquad \text{and} \qquad x^2_{i,t} = a_{t-2}.$$

$x^1_{i,t}$ is a private signal as in Section 3, except that we have appended time-subscripts. Following Theorem 2, a sentiment equilibrium in which $\phi(L)$ is a MA(1) can exist. In fact, Appendix B.3 shows that there exists a sentiment equilibrium in which the aggregate outcome at any date t is affected by the contemporaneous sentiment shock ϵ_t and lagged sentiment shock ϵ_{t-1}:

$$a_t = \phi_0 \epsilon_t + \phi_1 \epsilon_{t-1}.$$

The expressions describing ϕ_0 and ϕ_1 aren't particularly insightful and are relegated to Appendix B.3. This example shows that as long as Theorem 2 does not hold for $k = 1$, there exist sentiment equilibria in which sentiments can drive persistent aggregate fluctuations, albeit in a limited fashion. As was the case in Section 3.1, the sentiment equilibrium exists only if α is large enough.

Extending the example in Section 3.2 by using this strategy is not analytically tractable. While we can establish the existence of a sentiment equilibrium in which sentiments can generate MA(1) dynamics by assuming that agents can only observe the aggregate fundamental with a lag of $k = 2$ periods, we have to resort to numerical methods. From an applied macroeconomics point of view, in order to match observed dynamics, one would have to compute an equilibrium with large information lags (large k) to capture the persistence. However, doing so in an analytically tractable way is not trivial. Technically, there are two ways to solve for equilibria with a large k. First, one could apply the Kalman filter to perform the filtering. However, a large k also implies a large dimensional state space and forces one to resort to numerical methods. Using numerical methods greatly reduces our ability to characterize *all* possible sentiment equilibrium (there might be multiple). The lack of analytical tractability makes it difficult to establish the existence of sentiment equilibrium and to characterize its properties in these general settings in which the forecast errors abruptly become zero. Thus, this is not our preferred way forward.

The other possibility is to circumvent the large dimensionality of the state space by using frequency domain filtering methods such as the Wiener filter. However, one has to flip k roots outside the unit circle, which requires us to construct k orthogonal matrices by hand. For large k this is infeasible. Thus, we present an alternate strategy which allows one to relax the assumptions in Theorem 2 and at the same time retain analytical tractability while constructing equilibria with persistent sentiment-driven fluctuations.

4.2 A practical guide for constructing persistent sentiment-driven fluctuations

When Assumptions 1 and 2 are satisfied, the perfect observation of aggregate variables is only delayed by a finite number of periods, and this type of truncation forces the forecast errors to jump discretely to zero after a finite horizon. The fact that the number of periods after which agents' forecast error goes to zero is discrete makes it hard to have an analytical handle. A simple strategy to solve this problem would be to "*smooth*" out the decay of forecast errors. This can be accomplished by assuming that agents observe past realizations of aggregate outcomes and aggregate fundamentals with additional noise. The presence of such noise would prevent agents from perfectly inferring the exact realizations of the aggregate fundamental. However, such a strategy would lead to an environment in which agents observe fewer signals than shocks. While this in itself is not a problem, this does severely restrict analytical tractability in characterizing the equilibrium, since it involves dynamic signal extraction in a "non-square system" in a setting with endogenous signals (see Nimark (2017) and Huo and Takayama (2017) for a detailed discussion). Additionally, the reliance on numerical methods prevents us from characterizing the entire set of sentiment equilibria and their general properties. Thus, this strategy is also not ideal for our purposes, since we want to uncover more general conditions under which sentiments can have persistent effects. Thus, we adopt a different strategy which allows us to "*smooth*" out the decay of forecast errors while at the same time retaining analytical tractability.

Smoothing out the decay of forecast errors Suppose a latent stationary variable ξ_t is described by $\xi_t = \phi(L)e_t$, where $\phi(L)$ is invertible. Each period, a signal x_t is observed which provides information about the realizations of the sequence ξ^t. Assume that the signal takes the form

$$x_t = (L - \lambda)\xi_t, \qquad \lambda \in (-1, 1). \tag{26}$$

For $|\lambda| < 1$, the signal x_t puts little weight on the current realization of ξ_t and thus the agent is unable to infer ξ_t perfectly by observing the sequence of signals x^t.[21] The signal x_t can be broadly

[21] A higher λ implies a larger weight on the current ξ_t and increases the informativeness of the signal about ξ_t. Actually, when λ is larger than 1, the signal reveals the underlying shock perfectly.

interpreted as a moving average of the realizations of ξ_t. Appendix E shows that for $k \geq 0$, the forecast error about ξ_{t-k} at date t can be written as

$$\xi_{t-k} - \mathbb{E}[\xi_{t-k}|x^t] = \lambda^k \phi(\lambda)(1-\lambda^2) \sum_{s=0}^{\infty} \lambda^s e_{t-s}.$$

The forecast error decays gradually at a rate which is proportional to λ and only converges to zero asymptotically, i.e., the agent only learns about the actual realization of ξ_{t-k} asymptotically. Thus, a signal of the form (26) is a convenient way to model a smooth decay of the forecast error and avoids forecast errors vanishing with an undesirable discreteness, as in the previous section. Of course, this is not the most traditional way to model information lags; however, compared to the strategy of adding observation noise, this alternative signal structure allows us to generate smoothly decaying forecast errors and also affords analytical tractability.[22] Armed with this modeling device, we revisit the familiar examples from Section 3.

4.2.1 Persistent sentiment-driven fluctuations absent fundamental shocks

Let's begin with the environment studied in Section 3.1. We maintain the assumption that $z_{i,t}$ is i.i.d. across time and individuals. At each date t, agents observe a private signal $x^1_{i,t} = a_t + z_{i,t}$. In addition, each agent observes another signal which takes the form $x^2_{i,t} = (L-\lambda)a_t$, where $|\lambda| < 1$.[23]

As was the case in Section 3.1, the unique fundamental equilibrium of this economy is still given by $a_t = 0$ for all t and is unaffected by the signal $x^2_{i,t}$. However, the additional signal implies that the conditions in Theorem 2 are satisfied for sentiments to have persistent effects.

Proposition 1. *For $\alpha > 1$, there exists a unique sentiment equilibrium in which the dynamics of the aggregate outcome can be described by*

$$a_t = \lambda a_{t-1} + \phi\left(1-\lambda^2\right)\epsilon_t, \tag{27}$$

where ϕ is the variance of the aggregate outcome in the static sentiment equilibrium in Section 3.1 and is defined in (21).[24]

[22] This strategy shares some similarity with the confounding process in Rondina and Walker (2020). In Rondina and Walker (2020), the variable ξ_t itself may follow a non-invertible process. Instead, we introduce the non-invertible component in the signal which prevents agents from inferring the underlying shock e_t perfectly after finite time. To be clear, while the modeling strategy looks superficially similar to Rondina and Walker (2020), they *do not* study self-fulfilling sentiment fluctuations or how the presence of sentiments can affect the dynamic response of the economy to aggregate fundamentals, which we show next.

[23] In this example, $n_s = 2$. Again, since the aggregate fundamental is known to be fixed at zero for all time, we can set $n_\nu = 1$. Then, the dimension of $\mathbf{A}(L) = \begin{bmatrix}1 & L-\lambda\end{bmatrix}'$ is 2×1, and of $\mathbf{B}(L) = \begin{bmatrix}0 & 0\end{bmatrix}'$ is 2×1. Note that the rank condition (14) is satisfied in this example: $\operatorname{rank}\begin{bmatrix}\mathbf{A}(L) & \mathbf{B}(L)\end{bmatrix} = \operatorname{rank}\begin{bmatrix}1 & 0 \\ L-\lambda & 0\end{bmatrix} = 1 = \operatorname{rank}\begin{bmatrix}1 & 0 \\ L-\lambda & 0 \\ 1 & 0\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\mathbf{A}(L) & \mathbf{B}(L) \\ 1 & 0\end{bmatrix}$.

[24] Notice that again there are two sentiment equilibria, $\phi > 0$ and $\phi < 0$, since (21) only defines ϕ uniquely up to the sign. As has been the case throughout the paper, we focus on the equilibrium with $\phi > 0$. This is without loss of generality, since the variance and autocovariances of a are the same regardless of the sign of ϕ: both sentiment

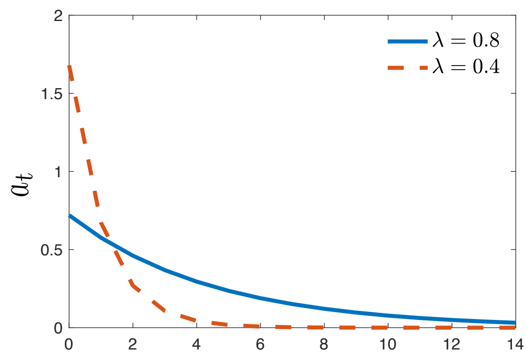


Figure 3: **IRF in sentiment equilibrium:** For the purpose of this figure, we set $\sigma_z = 2$, $\alpha = 1.5$.

Proof. See Appendix B.1. □

The first thing to notice is that setting $\lambda = 0$ implies that agents can observe a_{t-1} perfectly. In this case, (27) shows that the only sentiment equilibrium is given by $a_t = \phi\epsilon_t$, which is the same as the static sentiment equilibria in Section 3.1, thus confirming the claim in Theorem 2. The static example in Section 3.1 showed that sentiment shocks can amplify the unconditional volatility of aggregate outcomes. This dynamic extension shows that in addition, sentiment shocks can also generate persistence. A sentiment shock at date t induces persistent deviations of the aggregate action a_t from 0. In fact, the aggregate dynamics of a_t can be described by an AR(1) process with persistence λ and (unconditional) variance $\phi^2(1-\lambda^2)^2$. Notice that the volatility and persistence are closely related. Recall that the magnitude of λ is indicative of how informative the signal $x^2_{i,t}$ is about the current realization of a_t. Equation (27) shows that the more informative the signal is about current a_t, the more persistent is the effect of a sentiment shock. At the same time, a higher λ also reduces the impact effect of a sentiment shock on aggregate outcomes, since agents are relatively better informed about the true realization of a_t. Figure 3 plots the impulse response to a sentiment shock for two different values of λ. As discussed above, the impulse response with a lower (higher) λ has a larger (smaller) effect on impact but dies down faster (slower). While we concentrate on $\lambda \in (0,1)$ in the figure, the characterization continues to be true for $\lambda \in (-1,0)$ as well. This would result in an oscillatory impulse response.

This close relation between the persistence and size of the impact effect underscores the fact that the properties of sentiment-driven fluctuations are disciplined by equilibrium – the example shows that given an economic environment, a modeler who constructs a sentiment equilibrium with a particular unconditional volatility may not be allowed to independently choose the persistence. If instead sentiments were modeled as exogenous shocks, the modeler would have the freedom to choose both the volatility and persistence of the exogenous shock process.

equilibria feature the same autocovariance-generating function $\phi(L)\phi(L^{-1})$.

4.2.2 Persistent sentiment-driven fluctuations with fundamental shocks

In the context of the static example in Section 3.2, we showed that not only can sentiments drive aggregate fluctuations, they can also alter the effects aggregate fundamentals have on aggregate outcomes. In this section we extend the example to a dynamic setting and reaffirm that not only can sentiment shocks have persistent effects on aggregate outcomes, they can also alter how fundamental shocks affect aggregate outcomes. In particular, we show that sentiments can alter not just the impact effect of aggregate fundamentals but also the lagged effects.

Recall that the best response in this case can be written as: $a_{i,t} = \delta \mathbb{E}_{it}[\theta_t] + \gamma \mathbb{E}_{it}[a_t]$. Agents are assumed to observe three signals:

$$x^1_{i,t} = \theta_t + \varsigma_{i,t}, \qquad x^2_{i,t} = a_t - \theta_t, \qquad x^3_{i,t} = (L-\lambda)\theta_t.$$

We assume that dynamics of the aggregate fundamental θ_t can be described by $\theta_t = g(L)v_t$, where $v_t \sim N(0,1)$ denotes the innovation to aggregate fundamentals at date t.[25] The first two signals are the same as in Section 3.2, with time-subscripts appended. The new signal $x^3_{i,t}$ allows an agent to imperfectly observe the realization of the aggregate fundamental θ_t.[26]

In the fundamental equilibrium, the aggregate action is by definition only driven by the fundamental shock, $a_t = \psi(L)v_t$. Observing the signal $x^2_{i,t}$ allows each agent to infer the true past and current realizations of v_t (and hence of θ_t). This implies that agents have full information in equilibrium and agents can ignore the signal $x^3_{i,t}$ in equilibrium. Consequently, the fundamental equilibrium is identical to the full-information equilibrium in which the aggregate action a_t tracks θ_t:

$$a^f_t = \overline{\psi}\theta_t, \qquad \text{where} \qquad \overline{\psi} = \frac{\delta}{1-\gamma}. \tag{28}$$

Instead of allowing agents to observe past aggregate fundamentals perfectly, adding the signal $x^3_{i,t}$ to agents' information set relaxes Assumption 2 and permits sentiment equilibria in which the aggregate action a_t is affected by not just the current and past realizations of the aggregate fundamental but also by the current and past sentiment shocks: $a_t = \psi(L)v_t + \phi(L)\epsilon_t$. Since a_t also depends on the sequence ϵ^t, observing the signal $x^2_{i,t}$ no longer allows agents to infer the current realization of v_t. Instead, they must solve a dynamic signal extraction problem. Importantly, unlike in the fundamental equilibrium, agents utilize the signal $x^3_{i,t}$ in doing so.

Proposition 2. *For δ large enough, there exists a unique sentiment equilibrium $a_t = \psi(L)v_t + \phi(L)\epsilon_t$,*

[25] We assume that $g(L)$ is invertible and normalize $g(0) = 1$ without loss of generality.

[26] In this example, there are three signals ($n_s = 3$) and one aggregate fundamental ($n_\nu = 1$). So the dimension of $\mathbf{A}(L) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}'$ is 3×1 and of $\mathbf{B}(L) = \begin{bmatrix} 0 & -g(L) & (L-\lambda)g(L) \end{bmatrix}'$ is also 3×1. The rank condition (14) is

$$\text{rank}\begin{bmatrix} \mathbf{A}(L) & \mathbf{B}(L) \end{bmatrix} = \text{rank}\begin{bmatrix} 0 & 1 \\ 1 & g(L) \\ 0 & (L-\lambda)g(L) \end{bmatrix} = 2 = \text{rank}\begin{bmatrix} \mathbf{A}(L) & \mathbf{B}(L) \\ 1 & 0 \end{bmatrix} = \text{rank}\begin{bmatrix} 0 & 1 \\ 1 & g(L) \\ 0 & (L-\lambda)g(L) \\ 0 & 1 \end{bmatrix}.$$

where

$$\psi(L) = \overline{\psi} g(L) - \frac{\sigma_\varsigma^2 (1-\lambda^2)}{(1-\gamma) g(\lambda)} \frac{1}{1-\lambda L}, \tag{29}$$

$$\phi(L) = \pm \frac{(1-\lambda^2)\, \sigma_\varsigma \sqrt{(\gamma+\varphi-1) g(\lambda)^2 - \sigma_\varsigma^2}}{(1-\gamma) g(\lambda)} \frac{1}{1-\lambda L}, \tag{30}$$

where $\overline{\psi}$ *is defined in* (28).

Proof. See Appendix B.2. □

Equation (30) shows that a sentiment shock ϵ_t can generate persistent aggregate fluctuations. In fact, the impulse response of the aggregate action in response to a sentiment shock ϵ_t can be described by an AR(1), which is depicted graphically by the blue solid curve in Figure 4a[27]:

$$\frac{\partial \left| a_{t+s} \right|}{\partial \epsilon_t} = \sigma_\varsigma \left(1-\lambda^2\right) \frac{\sqrt{\gamma+\delta-1-\sigma_\varsigma^2 g(\lambda)^{-2}}}{1-\gamma} \lambda^s \qquad \text{for } s \geq t.$$

As in the previous example, there is a close relationship between the persistence and the impact effect. The greater the persistence λ, the smaller the impact effect $\sigma_\varsigma \left(1-\lambda^2\right) \frac{\sqrt{\gamma+\delta-1-\sigma_\varsigma^2 g(\lambda)^{-2}}}{1-\gamma}$.[28] Sentiment shocks also affect aggregate outcomes indirectly, by altering how fundamental shocks affect aggregate outcomes. To see this, using (29)-(30), we can write the evolution of a_t as

$$a_t - a_t^f = \lambda \left(a_{t-1} - a_{t-1}^f\right) + \omega_1 v_t \pm \omega_2 \epsilon_t, \qquad \text{where} \qquad a_t^f = \overline{\psi} \theta_t, \tag{31}$$

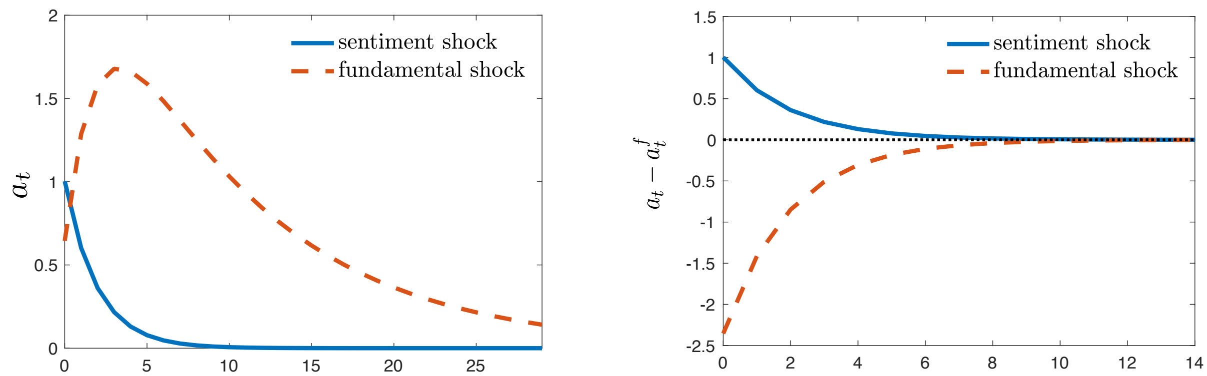
where $\omega_1 = -\frac{\sigma_\varsigma^2}{1-\gamma}\left(\frac{1-\lambda^2}{\lambda}\right)$ and $\omega_2 = \sigma_\varsigma \left(1-\lambda^2\right) \frac{\sqrt{\gamma+\varphi-1-\sigma_\varsigma^2 g(\lambda)^{-2}}}{1-\gamma}$. Equation (31) shows that while in the fundamental equilibrium a_t tracks $\overline{\psi}\theta_t$ perfectly, in the sentiment equilibrium, this is only true on average. In fact, the gap between a_t and $a_t^f = \overline{\psi}\theta$ only closes slowly, and this process can be described by an AR(1). It is important to note that while the figure restricts attention to the case with $0 < \lambda < 1$, the characterization also holds for $-1 < \lambda < 0$. However, the impulse response would feature oscillations in this case.

A fundamental shock v_t or a sentiment shock ϵ_t at date t can drive a persistent wedge between a_t and $a_t^f = \overline{\psi}\theta_t$. The blue-solid curve plots the difference between a_t in the sentiment equilibrium and a_t^f in response to a positive sentiment shock at date 0. Since $\frac{\partial a_t^f}{\partial \epsilon_t} = 0$, the blue line shows that in a sentiment equilibrium, a sentiment shock can drive a persistent wedge between a_t and a_t^f. The red-dashed curve in Figure 4 plots the persistent gap between a_t in the sentiment equilibrium and in the fundamental equilibrium following a positive shock to fundamentals at date 0. As can be seen, a_t responds less to the fundamental shock in the sentiment equilibrium than in the fundamental equilibrium ($a_t < a_t^f$). Over time, this gap becomes smaller before it finally vanishes.

[27] Again, we are focusing on the positive solution in (30), i.e., $\phi = \frac{(1-\lambda^2)\sigma_\varsigma \sqrt{(\gamma+\varphi-1)g(\lambda)^2-\sigma_\varsigma^2}}{(1-\gamma)g(\lambda)} \frac{1}{1-\lambda L}$. This is without loss of generality, since the autocovariance-generating function $\phi(L)\phi(L^{-1})$ is identical in both cases.

[28] Setting $\lambda = 0$ implies that agents can observe θ_{t-1} perfectly. Then Theorem 2 states that the only sentiment equilibrium can be the static one described in Section 3.2. Setting $\lambda = 0$ in equations (29)-(30) confirms this.

The red dashed curve in figure 4a depicts the impulse response of a_t in response to an aggregate shock in the sentiment equilibrium. The red-dashed curve exhibits a hump-shaped response to an aggregate fundamental shock. This is in contrast to the dynamics of a_t^f. Since a_t^f tracks θ_t, which is AR(1) in this figure, the largest effect on a_t^f is on impact, and then it monotonically declines. In contrast, in the sentiment equilibrium, since agents are unable to infer the true realization of θ at date 0 when the shock hits, they attribute a part of the observed fluctuations in the signals to sentiments, causing each of them to react less to the fundamental shock. This opens up a gap between the response of a_t in the fundamental equilibrium and in the sentiment equilibrium. Over time, agents learn that it was in fact a positive fundamental shock that hit at date 0 and consequently react more, endowing the impulse response with a hump-shape. As the agents learn over time, the difference between a_t and a_t^f vanishes.



(a) IRFs in the sentiment equilibrium

(b) persistent gap between a_t and $\overline{\psi}\theta_t$

Figure 4: **Properties of the sentiment equilibrium:** For the purpose of generating this figure, we further assume that θ_t follows an AR(1) process: $\theta_t = \rho\theta_{t-1} + v_t$. We set $\rho = 0.9$, $\lambda = 0.6$, $\sigma = 2$, $\delta = 1.5$, and $\gamma = 0.5$.

Hump-shaped impulse responses The hump-shaped response is an empirical regularity emphasized in the DSGE literature (e.g., Christiano et al. (2005), Smets and Wouters (2007)) in the context of many macroeconomic data series. Typically, these models generate a hump-shaped response by introducing ad hoc features such as habit formation or adjustment costs. Another potential way to generate hump-shaped responses is to model the presence of dispersed information. Since higher-order expectations are more anchored by the prior, they can explain the presence of additional inertia in an economy's response to shocks (e.g., Woodford (2002) and Angeletos and Huo (2021)).

This paper presents another logically distinct way which could explain the presence of additional inertia. The example above shows that sentiment equilibria can allow one to generate hump-shaped dynamics without any assumption of adjustment costs or even exogenously imposed information constraints or dispersed information. In our model environment, the information is complete in the fundamental equilibrium and so there is no more dispersed information after agents observe aggregate outcomes. However, in the sentiment equilibrium, the presence of the sentiment shock prevents agents from inferring the true realization of the fundamental, making information incomplete. Agents then have to infer the shocks and others' actions. Thus, sentiment shocks act as an *endogenous* source of noise which can generate additional inertia. This source of noise is absent by definition in the fundamental equilibrium.

Learnability The static examples in Section 3 showed that sentiment equilibria are stable under least-squares learning, while the fundamental equilibrium is not. Are the sentiment equilibria also stable under least-squares learning in the dynamic case? Appendix C.2 shows that this is in fact the case when we extend our analysis in Section 3.4 to our dynamic examples in Sections 4.2.1 and 4.2.2. For the example in Section 4.2.1, since the actual sentiment equilibrium features $a_t = \frac{(1-\lambda^2)\phi}{1-\lambda L}$, we assume that agents perceive that the aggregate action is described by the AR(1) in equilibrium but do not know the parameters: the perceived law of motion is $a_t = \frac{\omega_1}{1+\omega_1\omega_2^{-1}L}\epsilon_t$ where the agents are learning about ω_1 and ω_2. Appendix C.2.1 shows that in this context, the sentiment equilibrium again satisfies the property of E-stability. Similarly, for the example in Section 4.2.2, following the structure of the actual equilibrium in (29)-(30), we assume that $\widehat{\psi}(L) = \frac{\delta}{1-\gamma}g(L) + \frac{a}{1-bL}$ and $\widehat{\phi}(L) = \frac{c}{1-dL}$ where the agents are learning about a, b, c and d. Again, Appendix C.2.2 shows that the sentiment equilibrium is E-stable.

5 Conclusion

Within the class of the commonly used beauty contest games, we provide a characterization of necessary conditions for the existence of sentiment equilibria and when sentiments can drive persistent aggregate fluctuations. Through some illustrative examples, we show that sentiment equilibria are stable under least-squares learning, while the fundamental equilibrium is not. We also show that sentiments can also induce additional inertia in the response of aggregate variables to fundamental shocks (hump-shaped response). This characterization serves as a guide for a growing literature in the field of macroeconomics that is trying to theoretically and quantitatively evaluate the importance of sentiments or correlated equilibria in understanding aggregate fluctuations.

While we focus on static beauty contest games in this paper,[29] the same notion of sentiment equilibrium can be extended to environments in which agents' decisions are dynamic (they depend on the expectations of fundamentals and aggregate outcomes in the future), as in Allen et al. (2006), Nimark (2017) and Angeletos and Huo (2021). Another interesting direction for future research is to explore the properties of the sentiment equilibrium when agents acquire information endogenously (Benhabib et al., 2016). We believe these are potentially fruitful paths to explore going forward.

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[29]By static, we mean that the agents' best response function at any date t only depends on fundamentals and aggregate actions in that period. All the dynamics arise due to the information set.

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Appendix

A Proofs

A.1 Restricting attention to equilibria with invertible $\phi(L)$

Theorem 3. *If $a_t = \boldsymbol{\psi}(L)\nu_t + \phi(L)\epsilon_t$ is a sentiment equilibrium, then $a_t = \boldsymbol{\psi}(L)\nu_t + \widetilde{\phi}(L)\epsilon_t$ is also a sentiment equilibrium if*

$$\phi(L)\phi(L^{-1}) = \widetilde{\phi}(L)\widetilde{\phi}(L^{-1}).$$

Furthermore, there exists a sentiment equilibrium $\widetilde{\phi}(L)$ which is invertible.

Proof. Assume that there exists a sentiment equilibrium where a_t is given by

$$a_t = \boldsymbol{\psi}(L)\boldsymbol{\nu}_t + \phi(L)\epsilon_t.$$

The signal process can be represented by

$$\mathbf{x}_{i,t} = \mathbf{A}(L)\phi(L)\epsilon_t + (\mathbf{A}(L)\boldsymbol{\psi}(L) + \mathbf{B}(L))\boldsymbol{\nu}_t + \mathbf{C}(L)\boldsymbol{\zeta}_{i,t} \equiv \mathbf{M}(L)\begin{bmatrix}\epsilon_t \\ \boldsymbol{\nu}_t \\ \boldsymbol{\zeta}_{i,t}\end{bmatrix}.$$

Suppose $\widetilde{\phi}(L)$ satisfies

$$\widetilde{\phi}(L)\widetilde{\phi}(L^{-1}) = \phi(L)\phi(L^{-1}).$$

Define $\widetilde{\mathbf{x}}_{i,t}$ and $\widetilde{\mathbf{M}}(L)$ as

$$\widetilde{\mathbf{x}}_{i,t} = \mathbf{A}(L)\widetilde{\phi}(L)\epsilon_t + (\mathbf{A}(L)\boldsymbol{\psi}(L) + \mathbf{B}(L))\boldsymbol{\nu}_t + \mathbf{C}(L)\boldsymbol{\zeta}_{i,t} \equiv \widetilde{\mathbf{M}}(L)\begin{bmatrix}\epsilon_t \\ \boldsymbol{\nu}_t \\ \boldsymbol{\zeta}_{i,t}\end{bmatrix}.$$

It is easy to see that

$$\begin{aligned}&\mathbf{M}(L)\mathbf{M}'(L^{-1}) \\ =&\mathbf{A}(L)\phi(L)\phi(L^{-1})\mathbf{A}'(L^{-1}) + (\mathbf{A}(L)\boldsymbol{\psi}(L) + \mathbf{B}(L))(\mathbf{A}(L^{-1})\boldsymbol{\psi}(L^{-1}) + \mathbf{B}(L^{-1}))' + \mathbf{C}(L)\mathbf{C}'(L^{-1}) \\ =&\widetilde{\mathbf{M}}(L)\widetilde{\mathbf{M}}'(L^{-1}).\end{aligned}$$

This equality implies that the fundamental representation of $\mathbf{M}(L)$ and $\widetilde{\mathbf{M}}(L)$ is the same. Denoting the fundamental representation of $\mathbf{M}(L)$ as $\mathbf{B}(L)$, we have

$$\mathbf{M}(L)\mathbf{M}'(L^{-1}) = \widetilde{\mathbf{M}}(L)\widetilde{\mathbf{M}}'(L^{-1}) = \mathbf{B}(L)\mathbf{B}'(L^{-1}),$$

where $\mathbf{B}(L)$ is invertible.

Under the perceived law of motion of $a_t = \phi(L)\epsilon_t$, consider the following stochastic variable

$$f_{i,t} = \delta\theta_t + \gamma a_t + \alpha z_{i,t} = \gamma(\phi(L)\epsilon_t + \boldsymbol{\psi}(L)\boldsymbol{\nu}_t) + \delta\mathbf{h}(L)\mathbf{v}_t + \alpha\mathbf{g}(L)\mathbf{u}_{i,t} \equiv \mathbf{F}(L)\begin{bmatrix}\epsilon_t \\ \boldsymbol{\nu}_t \\ \boldsymbol{\zeta}_{i,t}\end{bmatrix},$$

and the best response requires that $a_{i,t} = \mathbb{E}[f_{i,t}|\mathbf{x}_i^t]$. By the Kolomogrov-Weiner projection formula, the forecast is given by

$$\mathbb{E}[f_{i,t}|\mathbf{x}_i^t] = \left[\mathbf{F}(L)\mathbf{M}'(L^{-1})\mathbf{B}'(L^{-1})\right]_+ \mathbf{B}(L)^{-1}\mathbf{x}_{i,t} \equiv \boldsymbol{\Pi}(L)\mathbf{x}_{i,t},$$

where $\boldsymbol{\Pi}(L)$ is the corresponding equilibrium policy rule. Since $\phi(L)$ and $\boldsymbol{\psi}(L)$ is a sentiment equilibrium by assumption, it follows that

$$\boldsymbol{\Pi}(L)\mathbf{A}(L) = 1, \quad \text{and} \quad \boldsymbol{\Pi}(L)\mathbf{B}(L) = \mathbf{0}.$$

Now we verify that $\widetilde{\phi}(L)$ and $\boldsymbol{\psi}(L)$ is also a sentiment equilibrium with policy rule $\boldsymbol{\Pi}(L)$. Under the perceived law of motion $a_t = \widetilde{\phi}(L)\epsilon_t + \boldsymbol{\psi}(L)\boldsymbol{\nu}_t$, define $\widetilde{f}_{i,t}$ as

$$\widetilde{f}_{i,t} = \delta\theta_t + \gamma a_t + \alpha z_{i,t} = \gamma(\widetilde{\phi}(L)\epsilon_t + \boldsymbol{\psi}(L)\boldsymbol{\nu}_t) + \delta\mathbf{h}(L)\mathbf{v}_t + \alpha\mathbf{g}(L)\mathbf{u}_{i,t} \equiv \widetilde{\mathbf{F}}(L)\begin{bmatrix}\epsilon_t \\ \boldsymbol{\nu}_t \\ \boldsymbol{\zeta}_{i,t}\end{bmatrix}.$$

The individual optimal response is

$$a_{i,t} = \mathbb{E}[\widetilde{f}_{i,t}|\widetilde{\mathbf{x}}_i^t] = \left[\widetilde{\mathbf{F}}(L)\widetilde{\mathbf{M}}'(L^{-1})\mathbf{B}'(L^{-1})\right]_+ \mathbf{B}(L)^{-1}\widetilde{\mathbf{x}}_{i,t}.$$

By construction, $\widetilde{\mathbf{F}}(L)\widetilde{\mathbf{M}}'(L^{-1}) = \mathbf{F}(L)\mathbf{M}'(L^{-1})$, it follows that $\mathbb{E}[\widetilde{f}_{i,t}|\widetilde{\mathbf{x}}_i^t] = \boldsymbol{\Pi}(L)\widetilde{\mathbf{x}}_{i,t}$. Therefore, $a_t = \widetilde{\phi}(L)\epsilon_t + \boldsymbol{\psi}(L)\boldsymbol{\nu}_t$ is also an equilibrium. By the Wold representation theorem, there always exists $\widetilde{\phi}(L)$ such that $\widetilde{\phi}(L)$ is invertible, which completes the proof. □

To illustrate the key idea in this proof, consider the following example. Agent i wants to forecast a_t by observing the following signal every period:

$$x_{i,t} = a_t + u_{i,t}, \quad u_{i,t} \sim N(0,1).$$

Supposing agents believe that $a_t = \epsilon_t$ with $\epsilon_t \sim N(0,1)$, then the optimal forecast of a_t is simply

$$\mathbb{E}_{i,t}[a_t] = \frac{1}{2}x_{i,t},$$

as a_t and $u_{i,t}$ have the same variance. Supposing instead that agent i believes that $a_t = \phi(L)\epsilon_t = \frac{L-\lambda}{1-\lambda L}\epsilon_t$ with $\lambda \in (-1,1)$, i.e., a_t follows a non-invertible process. Note that $\phi(L)\phi(L^{-1}) = 1$, which is the same as the autocovariance generating function of $a_t = \epsilon_t$. Even though a_t now follows a different process, the optimal forecast about a_t actually is the same as before

$$\mathbb{E}_{i,t}[a_t] = \frac{1}{2}x_{i,t}.$$

Moreover, based on the historical time series generated by $a_t = \epsilon_t$ and $a_t = \frac{L-\lambda}{1-\lambda L}\epsilon_t$, an econometrician would not be able to tell the dynamics apart in the two cases.

A.2 Proof of Lemma 1 and Theorem 1

Let $\mathbf{\Pi}(L)$ denote individual agent's policy rule. The individual action is $a_{i,t} = \mathbf{\Pi}(L)\boldsymbol{x}_{i,t}$, and it follows that the average action is

$$a_t = \int_i a_{i,t}di = \mathbf{\Pi}(L)\mathbf{A}(L)a_t + \mathbf{\Pi}(L)\mathbf{B}(L)\boldsymbol{\nu}_t,$$

since the idiosyncratic shocks average out to zero. In a sentiment equilibrium, the aggregate action is driven by both sentiment shocks and primitive shocks, $a_t = \phi(L)\epsilon_t + \boldsymbol{\psi}(L)\boldsymbol{\nu}_t$. The aggregation result above can be written as

$$\phi(L)\epsilon_t + \boldsymbol{\psi}(L)\boldsymbol{\nu}_t = \mathbf{\Pi}(L)\mathbf{A}(L)\phi(L)\epsilon_t + \mathbf{\Pi}(L)\mathbf{A}(L)\boldsymbol{\psi}(L)\boldsymbol{\nu}_t + \mathbf{\Pi}(L)\mathbf{B}(L)\boldsymbol{\nu}_t. \tag{32}$$

Condition (32) must be satisfied for any realizations of $(\epsilon_t, \boldsymbol{\nu}_t)$:

$$\phi(L)(1 - \mathbf{\Pi}(L)\mathbf{A}(L)) = 0 \quad \text{and} \quad \boldsymbol{\psi}(L) = \mathbf{\Pi}(L)\mathbf{A}(L)\boldsymbol{\psi}(L) + \mathbf{\Pi}(L)\mathbf{B}(L).$$

With $\phi(L) \neq 0$, it has to be the case that $\mathbf{\Pi}(L)\mathbf{A}(L) = 1$, and $\mathbf{\Pi}(L)\mathbf{B}(L) = 0$. Therefore, in a sentiment equilibrium, it is necessary that

$$\begin{bmatrix} \mathbf{A}'(L) \\ \mathbf{B}'(L) \end{bmatrix} \mathbf{\Pi}'(L) = \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}$$

. To allow a solution for $\mathbf{\Pi}(L)$, it is necessary that

$$\text{rank} \begin{bmatrix} \mathbf{A}'(L) \\ \mathbf{B}'(L) \end{bmatrix} = \text{rank} \begin{bmatrix} \mathbf{A}'(L) & 1 \\ \mathbf{B}'(L) & \mathbf{0} \end{bmatrix},$$

where the rank of a polynomial matrix $X(L)$ is defined as $\max_{z \in \mathbb{C}} \text{rank}\Big(\mathbf{X}(z)\Big)$. The rank condition in Theorem 1 follows by transposing the condition above.

A.3 Proof of Theorem 2

Consider an impulse response of the signals to an ϵ_t shock, where $\epsilon_0 = 1$, and $\epsilon_t = 0$ for $t \neq 0$. In $\phi(L) = \sum_{t=0}^{\infty} \phi_t L^t$, ϕ_t denotes the response of a at time t. To show that $\phi(L)$ is at most an MA(k) process, it is sufficient to show that the impulse response of a_t is zero from period k onwards.

By Theorem 3 in Appendix A.1, we only need to consider the case where $\phi(L)$ is invertible. If Assumption 1 and 2 are satisfied, $\mathbb{E}[\boldsymbol{\nu}_{t-\tau} \mid \mathbf{y}_i^t] = \boldsymbol{\nu}_{t-\tau}$ and agents also observe $a_{t-\tau} = \boldsymbol{\psi}(L)\boldsymbol{\nu}_{t-\tau} + \phi(L)\epsilon_{t-\tau}$ for $\tau \geq k$. As a result, agents observe $\phi(L)\epsilon_{t-\tau}$ perfectly for $\tau \geq k$. Because $\phi(L)$ is invertible, past sentiment shocks $\{\epsilon_{t-\tau}\}_{\tau=k}^{\infty}$ can be inferred perfectly.

Recall that the signal process is given by

$$\mathbf{x}_{i,t} \quad = \quad \mathbf{A}(L)a_t + \mathbf{B}(L)\boldsymbol{\nu}_t + \mathbf{C}(L)\boldsymbol{\zeta}_{i,t}.$$

In the impulse response experiment, only $\epsilon_0 = 1$, $\epsilon_t = 0$ at all other dates. Effectively, agents observe

$$\mathbf{x}_{i,t} \quad = \quad \mathbf{A}(L)\phi_t\epsilon_0,$$

where $\phi_t = 0$ for $t < 0$. With $t \geq k$, after subtracting the part $\mathbf{A}(L)\phi_t\epsilon_0$ which agents observe perfectly, the signals are all zero. It follows that the optimal forecasts for all other shocks have to be zero, $\mathbb{E}_{i,t}\boldsymbol{\zeta}_{i,t-\tau} = 0$, $\mathbb{E}_{i,t}\boldsymbol{\nu}_{t-\tau} = 0$, and $\mathbb{E}_{i,t}\epsilon_{t-\tau} = 0$ for $t \geq k$ and $\tau \geq 0$. Therefore, the impulse response with $t \geq k$ is given by

$$a_t = \phi_t = \int \Big\{\alpha\mathbb{E}_{i,t}[\mathbf{h}(L)\mathbf{u}_{i,t}] + \delta\mathbb{E}_{i,t}[\mathbf{g}(L)\mathbf{v}_t] + \gamma\mathbb{E}_{i,t}[\phi(L)\epsilon_t]\Big\}di = \gamma\phi_t\epsilon_0.$$

Given that $\gamma < 1$ and $\epsilon_0 = 1$, it has to be that $\phi_t = 0$ for $t \geq k$. It follows that $\phi(L) = \sum_{\tau=0}^{k-1} L^\tau\phi_\tau L^\tau$.

B Solving for equilibrium in Sections 4.2.1, 4.2.2 and 3.2 and the example in Section 4

B.1 Proof of results in Section 4.2.1

In a linear rational expectations equilibrium with only sentiment shock, the aggregate outcome can be written as $a_t = \phi(L)\epsilon_t$. Each agents' signals can be expressed as

$$\begin{bmatrix} x_{i,t}^1 \\ x_{i,t}^2 \end{bmatrix} = \begin{bmatrix} \phi(L) & \sigma_z \\ (L-\lambda)\phi(L) & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ \frac{z_{i,t}}{\sigma_z} \end{bmatrix} \qquad \Leftrightarrow \qquad \mathbf{x}_{i,t} = \mathbf{M}\left(L\right)\boldsymbol{e}_{i,t}.$$

Note that the determinant of $\mathbf{M}(L)$ is

$$\det[\mathbf{M}(L)] = \sigma_z(L-\lambda)\phi(L),$$

and there is one root inside the unit circle, 0 and λ. According to Wold's theorem, this mapping has an observationally equivalent invertible representation, which is given by

$$\mathbf{x}_{i,t} = \underbrace{\mathbf{M}\left(L\right)\mathbf{W}\mathbb{B}\left(L;\lambda\right)}_{\widetilde{\mathbf{M}}(L)}\underbrace{\mathbb{B}\left(L^{-1};\lambda\right)'\mathbf{W}'\boldsymbol{e}_{i,t}}_{\widetilde{\boldsymbol{e}}_{i,t}},$$

where $\mathbb{B}\left(L;\lambda\right)$ is a Blaschke matrix which flips the root from λ to λ^{-1} and $\mathbf{W}$ is a orthonormal matrix:

$$\mathbb{B}\left(L;\lambda\right) = \begin{bmatrix} \frac{L^{-1}-\lambda}{1-\lambda L^{-1}} & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{W} = \begin{bmatrix} \frac{\sigma_z}{\sqrt{\sigma_z^2+\phi(\lambda)^2}} & \frac{\phi(\lambda)}{\sqrt{\sigma_z^2+\phi(\lambda)^2}} \\ -\frac{\phi(\lambda)}{\sqrt{\sigma_z^2+\phi(\lambda)^2}} & \frac{\sigma_z}{\sqrt{\sigma_z^2+\phi(\lambda)^2}} \end{bmatrix}.$$

Rozanov (1967) shows how one can create the $\mathbf{W}$ matrix.[30] Briefly, the $\mathbf{W}$ matrix can be constructed by recursively using the Gram-Schmidt process. The first column of $\mathbf{W}$ is constructed by finding a vector $\{w_{11}, w_{21}\}'$ of magnitude 1 such that $w_{11}\phi(\lambda) + \sigma_z w_{21} = 0$. The second column of $\mathbf{W}$ can be constructed by finding a vector orthogonal to the first column of $\mathbf{W}$ and then normalizing it to have unit norm. Next, using the Kolmogrov-Weiner projection formulas:

$$\int \mathbb{E}_{i,t}a_t di = \phi(L) + \frac{\left(1-\lambda^2\right)\sigma_z^2\phi(\lambda)}{\left(1-\lambda L\right)\left(\sigma_z^2+\phi(\lambda)^2\right)} \qquad \text{and} \qquad \int \mathbb{E}_{i,t}z_{i,t}di \quad = \quad a_t - \int \mathbb{E}_{i,t}a_t di.$$

Equilibrium must satisfy $a_t = \alpha\int\mathbb{E}_{i,t}z_{i,t} + \gamma\int\mathbb{E}_{i,t}a_t$. As a result, it must be the case that

$$\phi(L) = \frac{\left(1-\lambda^2\right)\sigma_z^2\phi(\lambda)(\alpha-\gamma)}{(1-\gamma)(1-\lambda L)\left[\sigma_z^2+\phi(\lambda)^2\right]} \qquad \Rightarrow \qquad \phi(\lambda)\left[1 - \frac{\sigma_z^2(\alpha-\gamma)}{(1-\gamma)\left[\sigma_z^2+\phi(\lambda)^2\right]}\right] = 0. \tag{33}$$

[30] See also Hansen and Sargent (1991) and Kasa (2000) for an example of how to construct $\mathbf{W}$.

The second equation implies that either $\phi(\lambda) = 0$, in which case $\phi(L)$ also equals 0 (the fundamental equilibrium), or $\phi(\lambda) = \pm\sigma_z\sqrt{\frac{\alpha-1}{1-\gamma}}$. Plugging in the second expression for $\phi(\lambda)$ into $\phi(L)$ yields $\phi(L) = \sigma_z(1-\lambda^2)\sqrt{\frac{\alpha-1}{1-\gamma}}\frac{1}{1-\lambda L}$.

B.2 Proof of results in section 4.2.2 and 3.2

In a linear sentiment equilibrium, the aggregate outcome can be written as $a_t = \phi(L)\epsilon_t + \psi(L)v_t$. Each agent's signals can be expressed as

$$\begin{bmatrix} x_{i,t}^1 \\ x_{i,t}^2 \\ x_{i,t}^3 \end{bmatrix} = \begin{bmatrix} \sigma_\varsigma & g(L) & 0 \\ 0 & \psi(L) - g(L) & \phi(L) \\ 0 & (L-\lambda)g(L) & 0 \end{bmatrix} \begin{bmatrix} \frac{\varsigma_{i,t}}{\sigma_\varsigma} \\ v_t \\ \epsilon_t \end{bmatrix} \qquad \Leftrightarrow \qquad \mathbf{x}_{i,t} = \mathbf{M}(L)\,\boldsymbol{e}_{i,t}.$$

Note that the determinant of $\mathbf{M}(L)$ is

$$\det[\mathbf{M}(L)] = \sigma_\varsigma(L-\lambda)\phi(L)g(L),$$

and there is one root inside the unit circle, λ. According to Wold's theorem, this mapping has an observationally equivalent invertible representation, which is given by:

$$\mathbf{x}_{i,t} = \underbrace{\mathbf{M}(L)\,\mathbf{W}\mathbb{B}(L;\lambda)}_{\widetilde{\mathbf{M}}(L)}\underbrace{\mathbb{B}\left(L^{-1};\lambda\right)'\mathbf{W}'e_{i,t}}_{\widetilde{e}_{i,t}}.$$

Similar to the case in subsection 4.2.1, the $\mathbf{W}$ matrix can be constructed by recursively using the Gram-Schmidt process. The first column of $\mathbf{W}$ is constructed by finding a vector $\{w_{11}, w_{21}, w_{31}\}'$ of magnitude 1 such that $w_{21}g_\lambda + \sigma_\varsigma w_{11} = 0$ and $w_{21}\left(\psi_\lambda - g_\lambda\right) + w_{31}\phi_\lambda = 0$ where $\psi_\lambda = \psi(\lambda), g_\lambda = g(\lambda)$ and $\phi_\lambda = \phi(\lambda)$. The second column of $\mathbf{W}$ can be constructed by finding a vector orthogonal to the first column of $\mathbf{W}$ and then normalizing it to have unit norm. The third column can be constructed such that it is orthogonal to the first two columns and then normalizing it to have unit norm. Next, using the Kolomogrov-Weiner projection formulas:

$$\int_i \mathbb{E}_{i,t}\theta_t di = \left[g(L) - \frac{(1-\lambda^2)\,\sigma_\varsigma^2\phi_\lambda^2 g_\lambda}{\kappa(\lambda)}\frac{1}{1-\lambda L}\right]v_t - \frac{(1-\lambda^2)\,(g_\lambda - \psi_\lambda)\,\sigma_\varsigma^2 g_\lambda\phi_\lambda}{\kappa(\lambda)}\frac{1}{1-\lambda L}\epsilon_t,$$
$$\int_i \mathbb{E}_{i,t}a_t di = \left[\phi(L) - \frac{(1-\lambda^2)\,(g_\lambda - \psi_\lambda)\,\sigma_\varsigma^2 g_\lambda\phi_\lambda}{\kappa(\lambda)}\frac{1}{1-\lambda L}\right]\epsilon_t + \left[\psi(L) - \frac{(1-\lambda^2)\,\sigma_\varsigma^2 g_\lambda\phi_\lambda^2}{\kappa(\lambda)}\frac{1}{1-\lambda L}\right]v_t,$$

where

$$\kappa(\lambda) = -2\sigma_\varsigma^2 g_\lambda\psi_\lambda + g_\lambda^2\left(\phi_\lambda^2 + \sigma_\varsigma^2\right) + \sigma_\varsigma^2\left(\psi_\lambda^2 + \phi_\lambda^2\right).$$

Equilibrium requires that $a_t = \delta\int_i \mathbb{E}_{i,t}[\theta_t]di + \int_i \mathbb{E}_{i,t}[a_t]di$, which leads to

$$\phi(L) = \frac{(1-\lambda^2)\,\sigma_\varsigma^2(\gamma+\delta)g_\lambda\phi_\lambda(\psi_\lambda - g_\lambda)}{(1-\gamma)\kappa(\lambda)}\frac{1}{1-\lambda L},$$
$$\psi(L) = \frac{\delta}{1-\gamma}g(L) - \frac{(1-\lambda^2)\,\sigma_\varsigma^2(\gamma+\delta)g_\lambda\phi_\lambda^2}{(1-\gamma)\kappa(\lambda)}\frac{1}{1-\lambda L}.$$

Solving for ϕ_λ and ψ_λ by evaluating $\phi(L)$ and $\psi(L)$ at $L = \lambda$ yields the sentiment equilibrium:

$$\phi_\lambda = \pm\frac{\sigma_\varsigma\sqrt{(\gamma+\delta-1)g_\lambda^2 - \sigma_\varsigma^2}}{(1-\gamma)g_\lambda} \quad \text{and} \quad \psi_\lambda = \frac{\delta g_\lambda^2 - \sigma_\varsigma^2}{(1-\gamma)g_\lambda},$$

and the solution to the fundamental equilibrium

$$\phi_\lambda = 0 \quad \text{and} \quad \psi_\lambda = \frac{\delta g_\lambda}{1-\gamma}.$$

Substituting ϕ_λ and ψ_λ in to $\phi(L)$ and $\psi(L)$ yields the results in the main text. Note that for the sentiment equilibrium to exist, we need $(\gamma+\delta-1)g_\lambda^2 > \sigma_\varsigma^2$. This requires that δ be large enough.

Deriving the expressions in Section 3.2 Next, we can derive the expressions in the static example in Section 3.2. To do so, all we need to do is specialize the example above to the case in which θ_t is i.i.d., i.e., $g(L) = 1$ and $\lambda = 0$. With $\lambda = 0$, agents can observe θ_{t-1} through the signal $x^3_{i,t}$. Then, using Theorem 2, we know that the only sentiment equilibrium is static. Specializing the expressions, we have the fundamental equilibrium in which $\psi = \frac{\delta}{1-\gamma}$ and $\phi = 0$, i.e., $a^f_t = \frac{\delta}{1-\gamma}$. Similarly, the sentiment equilibrium features $\psi = \pm\frac{\sigma_\varsigma\sqrt{\gamma+\delta-(1+\sigma^2_\varsigma)}}{1-\gamma}$ and $\psi = \psi_\lambda = \frac{\delta-\sigma^2_\varsigma}{1-\gamma}$. These expressions are the same as in the main text in Section 3.2.

B.3 Deriving sentiment equilibrium when agents do not observe past aggregate actions

This section derives the equilibrium for the example in Section 4. The environment is the same as in Section 3.1, but we additionally assume that the idiosyncratic fundamental $z_{i,t}$ is given by an AR(1): $z_{i,t} = \rho z_{i,t-1} + u_{i,t}$. At date t, each agent receives a noisy private signal $x_{i,t}$, which can be written as $x_{i,t} = a_t + u_{i,t}$ and also the realization of the aggregate outcome from two periods ago, $a_{t-2} = L^2\phi(L)\epsilon_t$. An educated guess for the equilibrium dynamics of the aggregate action is $\phi(L) = \phi_0 + \phi_1 L$. Given this guess, the problem can be transformed into a static problem with the relevant information encoded in the following modified signals:

$$w^1_{i,t} = \phi_0\epsilon_{t-1} + u_{i,t-1} \qquad \text{and} \qquad w^2_{i,t} = (\phi_0\epsilon_t + \phi_1\epsilon_{t-1}) + u_{i,t}.$$

The covariance matrix of $w_{i,t} = [w^1_{i,t}, w^2_{i,t}]'$ can be written as $\Omega = \begin{bmatrix} \phi_0^2 + \sigma_u^2 & \psi_0\psi_1 \\ \psi_0\psi_1 & (\phi_0^2+\phi_1^2) + \sigma_u^2. \end{bmatrix}$. Then, using the Kalman filter, the sentiment equilibrium satisfies:

$$\phi_0 = \pm\sigma_u\sqrt{\frac{(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4\alpha^2\rho^2}}{1-\gamma}} \qquad \text{and} \qquad \phi_1 = \phi_0\frac{(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4\alpha^2\rho^2}}{2\alpha\rho}.$$

Again, as in Section 3.1, the sentiment equilibrium exists as long as α is large enough.

C Stability of sentiment equilibria under learning

C.1 Static sentiment equilibria

In this appendix, we show that the sentiment equilibria in sections 3.1 and 3.2 are stable under learning, while the corresponding fundamental equilibria are not.

C.1.1 Stability of sentiment equilibrium in Section 3.1

We start with the example in Section 3.1. Recall that the fundamental equilibrium in this case was $a = 0$, while the sentiment equilibrium was $a = \phi\epsilon$. Suppose agents do not know which equilibrium is being played and perceive that a is given by $a = \widehat{\phi}\epsilon_t$.[31] Given the perception that $a = \widehat{\phi}\epsilon$, agent i believes that the signal x_i is given by $x_i = \widehat{\phi}\epsilon + z_i$. Consequently, their expected value of z_i and a is given by $\widetilde{\mathbb{E}}_i z_i = \frac{\sigma^2_z}{\widehat{\phi}^2+\sigma^2_z}$ and $\widetilde{\mathbb{E}}_i a = \frac{\widehat{\phi}^2}{\widehat{\phi}^2+\sigma^2_z}$, where $\widetilde{\mathbb{E}}_i$ implies that the expectation is with respect to the perceived law of motion, which may be different from the actual law of motion. Then each agent's optional action can be written as

$$a_i = \frac{\gamma\widehat{\phi}^2 + \alpha\sigma^2_z}{\widehat{\phi}^2 + \sigma^2_z}x_i.$$

Averaging across all agents yields

$$a_i = \frac{\gamma\widehat{\phi}^2 + \alpha\sigma^2_z}{\widehat{\phi}^2 + \sigma^2_z}\widehat{\phi}\epsilon.$$

[31] Notice that this nests the fundamental equilibrium $\widehat{\phi} = 0$ as well as the sentiment equilibrium.

Then we can specify the T-map (Evans and Honkapohja, 2012) as $\mathcal{T}(\widehat{\phi}) = \widehat{\phi}\frac{\gamma\widehat{\phi}^2+\alpha\sigma_z^2}{\widehat{\phi}^2+\sigma_z^2}$. Then it can easily be seen that the equation

$$\widehat{\phi} = \mathcal{T}(\widehat{\phi})$$

has three solutions: $\widehat{\phi} = 0$ (the fundamental equilibrium) and $\widehat{\phi} = |\phi|$, where ϕ is defined in (21) (the sentiment equilibrium). To check for expectational stability, we start by linearizing the differential equation $\frac{d}{d\tau}\widehat{\phi} = \mathcal{T}(\widehat{\phi}) - \widehat{\phi}$ around each of the stationary points. This yields the linearized ODE:

$$\frac{d}{d\tau}\widehat{\phi} = \left(\mathcal{T}'(\widehat{\phi}) - 1\right)\widehat{\phi}.$$

Next, notice that

$$\mathcal{T}'\left(\widehat{\phi}\right) - 1 = \frac{\gamma\widehat{\phi}^2 + \alpha\sigma_z^2}{\widehat{\phi}^2 + \sigma_z^2} + 2\left(1-\alpha\right)\frac{\gamma\widehat{\phi}^2\sigma_z^2}{\left(\widehat{\phi}^2 + \sigma_z^2\right)^2} - 1.$$

Evaluating at $\widehat{\phi} = 0$ yields $\mathcal{T}'(0) - 1 = \alpha - 1 > 0$, which implies that the fundamental equilibrium is not stable under learning. In contrast, evaluating at $\phi = \sigma_z\sqrt{\frac{\alpha-1}{1-\gamma}}$ yields

$$\mathcal{T}'\left(\sigma_z\sqrt{\frac{\alpha-1}{1-\gamma}}\right) - 1 = -2\gamma\left(1-\gamma\right)\left(\frac{\alpha-1}{\alpha-\gamma}\right)^2 < 0,$$

since $\alpha > 1$ and $\gamma < 1$. Thus, the sentiment equilibrium is stable under learning.

C.1.2 Stability of sentiment equilibrium in Section 3.2

Recall that in Section 3.2, the fundamental equilibrium took the form $a = \frac{\delta}{1-\gamma}\theta$, while the sentiment equilibrium took the form $a = \frac{\delta-\sigma_\varsigma^2}{1-\gamma}\theta \pm \frac{\sigma_\varsigma\sqrt{\gamma+\delta-(1+\sigma_\varsigma^2)}}{1-\gamma}\epsilon$. Suppose again that agents do not know which equilibrium is being played and perceive that the process defining a is given by $a = \widehat{\psi}\theta + \widehat{\phi}\epsilon$. Given this perceived process defining a, the signals that each agent observes can be written as

$$x_i^1 = \theta + \varsigma_i \qquad \text{and} \qquad x_i^2 = (\widehat{\psi} - 1)\theta + \widehat{\phi}\epsilon.$$

Agents use these signals to filter the actual realization of θ and of a. Following the steps in Appendix B.2 (setting $g(L) = 1$ and $\lambda = 0$), one can derive the actual law of motion of a as

$$a = \mathcal{T}_\psi(\widehat{\psi}, \widehat{\phi})\theta + \mathcal{T}_\phi(\widehat{\psi}, \widehat{\phi})\epsilon,$$

where

$$\begin{aligned}
\mathcal{T}_\psi\left(\widehat{\psi}, \widehat{\phi}\right) &= -\frac{\sigma_\varsigma^2\left(\gamma+\delta\right)\widehat{\phi}^2}{\sigma_\varsigma^2\left(\widehat{\psi}-1\right)^2 + \widehat{\phi}^2\left(1+\sigma_\varsigma^2\right)} + \gamma\widehat{\psi} + \delta, \\
\mathcal{T}_\phi\left(\widehat{\psi}, \widehat{\phi}\right) &= \widehat{\phi}\frac{\sigma_\varsigma^2\left(\gamma\widehat{\psi}+\delta\right)\left(\widehat{\psi}-1\right) + \gamma\left(1+\sigma_\varsigma^2\right)\widehat{\phi}^2}{\sigma_\varsigma^2\left(\widehat{\psi}-1\right)^2 + \widehat{\phi}^2\left(1+\sigma_\varsigma^2\right)}.
\end{aligned}$$

Again, it is easy to see that the system of equations $\widehat{\psi} = \mathcal{T}_\psi(\widehat{\psi}, \widehat{\phi})$ and $\widehat{\phi} = \mathcal{T}_\psi(\widehat{\psi}, \widehat{\phi})$ has three roots. The first of these is at $\left(\widehat{\psi}, \widehat{\phi}\right) = \left(\frac{\delta}{1-\gamma}, 0\right)$, which is the fundamental equilibrium. The other two roots are given by $\left(\widehat{\psi}, \widehat{\phi}\right) = \left(\frac{\delta-\sigma_\varsigma^2}{1-\gamma}, \pm\frac{\sigma_\varsigma\sqrt{\gamma+\delta-(1+\sigma_\varsigma^2)}}{1-\gamma}\right)$, which are the sentiment equilibria. The last two roots exist when δ is large enough.

Again, to test the E-stability properties of the fundamental and sentiment equilibria, one must evaluate the eigenvalues of the Jacobian of

$$\begin{bmatrix} \mathcal{T}_\psi(\widehat{\psi}, \widehat{\phi}) - \widehat{\psi} \\ \mathcal{T}_\phi(\widehat{\psi}, \widehat{\phi}) - \widehat{\phi} \end{bmatrix}.$$

The Jacobian evaluated at the fundamental equilibrium is

$$J^{\text{fundamental}} = \begin{bmatrix} \gamma - 1 & 0 \\ 0 & \frac{1-\gamma}{\delta+\gamma-1} \end{bmatrix},$$

which clearly has one negative eigenvalue $\gamma - 1 < 0$ and one positive $\frac{1-\gamma}{\delta+\gamma-1} > 0$, Thus, the fundamental equilibrium is not stable under learning. Agents would not converge to the sentiment equilibrium unless they start from an initial point with $\widehat{\phi} = 0$ in the neighborhood around $\left(\frac{\delta}{1-\gamma}, 0\right)$. Next, evaluating the Jacobian at the sentiment equilibrium yields

$$J^{\text{sentiment}} = \begin{bmatrix} -2\left(\frac{1-\gamma}{\gamma+\delta}\right)\left(1+\sigma_\varsigma^2\right) & -\sigma_\varsigma\left(\frac{1-\gamma}{\gamma+\delta}\right)\frac{\gamma+\delta-2\left(1+\sigma_\varsigma^2\right)}{\sqrt{\gamma+\delta-(1+\sigma_\varsigma^2)}} \\ -2\left(\frac{1-\gamma}{\gamma+\delta}\right)\sigma_\varsigma\sqrt{\gamma+\delta-(1+\sigma_\varsigma^2)} & \gamma - 1 + 2\left(\frac{1-\gamma}{\gamma+\delta}\right)\sigma_\varsigma^2 \end{bmatrix}.$$

The eigenvalues of this matrix are $\gamma - 1$ and $-\frac{2(1-\gamma)}{\gamma+\delta}$, which are both negative as long as δ is large enough for the sentiment equilibrium to exist and $\gamma < 1$. Thus, the sentiment equilibrium is stable under learning.

C.2 Learning the equilibria in which sentiments have persistent effects

C.2.1 Stability of sentiment equilibria in Section 4.2.1

Recall that in Section 4.2.1, the fundamental equilibrium was the same as in the static case $a_t = 0$, and the sentiment equilibrium had $a_t = \lambda a_{t-1} + \phi(1-\lambda^2)\epsilon_t = \frac{\phi(1-\lambda^2)}{1-\lambda L}\epsilon_t$. We start by assuming that agents' perceived law of motion is given by

$$a_t = \widehat{\phi}(L) = \frac{\omega_1}{1+\omega_1\omega_2^{-1}L}\epsilon_t,$$

where the agents learn about ω_1 and ω_2. Here ω_1 controls the standard deviation of a_t while ω_2 affects the persistence. Given the perceived law of motion, the signals that each agent observes can be represented as

$$\begin{bmatrix} x_{i,t}^1 \\ x_{i,t}^2 \end{bmatrix} = \begin{bmatrix} \widehat{\phi}(L) & \sigma \\ (L-\lambda)\widehat{\phi}(L) & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ \frac{z_{i,t}}{\sigma} \end{bmatrix}.$$

Then, following the same steps as in Appendix B.1, we can derive the actual law of motion of a_t as:

$$a_t = \omega_1\omega_2\left\{\gamma\frac{1}{\omega_2 - \omega_1 L} - \frac{\left(\lambda^2-1\right)\sigma^2(\alpha-\gamma)\left(\omega_2-\omega_1\lambda\right)}{\omega_1^2\omega_2^2 + \sigma^2\left(\omega_2-\omega_1\lambda\right)^2}\frac{1}{1-\lambda L}\right\}\epsilon_t.$$

The T-map here can be expressed as

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} \omega_1\left(\gamma - \frac{\omega_2\left(\lambda^2-1\right)\sigma^2(\alpha-\gamma)(\omega_2-\omega_1\lambda)}{\omega_1^2 d^2+\sigma^2(\omega_2-\omega_1\lambda)^2}\right) \\ \omega_1\omega_2\left\{\gamma\frac{\omega_1}{\omega_2^2} - \frac{\left(\lambda^2-1\right)\sigma^2(\alpha-\gamma)(\omega_2-\omega_1\lambda)}{\omega_1^2\omega_2^2+\sigma^2(\omega_2-\omega_1\lambda)^2}\lambda\right\} \end{bmatrix},$$

where given (ω_1, ω_2), the first equation is a mapping into the *new* ω_1 and the second is the mapping into the *new* ω_2. The three roots of this system correspond to the fundamental and sentiment equilibria. Next, evaluating the Jacobian of $\mathcal{T}(\omega_1,\omega_2) - (\omega_1,\omega_2)$ at the sentiment equilibrium yields

$$J = \begin{bmatrix} \frac{\lambda^2[\alpha(3\gamma-2)+(\gamma-6)\gamma+4]-(2-\alpha)(1-2\gamma)-\gamma}{\left(\lambda^2-1\right)(\alpha-\gamma)} - 1 & -\frac{(\gamma-1)\lambda(\alpha+\gamma-2)}{\left(\lambda^2-1\right)(\alpha-\gamma)} \\ \frac{(\gamma-1)\lambda\left(2\lambda^2-1\right)(\alpha+\gamma-2)}{\left(\lambda^2-1\right)(\alpha-\gamma)} & \frac{\lambda^2(\alpha+\gamma(3-2\gamma)-2)+\gamma(\gamma-\alpha)}{\left(\lambda^2-1\right)(\alpha-\gamma)} - 1 \end{bmatrix},$$

which has eigenvalues $\gamma - 1, -\frac{2(\alpha-1)(1-\gamma)}{\alpha-\gamma}$, both of which are negative as long as $\gamma < 1$ and $\alpha > 1$, the second of which is needed for the sentiment equilibrium to exist.

C.2.2 Stability of sentiment equilibria in Section 4.2.2

In the context of the example in section 4.2.2, we conduct the stability analysis under the assumption that θ_t follows an AR(1), i.e., $g(L) = \frac{1}{1-\rho L}$. Recall that the sentiment equilibrium takes the form

$$\psi(L) = \frac{\delta}{1-\gamma}\frac{1}{1-\rho L} - \frac{\sigma_\varsigma^2(1-\lambda^2)(1-\rho\lambda)}{1-\gamma}\frac{1}{1-\lambda L},$$

$$\phi(L) = \pm\sigma_\varsigma \frac{\left(1-\lambda^2\right)(1-\rho\lambda)\sqrt{(\gamma+\varphi-1)(1-\rho\lambda)^{-2}-\sigma_\varsigma^2}}{1-\gamma}\frac{1}{1-\lambda L}.$$

Agents now do not know which equilibrium is being played and entertain the perceived laws of motions,

$$\widehat{\phi}(L) = \frac{a}{1-\frac{b}{a}L} \qquad \text{and} \qquad \widehat{\psi}(L) = \frac{\delta}{1-\gamma}\frac{1}{1-\rho L} + \frac{c}{1-\frac{d}{c}L},$$

where the agents must learn about a, b, c, d. Then, following the steps in Appendix B.2, we can write the T-map as

$$\mathcal{T}_\phi\left(\widehat{\phi}(L)\right) = \gamma\frac{a^2}{a-bL} + \Omega(\lambda)^{-1} a\left(\lambda^2-1\right)\sigma_\varsigma^2(\gamma+\delta)\left[\frac{1-\gamma-\delta}{(1-\lambda\rho)(1-\gamma)} - \frac{c^2}{c-d\lambda}\right]\frac{1}{1-\lambda L},$$

and

$$\mathcal{T}_\psi\left(\widehat{\psi}(L)\right) = \frac{\delta}{1-\gamma}\frac{1}{1-\rho L} + \frac{\left(\lambda^2-1\right)(\delta+\gamma)\sigma_\varsigma^2}{(a-b\lambda)\Omega(\lambda)}a^3\frac{1}{1-\lambda L} + \gamma\frac{c^2}{c-dL},$$

where

$$\begin{aligned}\Omega(\lambda) &= (1-\lambda\rho)\left(\frac{a-b\lambda}{a}\right)\left\{\sigma_\varsigma^2\frac{a^4}{(a-b\lambda)^2} + \sigma_\varsigma^2\left[\frac{\delta}{(1-\gamma)(1-\lambda\rho)} + \frac{c^2}{c-d\lambda}\right]^2\right\} \\ &\quad + (1-\lambda\rho)\left(\frac{a-b\lambda}{a}\right)\left\{\frac{1}{(1-\lambda\rho)^2}\left[\frac{a^4}{(a-b\lambda)^2} + \sigma_\varsigma^2\right] - \frac{2\sigma_\varsigma^2}{1-\lambda\rho}\left[\frac{\delta}{(1-\gamma)(1-\lambda\rho)} + \frac{c^2}{c-d\lambda}\right]\right\}.\end{aligned}$$

Given the forms, $\widehat{\phi}(L) = \frac{a}{1-\frac{b}{a}L}$ and $\widehat{\psi}(L) = \frac{\delta}{1-\gamma}\frac{1}{1-\rho L} + \frac{c}{1-\frac{d}{c}L}$, we can simplify the T-maps as follows:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} \gamma a + \Omega(\lambda)^{-1} a\left(\lambda^2-1\right)\sigma_\varsigma^2(\gamma+\delta)\left[\frac{1-\gamma-\delta}{(1-\lambda\rho)(1-\gamma)} - \frac{c^2}{c-d\lambda}\right] \\ \gamma b + \Omega(\lambda)^{-1} a\left(\lambda^2-1\right)\sigma_\varsigma^2(\gamma+\delta)\left[\frac{1-\gamma-\delta}{(1-\lambda\rho)(1-\gamma)} - \frac{c^2}{c-d\lambda}\right]\lambda \\ \gamma c + \frac{\delta}{1-\gamma} + \frac{\left(\lambda^2-1\right)(\delta+\gamma)\sigma_\varsigma^2}{(a-b\lambda)\Omega(\lambda)}a^3 \\ \gamma d + \frac{\delta}{1-\gamma}\rho + \frac{\left(\lambda^2-1\right)(\delta+\gamma)\sigma_\varsigma^2}{(a-b\lambda)\Omega(\lambda)}a^3\lambda \end{bmatrix}.$$

Finally, the eigenvalues of the Jacobian of $\mathcal{T}(a,b,c,d)-(a,b,c,d)$ evaluated at the sentiment equilibrium are $\gamma-1, \gamma-1, \gamma-1$ and $-\frac{2(1-\gamma)}{\gamma+\delta}$. All are negative as long as the sentiment equilibria exist and $\gamma < 1$. Thus, even in the case where sentiments can drive persistent fluctuations, the sentiment equilibria are stable under learning.

D Policy can be used to prevent sentiment equilibria

Consider a standard monetary economy in which the representative household's utility function can be written as $\ln C_t - N_t$. The date t budget constraint of the household can be written as

$$P_t C_t = P_t w_t N_t,$$

where P_t denotes the aggregate price, w_t denotes the real wage, C_t denotes consumption of the final good and N_t denotes hours worked. The optimal choice of hours can be written as

$$w_t = C_t = Y_t,$$

where we have used the fact that goods market clearing implies $C_t = Y_t$, where Y_t denotes aggregate output. The monetary authority controls nominal expenditures $Q_t = P_t Y_t$, which we describe by some exogenous stationary process $Q_t = g(L)v_t$. This is the equivalent of the aggregate fundamental in terms of the nomenclature of the paper.

There is a unit mass of monopolistically competitive firms, each of whom produces a single variety $i \in [0,1]$. The final good is simply a CES aggregate of all varieties: $Y_t = \left[\int_0^1 Y_{i,t}^{\frac{\upsilon-1}{\upsilon}} di\right]^{\frac{\upsilon}{\upsilon-1}}$, where $\upsilon > 1$ denotes the elasticity of substitution between varieties. Consequently, the demand curve facing firm i is simply

$$Y_{i,t} = \left(\frac{P_{i,t}}{P_t}\right)^{-\upsilon} Y_t,$$

where $Y_{i,t}$ denotes demand for variety i. Each firm also uses a decreasing returns-to-scale production function which utilizes labor to produce output: $Y_{i,t} = Z_{i,t} L_{i,t}^{\mu}$, where $\mu < 1$ controls the curvature of the production function. The random variable $Z_{i,t}$ denotes idiosyncratic productivity (idiosyncratic fundamental in the nomenclature of the paper) and $N_{i,t}$ denotes labor demand by firm i. Each firm sets price at any date t by solving the following profit maximization problem:

$$\max_{P_{i,t}} \mathbb{E}_{i,t}\Big\{ \left(\frac{P_{i,t}}{P_t}\right) Y_{i,t} - (1-\tau_{i,t}) w_t N_{i,t} \Big\},$$

subject to

$$Y_{i,t} = Z_{i,t} L_{i,t}^{\mu} \qquad \text{and} \qquad Y_{i,t} = \left(\frac{P_{i,t}}{P_t}\right)^{-\upsilon} Y_t,$$

where $(1-\tau_{i,t})$ is a firm specific payroll subsidy. Also, notice that we have not taken a stand on what the firm's information set is. The optimal choice of $P_{i,t}$ must satisfy

$$\mathbb{E}_{i,t}\left\{ \left(\frac{P_{i,t}}{P_t}\right)^{1-\upsilon} + \frac{\upsilon}{\mu(1-\upsilon)}(1-\tau_{i,t})\left(\frac{Q_t}{P_t Z_{i,t}}\right)^{\frac{1}{\mu}} \left(\frac{P_{i,t}}{P_t}\right)^{-\frac{\upsilon}{\mu}} \right\} Y_t = 0. \tag{34}$$

Next, assume that the payroll subsidy takes the form

$$1 - \tau_{i,t} = (1-\tau)\, e^{\tau_q q_t - \tau_z z_{i,t}},$$

where τ denotes the level of the subsidy in steady state. τ_q denotes how responsive the subsidy is to changes in nominal expenditure relative to steady state, $q_t = \ln Q_t - \ln Q$, where Q denotes steady state nominal expenditure. $\tau_q > 0 (< 0)$ implies that all firms get a higher (lower) subsidy when nominal expenditures are high. Similarly, τ_z controls how responsive the subsidy is to realizations of $Z_{i,t}$ relative to the average $Z = 1$. In particular, $\tau_z > 0$ implies that firms that draw a low productivity $Z_{i,t} < 1$ get a larger subsidy, while a policy with $\tau_z < 0$ subsidizes firms with high productivity realizations.

Log-linearizing (34) around the deterministic steady state yields

$$p_{i,t} = \underbrace{\frac{(1-\mu)(1+\upsilon)}{(1-\upsilon)\mu+\upsilon}}_{\gamma} \mathbb{E}_{i,t} p_t + \underbrace{\left(\frac{\tau_q \mu + 1}{(1-\upsilon)\mu+\upsilon}\right)}_{\delta} \mathbb{E}_{i,t} q_t + \underbrace{\left(-\frac{\tau_z \mu + 1}{(1-\upsilon)\mu+\upsilon}\right)}_{\alpha} \mathbb{E}_{i,t} z_{i,t},$$

where $p_{i,t} = \ln P_{i,t} - \ln P$ and $p_t = \ln P_t - \ln P$. Notice that this looks identical to the best response function considered in the paper with $\gamma = \frac{(1-\mu)(1+\upsilon)}{(1-\upsilon)\mu+\upsilon}$, $\delta = \frac{\tau_q\mu+1}{(1-\upsilon)\mu+\upsilon}$ and $\alpha = -\frac{\tau_z\mu+1}{(1-\upsilon)\mu+\upsilon}$. Further notice that by choosing the policy parameters τ_q and τ_z, the policymaker can affect the values of δ and α respectively. In particular,

$$\frac{d\alpha}{d\tau_z} = -\frac{\mu}{\mu + \upsilon(1-\mu)} < 0.$$

Thus, a higher τ_z makes the effective response to changes in idiosyncratic productivity smaller. As discussed in the paper, α needs to be large in order for a sentiment equilibrium to exist. Thus, by appropriately adjusting τ_z, the planner can choose to eliminate the sentiment equilibrium. Similarly, by affecting τ_q, the planner can make the effective δ small enough so that the sentiment equilibrium does not exist.

E Smoothing out forecast errors

The signal is $x_t = (L-\lambda)\phi(L)e_t$. The fundamental representation of this stochastic process is

$$x_t = \phi(L)(1-\lambda L)\frac{L-\lambda}{1-\lambda L}e_t \equiv \phi(L)(1-\lambda L)w_t.$$

The forecast about $L^k\phi(L)e_t$ is

$$\begin{aligned}
\mathbb{E}_t[L^k\phi(L)e_t] &= \mathbb{E}_t\left[L^k\phi(L)\phi(L^{-1})(L^{-1}-\lambda)\frac{1}{\phi(L^{-1})(1-\lambda L^{-1})}\right]_+\frac{1}{\phi(L)(1-\lambda L)}(L-\lambda)\phi(L)e_t \\
&= \mathbb{E}_t\left[L^k\phi(L)\frac{1-\lambda L}{L-\lambda}\right]_+\frac{L-\lambda}{(1-\lambda L)}e_t \\
&= \mathbb{E}_t\left[L^k\phi(L)\frac{1-\lambda L}{L-\lambda}-\lambda^k\phi(\lambda)\frac{1-\lambda^2}{L-\lambda}\right]\frac{L-\lambda}{(1-\lambda L)}e_t \\
&= L^k\phi(L)e_t-\lambda^k\phi(\lambda)\frac{1-\lambda^2}{1-\lambda L}e_t.
\end{aligned}$$

It follows that the forecast error is then given by $\lambda^k\phi(\lambda)\frac{1-\lambda^2}{1-\lambda L}e_t$.