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ToTEM III: The Bank of Canada's Main DSGE Model for Projection and Policy Analysis

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Contents

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Abstract

We present a technical description of the second large-scale update to the Terms-of-Trade Economic Model: ToTEM III. This updated version of the model replaced ToTEM II in 2017. ToTEM III's structure includes key aspects of household indebtedness and improved modelling of the housing market. These new features allow Bank staff to address a broader range of economic issues. Moreover, the model is estimated using a Bayesian methodology with informative priors and a larger set of observable variables, including improved measures of the factors explaining non-commodity exports. These enhancements in the model structure and estimation have contributed to significant improvements in the empirical properties of the model. We also compare the new model's responses to key macroeconomic shocks with those of ToTEM II and explore two important policy applications.

Topics: Business fluctuations and cycles, Economic models, Housing, Interest rates, Monetary policy

JEL codes: E, E1, E2, E17, E20, E30, E40, E50, E62, E65, F40, F41, G51

Introduction

The Terms-of-Trade Economic Model (ToTEM) is the Bank of Canada's main dynamic stochastic general equilibrium (DSGE) model. This model has played an important role in analyzing the Canadian economy for about 15 years. More specifically, ToTEM has been used to:

- conduct economic projections for Canada
- quantify the impact of risks to the domestic outlook
- assess the effects of economic developments on the Canadian economy
- evaluate economic policies

To expand the range of economic issues that can be analyzed with ToTEM and improve the model's ability to explain Canadian macroeconomic data, the model has undergone two large-scale updates aimed at incorporating advances in economic modelling. The first update (ToTEM II) was launched in 2011 and introduced multiple interest rates and a more general structure for price- and wage-setting behaviour.[1] The second update (ToTEM III) took place in 2017. Its key features include:

- the introduction of household debt
- a more detailed modelling of the housing market
- an enhanced estimation method
- improved measures of the determinants of non-commodity exports

The remainder of this introduction elaborates on each of these four features before laying out a road map for the rest of the technical report.

While household debt has already been part of two of the Bank's models that focus on financial stability issues (the Macroprudential and Monetary Policy Model [MP2] and the Household Risk Assessment Model [HRAM]), ToTEM III constitutes the first effort to add household debt into the

[1]See J. Dorich et al. (2013).

Bank's main DSGE model.[2] In particular, by following Alpanda and Zubairy (2017), ToTEM III integrates four notable elements associated with household indebtedness. First, the model includes borrower households who differ from saver households in ToTEM II in their degree of impatience: borrower households have a relatively stronger preference for present consumption over future consumption. In equilibrium, this leads to a situation in which borrowers finance some of their spending using loans supplied by savers. Second, only a fraction of a given household's debt is repaid in a given period. This allows us to differentiate between the stock and flow of household debt. Third, the model incorporates two forms of collateralized household debt observed in the data, which together account for more than 80 percent of total household debt in Canada. In particular, a collateral constraint in ToTEM III links new loans to borrowers with two distinct components:

- the value of their new housing investments multiplied by the current loan-to-value ratio, which is meant to capture residential mortgages
- a given share of their current home equity, which aims to capture home equity lines of credit (HELOCs)

Fourth, we distinguish between the interest rate paid on new loans and the effective interest rate paid on total household debt. This distinction plays a critical role in enabling the model to capture the mix of fixed- and variable-rate collateralized household debt observed in the data.

ToTEM III also includes a more elaborate structure of the housing market. On the demand side of this market, the main innovation is that borrowers now contribute to overall housing demand, allowing mortgage debt and HELOCs to influence house prices. Moreover, the model captures impact from house prices on household debt to the extent that house prices affect the borrowers' collateral constraint through their effect on housing investment and home equity. On the supply side of the housing market, the model features a new sector specifically oriented toward producing residential investment goods. This addition allows ToTEM III to track the residential investment price deflator, which was not the case in ToTEM II.[3]

Another area where ToTEM III improves significantly on earlier versions of the model relates to how the model has been estimated. While most parameters in ToTEM II were estimated using classical maximum likelihood methods, the majority of parameters in ToTEM III have been estimated using a Bayesian methodology with informative priors. This new method helps to increase the speed of the estimation process and to incorporate information from microdata and other sources. Alongside the model changes described above, this new estimation approach has contributed to significant improvements in the model's in-sample goodness-of-fit.

[2]Information about MP2 and HRAM can be found in Alpanda, Cateau and Meh (2014) and Peterson and Roberts (2016).

[3]ToTEM II assumes that residential investment goods are produced and supplied by the same sector that produces consumption goods, implying no difference between the in-model price deflators for consumption and residential investment.

Finally, we note that ToTEM III features significant improvements in the measurement of the determinants of Canadian non-commodity exports. More specifically, ToTEM III includes Bank staff's developed Canadian Effective Exchange Rate (CEER) and Global Real Economic Activity for Canadian Exports (GRACE) series, which respectively provide better measures of the real exchange rate and level of foreign activity.[4] With these new measures, ToTEM III better explains the recent evolution of non-commodity exports.

Our discussion of ToTEM III proceeds as follows. Chapter 1 elaborates on the additions and modifications to the model structure relative to ToTEM II. Chapter 2 describes key elements of the model's estimation. Chapter 3 presents a comparison of ToTEM III's responses to key macroeconomic shocks against those of ToTEM II. Chapter 4 provides two applications in which we use ToTEM III to analyze relevant issues concerning the Canadian economy. Chapter 5 concludes.

[4]See Barnett, Charbonneau and Poulin-Bellisle (2016) and Binette, Chernis and De Munnik (2017).

Chapter 1

Model description

In this section, we present the model's most important new elements: collateralized household debt and a specific sector for producing residential investment goods. In particular, we explain how the household block and the production of finished goods have been modified to account for these new elements. Moreover, we describe updates to the:

- approach used to achieve stationarity of the net foreign assets (NFA) position
- specification of the monetary policy rule and the fiscal policy rules on government spending and transfers

Relative to ToTEM II, the structure of the model's import and commodity sectors is unchanged, as are the rest-of-world block and the specification of foreigners' demand for Canadian non-commodity exports. The quantitative implications of model changes are discussed in Chapter 3.

1.1 Households

ToTEM III includes four types of consumers. As in ToTEM II, there are two types of lifetime-income consumers, unrestricted and restricted, as well as current-income consumers (i.e., hand-to-mouth households). In addition, ToTEM III introduces borrowers. Borrowers differ from lifetime-income households in their degree of impatience, with the former placing a relatively greater weight on present consumption than on future consumption. In equilibrium, this leads to a situation where borrowers finance some of their spending by obtaining loans from lifetime-income households. Therefore, lifetime-income households in ToTEM III have one more asset than in ToTEM II through which they can save. In terms of their preferences, they remain modelled as in ToTEM II, except that their housing is composed of residential structures and land. Current-income consumers remain modelled as in ToTEM II.

We assume that lifetime-income consumers and borrowers each provide a different type of labour. More specifically, both groups supply differentiated labour across a continuum of types. The economy has two types of unions: one formed by lifetime-income consumers and one formed by borrowers. Within each type, there is a continuum of unions, each representing a type of differentiated labour. Each union sets its wage rate for its members and faces nominal wage rigidities. The model assumes that the current-income consumers receive the wage rate received by the lifetime-income consumers.

We begin by presenting the optimization problem of the borrowers. Then we explain how the optimization problems of both types of lifetime-income consumers differ from the ones in ToTEM II.

1.1.1 Borrowers

The period t utility function for this type of consumer is:

$$\begin{aligned} U_t^{borr} &= \frac{\mu}{\mu-1}\left(C_t^{borr}-\xi C_{t-1}^{borr}\right)^{\frac{\mu-1}{\mu}} \exp\left(\frac{\eta(1-\mu_L)}{\mu_L(1+\eta)}\eta_L \int_0^1 \left(E_t N_t^{borr}(h)\right)^{\frac{1+\eta}{\eta}} dh\right) \\ &+\zeta_t^{hl,borr}\frac{\mu_{HL}}{\mu_{HL}-1}\left(HL_t^{borr}-\xi_{HL}HL_{t-1}^{borr}\right)^{\frac{\mu_{HL}-1}{\mu_{HL}}}, \end{aligned} \tag{1.1}$$

where C_t^{borr} is total consumption expenditure of borrowers, E_t is their work effort, $N_t^{borr}(h)$ is the hours of type-h work, HL_t^{borr} is the stock of housing of borrowers and $\zeta_t^{hl,borr}$ is a housing preference shock. The parameters that appear in the utility function are the following: μ and μ_{HL} are intertemporal elasticities of substitution for consumption and housing, respectively; ξ and ξ_{HL} are habit-persistence parameters; and η is the wage elasticity of labour supply. Along the lines of Basant Roi and Winlaw (2007), we assume that the stock of housing is a composite of residential structures and land. We capture this assumption using the following relationship:

$$HL_t^{borr} = \zeta_t^{HL\frac{\sigma_{HL}}{\sigma_{HL}-1}}\left[s_{RS}^{\frac{1}{\sigma_{HL}}}RS_t^{borr\frac{\sigma_{HL}-1}{\sigma_{HL}}}+(1-s_{RS})^{\frac{1}{\sigma_{HL}}}LAND_t^{borr\frac{\sigma_{HL}-1}{\sigma_{HL}}}\right]^{\frac{\sigma_{HL}}{\sigma_{HL}-1}}, \tag{1.2}$$

where RS_t^{borr} is the residential structure of borrowers, $LAND_t^{borr}$ is land of borrowers and ζ_t^{HL} is a shock to the housing technology. The parameter σ_{HL} denotes the elasticity of substitution between land and residential structures. When $\sigma_{HL}=1$, equation (1.2) becomes a Cobb-Douglas function. The parameter s_{RS} governs the relative importance of residential structures in housing. ToTEM II assumed that housing was composed only of residential structures ($s_{RS}=1$) and that there were no shocks to the housing technology. The two-factor housing technology introduced in ToTEM III allows for a distinction between the price of new residential structures (P_t^{RS}) and the price of housing (P_t^H), implying that the price of housing is affected by the prices of residential structures and land (P_t^{LAND}) according to:

$$P_t^H = \left[s_{RS}(P_t^{RS})^{1-\sigma_{HL}} + (1-s_{RS})(P_t^{LAND})^{1-\sigma_{HL}}\right]^{\frac{1}{1-\sigma_{HL}}} . \tag{1.3}$$

We assume that residential structures depreciate at a rate δ_{RS} and are costly to install. In particular, the law of motion for residential structures is:

$$RS_{t+1}^{borr} = (1-\delta_{RS})RS_t^{borr} + I_t^{RS,borr} - \frac{\chi_{RS}}{2} I_t^{RS,borr} \left(\frac{I_t^{RS,borr}}{I_{t-1}^{RS,borr}} - 1 \right)^2 , \tag{1.4}$$

where $I_t^{RS,borr}$ is residential investment of borrowers and χ_{RS} regulates the size of the residential investment adjustment costs.

The aggregate supply of residential land is fixed. Households can trade land but face a cost when adjusting their stock. The law of motion for land is:

$$LAND_{t+1}^{borr} = LAND_t^{borr} + I_t^{LAND,borr} - \frac{\chi_{LAND}}{2} \left(I_t^{LAND,borr} - I_{t-1}^{LAND,borr} \right)^2 , \tag{1.5}$$

where $I_t^{LAND,borr}$ is the household's investment in land and χ_{LAND} measures the size of the land investment adjustment costs.

We follow Alpanda and Zubairy (2017) in modelling the key elements associated with household debt arising from borrowing. The stock of household debt evolves according to:

$$D_t^{borr} = (1-\kappa) D_{t-1}^{borr} + L_t^{borr}, \tag{1.6}$$

where D_t^{borr} is the stock of household debt and L_t^{borr} is the amount of new loans. The parameter κ is the amortization rate that determines the amount of principal payments out of the stock of debt. The sum of these payments and interest payments determine the debt service M_t^{borr}, which is given by:

$$M_t^{borr} = \left(R_{t-1}^{M,borr} + \kappa \right) D_{t-1}^{borr}, \tag{1.7}$$

where $R_t^{M,borr}$ represents the *effective* interest rate faced by the borrowers.

New loans carry an interest rate R_t^L, which corresponds to the current long-term household rate. Moreover, a fraction Φ of existing loans are renegotiated at this rate in each period. Then, the effective interest rate borrowers face is:

$$R_t^{M,borr} = (1-\Phi)\left(1-\frac{L_t^{borr}}{D_t^{borr}}\right)R_{t-1}^{M,borr} + \left[\frac{L_t^{borr}}{D_t^{borr}} + \Phi\left(1-\frac{L_t^{borr}}{D_t^{borr}}\right)\right]R_t^L. \tag{1.8}$$

For a given amortization rate, Φ governs the relative importance of R_t^L in determining the effective interest rate. In the extreme case where $\Phi = 1$, all debt is effectively one-period debt and $R_t^{M,borr} = R_t^L$. When the amortization rate is equal to 1, all debt is also one-period debt. In both of these cases, the interest rate on household debt changes every quarter, implying that all household debt is at a variable rate. This pattern of the interest rate on household debt is the one followed in standard models with borrowers, such as in Iacoviello (2005). By allowing both the amortization and renegotiation rates to be in line with the data (both below 1), we introduce long-term household debt, which is not a common feature in most models with household indebtedness.

New loans are subject to a collateral constraint specified as the sum of two components:

$$L_t^{borr} = \alpha_{ltv}\left(P_t^{RS} I_t^{RS,borr} + P_t^{LAND} I_t^{LAND,borr}\right) + \alpha_{heloc}\left(P_t^H HL_t^{borr} - (1-\kappa)D_{t-1}^{borr}\right), \tag{1.9}$$

where α_{ltv} is the loan-to-value (LTV) ratio on new mortgages and α_{heloc} is the fraction of home equity that is withdrawn. The first component captures the flow of residential mortgages, which are defined as the value of the borrowers' new housing investments multiplied by the current LTV ratio. The second component aims to capture new loans from HELOCs, which borrowers in the model are assumed to collateralize against a constant share of their home equity. Therefore, ToTEM III accounts for two forms of collateralized household debt, which together represent more than 80 percent of total household debt in Canada.

The specification of the collateral constraint differs from most considered in the literature because the LTV ratio is applied on new mortgage loans instead of on all the debt. This feature allows the model to be more in line with the data, which we view as valuable enhancement of the model.

The budget constraint that a representative borrower faces is:

$$\begin{aligned} P_{c,t}^{tot} C_t^{borr} + P_t^{RS} I_t^{RS,borr} + P_t^{LAND} I_t^{LAND,borr} + M_t^{borr} &= (1-\tau_{w,t})\int_0^1 W_t^{borr}(h) N_t^{borr}(h)dh \\ &\quad + L_t^{borr} + TF_t^{borr} + Div_t^{borr}, \end{aligned} \tag{1.10}$$

where $P_{c,t}^{tot}$ is the aggregate consumption price level, $\tau_{w,t}$ is the labour income tax rate, $W_t^{borr}(h)$ is the nominal wage rate received by the borrower's labour type h, TF_t^{borr} is the level of nominal transfers received from the government by the borrowers and Div_t^{borr} denotes the dividends received from firms.

The borrower household seeks to maximize the objective function:

$$\mathbf{E}_t \sum_{s=t}^{\infty} \beta_{t,s}^{borr} \mathbb{U}_s^{borr}, \tag{1.11}$$

with $\beta_{t,s}^{borr} \equiv \prod_{v=t}^{s-1} \beta_v^{borr}$ and $\beta_{t,t}^{borr} \equiv 1$, subject to (1.2) and (1.4) to (1.10). As in Iacoviello (2005), borrowers are relatively more impatient than savers, with their discount factor (β_v^{borr}) being lower than that of both types of lifetime-income consumers (β_v^l). This assumption guarantees a positive flow of funds between lifetime-income consumers and borrowers. The borrowers' optimization problem yields the first-order conditions that characterize the borrowers' economic decisions, which can be found in Appendix A. Below we provide linearized equations summarizing the borrowers' consumption and wage setting decisions.

Consumption of borrowers. By combining the borrowers' linearized optimality conditions for the choices of consumption, debt and loans, we obtain the following expression for borrowers' consumption:

$$\begin{aligned}\widehat{C}_t^{borr} &= \frac{1}{1+\xi} E_t \widehat{C}_{t+1}^{borr} + \frac{\xi}{1+\xi} \widehat{C}_{t-1}^{borr} - \Gamma \Delta \widehat{N}_{t+1}^{borr} + \mu \Lambda E_t \Delta \widehat{p}_{c,t+1}^{net} - \frac{\mu \Lambda \overline{\tau_c}}{(1-\overline{\tau_c})} \Delta \widehat{\tau}_{c,t+1} \\ &+ \frac{\mu \Lambda \overline{\pi^c}}{(1+\overline{\pi^c})} E_t \widehat{\pi}_{t+1}^c + \frac{\beta_{borr}}{1+\overline{\pi^c}} (1-\kappa)(1-\Phi) \Omega^{R,borr} \overline{R} \Xi E_t \widehat{R}_{t+1}^L \\ &+ \frac{\beta_{borr}}{1+\pi^c} (1-\kappa) \overline{\mu^B} (1-\alpha_{heloc}) \Xi E_t \widehat{\mu}_{t+1}^B - \overline{\Omega^{R,borr}} \overline{R} \Xi \widehat{R}_t^L - \overline{\mu}^B \Xi \widehat{\mu}_t^B, \end{aligned} \tag{1.12}$$

where hats denote log-deviations from steady state, except for the cases of $\widehat{R}_t^L$, $\widehat{\tau}_{c,t+1}$ and $\widehat{\pi}_{t+1}^c$, where a hat denotes absolute deviations and bars denote steady state values. N_t^{borr} is the aggregate labour of borrowers. The variable $p_{c,t+1}^{net}$ is the net consumption price level such that $p_{c,t+1}^{net} = \frac{p_{c,t+1}^{tot}}{(1+\tau_{c,t})}$, where $\tau_{c,t}$ is the consumption tax rate. The variable μ_t^B is the Lagrange multiplier associated with the collateral constraint and measures the degree of tightness of this constraint. $\Gamma, \Lambda, \Omega^{R,borr}$ and Ξ are strictly positive and defined as follows:

$$\begin{aligned}\Gamma &= \mu \Lambda (1-\mu_L) \eta_L (\overline{N^{borr}})^{\frac{1+\eta}{\eta}} \\ \Lambda &= (1-\xi)/(1+\xi) \\ \Omega^{R,borr} &= \beta_{borr}/(1+\pi^c - \beta_{borr}(1-\Phi)(1-\kappa)) \\ \Xi &= \mu \Lambda / (1-\overline{\mu^B}).\end{aligned}$$

Equation (1.12) shows that, everything else equal, borrowers' consumption falls in response to an increase in the lending rate. A tightening of the borrowing constraint (due, for instance, to a decline in house prices or a reduction in the maximum LTV ratio) also reduces borrowers' consumption.

Wage setting for borrowers. Borrowers have a continuum of unions, each representing one type of differentiated labour. As in ToTEM II, we assume the existence of two different types of unions: a fraction Ω of rule-of-thumb (RT) unions and a fraction $1-\Omega$ of forward-looking (FL) unions. Each type of union sets the wage rate for its members in the same way as in ToTEM II. This implies that the borrowers' average wage inflation π_t^{wborr} is given by:

$$\pi_t^{wborr} = \frac{(1-\theta_w)\gamma_w\Omega}{\phi}\pi_{t-1}^{wborr} + \frac{\beta^{borr}\theta_w}{\phi}E_t\pi_{t+1}^{wborr} - \tilde{\lambda}\left(\widehat{w}_t^{borr} - \hat{p}_{c,t}^{net} - \widehat{\tau}_{w,t} - \widehat{mrs}_t^{borr}\right) + \varepsilon_t^w \ , \quad (1.13)$$

where w_t^{borr} is the logarithm of the average wage received by borrowers, mrs_t^{borr} is the logarithm of the borrowers' marginal rate of substitution and ε_t^w is a linear combination of the inflation target and the wage markup shock. The coefficients in the equation are given by:

$$\begin{aligned}
\phi &= \theta_w + \Omega(1-\theta_w)(1+\gamma_w\beta^{borr}\theta_w) \\
\tilde{\lambda} &= \frac{(1-\Omega)(1-\beta^{borr}\theta_w)(1-\theta_w)\eta}{(\eta+\varepsilon_w)\phi}.
\end{aligned}$$

Equation (1.13) is analogous to equation (1.52) in Dorich et al. (2013).

1.1.2 Lifetime-income households

In ToTEM III, we modify ToTEM II's specification of both types of lifetime-income households in two dimensions. First, we allow both types of households to provide loans to the borrowers. Second, for both types of households, housing is composed of residential investment and land. Moreover, unlike ToTEM II, we include a country risk premium to achieve NFA stationarity. This change results in greater consistency between ToTEM and the open-economy model used to assess the Canadian neutral rate.[5]

[5] See Mendes (2014).

Unrestricted lifetime-income households

The period t utility function for unrestricted lifetime-income households is:

$$\begin{aligned} U_t^{ul} &= \frac{\mu}{\mu-1}\left(C_t^{ul}-\xi C_{t-1}^{ul}\right)^{\frac{\mu-1}{\mu}} \exp\left(\frac{\eta(1-\mu_L)}{\mu_L(1+\eta)}\eta_L \int_0^1 \left(E_t N_t^{ul}(h)\right)^{\frac{1+\eta}{\eta}} dh\right) \\ &\quad +\zeta_t^{hl,ul}\frac{\mu_{HL}}{\mu_{HL}-1}\left(HL_t^{ul}-\xi_{HL}HL_{t-1}^{ul}\right)^{\frac{\mu_{HL}-1}{\mu_{HL}}} \\ &\quad +\zeta_t^{inv,ul}\frac{\mu_{INV}}{\mu_{INV}-1}\left(INV_t^{ul}-\xi_{INV}INV_{t-1}^{ul}\right)^{\frac{\mu_{INV}-1}{\mu_{INV}}}, \end{aligned} \tag{1.14}$$

where the variables entering the function are analogous to those presented in the borrower's optimization problem, except for inventories (INV_t^{ul}). The unrestricted lifetime-income households' budget constraint is given by:

$$\begin{aligned} &(1+\tau_{c,t})P_{c,t}^{tot}C_t^{ul} + P_t^{RS}I_t^{RS,ul} + P_t^{LAND}I_t^{LAND,ul} + P_t^c I_t^{INV,ul} + \frac{B_t^{ul}}{(1+R_{RF,t})} \\ &+ \frac{e_t B_t^{*,ul}}{(1+R_t^*)(1+\vartheta_t)} + \frac{(1+\phi_t)B_t^{20,ul}}{(1+R_t^{20})^{20}} + L_t^{ul} = B_{t-1}^{ul} + e_t B_{t-1}^{*,ul} + B_{t-20}^{20,ul} + M_t^{ul} \\ &\qquad + (1-\tau_{w,t})\int_0^1 W_t^{ul}(h)N_t^{ul}(h)dh + TF_t^{ul} + Div_t^{ul}, \end{aligned} \tag{1.15}$$

where P_t^c is the price level in the core consumption sector, $I_t^{INV,ul}$ is the investment in inventories, B_t^{ul} is domestic short-term bonds, $R_{RF,t}$ is the short-term risk-free interest rate, e_t is the nominal exchange rate, $B_t^{*,ul}$ is foreign short-term bonds, R_t^* is the foreign short-term interest rate, ϑ_t is a country-specific risk premium, ϕ_t is the transaction cost of trading domestic long-term bonds, $B_t^{20,ul}$ is domestic long-term bonds, R_t^{20} is the domestic long-term interest rate, L_t^{ul} is new loans provided to borrowers by unrestricted households and M_t^{ul} is the sum of principal and interest payments received by unrestricted households for loans that they have provided. The remaining variables in the budget constraint are analogous to those entering the borrower's budget constraint.

The country-specific risk premium ϑ_t is related to Canada's NFA position. A decline in NFA (i.e., an increase in net foreign debt) will increase the country-specific risk premium and restrict domestic households' ability to borrow abroad. This is a standard technique to achieve the stationarity of NFA in open economy DSGE models.[6]

The specific functional form assumed for ϑ_t is the same as that used in ToTEM I (Murchison and Rennison 2006), relating ϑ_t to the ratio of aggregate net foreign bond holdings $e_t B_t^*$ to nominal

[6]See Schmitt-Grohé and Uribe (2003).

gross domestic product (GDP), $P_t Y_t$,

$$\vartheta_t = \varsigma \left[\exp\left(-\frac{e_t B_t^*}{P_t Y_t} \right) - 1 \right] + \varepsilon_t^{\vartheta}, \tag{1.16}$$

where $\varepsilon_t^{\vartheta}$ is an exogenous shock defined as $\varepsilon_t^{\vartheta} = \rho \varepsilon_{t-1}^{\vartheta} + \upsilon_t^{\vartheta}$.

The unrestricted lifetime-income household seeks to maximize the objective function:

$$\mathbf{E}_t \sum_{s=t}^{\infty} \beta_{t,s}^{ul} \mathbb{U}_s^{ul}, \tag{1.17}$$

with $\beta_{t,s}^{ul} \equiv \prod_{v=t}^{s-1} \beta_v^l$ and $\beta_{t,t}^{ul} \equiv 1$, subject to (1.15) and conditions analogous to (1.2) and (1.4) through (1.8), rewritten for unrestricted lifetime-income households rather than for borrowers. This optimization problem yields the first-order conditions that characterize unrestricted lifetime-income consumers' economic decisions (for a full summary see Appendix A).

Uncovered interest parity

The first-order conditions of the unrestricted lifetime-income household problem imply a standard uncovered interest parity condition such that:

$$\widehat{e}_t = E_t \widehat{e}_{t+1} + \widehat{R}_t^* - \widehat{R}_{RF,t} + \widehat{\vartheta}_t \ , \tag{1.18}$$

where $\widehat{R}_{RF,t} = \ln(1 + R_{RF,t}) - \ln(1 + \overline{R})$ and $\widehat{R}_t^* = \ln(1 + R_t^*) - \ln(1 + \overline{R^*})$. $\widehat{\vartheta}_t$ is given by the following expression:

$$\widehat{\vartheta}_t = \varsigma \left[\left(-\frac{e_t B_t^*}{P_t Y_t} \right) \right] + \widehat{\varepsilon}_t^{\vartheta}, \widehat{\varepsilon}_t^{\vartheta} = \rho \widehat{\varepsilon}_{t-1}^{\vartheta} + \widehat{\upsilon}_t^{\vartheta}. \tag{1.19}$$

The nominal exchange rate increases (indicating a depreciation) if either the country-specific risk-premium rises or the foreign bond yield R_t^* rises relative to the domestic bond yield $R_{RF,t}$. The shock to the country-specific risk premium, $\widehat{\varepsilon}_t^{\vartheta}$, can be thought of as an exogenous exchange rate shock.

Following ToTEM I and II, ToTEM III uses a modified version of the standard uncovered interest parity (UIP) condition above to better fit the exchange rate dynamics. The modified UIP condition satisfies:

$$\widehat{e}_t = \varkappa \widehat{e}_{t-1} + (1-\varkappa)\left[E_t \widehat{e}_{t+1} + \widehat{R}^*_t - \widehat{R}_{RF,t} + \widehat{\vartheta}_t\right] \ , \tag{1.20}$$

where $\varkappa$ is the intrinsic persistence of the real exchange rate.

Restricted lifetime-income households

Restricted lifetime-income households differ from the unrestricted ones in two aspects. First, restricted lifetime-income households do not trade in any foreign or domestic short-term bonds. Second, restricted lifetime-income households do not pay a transaction cost when trading domestic long-term bonds. As in ToTEM II, these differences allow movements in the domestic term premium to influence the consumption of restricted lifetime-income consumers. Moreover, these differences imply that the budget constraint faced by the restricted lifetime-income households is given by:

$$\begin{aligned}&(1+\tau_{c,t})P^{tot}_{c,t}C^{rl}_t + P^{RS}_t I^{RS,rl}_t + P^{LAND}_t I^{LAND,rl}_t + P^c_t I^{INV,rl}_t \\ &+ \frac{B^{20,rl}_t}{(1+R^{20}_t)^{20}} + L^{rl}_t = B^{rl}_{t-1} + e_t B^{*,rl}_{t-1} + B^{20,rl}_{t-20} + M^{rl}_t \\ &\qquad + (1-\tau_{w,t})\int_0^1 W^{rl}_t(h) N^{rl}_t(h) dh + TF^{rl}_t + Div^{rl}_t.\end{aligned} \tag{1.21}$$

The period t utility function for restricted lifetime-income households is:

$$\begin{aligned}U^{rl}_t \ =\ & \frac{\mu}{\mu-1}\left(C^{rl}_t - \xi C^{rl}_{t-1}\right)^{\frac{\mu-1}{\mu}} \exp\left(\frac{\eta(1-\mu_L)}{\mu_L(1+\eta)}\eta_L \int_0^1 \left(E_t N^{rl}_t(h)\right)^{\frac{1+\eta}{\eta}} dh\right) \\ &+\zeta^{hl,rl}_t \frac{\mu_{HL}}{\mu_{HL}-1}\left(HL^{rl}_t - \xi_{HL} HL^{rl}_{t-1}\right)^{\frac{\mu_{HL}-1}{\mu_{HL}}} \\ &+\zeta^{inv,rl}_t \frac{\mu_{INV}}{\mu_{INV}-1}\left(INV^{rl}_t - \xi_{INV} INV^{rl}_{t-1}\right)^{\frac{\mu_{INV}-1}{\mu_{INV}}} \ ,\end{aligned} \tag{1.22}$$

where the variables entering this function are analogous to those presented in the unrestricted lifetime-income's optimization problem.

The restricted lifetime-income household seeks to maximize the objective function:

$$\mathbf{E}_t \sum_{s=t}^{\infty} \beta^{rl}_{t,s} \mathbb{U}^{rl}_s, \tag{1.23}$$

with $\beta_{t,s}^{rl} \equiv \beta_{t,s}^{ul} = \prod_{v=t}^{s-1} \beta_v^l$ and $\beta_{t,t}^{rl} \equiv 1$, subject to (1.21) and conditions analogous to (1.2) and (1.4) through (1.8), rewritten for restricted lifetime-income households rather than for borrowers.

1.2 Finished-good sectors

A new sector specifically for producing residential investment goods has been added to the existing four sectors of production of finished goods (consumption, business investment, government and non-commodity exports). This allows the model to track the residential investment price deflator, which was not the case in ToTEM II.[7] The production structure in the new sector is the same as the one in the other four sectors. This implies that the structural equations describing the dynamics of the new, finished-good sector have the same form as well. We allow some of the parameters in these equations to be sector-specific. In this way, we capture particular characteristics of the supply of residential investment goods. For instance, we take into account the fact that residential investment goods have different import content and frequency of price re-optimization.[8] The supply of residential investment goods can be summarized in the following linearized aggregate supply curve:

$$\widehat{\pi}_t^{IRS} = (1-\theta_{IRS})\,\omega_{IRS}\gamma_{IRS}\phi_{IRS}^{-1}\widehat{\pi}_{t-1}^{IRS} + \beta^l\theta_{IRS}\phi_{IRS}^{-1}E_t\left[\widehat{\pi}_{t+1}^{IRS}\right] + \tilde{\lambda}_{IRS}\left(\widehat{rmc}_t^{IRS} - \widehat{p}_t^{IRS}\right), \quad (1.24)$$

where π_t^{IRS} and rmc_t^{IRS} denote the percentage change in the price of residential investment goods and the real marginal cost of producing residential investment goods, respectively. θ_{IRS} is the probability that price setters in the residential investment sector index their prices to the inflation target, ω_{IRS} is the share of rule-of-thumb price setters in the residential investment sector and γ_{IRS} is the degree of indexation to lagged inflation used by rule-of-thumb price setters. ϕ_{IRS} and $\tilde{\lambda}_{IRS}$ are given by:

$$\begin{aligned}
\phi_{IRS} &= \theta_{IRS} + \omega_{IRS}\,(1-\theta_{IRS})\,(1+\gamma_{IRS}\beta\theta_{IRS}) \\
\tilde{\lambda}_{IRS} &= (1-\omega_{IRS})\,(1-\theta_{IRS})\left(1-\beta^l\theta_{IRS}\right)\phi_{IRS}^{-1}.
\end{aligned}$$

1.3 Monetary policy

We assume that the central bank sets the short-term risk-free interest rate $R_{RF,t}$ according to an augmented Taylor rule. This rule includes the deviation of expected four-quarter-ahead year-over-

[7] In ToTEM II, residential investment goods are produced and supplied by the consumption sector, so the consumption price deflator does not differ from the residential investment price deflator.

[8] The import content affects the specification of $\widehat{rmc}_t^{IRS}$ in the model.

year core inflation from the inflation target $\bar{\pi}$, the output gap $\widehat{y}_t$ and $R_{RF,t-1}$ as follows:

$$R_{RF,t} = \Theta_R R_{RF,t-1} + (1-\Theta_R)\left(\bar{r} + \bar{\pi} + \Theta_\pi\left(E_t\left[\frac{1}{4}\sum_{j=1}^{4}\pi_{t+j}\right] - \bar{\pi}\right) + \Theta_y \widehat{y}_t\right), \tag{1.25}$$

where $\bar{r}$ is the steady-state short-term risk-free real interest rate, Θ_R is the interest rate smoothing parameter, Θ_π is the sensitivity of the short-term risk-free interest rate to core inflation's deviation from the target and Θ_y is the sensitivity of the short-term risk-free interest rate to the output gap.

The specification of this policy rule differs from the one assumed in ToTEM II in two ways. First, the policy rule in ToTEM III uses the year-over-year core inflation, a measure that is less volatile than the one used in ToTEM II: expected two-period-ahead quarterly inflation.

Second, the inflation target is assumed to be constant in ToTEM III because the model is estimated from 1995Q1 to 2015Q4, a period in which Canadian monetary policy operated around an explicit inflation target of 2 percent. In contrast, the target is assumed to be time-varying in ToTEM II because the model was estimated from 1980Q1 to 2012Q2, a sample including a long period with trend inflation.

1.4 Fiscal policy

Tax rules are specified in the same way as in ToTEM II. Government spending and transfer rules differ from those in ToTEM II in two aspects. First, these rules are specified for the levels of real government spending and transfers. In contrast, these rules in ToTEM II were set for government spending and transfers as a percentage of GDP. Second, both government spending and transfers are allowed to increase in response to a decline in labour. In this way, ToTEM III allows for counter-cyclical fiscal policy. More specifically, real government spending and transfers are determined according to:

$$\ln g_t = \rho_g \ln g_{t-1} + (1-\rho_g)\left(\Theta_H^g \widehat{H}_t + \ln \overline{g}\right) + v_t^g, v_t^g = \rho_{v,g} v_{t-1}^g + \varepsilon_t^g \tag{1.26}$$

$$\ln tf_t = \rho_{tf} \ln tf_{t-1} + (1-\rho_{tf})\left(\Theta_H^{tf} \widehat{H}_t + \ln \overline{tf}\right) + v_t^{tf}, v_t^{tf} = \rho_{v,tf} v_{t-1}^{tf} + \varepsilon_t^{tf}, \tag{1.27}$$

where g_t is real government spending, $\overline{g}$ is the steady-state level of government spending, $\widehat{H}_t$ is the log-deviation of total hours worked from their steady state (hours gap), Θ_H^g is the sensitivity of real government spending to movements in the hours gap, tf_t is real transfers, $\overline{tf}$ is the steady-state level of transfers and Θ_H^{tf} is the sensitivity of real transfers to movements in the hours gap. Lastly, $\varepsilon_t^g \sim iid(0, \sigma_g^2)$ and $\varepsilon_t^{tf} \sim iid(0, \sigma_{tf}^2)$.

Chapter 2

Model estimation

We estimate most of ToTEM III's parameters using Bayesian methods, which allow us to incorporate information from microdata and other sources. In contrast, the majority of parameters in ToTEM II were estimated using classical maximum likelihood methods. In this section, we:

- describe the data used in the estimation
- outline the estimation strategy
- discuss estimated parameters
- assess the forecasting performance of the model

2.1 Data

To estimate the model's Canadian block, we use 50 quarterly data series that run from 1995Q1 to 2015Q4. We think that this period is appropriate because the inflation rate in Canada has been stable around the 2 percent target since 1995. The series used for the estimation cover 40 Canadian economic variables and 10 rest-of-the-world (ROW) economic variables (see Table 2.1). We use the latter set of variables to estimate the ROW block, which we estimate independently of the Canadian block.[9] Since Canada is a small open economy, it is reasonable to assume that Canadian data are not important for identifying the dynamics of ROW economic variables. All variables are detrended before estimation (see Appendix B for details on detrending of key macroeconomic variables).

As in ToTEM II, real GDP and its components are obtained from Canadian National Accounts. However, in ToTEM III, total exports are divided in two observable variables: commodity exports and non-commodity exports. The relative prices in each production sector are computed as in ToTEM II. Nonetheless, the relative price of residential investment and the split of the relative price

[9] We use data from 1981Q1 to 2015Q4 to estimate the ROW block.

of exports (commodity and non-commodity) have been added to the set of observable variables in ToTEM III. The relative price of housing is constructed as in ToTEM II.[10] In contrast, the core consumer price index (core CPI) is constructed differently in ToTEM III, motivated by the findings of Bank staff research during the process of the 2016 renewal of the Bank of Canada's inflation-control target. More specifically, we measure core CPI inflation as the average of the Bank's current three preferred core measures: CPI-common, CPI-median and CPI-trim (Khan, Morel and Sabourin 2015).[11] As explained in Schembri (2017), these measures of core inflation are better than CPIX (the measure used in ToTEM II) for identifying persistent movements in inflation that reflect the dynamics of relevant macroeconomic variables for the conduct of monetary policy.

Collateralized household debt, as a share of households' disposable income, is introduced as an observable variable in ToTEM III. In the model, collateralized household debt consists of residential mortgage credit and HELOCs. We compute the former using the Bank's measure of residential mortgage credit, while we proxy the latter using all lines of credit outstanding at chartered banks.[12] Households' disposable income is the sum of employee compensation, net mixed income, property income and net transfers to other sectors (mainly transfers from the government net of taxes paid to the government). Households' disposable income is also an observable variable.[13]

The introduction of household debt in the model also requires a modification in the data sources for interest rates. Because the spread between the long-term household rate and the risk-free rate is endogenous in ToTEM III, using the long-term household rate directly is more convenient than using the spread between this rate and the long-term risk-free rate.[14] Motivated by this change, we also use short- and long-term business rates directly instead of their respective risk premiums, which continue to be exogenous as in ToTEM II. Given that the short-term household rate and the short-term risk-free rate exhibit a high degree of correlation, we assume that they are equal in the model for simplicity. The measurement of all Canadian and ROW short- and long-term rates is the same as in ToTEM II.

[10]The time series for house prices is constructed as follows. We use nominal growth of house prices from the Canadian Real Estate Association's Multiple Listing Service to construct a house price index from 1995Q1 to 1998Q4. From 1999Q1 to 2015Q4, this index is constructed using the nominal growth of the Teranet 11-city average house price index. The real house price is calculated by dividing the nominal house price index by core CPI.

[11]The growth rate of CPI-common is equivalent to a factor model-based measure of the common component of changes in Canadian consumer prices (Khan, Morel and Sabourin 2013). The growth rate of CPI-median equals the price change located at the median (in terms of CPI basket weights) of the distribution of price changes. CPI-trim growth is equal to a trimmed mean of consumer price inflation, which omits monthly growth of prices that is beyond the tails of the distribution.

[12]The Bank's measure of residential mortgage credit is recorded in CANSIM series V122746, while all lines of credit outstanding at chartered banks are recorded in CANSIM series V36869. Both are monthly series, with the month-end values used as the quarterly observations.

[13]The measure of disposable income is recorded in CANSIM series V62305980.

[14]Using the spread as observable variable would require an additional equation in the model for the estimation. Since this addition would not bring any benefit, we prefer to change the observable variable related to the long-term household rate.

Table 2.1: Observable variables in ToTEM III estimation

Category	Variables
GDP and components	GDP, consumption, business investment, residential investment, government expenditures, imports, commodity exports, non-commodity exports, inventory investment
Prices	Core CPI inflation, GDP deflator, consumption deflator, business investment deflator, residential investment deflator, government expenditures deflator, commodity exports deflator, non-commodity exports deflator, imports deflator, real house price, total CPI deflated by core CPI
Interest rates and exchange rates	Short-term risk-free interest rate, long-term risk-free interest rate, short-term business interest rate, long-term business interest rate, long-term household interest rate, ROW/Canada real effective exchange rate, CEER
Labour, output and income	Total hours worked, output gap, real disposable income, real wages
Fiscal variables	Real transfers, real income tax revenue, real consumption tax revenue, ratio of government debt to GDP
Balance of payment and degree of global openness	Ratio of net foreign assets to GDP, ratio of current account balance to GDP, global trade openness
Household balance sheet variables	Ratio of financial wealth to disposable income, ratio of household debt to disposable income
Rest-of-the-world variables	GRACE, ROW inflation, ROW commodity price, ROW energy commodity price, ROW energy price spread, ROW non-energy commodity price, ROW export price, ROW output gap, ROW short-term risk-free rate, ROW long-term risk-free rate

In addition to the ROW/Canada real effective exchange rate used in ToTEM II, we introduce CEER as an observable variable in ToTEM III.[15] This measure only replaces the ToTEM II measure of the exchange rate in the demand for Canadian non-commodity exports. The ToTEM II measure of the exchange rate remains in use in the rest of the model (for instance, in the real UIP condition). The addition of CEER is motivated by evidence presented in Alexander, Cayen and Proulx (2017) and Binette, Chernis and De Munnik (2017), both suggesting that movements in CEER better explain the recent dynamics of non-commodity exports.

NFAs, the current account balance, fiscal variables and ROW variables are measured as in ToTEM II, with the exception of the measure of global foreign activity. ToTEM III uses the global real activity for Canadian non-commodity exports (GRACE NCX) to measure foreign activity in the demand equation for non-commodity exports. The choice of this measure is supported by the findings of Binette, Chernis and De Munnik (2017) that GRACE NCX better accounts for the evolution of non-commodity experts than ToTEM II's measure.

2.2 Estimation strategy

Following the methodology used for ToTEM II, we calibrate the parameters that primarily determine the steady state. In contrast, we use real-time data to estimate the monetary policy rule and Bayesian methods to estimate the remaining parameters that do not affect the steady state of the model. In this subsection, we first describe the calibration strategy, then outline the estimation of the monetary policy rule and conclude with a brief description of the Bayesian estimation procedure.

2.2.1 Calibration

As in ToTEM II, we divide the parameters that affect the steady state into two groups:

- parameters chosen to match information from microdata
- parameters chosen to match historical averages in Canadian aggregate data from 1995Q1 to 2015Q4

The former group consists of parameters such as the depreciation rate of capital, the LTV ratio, the amortization rate of household debt and the share of loans that are renegotiated every quarter. The latter group includes parameters such as the discount factors of lifetime-income and borrower households, the shares of labour, capital and imports in the production functions, the steady-state sector-specific productivity, the marginal tax rates and the fraction of home equity that is withdrawn for household spending.

[15] CEER is an exchange rate indicator based on competition-based weights for individual exchange rates that are updated annually (see Barnett, Charbonneau and Poulin-Bellisle 2016).

The depreciation rate is equal across sectors and set to 0.031, which implies an annual depreciation rate of 12.4 percent. This value is in line with estimated depreciation rates by Statistics Canada (2007). The share of borrowers of all households is set to 45 percent, which is in line with estimates obtained by Guerrieri and Iacoviello (2017). The LTV ratio α_{ltv} is set to 0.78 to match the average LTV ratio on new mortgages.[16] The amortization rate κ is set to 0.0161, implying that half the principal is paid off in 10.7 years. This duration is consistent with an average amortization period of 21.4 years. The share of loans that are renegotiated every period Φ is set to 0.093 to match the fact that collateralized household debt is renegotiated on average once every 2.7 years.[17]

The discount factor of lifetime-income households β^l is set at 0.9940, in line with the average of the trend for the real risk-free short-term rate from 1995 to 2015. The borrowers' discount factor β^{borr} is set at 0.9891. This value is chosen such that the steady-state spread between the long-term rate faced by households and the risk-free short-term interest rate is 200 basis points (bps), which is the sample average. The labour-share parameter in the constant elasticity of substitution (CES) production functions is the same across sectors, chosen to match the average labour income share. The capital-share parameter in the CES production functions is also the same across sectors, chosen to replicate the average ratio of nominal investment to GDP.

In contrast, the import-share parameters in the CES production functions are sector-specific, set to match the import concentration in each sector. For each sector, steady-state productivity is chosen to replicate average relative prices in the data. In the commodity sector, the fixed-factor land used in production is set to match the average ratio of commodity production to GDP. The steady-state tax rates on labour income and consumption expenditures are chosen to match the averages of the ratio of revenue from labour income tax to GDP and of the ratio of revenue from consumption tax to GDP, respectively. The fraction of home equity that is withdrawn is set at 5 percent in order to match the average ratio of collateralized household debt to disposable income. Table 2.2 presents the data targets for key macroeconomic variables. All the targets, except those for the import concentrations, reflect historical averages in the Canadian data from 1995 to 2015. The targets for the import concentration in every sector are set using Statistics Canada's 2011 vintage input-output tables for the Canadian economy.

[16] The average LTV is calculated using data from the Office of the Superintendent of Financial Institutions on the LTV ratio for new mortgages from 2010 to 2016, compiled from regulatory filings.

[17] Using data from 2010 to 2016 on the average renegotiation period of new mortgages from the Office of the Superintendent of Financial Institutions, we estimate that these mortgages are renegotiated once every 3.05 years. In contrast, HELOCs are renegotiated every quarter. Taking into account these frequencies of renegotiation, and the fact that mortgages and HELOCs represent 87.5 and 12.5 percent of total collateralized household debt, respectively (based on the 1995–2015 sample averages), collateralized household debt is refinanced on average once every 2.7 years.

Table 2.2: Targets for key macroeconomic variables

Import concentration in consumption sector	0.24
Import concentration in business investment sector	0.39
Import concentration in residential investment sector	0.22
Import concentration in government sector	0.13
Ratio of residential investment to GDP	0.06
Ratio of government expenditures to GDP	0.24
Ratio of commodity exports to GDP	0.15
Ratio of business investment to GDP	0.12
Ratio of imports to GDP	0.34
Ratio of exports to GDP	0.36
Ratio of value of commodity production to GDP	0.19
Relative price of government goods	0.98
Relative price of business investment goods	1.04
Relative price of imported goods	1.12
Relative price of exported goods	1.03
Ratio of consumption tax revenue to GDP	0.18
Ratio of labour income tax revenue to GDP	0.17
Ratio of government debt to GDP	0.57
Ratio of nominal labour income to GDP	0.53
Ratio of collaterized household debt to disposable income	0.93
Real risk-free short-term rate	2.4%
Spread between household long-term rate and risk-free rate	200 bps

2.2.2 Estimation of the monetary policy rule

The monetary policy rule in ToTEM III is not estimated jointly with the rest of the model. Rather, we have set the parameters of the policy rule equal to those from a single-equation estimate of the Taylor rule using real-time data from 1993Q4 to 2015Q4. This dataset consists of real-time estimates of the contemporaneous output gap and deviations of four-quarter ahead forecasts of year-over-year CPIX inflation from the two percent target.[18] Using these estimates for the monetary policy rule in ToTEM III has the advantage of ensuring that the rule is consistent with the information set available at the time interest rate decisions were made. Such estimates based on real-time data can differ considerably from those based on ex post revised data, as pointed out by Orphanides (2001).

Table 2.3 shows that the policy rule in ToTEM III exhibits a moderate degree of interest rate smoothing. Moreover, the weight on inflation gap is significantly larger than that on output gap.

Table 2.3: ToTEM III policy rule estimates

Parameter	Description	Estimates
Θ_R	Policy rule—smoothing	0.85
Θ_π	Policy rule—inflation	4.65
Θ_y	Policy rule—output gap	0.4

2.2.3 Bayesian estimation

A two-step approach is followed to reduce the number of parameters that are estimated simultaneously. In the first step, we estimate the ROW block in isolation. In the second step, we estimate the Canadian block conditional on the posterior mean estimates for the parameters of the ROW block, the steady state of the Canadian block and the estimated values of the monetary policy rule.

Since the ROW block of equations is fairly small and the set of parameters is not large, we estimate the posterior distribution for the ROW block parameters using a standard random-walk Metropolis-Hastings algorithm.[19] In contrast, the large number of parameters and equations in the Canadian block pose some challenges. In particular, applying maximization methods based on second derivatives or approximating the covariance matrix with an inverse Hessian is not feasible. For this reason, we estimate the posterior distribution for the Canadian block parameters using an adaptive Metropolis-Hastings algorithm similar to that used in Haario, Saksman and Tamminen

[18]The data for the output gap are constructed in the following way. Before 2006, the data consist of the Bank staff assessment of the output gap. From 2006 onward, the data represent Governing Council's real-time view of the output gap. The real-time inflation forecasts were compiled from real-time Bank of Canada staff projection databases.

[19]See, for example, Herbst and Schorfheide (2015) for an introduction to the Metropolis-Hastings algorithm and its application to DSGE estimation.

(2001) and Roberts and Rosenthal (2009). Appendix C presents details of the adaptive Metropolis-Hastings algorithm.

Following the literature, we use beta priors for most parameters that economic theory suggests a range between 0 and 1.[20] For example, the Calvo probabilities that describe the price setting in each production sector are assumed to have a beta prior, with mean 0.5 and standard deviation 0.1. This mean is consistent with a prior that firms re-optimize their prices on average once every two quarters. We also follow the literature when setting priors for parameters that should be non-negative according to economic theory. More specifically, we assume gamma priors for parameters such as the semi-elasticity of the country risk premium with respect to NFA/GDP and input adjustment costs parameters. For the parameters describing the persistence of the shocks processes, we usually assign beta priors with a mean equal to 0.5 and a standard deviation equal to 0.2. These priors avoid pile-ups of the persistence parameter estimates at unity. For the standard deviations of the structural shocks, we generally assign diffuse inverse-gamma priors, as in Smets and Wouters (2007). Table 2.4, Table 2.5 and Table 2.6 report the prior distributions used for key structural parameters in the Canadian block of the model.

2.3 Parameter estimates

In this subsection, we discuss the posterior estimates of key structural parameters in the Canadian block of the model and their implications for the dynamics of prices, wages, investment, imports and the exchange rate.

For all estimated parameters, we report the prior and posterior distributions for ToTEM III, including 90 percent confidence intervals for the posteriors.[21] We also report, for comparison, the maximum-likelihood point estimates of the ToTEM II parameters.

2.3.1 Phillips curve parameters

Table 2.4 reports Phillips curve parameters (Calvo probabilities, degree of indexation and shares of rule-of-thumb (RoT) agents) in different sectors. Our estimates reveal considerable heterogeneity in firms' pricing behaviour across sectors. The Calvo probabilities range from 0.35 in the import sector to 0.84 in the consumption sector, whereas the shares of RoT price setters range from 0.12 in the non-commodity exports sector to 0.54 in the consumption sector. These results clearly show

[20]With a few exceptions, our approach to setting priors is in line with the Bayesian DSGE estimation literature starting with Smets and Wouters (2003, 2007). Fernandez-Villaverde (2010), Schorfheide (2011) and Guerron-Quintana and Nason (2012) survey the literature on Bayesian methods for estimation of DSGEs. Guerron-Quintana and Nason (2012) specifically underline the desirability of the beta distribution for parameters assumed by economic theory to be between zero and one (p. 15) as well as the general desirability of priors that are easy to understand.

[21]The statistics for ToTEM III are based on 500,000 draws from the posterior distribution.

Table 2.4: Priors and posteriors for ToTEM III Phillips curve parameters

Prior distributions for Phillips curve parameters				
Parameter	Sector	Form	Mean	Standard deviation
Calvo	Output prices	Beta	0.5	0.1
	Wages	Uniform	0.5	0.29
Indexation	All sectors	Beta	0.5	0.15
RoT share	Output prices	Beta	0.3	0.1
	Wages	Uniform	0.5	0.29
Posterior estimates of parameters				
Parameter	Description	ToTEM II	ToTEM III	
			Mean	Confidence interval
Calvo probabilities				
θ_C	Consumption	0.75	0.837	[0.783,0.892]
θ_{IRS}	Residential investment		0.634	[0.459,0.736]
θ_I	Business investment	0.399	0.535	[0.433,0.640]
θ_G	Government expenditure	0.783	0.684	[0.572,0.818]
θ_X	Exports	0.451	0.425	[0.341,0.542]
θ_E	Energy	0.945	0.508	[0.343,0.671]
θ_{NE}	Non-energy	0.934	0.464	[0.316,0.610]
θ_M	Imports	0.864	0.350	[0.241,0.460]
θ_W	Wages	0.590	0.488	[0.334,0.662]
Degrees of indexation				
γ_C	Consumption	0.058	0.506	[0.261,0.754]
γ_{IRS}	Residential investment		0.429	[0.203,0.683]
γ_I	Business investment	0.562	0.495	[0.252,0.729]
γ_G	Government expenditure	0.489	0.469	[0.238,0.714]
γ_X	Exports	0.006	0.504	[0.262,0.746]
γ_E	Energy	0.573	0.496	[0.254,0.742]
γ_{NE}	Non-energy	0.572	0.493	[0.245,0.745]
γ_M	Imports	0.736	0.552	[0.294,0.786]
γ_W	Wages	0.109	0.422	[0.202,0.664]
Weights of rule-of-thumb price setting				
ω_C	Consumption	0.48	0.542	[0.352,0.707]
ω_{IRS}	Residential investment		0.401	[0.217,0.576]
ω_I	Business investment	0.3	0.267	[0.133,0.422]
ω_G	Government expenditure	0.3	0.387	[0.195,0.582]
ω_X	Exports	0.3	0.124	[0.057,0.210]
ω_E	Energy	0.3	0.305	[0.152,0.475]
ω_{NE}	Non-energy	0.3	0.241	[0.120,0.391]
ω_M	Imports	0.3	0.203	[0.102,0.325]
ω_W	Wages	0.3	0.530	[0.312,0.731]

that the consumption sector has the flattest Phillips curve.

The estimation reinforces the idea that it is relevant to account for the differences in price setting across these two sectors, which was not the case in ToTEM II. For example, the Calvo probability and the share of rule-of-thumb price setters in the residential investment sector are considerably lower than those in the consumption sector. In contrast, the degree of heterogeneity in the indexation parameters across sectors is small, which reflects that there is little updating of the prior distribution for all indexation parameters.

Moreover, ToTEM III's estimated Phillips curve in the consumption sector is considerably flatter than that in ToTEM II. More specifically, the coefficient associated with the real marginal cost in the consumption sector is about 40 percent of that estimated in ToTEM II. This result is mainly explained by a higher estimate of the Calvo probability in ToTEM III (0.84 versus 0.75 in ToTEM II). Moreover, the share of RoT price setters is slightly higher in ToTEM III. Combined with a higher estimated degree of indexation to lagged inflation, this result leads to a higher weight on lagged inflation in ToTEM III. Despite this increase, the Phillips curve in the consumption sector is almost entirely forward-looking. The following linearized equations illustrate the differences in the dynamics of core inflation π_t^C in ToTEM III and ToTEM II:

$$\pi_t^C = 0.0466\pi_{t-1}^C + 0.8641 E_t \pi_{t+1}^C + 0.0045\widehat{rmc}_t^C + \varepsilon_t^p, \quad \text{(ToTEM III)} \tag{2.1}$$

$$\pi_t^C = 0.0079\pi_{t-1}^C + 0.8501 E_t \pi_{t+1}^C + 0.0129\widehat{rmc}_t^C + \varepsilon_t^p, \quad \text{(ToTEM II)} \tag{2.2}$$

where rmc_t^C is real marginal cost in the consumption sector and ε_t^p is a linear combination of the inflation target and the deviation of the markup from its steady state.

Wage inflation dynamics in ToTEM III are also characterized by a Phillips curve that is flatter than that in ToTEM II. The lower sensitivity of wage inflation to the gap between the real wage and the marginal rate of substitution reflects the larger share of RoT wage setters (0.53 versus 0.3 in ToTEM II) and the lower wage markup, which were partially offset by an estimated lower Calvo probability (0.49 versus 0.59 in ToTEM II). Moreover, a larger amount of indexation in ToTEM III (0.42 versus 0.11 in ToTEM II) results in more backward-looking wage setting. The different estimates result in different wage inflation dynamics in ToTEM III and ToTEM II, which are captured in the following linearized equations:

$$\widehat{\pi}^w_t = 0.1404\widehat{\pi}^w_{t-1} + 0.5957E_t\widehat{\pi}^w_{t+1} - 0.0021\left[\widehat{w}_t - \widehat{p}^{tot}_{c,t} - \tau_{w,t} - \widehat{mrs}_t\right], \quad \text{(ToTEM III)} \tag{2.3}$$

$$\widehat{\pi}^w_t = 0.0345\widehat{\pi}^w_{t-1} + 0.6575E_t\widehat{\pi}^w_{t+1} - 0.0027\left[\widehat{w}_t - \widehat{p}^{tot}_{c,t} - \tau_{w,t} - \widehat{mrs}_t\right], \quad \text{(ToTEM II)} \tag{2.4}$$

where $p^{tot}_{c,t}$ is the price of aggregate consumption, $\tau_{w,t}$ is the labour income tax rate and mrs_t is the marginal rate of substitution between consumption and leisure. In contrast with core CPI inflation and wage inflation, import price inflation dynamics are associated with a steeper Phillips curve than that in ToTEM II. This is mainly explained by an estimated Calvo probability in ToTEM III that is considerably lower than that in ToTEM II. The lower estimated Calvo probability also implies that import price inflation is less forward-looking in ToTEM III. The different dynamics of import price inflation π^m_t in both models are summarized in the following linearized equations:

$$\pi^m_t = 0.1439\pi^m_{t-1} + 0.6853E_t\pi^m_{t+1} + 0.6668\widehat{p}^m_t + \varepsilon^m_t, \quad \text{(ToTEM III)} \tag{2.5}$$

$$\pi^m_t = 0.0324\pi^m_{t-1} + 0.9213E_t\pi^m_{t+1} + 0.0147\widehat{p}^m_t + \varepsilon^m_t, \quad \text{(ToTEM II)} \tag{2.6}$$

where p^m_t is the logarithm of the ratio of the price paid by importers over the price paid by end-users.

2.3.2 Adjustment costs and uncovered interest rate parity parameters

In ToTEM III, the sizes of adjustment costs faced by firms for changing imports and business investment are significantly lower than those in ToTEM II (Table 2.5). In particular, the size of each of these costs is about 50 percent lower than in ToTEM II. Everything else equal, this implies that in ToTEM III there is a greater responsiveness of imports and business investment to economic shocks. The sizes of adjustment costs for changing hours, capital and commodities are roughly the same as those in ToTEM II. The sizes of the investment adjustment costs faced by households for residential investment and inventory investment are considerably larger than those in ToTEM II.

The estimated weight on the lagged real exchange rate in the UIP condition is 0.504, which is more than three times greater than the one estimated in ToTEM II. This implies that ToTEM III introduces considerably more intrinsic persistence in the real exchange rate than ToTEM II included. The estimated weight is roughly as high as the one in ToTEM, with both models including a country-specific risk premium. In contrast, such feature is absent in the UIP condition in ToTEM II. This suggests that the higher weight in ToTEM III seems to be associated with reintroducing the country-specific risk premium into the model. The semi-elasticity of the country-specific risk premium with

Table 2.5: Prior and posterior distributions for adjustment costs and uncovered interest rate parity parameters

Parameter	Description	ToTEM II	ToTEM III				
			Prior			Posterior	
			Form	Mean	SD	Mean	Conf. interval
Input adjustment costs faced by firms							
χ^K	Capital	9.099	Gamma	10	4	11.79	[6.260, 18.37]
χ^H	Hours	0.967	Gamma	1	0.4	1.025	[0.743, 1.358]
χ^M	Imports	2.032	Gamma	2	0.8	0.994	[0.618, 1.415]
χ^{COM}	Commodities	9.849	Gamma	10	4	9.573	[4.406, 16.70]
χ^I	Business investment	16.51	Gamma	15	4	8.207	[4.863, 13.36]
Input adjustment costs faced by households							
χ^{IRS}	Residential investment	0.320	Gamma	10	2	6.004	[4.555, 7.696]
χ^{LAND}	Land		Gamma	1	0.4	0.442	[0.268, 0.660]
χ^{INV}	Inventories	0.644	Gamma	3	0.8	2.163	[1.168, 3.396]
UIP parameters							
$\varkappa$	Weight of lag in UIP	0.1585	Beta	0.20	0.16	0.504	[0.403, 0.573]
ς	Elast. of risk premium	0	Gamma	0.05	0.01	0.050	[0.036, 0.067]

respect to the ratio of NFA to GDP is estimated to be 0.05. Everything else equal, this means that a reduction by one percentage point of the NFA-to-GDP ratio would cause a depreciation in the real exchange rate by 0.05 percent.

2.3.3 Fiscal policy

Table 2.6 reports the sensitivity of government spending and transfers to changes in the hours worked gap, along with the persistence parameters of the fiscal policy processes. We assume priors that impose the parameters Θ_H^g and Θ_H^{tf} to be negative, allowing for counter-cyclicality. Our posterior estimates reveal some degree of counter-cyclicality in both government spending and transfers. Moreover, the intrinsic persistence in government spending is estimated to be considerably larger than that of transfers.

2.4 Forecasting performance

In this subsection, we compare the forecasting performance of ToTEM III with that of ToTEM II. To do so, we compute the percentage difference in the models' root mean squared errors (RMSE) for one-quarter-, one-year- and two-year-ahead pseudo-out-of-sample forecasts.[22] We compute the

[22]It is not feasible to obtain a meaningful comparison of ToTEM III and ToTEM II in terms of goodness of in-sample fit as in Smets and Wouters (2003) or Ahn and Schorfheide (2007), where they compute the marginal likelihood (the average likelihood given a prior distribution). Since both models are estimated over very different

Table 2.6: Prior and posterior distributions for fiscal policy parameters

Persistence parameters						
Parameter	Description	Prior			Posterior	
		Form	Mean	SD	Mean	Conf. interval
ρ_g	Gov. expenditure process	Beta	0.75	0.1	0.830	[0.716, 0.909]
ρ_{tf}	Transfer process	Beta	0.5	0.2	0.345	[0.049, 0.715]
Sensitivity of policy to hours gap						
Parameter	Description	Prior			Posterior	
		Form	Lower bound	Upper bound	Mean	Conf. interval
Θ_H^g	Gov. expenditure sensitivity	Uniform	-100	0	-2.329	[-3.591,-1.054]
Θ_H^{tf}	Transfers sensitivity	Uniform	-100	0	-0.572	[-1.372,-0.051]

RMSEs over the estimation sample for the domestic block (1995Q1 to 2015Q4).

Table 2.7 reports the results for key macroeconomic variables common to both models. The percentage differences are reported in bold type when they are significant at the 10 percent level, using a standard Diebold-Mariano test as in Harvey, Leybourne and Newbold (1997). Overall, the table shows that ToTEM III dominates ToTEM II in terms of forecasting performance. This improvement partly reflects the enhanced estimation method and the richer structure of ToTEM III.

Table 2.7: Percentage change of RMSE of ToTEM III relative to ToTEM II, 1995Q1–2015Q4

Series	1Q ahead	4Q ahead	8Q ahead
GDP	-8	-4	**-25**
Consumption	-8	-16	**-32**
Business investment	-5	-11	-17
Residential investment	**-26**	**-33**	**-40**
Exports	-4	-20	**-30**
Commodity exports	-5	-18	**-31**
Non-commodity exports	**-6**	-15	**-21**
Imports	**-8**	-5	-10
Core inflation	**-28**	**-20**	-11
Short-term risk-free interest rate	-9	-17	-20
Nominal exchange rate	-1	-7	**-27**
Real house price	36	-3	-7
Bold figures are significant at the 10% level.			

ToTEM III significantly outperforms ToTEM II in forecasting residential investment at all horizons. For example, the RMSE for the two-year-ahead forecast for this variable falls by 40 percent

time periods and based on different number of observables (29 for ToTEM II, 50 in ToTEM III), the likelihood functions for the two models are not comparable.

relative to that of ToTEM II. Moreover, the reduction in RMSEs at all horizons is significant at the 10 percent level. The improvement in the ability to forecast residential investment results in part from a more detailed modelling of the supply and demand in the housing market. The more elaborate structure of this market in ToTEM III also helps improve the house price forecasts one and two years ahead. However, ToTEM II performs better when predicting house prices one quarter ahead. This is, in part, due to the fact that ToTEM II features a house price reduced-form equation with high persistence, which forecasts the short-term better than the structural relationships embedded in ToTEM III.

The forecasting performance for consumption is also improved significantly, with the largest improvement taking place at the two-year horizon. The better ability to forecast consumption with ToTEM III is partly explained by the fact that this model takes explicit account of the role of collateralized household debt in shaping consumption decisions. There are also significant improvements in longer-term forecasts of total exports (both commodity and non-commodity exports). For example, the RMSE for the two-year-ahead forecast of non-commodity exports is 21 percent lower in ToTEM III. The improvement is due in part to the adoption of better measures of the determinants of non-commodity exports (using GRACE to measure foreign activity and CEER to account for the real exchange rate). The eight-quarter-ahead forecast error for the nominal exchange rate is reduced by 27 percent, profiting from improved estimation that uses more information to capture persistence in the exchange rate data.

The improvement in forecasting imports is relatively modest. For the two-year-ahead forecasts of business investment and the nominal exchange rate, the RMSEs are considerably lower than those obtained with ToTEM II. Overall, ToTEM III significantly improves the real GDP forecast at the two-year horizon. The forecasting performance for core inflation and the short-term risk-free interest rate also improves. ToTEM III's forecasts for core inflation are particularly better than those of ToTEM II at the one-quarter and one-year horizons, whereas the interest rate forecasts are considerably improved at the one- and two-year horizons.

Chapter 3

Changes to model properties

This section reports ToTEM III's impulse responses to some key exogenous disturbances and discusses their effects on important Canadian macroeconomic variables, highlighting differences from ToTEM II. We consider two groups of shocks, domestic and ROW. On the domestic side, we include shocks to:

- monetary policy
- the term premium
- the exchange rate
- real house prices
- productivity

We also examine shocks originating abroad, including:

- foreign demand shock
- shock to world energy price

We control for a few important factors to calculate the impulse responses reported here. We first modify the baseline steady state assumed for the estimation to account for the fact that the Canadian economy will likely fluctuate for some time around a steady state with a higher degree of household indebtedness. More specifically, we set the steady state ratio of collateralized household debt to disposable income equal to 140 percent, which is roughly in line with the historical average over the period following the global financial crisis. To match this alternative steady state, we increase the LTV ratio, the fraction of home equity that is withdrawn, the steady-state ratio of the value of residential structures to disposable income and the steady-state ratio of total housing assets to disposable income (see Table 3.1).

Table 3.1: Selected parameters and target ratios in the baseline versus high-debt calibration

	Baseline	*High-debt*
Ratio of household debt to disposable income	0.93	1.40
Loan-to-value ratio	0.78	0.95
Withdrawal rate of home equity	0.05	0.08
Ratio of value of residential structures to disposable income	1.54	1.74
Ratio of total housing assets to disposable income	3.22	4.16

When comparing the impulse responses between ToTEM III and ToTEM II, we also focus on the implications of the differences in the Canadian blocks between these two models. For this purpose, we follow three steps. First, we impose ToTEM III's monetary policy rule, ROW block and serially correlated error processes in ToTEM II. Second, in the case of transitory shocks, we scale the size of shocks such that the peak/trough impact on the associated variable is the same in both models. For example, for a contractionary monetary policy shock, we choose its size such that the peak impact on the short-term nominal interest rate is the same in both models. Third, in the case of a permanent shock, we choose its size to match the long-run impact of the associated observable input. For instance, the size of the productivity shock is such that the long-run impact on output is the same in the two models.

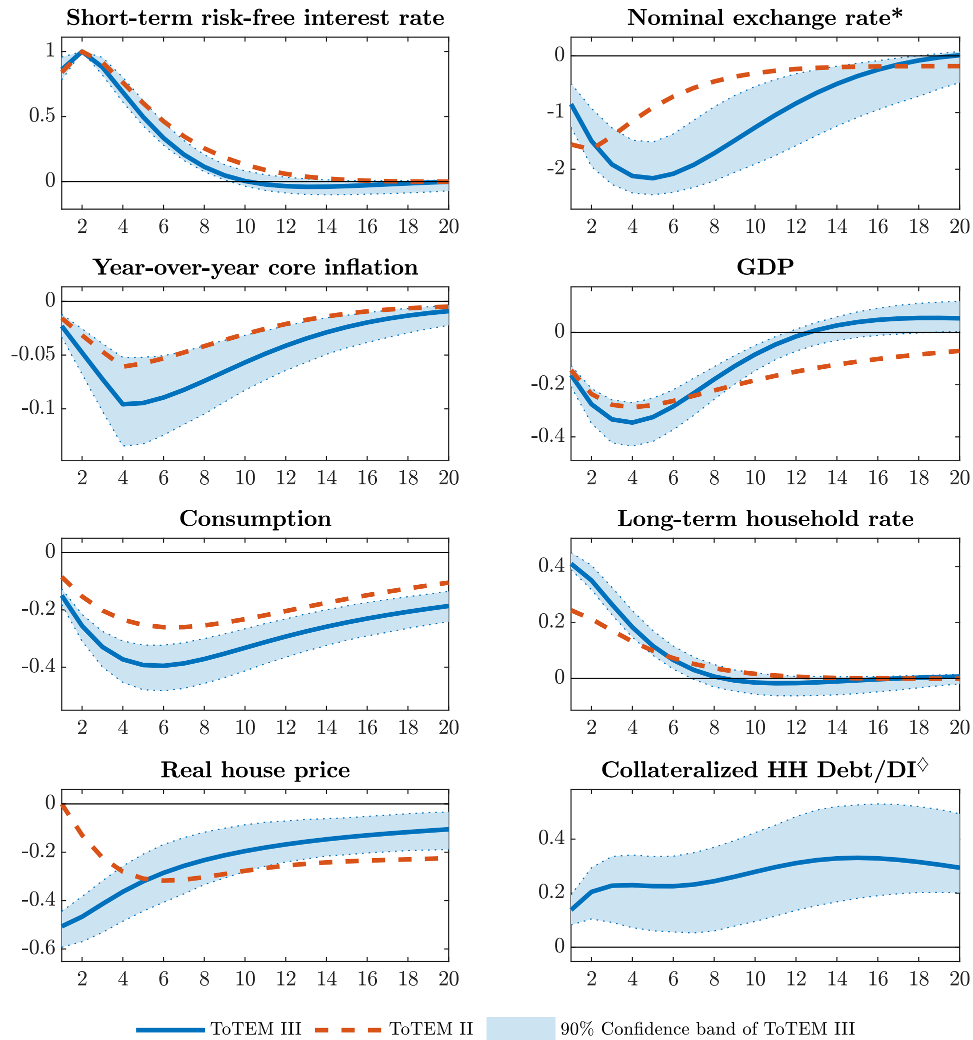
We also report confidence bands for ToTEM III's impulse responses, which are calculated as follows. We first take a representative sample of 1,000 draws from the full posterior distribution of the dynamic (domestic) parameters. We then solve ToTEM III conditional on each vector of the sample distribution of parameters to compute the impulse responses. The confidence bands in the figures below report the 5th and 95th percentile of the impulse response at each horizon.

3.1 Monetary policy shock

Figure 3.1 presents the effects of a 100 bps transitory increase in the short-term risk-free interest rate. On impact, the Canadian dollar in ToTEM III appreciates by about 1 percent, which is roughly two-thirds of its response in ToTEM II. This lower short-run impact mainly reflects higher intrinsic persistence of the exchange rate in ToTEM III. After the initial impact, the appreciation of the Canadian dollar remains larger in ToTEM III starting at the third quarter after the shock. This larger response of the exchange rate and steeper Phillips curve for import prices in ToTEM III mainly explain the larger inflation response in this model.

At the trough, the response of real GDP in ToTEM III is larger than that in ToTEM II. Several factors explain this result. First, business investment is more responsive in ToTEM III, given lower estimates of investment adjustment costs in this model. Second, consumption responds more as well, which reflects the stronger response of long-term household rates and the larger sensitivity of consumption in a high-debt environment that is present in ToTEM III but absent in ToTEM II. Third, since inflation falls more in ToTEM III, real risk-free short-term and effective long-term household rates increase more in this model than in ToTEM II, generating an extra downward pressure on business investment and consumption. Lastly, the counter-cyclical response of government spending in ToTEM III provides a partial offset to the other three factors.

Figure 3.1: Response to a monetary policy shock of 100 basis points (%)



* A positive response indicates depreciation of the Canadian dollar
◊ Ratio of collateralized household debt to disposable income

Over the first four quarters after the monetary shock, house prices decline more in ToTEM III than in ToTEM II. This larger decline is associated with the stronger increase of long-term household rates in ToTEM III. The reduction in house prices in ToTEM III contributes to the decline in collateralized household debt. However, since disposable income decreases more than collateralized household debt after the monetary policy shock, the ratio of collateralized household debt to disposable income modestly rises in response to the contractionary monetary policy shock (by about 0.4 percentage points four years after the shock). This outcome is in line with evidence provided by Alpanda and Zubairy (2017) for the United States.

3.2 Term premium shock

In ToTEM III, consumption and residential investment respond less than in ToTEM II to an increase in the domestic term premium (Figure 3.2). This result is mainly explained by two differences between ToTEM III and ToTEM II. First, the share of restricted lifetime-income households in ToTEM III is estimated to be 50 percent, which is smaller than the estimated share of 88 percent in ToTEM II. Second, the pass-through from the term premium to the long-term household rate is estimated to be weaker in ToTEM III. These two differences, together with more counter-cyclical fiscal policy in ToTEM III, lead to a smaller response of output.

Since the rise in the term premium is contractionary, short-term risk-free rates decline in both models to stabilize inflation. The magnitude of these declines is roughly similar in both models. As a result of this monetary policy response, the effects of the term premium shock on year-over-year core inflation are modest in both models. The impact of a term premium shock on collateralized household debt to disposable income is also small. On impact, this ratio rises by about 0.1 percentage points.

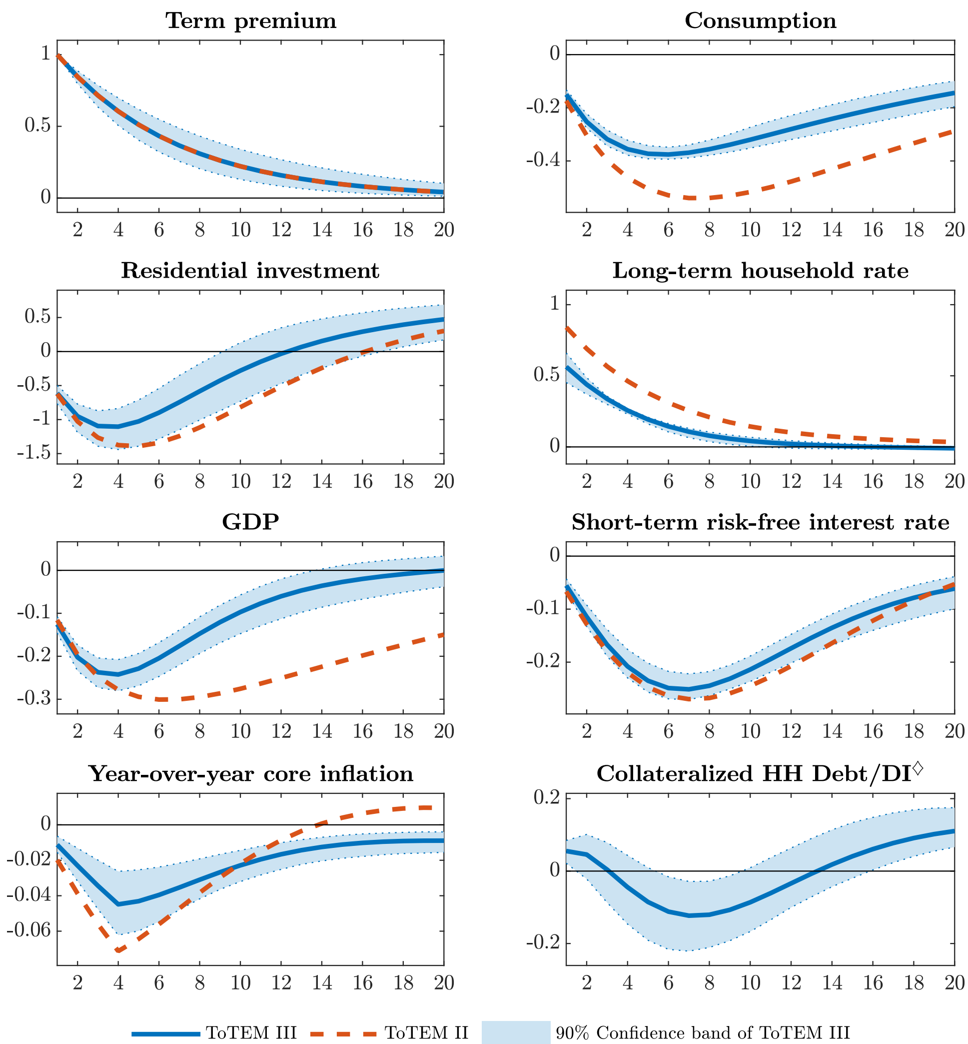
3.3 Exchange rate shock

Since the exchange rate exhibits a higher intrinsic persistence in ToTEM III than in ToTEM II, the response of the Canadian dollar to an exchange rate shock takes longer to reach its peak and is more persistent in ToTEM III (Figure 3.3). Core inflation is more sensitive to exchange rate movements in ToTEM III, mainly due to a steeper Phillips curve for import prices in this model compared with ToTEM II. In response to a 10 percent depreciation, the peak impact on year-over-year core inflation is about 25 bps, which is roughly five times the peak impact in ToTEM II and more in line with estimates using other approaches.[23] As a result of greater inflationary pressures in ToTEM III, the short-term risk-free interest rate rises more in ToTEM III than in ToTEM II.

Import prices increase considerably more in ToTEM III in response to a depreciation of the Canadian dollar. This result reflects substantially less price rigidity in import prices in ToTEM III. Given its high import content, business investment becomes more costly in response to a weaker Canadian dollar in ToTEM III. This higher cost, together with lower business investment adjustment

[23] Using Canadian data from 1995 to 2013, Savoie-Chabot and Khan (2015), for example, estimate that a 10 percent depreciation in the Canadian dollar leads to a peak impact of 0.3 percentage points on CPIX inflation.

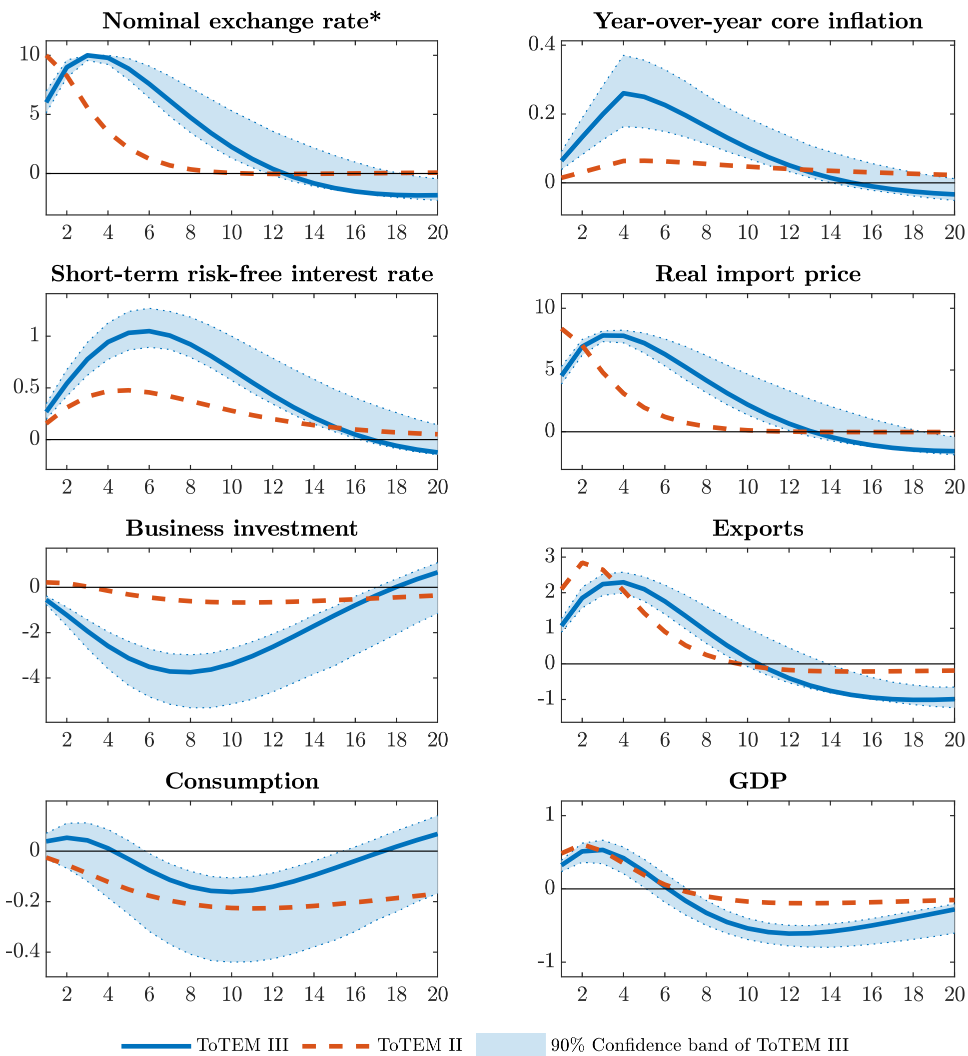
Figure 3.2: Response to a 100 basis point rise in the term premium (%)



◊ Ratio of collateralized household debt to disposable income costs, helps explain the larger reduction in business investment in ToTEM III in response to a depreciation of the Canadian dollar.

Overall, GDP experiences a short expansion of similar magnitude in both models followed by a prolonged decline that is more pronounced in ToTEM III. The drivers of the expansion and decline differ in both models. In ToTEM III, exports and consumption contribute to the expansion. In contrast, exports are the only driver of the expansion in ToTEM II. Business investment, consumption and exports explain the decline in ToTEM III, whereas consumption accounts for most of the decline in ToTEM II.

Figure 3.3: Response to a 10 percent shock to the nominal exchange rate (%)

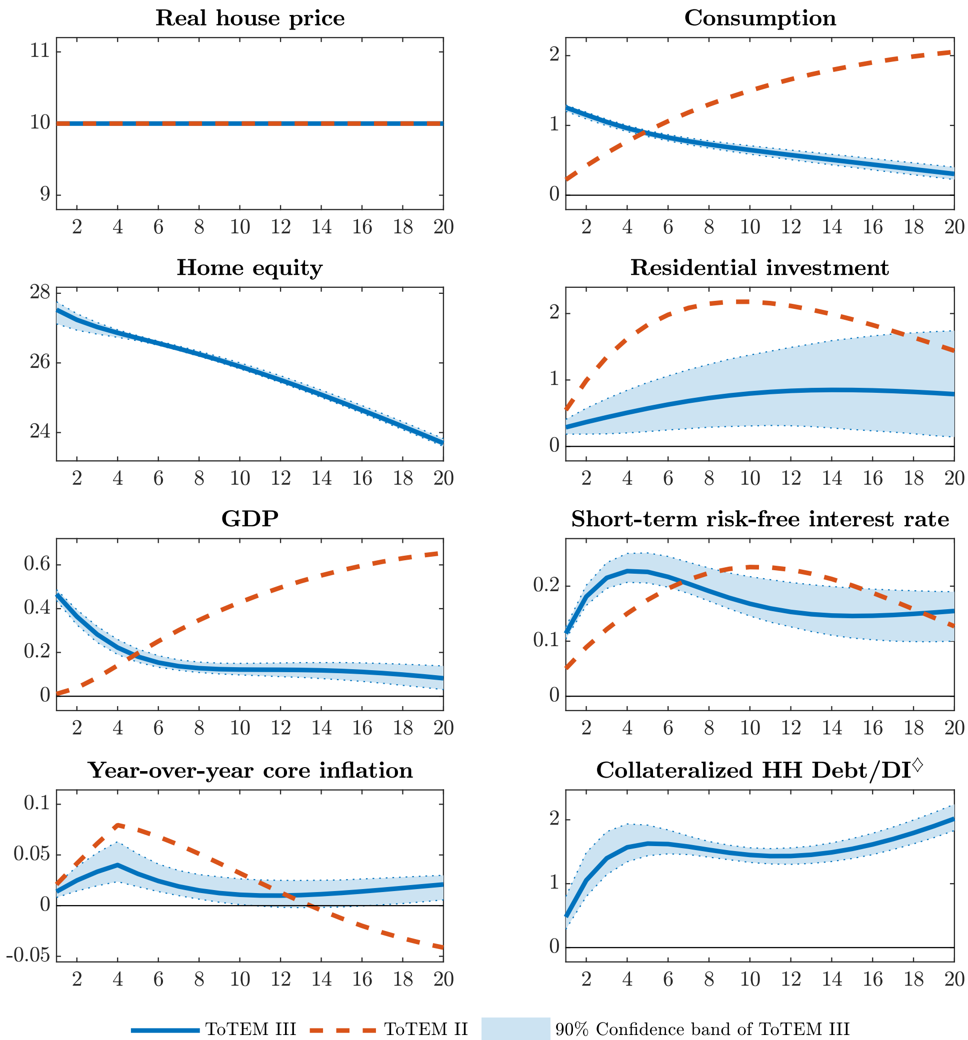


* A positive response indicates depreciation of the Canadian dollar

3.4 House price shock

Estimates from ToTEM III reveal that house price shocks are significantly more persistent than those estimated in ToTEM II. Figure 3.4 compares the responses of key variables in both models subject to a permanent 10 percent rise in real house prices. We impose the same house price profile and, thus, remove any differences arising from the estimated shock persistence. This allows us to focus on how the distinct differences in structure and parameter estimates of the models affect the propagation of house price shocks.

Figure 3.4: Response to a permanent 10 percent rise in the real house price (%)



◊ Ratio of collateralized household debt to disposable income

The positive consumption response in ToTEM III is larger than that in ToTEM II over the first year after the shock and smaller afterward. These differences are primarily explained by the distinct structure of the models. In ToTEM III, borrowers are the ones demanding more consumption, mainly due to an increase in home equity. In contrast, in ToTEM II, about 95 percent of households increase consumption because of higher net financial wealth. Since the implicit estimated elasticities of consumption to home equity and net financial wealth are different and the paths of these variables in response to house price shocks differ as well, the consumption responses in the models are not the same.

The response of residential investment in ToTEM III is significantly lower than the one in ToTEM II. This result is also explained by the different structure of the models. In ToTEM III, house price shocks influence residential investment through their effect on household debt, while in ToTEM II they do so through their effect on net financial wealth. Driven primarily by the dynamics of consumption, the response of real GDP in ToTEM III is also greater over the first year after the shock and smaller afterward. This results in a greater response of the output gap over the first year after the shock in ToTEM III, which leads to the larger response of the short-term risk-free interest rate over the same horizon in that model. The response of year-over-year core inflation is relatively modest in both models, reflecting comparable ability of monetary policy to stabilize inflation.

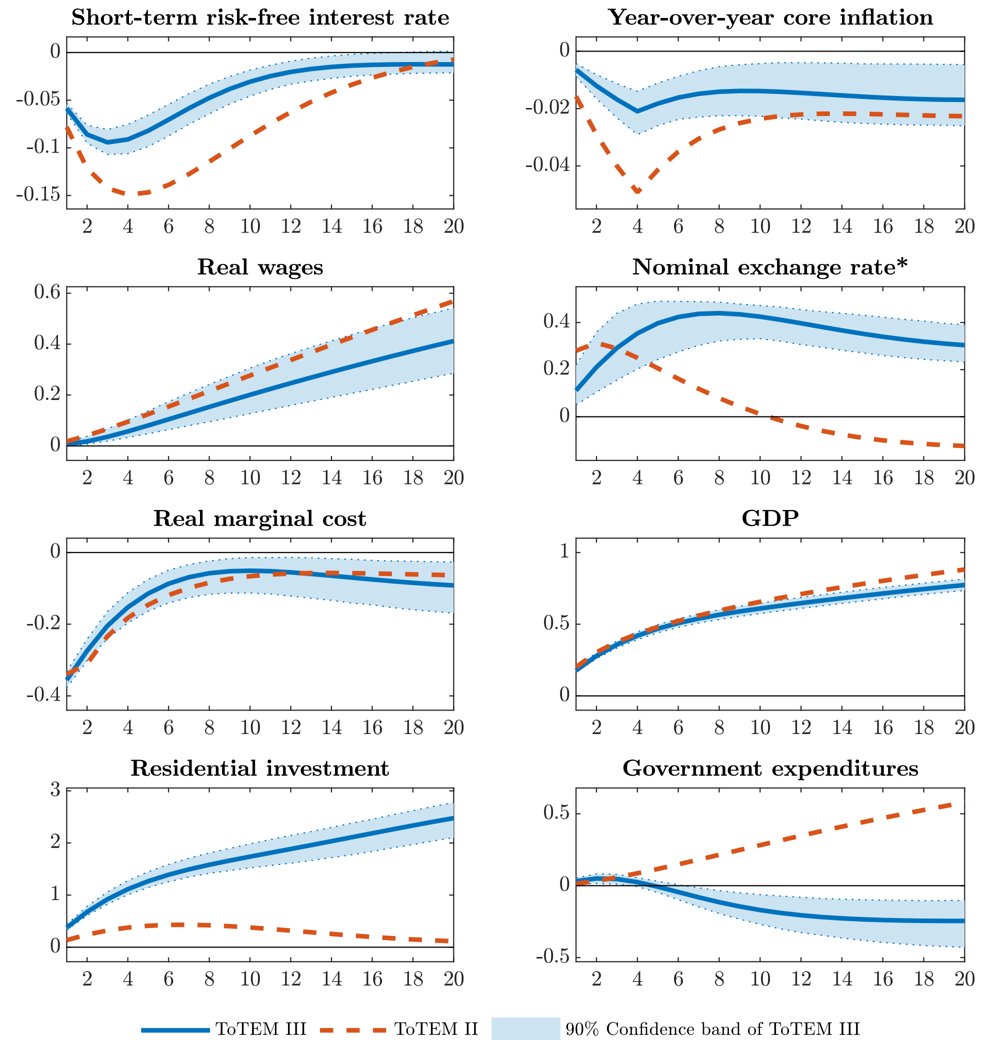
Positive house price shocks in ToTEM III cause an increase in home equity, which enhances the ability of borrowers to obtain new loans. As a result, the ratio of collateralized household debt to disposable income also rises in response to the positive house price shock. More specifically, a 10 percent permanent house price shock increases the ratio of collateralized household debt to disposable income by about 2 percentage points 20 quarters after the shock.

3.5 Productivity shock

After a one percent permanent productivity shock (see Figure 3.5), the response of the short-term risk-free interest rate is more muted in ToTEM III than in ToTEM II. The weaker response of core inflation in ToTEM III largely explains this result because of its flatter Phillips curve in the consumption sector. Real wages also increase less in response to the positive productivity shock in ToTEM III because of the flatter wage Phillips curve. In contrast, on average, the exchange rate responds significantly more in ToTEM III over the first five years after the shock. The exchange rate response is also more persistent in ToTEM III and exerts upward pressure mainly on import prices. On net, the smaller increase of real wages in ToTEM III mostly offsets the larger rise in import prices in this model, leaving the response of the real marginal cost in the consumption sector relatively unchanged with respect to the one observed in ToTEM II.

The GDP response in ToTEM III is very comparable to that under ToTEM II following the productivity shock. Regarding the components of GDP, the main difference between the two models manifests in the response of residential investment, which rises by a significantly higher amount in ToTEM III. This larger increase is partly driven by the differences in the response of government

Figure 3.5: Response to a one percent permanent productivity shock (%)



* A positive response indicates depreciation of the Canadian dollar

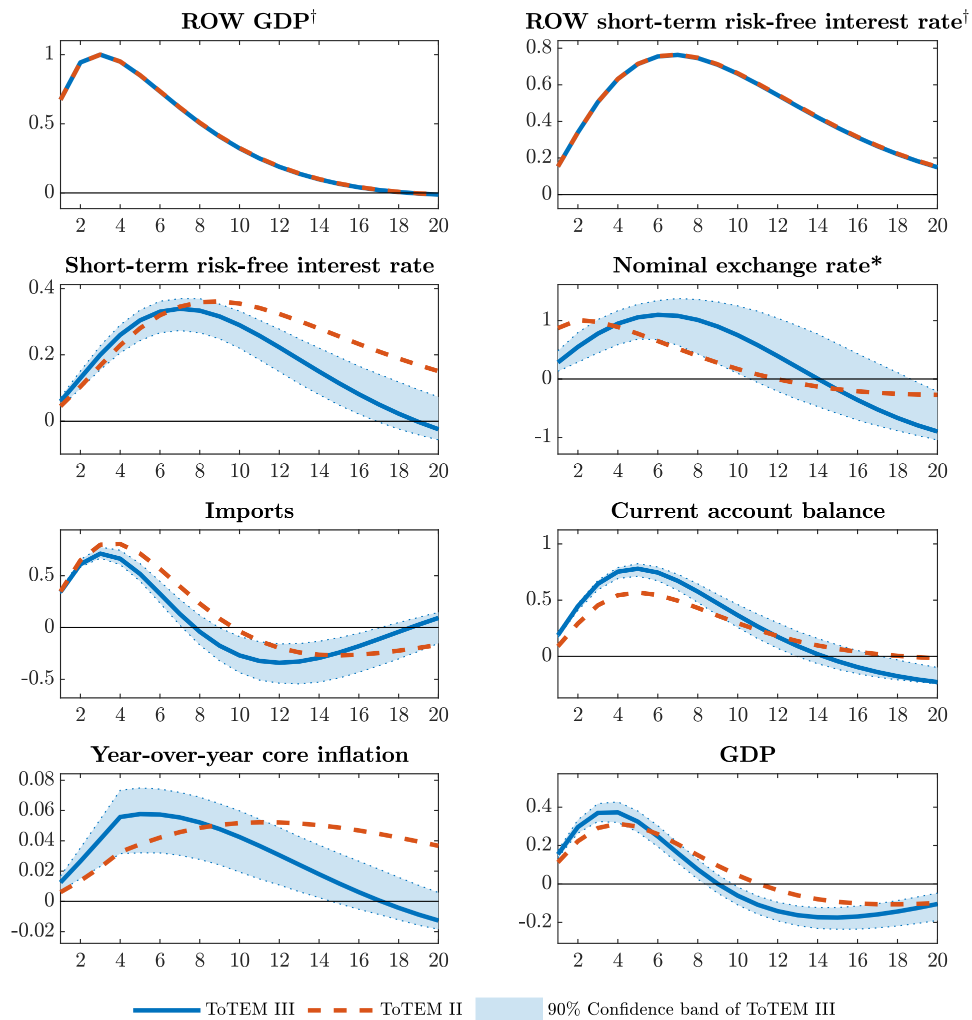
spending. In ToTEM III, government spending decreases in response to positive productivity shocks, leading to a reduction of expected taxes in the future. As a result, lower future taxes lead to an increase in the net present value of household income, thereby positively influencing residential investment. In contrast, in ToTEM II, government spending increases, which results in negative downward pressure on residential investment.

3.6 Foreign demand shock

In response to a shock that increases foreign GDP by one percent, the foreign short-term risk-free interest rate rises more than the domestic rate in both models, leading to a depreciation of the Canadian dollar. Given that the domestic short-term risk-free rate rises by about the same amount over the first five quarters in both models but less in ToTEM III from the sixth quarter onward (Figure 3.6), the Canadian dollar depreciates more in ToTEM III. This larger depreciation results in a smaller increase of imports in ToTEM III, which explains why the current account balance rises more in this model.

The response of core inflation is roughly similar over the first eight quarters after the shock. After the eighth quarter, year-over-year core inflation is higher in ToTEM II, which explains the stronger response of the short-term interest rate from the sixth quarter in ToTEM II. The response of GDP is fairly similar in both models.

Figure 3.6: Response to a one percent increase in ROW GDP (%)



* A positive response indicates depreciation of the Canadian dollar

† ROW stands for rest of the world.

3.7 World energy price shock

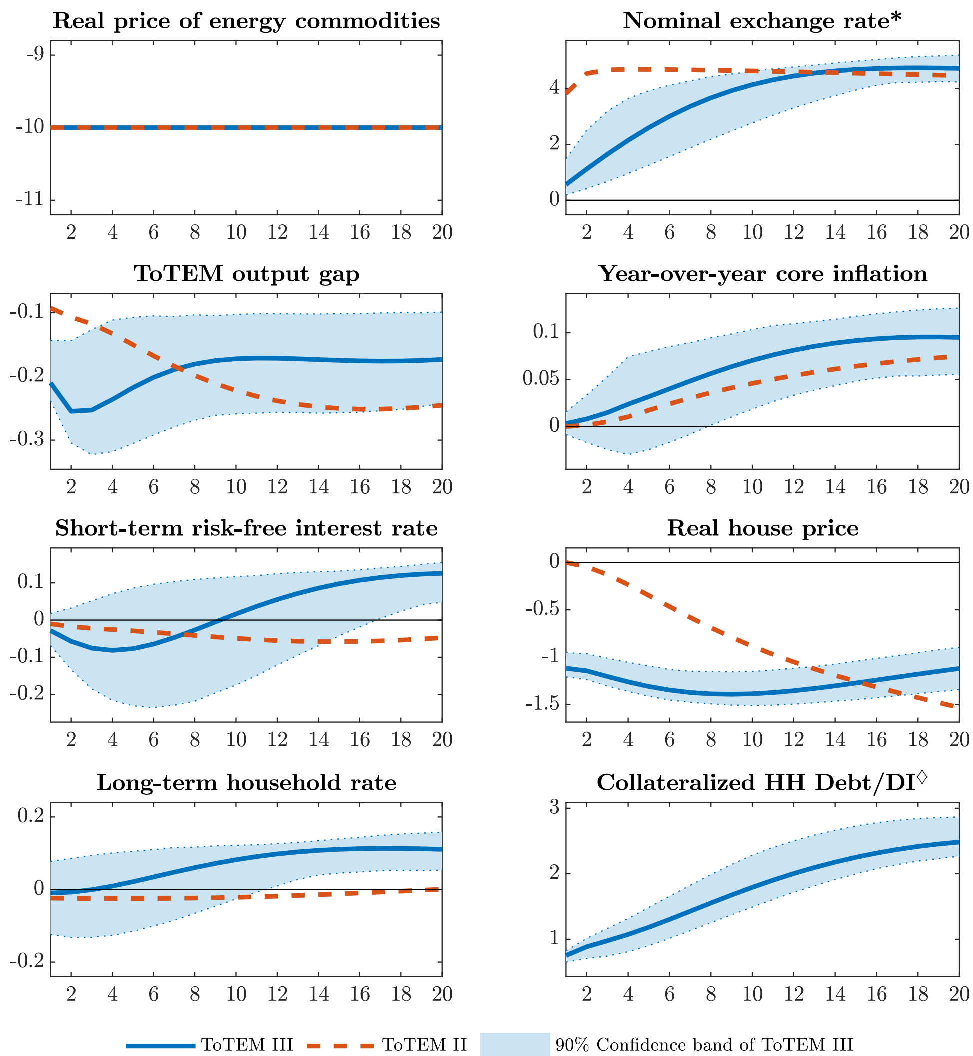
Figure 3.7 illustrates the effects of a permanent decline in the world price of energy commodities due to a supply disruption that arises in other countries that produce energy commodities, while Canada's supply capacity of energy commodities remains unchanged. Contrary to ToTEM II, the Canadian dollar gradually depreciates, reflecting higher intrinsic persistence in the UIP equation in ToTEM III. This partly explains why the output gap declines considerably more than in ToTEM II in the first year after the shock. In contrast, by the end of the fifth year after the shock, the output gap declines less in ToTEM III, reflecting the counter-cyclical nature of government spending in this model, which is not the case in ToTEM II.

As in ToTEM II, core inflation slightly rises in ToTEM III. Given the initial larger decline of the output gap in ToTEM III, the short-term risk-free interest rate initially declines more in ToTEM III than in ToTEM II during the first year and a half after the shock. By the end of the fifth year, the interest rate rises in ToTEM III, whereas it declines in ToTEM II. The interest rate response to a stronger expected deviation of inflation from target mainly explains this difference.

House prices decline in both models, but their dynamics differ. For example, the reduction of house prices on impact in ToTEM III is considerably greater than the one in ToTEM II. This difference, in part, reflects the higher long-term household rate that curtails housing demand in ToTEM III.

The decline in house prices in ToTEM III contributes to a reduction in collateralized household debt. Nonetheless, since disposable income declines considerably more than collateralized household debt, the ratio of collateralized household debt to disposable income increases by about 3 percentage points by the end of the fifth year after the shock.

Figure 3.7: Response to a 10 percent permanent decrease in world energy prices (%)



* A positive response indicates depreciation of the Canadian dollar

◊ Ratio of collateralized household debt to disposable income

Chapter 4

Applications

We consider two applications of ToTEM III for analyzing key issues for the Canadian economy:

- evaluating how the degree of household indebtedness affects the impact of higher interest rates on consumption in Canada
- assessing the effectiveness of regulating the LTV ratio in mitigating financial vulnerabilities

4.1 The impact of higher interest rates on consumption—the role of household indebtedness

Canadian household indebtedness has been considerably high, raising concerns over the response of the Canadian economy to increases in interest rates. To explore this issue, we compare how consumption and GDP respond to a 50 bp increase in the interest rate in two versions of ToTEM III that have different steady-state values for the ratio of collateralized household debt to disposable income.[24] In one version (baseline), the steady-state ratio is 90 percent, which is line with the historical average of the ratio. In the other version (higher-debt), the steady-state ratio is 140 percent, which is close to the average ratio observed over the period following the global financial crisis. Table 3.1 presents the details on the calibration of each steady state.

Overall, the impulse responses over the first eight quarters show that average consumption during this period falls 0.1 percentage points more in the higher-debt version than in the baseline version, amounting to an amplification effect on consumption of around 28 percent (Table 4.1). The higher steady state of collateralized household debt to disposable income also amplifies the effect on output by about 9 percent over the first eight quarters. The amplification effects are mainly driven by the variations in the responses of borrowers' consumption. Two factors explain the amplification.

[24]The 50 bp increase persists for eight quarters and then gradually dissipates.

First, in the higher-debt version, a larger withdrawal rate for home equity implies that borrower's consumption falls by more in response to declines in house prices. Second, higher indebtedness implies that interest payments rise more in the higher-debt version, which exert more downward pressure on the borrowers' consumption in the higher-debt version.

Table 4.1: Impact of increase in policy rate by 50 basis points

	Average impact over the first 8 quarters			Average impact over the first 12 quarters		
	Baseline	High debt	% difference	Baseline	High debt	% difference
Consumption	-0.38	-0.49	27.50	-0.44	-0.53	21.50
Output	-0.39	-0.43	8.72	-0.35	-0.37	7.95

4.2 Assessing the effectiveness of regulating the loan-to-value ratio

Two financial vulnerabilities that have raised financial stability concerns in Canada are:

- an elevated level of household indebtedness
- imbalances in the housing market

Different policy tools to mitigate these vulnerabilities have been discussed and analyzed.[25] Among these tools, one that has received a lot of attention is the regulatory LTV ratio. In this application, we assess the impact of permanently reducing the LTV ratio by five percentage points. Because we are interested in isolating the effects of regulating the LTV ratio, we calculate these under the assumption that policy rate and government spending do not react to the change in the LTV ratio.

Figure 4.1 shows the dynamics of key macroeconomic variables following the lower LTV ratio. A reduction of LTV implies a tightening of the credit constraint of borrowers, which results in lower demand for new loans. On impact, new loans decline by about eight percent. Because new loans represent a relatively small fraction of collateralized household debt, the impact on the latter is considerably smaller. Five years after the LTV regulation takes place, collateralized household debt is only one percent lower. Given that the reduction in disposable income is relatively small, the ratio of collateralized household debt to disposable income also falls by about one percentage point after five years.

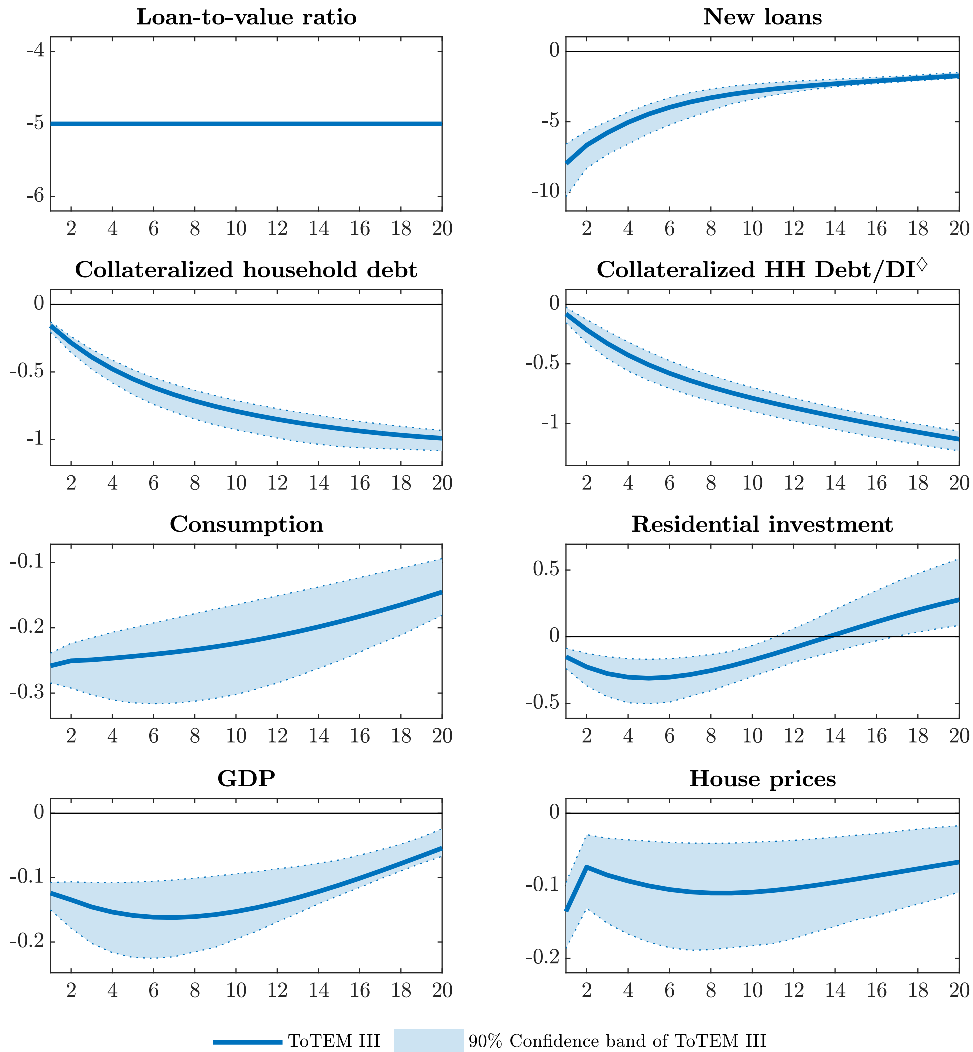
Borrowers' consumption and housing demand fall following the tightening of the LTV policy. In contrast, lifetime-income households respond to the reduced demand for loans by increasing

[25] See Alpanda, Cateau and Meh (2014, 2018).

their own demand for consumption and housing. Overall, both aggregate consumption and housing modestly fall. As a result, GDP also experiences a relatively small decline.

The net lower demand for housing leads to a decline in the demand for residential investment and land. This causes a reduction of their relative prices and, consequently, lower house prices. In particular, house prices fall by about 0.1 percent with such impact remaining persistent for the next five years.[26]

Figure 4.1: Response to a five percent permanent decrease in the loan-to-value ratio (%)



◇ Ratio of collateralized household debt to disposable income

[26]This impact is considerably smaller than that suggested in other models used at the Bank, such as MP2, reflecting different modelling approaches.

Chapter 5

Concluding remarks

In this report, we describe the most important changes in ToTEM III, which include modifications to the model's structure and improvements in the estimation of the model's parameters. ToTEM III's structure now incorporates borrowers and a more elaborate modelling of the housing market. These new features allow Bank staff to explore a broader range of policy questions. Moreover, the new structure of the model and the enhanced estimation of the model's parameters have contributed to a significant improvement of the forecasting performance of the model.

Since ToTEM III was completed in 2017, its new features have been used to address many relevant policy questions. For example, the presence of borrowers in the model has allowed staff to assess how higher household indebtedness affects the sensitivity of consumption to interest rates.[27] ToTEM III was also used to assess the impact of macroprudential policy, such as a reduction in the LTV ratio. In addition to these two applications discussed in this report, staff also used the model in 2018 to shed light on the sources of weakness in non-commodity exports.[28] More recently, staff used the model to compare the implications of alternative monetary policy frameworks on the evolution of household debt.[29]

Future work on ToTEM III's development will concentrate on incorporating policy-relevant advances of economic modelling as well as improving the model's forecasting ability. Among many potential extensions, some alternative forms of expectations, a more elaborate labour market as well as introducing a banking sector constitute our medium-term modelling effort.

[27]See Bank of Canada *Monetary Policy Report* (January 2018).
[28]See Dorich, Lepetyuk and Swarbrick (2018).
[29]See Dorich, Mendes and Zhang (forthcoming).

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Appendix A

Model

A.1 Households' problems and first-order conditions

A.1.1 Borrowers

The borrower households' problem is to choose their consumption, investment (as well as corresponding stock) in residential structures and land, new borrowing, stock of debt, and the effective mortgage rate they pay on new loans:

$$\left[C_t^{borr}, I_t^{RS,borr}, I_t^{LAND,borr}, RS_t^{borr}, LAND_t^{borr}, L_t^{borr}, D_t^{borr}, R_t^{M,borr}\right]$$

to maximize their expected utility.

The problem they face is the following:

$$
\begin{aligned}
&\mathcal{L}_t^{borr} = \\
&\quad E_t \sum_{s=t}^{\infty} \beta_{t,s}^{borr} \Bigg[\frac{\mu}{\mu-1} \left(C_s^{borr} - \xi C_{s-1}^{borr} \right)^{\frac{\mu-1}{\mu}} \exp\left(\frac{\eta(1-\mu_L)}{\mu_L(1+\eta)} \eta_L \int_0^1 \left(E_s N_s^{borr}(h) \right)^{\frac{1+\eta}{\eta}} dh \right) \\
&\quad + \zeta_s^{hl,borr} \frac{\mu_{HL}}{\mu_{HL}-1} \left(HL_s^{borr} - \xi_{HL} HL_{s-1}^{borr} \right)^{\frac{\mu_{HL}-1}{\mu_{HL}}} \Bigg] \\
&\quad - \Lambda_s^{borr} \Bigg((1+\tau_{c,s}) P_{c,s}^{tot} C_s^{borr} + P_s^{RS} I_s^{RS,borr} + P_s^{LAND} I_s^{LAND,borr} + \left(R_{s-1}^M + \kappa \right) D_{s-1}^{borr} \\
&\quad - \left((1-\tau_{w,s}) \int_0^1 W_s^{borr}(h) N_s^{borr}(h) dh + Tr_s^{borr} + Div_s^{borr} + L_s^{borr} \right) \Bigg) \\
&\quad - M_{B,s} \left(\alpha_{ltv} \left(P_s^{RS} I_s^{RS,borr} + P_s^{LAND} I_s^{LAND,borr} \right) + \alpha_{heloc} \left(P_s^H HL_s^{borr} - (1-\kappa) D_{s-1}^{borr} \right) - L_s^{borr} \right) \\
&\quad - \Omega_s^{d,borr} \left(D_s^{borr} - (1-\kappa) D_{s-1}^{borr} - L_s^{borr} \right) \\
&\quad - \Omega_s^{R,borr} \left(R_s^M D_s^{borr} - (1-\Phi)(1-\kappa) D_{s-1}^{borr} R_{s-1}^M - \left[L_s^{borr} + \Phi(1-\kappa) D_{s-1}^{borr} \right] R_s^L \right) \\
&\quad - \Psi_s^{RS,borr} \left(RS_{s+1}^{borr} - (1-\delta_{RS}) RS_s^{borr} - I_s^{RS,borr} + \frac{\chi_{RS}}{2} I_s^{RS,borr} \left(I_s^{RS,borr} / I_{s-1}^{RS,borr} - 1 \right)^2 \right) \\
&\quad - \Psi_s^{LAND,borr} \left(LAND_{s+1}^{borr} - LAND_s^{borr} - I_s^{LAND,borr} + \frac{\chi_{LAND}}{2} \left(I_s^{LAND,borr} - I_{s-1}^{LAND,borr} \right)^2 \right) \Big\}
\end{aligned}
\tag{A.1}
$$

To simplify deriving the first-order conditions, we re-state the process for the effective mortgage rate as follows. Recall that the effective interest rate on debt owed by borrowers is:

$$
R_t^{M,borr} = (1-\Phi) \left(1 - \frac{L_t^{borr}}{D_t^{borr}} \right) R_{t-1}^M + \left[\frac{L_t^{borr}}{D_t^{borr}} + \Phi \left(1 - \frac{L_t^{borr}}{D_t^{borr}} \right) \right] R_t^L \tag{A.2}
$$

or, equivalently,

$$
R_t^{M,borr} D_t^{borr} = (1-\Phi) \left(D_t^{borr} - L_t^{borr} \right) R_{t-1}^M + \left[L_t^{borr} + \Phi \left(D_t^{borr} - L_t^{borr} \right) \right] R_t^L. \tag{A.3}
$$

Using the process for stock of household debt, we can re-write the above as:

$$
R_t^{M,borr} D_t^{borr} = (1-\Phi)(1-\kappa) D_{t-1}^{borr} R_{t-1}^M + \left[L_t^{borr} + \Phi(1-\kappa) D_{t-1}^{borr} \right] R_t^L. \tag{A.4}
$$

First-order conditions

The first-order conditions of the borrowers' problem are as follows.

Consumption:

$$\lambda_t^{borr} = \Lambda_t^{borr}(1+\tau_{c,t})P_{c,t}^{tot} = \left(C_t^{borr} - \xi C_{t-1}^{borr}\right)^{-\frac{1}{\mu}} \exp\left(\frac{\eta(1-\mu)}{\mu(1+\eta)}\eta_L \int_0^1 \left(E_t N_t^{borr}(h)\right)^{1+\frac{1}{\eta}} dh\right) \quad \text{(A.5)}$$

Investment in residential structures:

$$\begin{aligned}\Lambda_t^{borr} p_t^{RS}\left(1-\mu_t^B \alpha_{ltv}\right) &= \beta_{borr}\Psi_{t+1}^{RS,borr}\chi_{RS}\left(\frac{I_{t+1}^{RS,borr}}{I_t^{RS,borr}}-1\right)\left(\frac{I_{t+1}^{RS,borr}}{I_t^{RS,borr}}\right)^2 \\ &+ \Psi_t^{RS,borr}\left(1-\left(\frac{\chi_{RS}}{2}\right)\left(\frac{I_t^{RS,borr}}{I_{t-1}^{RS,borr}}-1\right)^2 - \chi_{RS}\left(\frac{I_t^{RS,borr}}{I_{t-1}^{RS,borr}}-1\right)\left(\frac{I_t^{RS,borr}}{I_{t-1}^{RS,borr}}\right)\right) \quad \text{(A.6)}\end{aligned}$$

Investment in residential land:

$$\begin{aligned}\Lambda_t^{borr} p_t^{LAND}\left(1-\mu_t^B \alpha_{ltv}\right) &= \beta_{t,t+1}^{borr}\chi_{LAND}\left(I_{t+1}^{LAND,borr} - I_t^{LAND,borr}\right) \\ &+ \Psi^{LAND,borr}\left(1-\chi_{LAND}\left(I_t^{LAND,borr} - I_{t-1}^{LAND,borr}\right)\right) \quad \text{(A.7)}\end{aligned}$$

Residential structures:

$$\begin{aligned}\beta_{t,t+1}^{borr}\left(E_t HL_{t+1}^{borr} - \xi_{HL} HL_t^{borr}\right)^{\frac{-1}{\mu_{HL}}}\left(\frac{E_t HL_{t+1}^{borr} s_{RS}}{E_t RS_{t+1}^{borr}}\right)^{\frac{1}{\sigma_{HL}}} &= \Psi_t^{RS,borr} \\ - \beta_{t,t+1}^{borr}\left(\Psi_{t+1}^{RS,borr}(1-\delta_{RS}) + E_t\left(M_{t+1}^B \alpha_{heloc} P_t^H \left(\frac{E_t\ HL_{t+1}^{borr} s_{RS}}{E_t RS_{t+1}^{borr}}\right)^{\frac{1}{\sigma_{HL}}}\right)\right) & \quad \text{(A.8)}\end{aligned}$$

Residential land:

$$\begin{aligned}\beta_{t,t+1}^{borr}\left(E_t HL_{t+1}^{borr} - \xi_{HL} HL_t^{borr}\right)^{\frac{-1}{\mu_{HL}}}\left(\frac{E_t HL_{t+1}^{borr}(1-s_{RS})}{E_t LAND_{t+1}^{borr}}\right)^{\frac{1}{\sigma_{HL}}} &= \Psi_t^{LAND,borr} \\ - \beta_{t,t+1}^{borr}\left(E_t \Psi_{t+1}^{LAND,borr} + E_t\left(M_{t+1}^B \alpha_{heloc} P_t^H \left(\frac{HL_{t+1}^{borr}(1-s_{RS})}{LAND_{t+1}^{borr}}\right)^{\frac{1}{\sigma_{HL}}}\right)\right) & \quad \text{(A.9)}\end{aligned}$$

New loans:

$$\left(1-\mu_t^B\right) = \omega_t^{d,borr} + \omega_t^{R,borr} R_t^F, \quad \text{(A.10)}$$

where $\omega^{d,borr} = \Omega^{d,borr}/\Lambda^{borr}$, $\omega^{R,borr} = \Omega^{R,borr}/\Lambda^{borr}$ and $\mu^B = M_B/\Lambda^{borr}$.

Household debt:

$$\beta^{borr}_{t,t+1}\frac{E_t\lambda^{borr}_{t+1}}{\lambda^{borr}_t}\frac{1}{(1+\pi^c_{t+1})}\frac{(1+\tau_{c,t})p^{tot}_{c,t}}{(1+\tau_{c,t+1})p^{tot}_{c,t+1}}\Bigg(R^{M,borr}_t+\kappa+(1-\kappa)$$
$$\Bigg(\ \omega^{d,borr}_{t+1}+\omega^{R,borr}_{t+1}\Big((1-\Phi)R^{M,borr}_t+\Phi R^L_{t+1}\Big)\Big)+\mu^B_{t+1}\alpha_{heloc}(1-\kappa)\Bigg)=\omega^{d,borr}_t+\ \omega^{R,borr}_t R^{M,borr}_t \tag{A.11}$$

Effective mortgage rate:

$$\omega^{R,borr}_t=\beta_{borr}\frac{E_t\lambda^{borr}_{t+1}}{\lambda^{borr}_t}\frac{p^C_t}{p^C_{t+1}}\frac{1}{1+\pi^C_{t+1}}\Big(1+(1-\Phi)(1-\kappa)\omega^{R,borr}_{t+1}\Big) \tag{A.12}$$

A.1.2 Unrestricted lifetime-income households

The unrestricted lifetime-income households choose their consumption, investment in residential structures and land, the stock of residential structures and land, and their stock of inventories. In addition, they choose their holding of (domestic and ROW) short-term bonds and long-term domestic bonds, new lending to borrowers, stock of debt, and the effective mortgage rate they charge on the loans to borrowers:[30]

$$\Big[C^{ul}_t,I^{RS,ul}_t,I^{LAND,ul}_t,I^{INV,ul}_t,RS^{ul}_t,LAND^{ul}_t,INV^{ul}_t,B^{ul}_t,B^{*,ul}_t,B^{20,ul}_t,L^{ul}_t,D^{ul}_t,R^M_t\Big]$$

[30]Note that in this context, R^M_t represents the effective mortgage interest rate charged by lenders. In equilibrium, this would be equivalent to the mortgage rate faced by borrowers: $R^M_t=R^{M,borr}_t$.

to maximize their expected utility.

The problem they face is the following:

$$
\begin{aligned}
&\mathcal{L}_t^{ul} = \\
&E_t \sum_{s=t}^{\infty} \beta_{t,s}^{ul} \Bigg[\frac{\mu}{\mu-1} \left(C_s^{ul} - \xi C_{s-1}^{ul}\right)^{\frac{\mu-1}{\mu}} \exp\left(\frac{\eta(1-\mu_L)}{\mu_L(1+\eta)} \eta_L \int_0^1 \left(E_s N_s^{ul}(h)\right)^{\frac{1+\eta}{\eta}} dh \right) \\
&+ \zeta_s^{hl,ul} \frac{\mu_{HL}}{\mu_{HL}-1} \left(HL_s^{ul} - \xi_{HL} HL_{s-1}^{ul}\right)^{\frac{\mu_{HL}-1}{\mu_{HL}}} + \zeta_s^{inv,ul} \frac{\mu_{INV}}{\mu_{INV}-1} \left(INV_s^{ul} - \xi_{INV} INV_{s-1}^{ul}\right)^{\frac{\mu_{INV}-1}{\mu_{INV}}} \Bigg] \\
&- \Lambda_s^{ul} \Bigg((1+\tau s) P_{c,t}^{tot} C_s^{ul} + P_s^{RS} I_s^{RS,ul} + P_s^{LAND} I_s^{LAND,ul} + P_s^c I_s^{INV,ul} + \frac{B_s^{ul}}{(1+R_{RF,s})} + \\
&\frac{e_s B_s^{*,ul}}{(1+R_s{}^*)} + \frac{(1+\phi_s) B_s^{20,ul}}{(1+R_s^{20})^{20}} + L_s^{ul} - \\
&(B_{s-1}^{ul} + e_s B_{s-1}^{*,ul} + B_{s-20}^{20,ul} + \left(R_{s-1}^M + \kappa\right) D_{s-1}^{ul} + (1-\tau_{w,s}) \int_0^1 W_s^{ul}(h) N_s^{ul}(h) dh + Tr_s^{ul} + Div_s^{ul}) \Bigg) \\
&- \Omega_s^{d,ul} \left(D_s^{ul} - (1-\kappa) D_{s-1}^{ul} - L_s^{ul}\right) \\
&- \Omega_s^{R,ul} \left(R_s^M D_s^{ul} - (1-\Phi)(1-\kappa) D_{s-1}^{ul} R_{s-1}^M - \left[L_s^{ul} + \Phi(1-\kappa) D_{s-1}^{ul}\right] R_s^L\right) \\
&- \Psi_s^{RS,ul} \left(RS_{s+1}^{ul} - (1-\delta_{RS}) RS_s^{ul} - I_s^{RS,ul} + \frac{\chi_{RS}}{2} I_s^{RS,ul} \left(\frac{I_s^{RS,ul}}{I_{s-1}^{RS,ul}} - 1 \right)^2 \right) \\
&- \Psi_s^{LAND,ul} \left(LAND_{s+1}^{ul} - LAND_s^{ul} - I_s^{LAND,ul} + \frac{\chi_{LAND}}{2} \left(I_s^{LAND,ul} - I_{s-1}^{LAND,ul} \right)^2 \right) \\
&- \Psi_s^{INV,ul} \left(INV_{s+1}^{ul} - (1-\delta_{INV}) INV_s^{ul} - I_s^{INV,ul} + \frac{\chi_{INV}}{2} I_s^{INV,ul} \left(\frac{I_s^{INV,ul}}{I_{s-1}^{INV,ul}} - 1 \right)^2 \right)
\end{aligned}
\tag{A.13}
$$

First-order conditions

The first-order conditions of the unrestricted households' problem are as follows.

Consumption:

$$
\lambda_t^{ul} = \Lambda_t^{ul}((1+\tau_{c,t}) P_{c,t}^{tot} = (C_t^{ul} - \xi C_{t-1}^{ul})^{\frac{-1}{\mu}} \exp\left(\frac{\eta(1-\mu)}{\mu(1+\eta)} \eta_L \int_0^1 \left(E_t N_{ht}^{ul}\right)^{\frac{1+\eta}{\eta}} dh \right) \tag{A.14}
$$

Investment in residential structures:

$$\Lambda_t^{ul} P_t^{RS} = \beta_{t,t+1}^{ul} E_t \Psi_{t+1}^{RS,ul} \chi_{RS} \left(\frac{E_t I_{t+1}^{RS,ul}}{I_t^{RS,ul}} - 1 \right) \left(\frac{E_t I_{t+1}^{RS,ul}}{I_t^{RS,ul}} \right)^2 + \Psi_t^{RS,ul} \left(1 - \left(\frac{\chi_{RS}}{2} \right) \left(\frac{I_t^{RS,ul}}{I_{t-1}^{RS,ul}} - 1 \right)^2 - \chi_{RS} \left(\frac{I_t^{RS,ul}}{I_{t-1}^{RS,ul}} - 1 \right) \left(\frac{I_t^{RS,ul}}{I_{t-1}^{RS,ul}} \right) \right) \quad \text{(A.15)}$$

Investment in residential land:

$$\Lambda_t^{ul} P_t^{LAND} = \beta_{t,t+1}^{ul} \chi_{LAND} \left(E_t I_{t+1}^{LAND,ul} - I_t^{LAND,ul} \right) + \Psi_t^{LAND,ul} \left(1 - \chi_{LAND} \left(I_t^{LAND,ul} - I_{t-1}^{LAND,ul} \right) \right) \quad \text{(A.16)}$$

Residential structures:

$$\beta_{t,t+1}^{ul} \left(E_t HL_{t+1}^{ul} - \xi_{HL} HL_t^{ul} \right)^{\frac{-1}{\mu_{HL}}} \left(\frac{E_t HL_{t+1}^{ul} s_{RS}}{E_r RS_{t+1}^{ul}} \right)^{\frac{1}{\sigma_{HL}}} = \Psi_t^{RS,ul} - \beta_{t,t+1}^{ul} E_t \Psi_{t+1}^{RS,ul} (1 - \delta_{RS}) \quad \text{(A.17)}$$

Residential land:

$$\beta_{t,t+1}^{ul} \left(E_t HL_{t+1}^{ul} - \xi_{HL} HL_t^{ul} \right)^{\frac{-1}{\mu_{HL}}} \left(\frac{E_t HL_{t+1}^{ul} (1 - s_{RS})}{E_t \ LAND_{t+1}^{ul}} \right)^{\frac{1}{\sigma_{HL}}} = \Psi_t^{LAND,ul} - \beta_{t,t+1}^{ul} E_t \Psi_{t+1}^{LAND,ul} \quad \text{(A.18)}$$

Inventories:

$$\beta_{t,t+1}^{ul} \left(E_t INV_{t+1}^{ul} - \xi_{INV} INV_t^{ul} \right)^{\frac{-1}{\mu_{INV}}} = \Psi_t^{INV,ul} - \beta_{t,t+1}^{ul} E_t \Psi_{t+1}^{INV,ul} (1 - \delta_{INV}) \quad \text{(A.19)}$$

Investment in inventories:

$$\Lambda_t^{ul} P_t^c = \beta_{t,t+1}^{ul} E_t \Psi_{t+1}^{INV,ul} \chi_{INV} \left(\frac{E_t I_{t+1}^{INV,ul}}{I_t^{INV,ul}} - 1 \right) \left(\frac{E_t I_{t+1}^{INV,ul}}{I_t^{INV,ul}} \right)^2$$
$$+ \Psi_t^{INV,ul} \left(1 - \left(\frac{\chi_{INV}}{2} \right) \left(\frac{I_t^{RS,ul}}{I_{t-1}^{RS,ul}} - 1 \right)^2 - \chi_{INV} \left(\frac{I_t^{INV,ul}}{I_{t-1}^{INV,ul}} - 1 \right) \left(\frac{I_t^{INV,ul}}{I_{t-1}^{INV,ul}} \right) \right) \quad \text{(A.20)}$$

Domestic short-term bonds:

$$\frac{1}{(1+R_{RF,t})} = \beta_{t,t+1}^{ul} \frac{E_t \Lambda_{t+1}^{ul}}{\Lambda_t^{ul}} = \beta_{t,t+1}^{ul} \frac{1}{E_t(1+\pi_{t+1}^c)} \frac{(1+\tau_{c,t})}{(1+\tau_{c,t+1})} \frac{p_{c,t}^{tot}}{p_{c,t+1}^{tot}} \frac{E_t \lambda_{t+1}^{ul}}{\lambda_t^{ul}} \quad \text{(A.21)}$$

Foreign short-term bonds:

$$\frac{1}{(1+R_{RF,t})} \frac{e_t}{e_{t+1}} = \beta_{t,t+1}^{ul} \frac{E_t \Lambda_{t+1}^{ul}}{\Lambda_t^{ul}} = \beta_{t,t+1}^{ul} \frac{1}{(1+\pi_{t+1}^c)} \frac{(1+\tau_{c,t})}{(1+\tau_{c,t+1})} \frac{p_{c,t}^{tot}}{p_{c,t+1}^{tot}} \frac{E_t \lambda_{t+1}^{ul}}{\lambda_t^{ul}} \quad \text{(A.22)}$$

Domestic long-term bonds:

$$\frac{(1+\phi_t)}{(1+R_t^{20})^{20}} = \beta_{t,t+20}^{ul} \frac{E_t \Lambda_{t+20}^{ul}}{\Lambda_t^{ul}} = \beta_{t,t+20}^{ul} \frac{1}{E_t \prod_{j=1}^{20} (1+\pi_{t+j}^c)} \frac{(1+\tau_{c,t})}{(1+\tau_{c,t+20})} \frac{p_{c,t}^{tot}}{p_{c,t+20}^{tot}} \frac{E_t \ \ \lambda_{t+20}^{ul}}{\lambda_t^{ul}} \quad \text{(A.23)}$$

New loans:

$$1 = \omega_t^{d,ul} + \omega_t^{R,ul} R_t^L, \quad \text{(A.24)}$$

where $\omega^{d,ul} = \Omega^{d,ul}/\Lambda^{ul}$ and $\omega^{R,ul} = \Omega^{R,ul}/\Lambda^{ul}$.

Debt:

$$\beta_{t,t+1}^{ul} \frac{E_t \lambda_{t+1}^{ul}}{\lambda_t^{ul}} \frac{1}{(1+\pi_{t+1}^c)} \frac{(1+\tau_{c,t}) p_{c,t}^{tot}}{(1+\tau_{c,t+1}) p_{c,t+1}^{tot}} \left(R_t^M + \kappa + (1-\kappa) \left(\omega_{t+1}^{d,ul} + \omega_{t+1}^{R,ul} \left((1-\Phi) R_t^M + \Phi R_{t+1}^L \right) \right) \right)$$
$$= \omega_t^{d,ul} + \omega_t^{R,ul} R_t^M \quad \text{(A.25)}$$

Effective mortgage rate:

$$\omega_t^{R,ul} = \beta_{t,t+1}^{ul} \frac{E_t \lambda_{t+1}^{ul}}{\lambda_t^{ul}} \frac{1}{(1+\pi_{t+1}^c)} \frac{(1+\tau_{c,t}) p_{c,t}^{tot}}{(1+\tau_{c,t+1}) p_{c,t+1}^{tot}} \left(1 + (1-\Phi)(1-\kappa) \omega_{t+1}^{R,ul} \right) \quad \text{(A.26)}$$

A.1.3 Restricted lifetime-income households

Restricted lifetime-income households face similar problems to unrestricted households except on holding assets. Restricted households do not hold (foreign or domestic) short-term bonds. They only hold domestic long-term bonds and do not have to pay any transaction cost when they trade these bonds.

In each period, a representative restricted lifetime-income household seeks to maximize an intertemporal utility function identical to that of an unrestricted agent:

$$\begin{aligned} E_0 \sum_{t=0}^{\infty} \beta_t^l \Bigg[& \frac{\mu}{\mu-1} \left(C_t^{rl} - \xi C_{t-1}^{rl} \right)^{\frac{\mu-1}{\mu}} \exp\left(\frac{\eta(1-\mu_L)}{\mu_L(1+\eta)} \eta_L \int_0^1 \left(E_t N_t^{rl}(h) \right)^{\frac{1+\eta}{\eta}} dh \right) \\ & + \zeta_t^{hl,rl} \frac{\mu_{HL}}{\mu_{HL}-1} \left(HL_t^{rl} - \xi_{HL} HL_{t-1}^{rl} \right)^{\frac{\mu_{HL}-1}{\mu_{HL}}} \\ & + \zeta_t^{inv,rl} \frac{\mu_{INV}}{\mu_{INV}-1} \left(INV_t^{rl} - \xi_{INV} INV_{t-1}^{rl} \right)^{\frac{\mu_{INV}-1}{\mu_{INV}}} \Bigg] \end{aligned} \tag{A.27}$$

subject to the same constraints, except that the budget constraint of restricted households is given by:

$$\begin{aligned} & (1+\tau_{c,t}) P_{c,t}^{tot} C_t^{rl} + P_t^{RS} I_t^{RS,rl} + P_t^{LAND} I_t^{LAND,rl} + P_t^c I_t^{INV,rl} \\ & + \frac{B_t^{20,rl}}{(1+R_t^{20})^{20}} + L_t^{rl} = B_{t-20}^{20,rl} + M_t^{rl} \\ & + (1-\tau_{w,t}) \int_0^1 W_t^{rl}(h) N_t^{rl}(h) dh + Tr_t^{rl} + Div_t^{rl} \end{aligned} \tag{A.28}$$

.

The restricted lifetime-income household's problem is to choose:

$$\left[C_t^{rl}, I_t^{RS,rl}, I_t^{LAND,rl}, I_t^{INV,rl}, RS_t^{rl}, LAND_t^{rl}, INV_t^{ul}, B_t^{20,rl}, L_t^{rl}, D_t^{rl}, R_t^M \right]$$

to maximize:

$$
\begin{aligned}
&\mathcal{L}_t^{rl} = \\
&\quad E_t \sum_{s=t}^{\infty} \beta_{t,s}^{ul} \Bigg[\frac{\mu}{\mu-1} \left(C_s^{rl} - \xi C_{s-1}^{rl} \right)^{\frac{\mu-1}{\mu}} \exp\left(\frac{\eta(1-\mu_L)}{\mu_L(1+\eta)} \eta_L \int_0^1 \left(E_s N_s^{rl}(h) \right)^{\frac{1+\eta}{\eta}} dh \right) \\
&\quad + \zeta_s^{hl,rl} \frac{\mu_{HL}}{\mu_{HL}-1} \left(HL_s^{rl} - \xi_{HL} HL_{s-1}^{rl} \right)^{\frac{\mu_{HL}-1}{\mu_{HL}}} + \zeta_s^{inv,rl} \frac{\mu_{INV}}{\mu_{INV}-1} \left(INV_t^{rl} - \xi_{INV} INV_{t-1}^{rl} \right)^{\frac{\mu_{INV}-1}{\mu_{INV}}} \Bigg] \\
&\quad - \Lambda_s^{rl} \Bigg((1+\tau_{c,s}) P_{c,s}^{tot} C_s^{rl} + P_s^{RS} I_s^{RS,rl} + P_s^{LAND} I_s^{LAND,rl} + P_t^c I_s^{INV,rl} + \frac{B_s^{20,rl}}{(1+R_s^{20})^{20}} + L_s^{rl} \\
&\quad - \left(B_{s-20}^{20,rl} + \left(R_{s-1}^M + \kappa \right) D_{s-1}^{rl} + (1-\tau_{w,s}) \int_0^1 W_s^{rl}(h) N_s^{rl}(h) dh + Tr_s^{rl} + Div_s^{rl} \right) \Bigg) \\
&\quad - \Omega_s^{d,rl} \left(D_s^{rl} - (1-\kappa) D_{s-1}^{rl} - L_s^{rl} \right) \\
&\quad - \Omega_s^{R,rl} \left(R_s^M D_s^{rl} - (1-\Phi)(1-\kappa) D_{s-1}^{rl} R_{s-1}^M - \left[L_s^{rl} + \Phi(1-\kappa) D_{s-1}^{rl} \right] R_s^L \right) \\
&\quad - \Psi_s^{RS,rl} \left(RS_{s+1}^{rl} - (1-\delta_{RS}) RS_s^{rl} - I_s^{RS,rl} + \frac{\chi_{RS}}{2} I_s^{RS,rl} \left(\frac{I_s^{RS,rl}}{I_{s-1}^{RS,rl}} - 1 \right)^2 \right) \\
&\quad - \Psi_s^{LAND,rl} \left(LAND_{s+1}^{rl} - LAND_s^{rl} - I_s^{LAND,rl} + \frac{\chi_{LAND}}{2} \left(I_s^{LAND,rl} - I_{s-1}^{LAND,rl} \right)^2 \right) \\
&\quad - \Psi_s^{INV,rl} \left(INV_{s+1}^{rl} - (1-\delta_{INV}) INV_s^{rl} - I_s^{INV,rl} + \frac{\chi_{INV}}{2} I_s^{INV,rl} \left(\frac{I_s^{INV,rl}}{I_{s-1}^{INV,rl}} - 1 \right)^2 \right) \}
\end{aligned} \tag{A.29}
$$

First-order conditions

The first-order conditions for the restricted lifetime-income consumers' problems are identical to that of the unrestricted households' problem, except that there are no first-order conditions for (domestic or foreign) short-term bonds, and the first-order condition for long-term bonds satisfies:

$$
\frac{1}{(1+R_t^{20})^{20}} = \beta_{t,t+20}^{rl} \frac{E_t \Lambda_{t+20}^{rl}}{\Lambda_t^{rl}} = \beta_{t,t+20}^{rl} \frac{1}{E_t \prod_{j=1}^{20} (1+\pi_{t+j}^c)} \frac{(1+\tau_{c,t})}{(1+\tau_{c,t+20})} \frac{p_{c,t}^{tot}}{p_{c,t+20}^{tot}} \frac{E_t \ \lambda_{t+20}^{rl}}{\lambda_t^{rl}}. \tag{A.30}
$$

Appendix B

Estimation

B.1 Detrending of data

As with ToTEM II, we remove trends from the level data in ToTEM III and use detrended data in estimation. GDP and its components are detrended using an LRX filter (Laxton and Tetlow 1992). The trend of consumption is obtained as a residual of trends of GDP and other GDP components. Core CPI inflation is detrended with a constant trend of the Bank of Canada's annualized inflation target (at 2 percent throughout the sample period).

Most real price deflators are also detrended with an LRX filter. The most important exception is the real house price, whose trend is assumed proportional to trends of real disposable income and long-term interest rates, with weights equal to those used in Bauer's (2014) work on calculating the probability of a house price correction.

To detrend interest rates and household debt, we take the following approach.

Short-term risk-free rate: In 2000Q1 and after, the trend for the short-term risk-free rate is assumed to be equal to the Bank's estimate for the real neutral short-term risk-free rate in that period. In practice, this amounts to the trend's being set such that the annualized real short-term risk-free rate linearly declines from 2.75 percent in 2002Q1 to 1.75 percent in 2014Q1, where it remains through 2015Q4.

In periods before 2000Q1, the real trend short-term risk-free rate is calculated by estimating a univariate state space model of the rate on data starting in 1976Q1. The rate is assumed to have a permanent component (modelled as a random walk) and a temporary component (modelled as an AR(1)). The trend of the real risk-free rate is then assumed to equal the estimated permanent component.

To obtain the nominal trend for the risk-free rate, we simply add the constant value 2 percent which is the Bank of Canada's inflation target.

Long-term risk-free rate: A constant premium is added to the trend for the short-term risk-free rate to obtain the trend for the long-term risk-free rate, estimated from 1981Q1 to 2015Q4. There is no evidence of persistent changes in the risk-free term premium over the period from 1981Q1 to 2015Q4.

Market interest rates: For market interest rates (short- and long-term business rates and the long-term household rate), a similar approach is taken as with the real short-term risk-free rate. We estimate the premium of each rate using a univariate state-space model, which decomposes each premium into a permanent component (following a random walk without drift) and a temporary component (following an AR(1)). The trends of the premiums are then assumed equal to the estimated permanent components and added to the risk-free trends to get the trends for the rates.

Household debt to disposable income: To get a trend for the ratio of household debt to disposable income, we use the fact that the ratio DSR of debt-service costs to disposable income is constant in the steady state. Specifically, we assume:

$$\overline{DSR} = \frac{\overline{m_M}}{\overline{DI}} + \frac{\overline{m_{LOC}}}{\overline{DI}} \tag{B.1}$$

$$\frac{\overline{m_M}}{\overline{DI}} = \left(\frac{\overline{R_M}\left(1+\overline{R_M}\right)^n}{\left(1+\overline{R_M}\right)^n - 1}\right)\left(\frac{\overline{d_M}}{\overline{DI}}\right) \tag{B.2}$$

$$\frac{\overline{m_{LOC}}}{\overline{DI}} = \overline{R_{LOC}}\frac{\overline{d_{LOC}}}{\overline{DI}} \tag{B.3}$$

so that

$$\frac{\overline{d_M}}{\overline{DI}} = \frac{\overline{m_M}}{\overline{DI}}\left(\frac{\left(1+\overline{R_M}\right)^n - 1}{\overline{R_M}\left(1+\overline{R_M}\right)^n}\right) \tag{B.4}$$

$$\frac{\overline{d_{LOC}}}{\overline{DI}} = \frac{\overline{m_{LOC}}}{\overline{DI} \times \overline{R_{LOC}}}, \tag{B.5}$$

where DI is disposable income, m_M and m_{LOC} are total debt-service payments on mortgage debt and lines of credit, d_M is total residential mortgage debt with an average interest rate of R_M, and d_{LOC} is total balances on HELOCs with an average interest rate of R_{LOC}. Mortgage payments are set to conform to an amortization period of n quarters (in the results reported in Chapter 2, $n = 100$). As HELOCs are revolving lines, we assume that there are no re-payments of principal, so that debt service costs comprise of interest payments only.

The trend of the mortgage debt service ratio $\frac{\overline{m_M}}{\overline{DI}}$ is set to 7.06 percent, based on data from 1990Q1 to 2016Q4. Specifically, this trend is derived from the simple average of the mortgage (interest and principal) debt service ratio, multiplied in each period by the share of mortgage debt. This share is defined as a ratio of residential mortgage credit (reported by the Bank of Canada) to household mortgage debt (reported by Statistics Canada in the National Balance Sheet Accounts (NBSA)).

While in ToTEM III the effective mortgage rate R_M equals the rate on new mortgages R_F in steady state, the effective mortgage interest rate implied by NBSA debt service data is consistently higher than the long-term household rate and has a time-varying premium. To get the trend value for R_M, a trend of the premium of the NBSA-implied effective mortgage rate to the long-term household rate is estimated with a univariate state space model, as is done with the market premiums over risk-free rates. The trend of the premium is then assumed equal to the estimated permanent component and is added to the trend of the long-term household rate to get the trend for R_M. The R_M trend is then used to calculate the trend for $\overline{\frac{d_M}{DI}}$.

The actual and trend level for the HELOC interest rate R_{LOC} is assumed to be equal to the actual and trend of the short-term risk-free interest rate plus a fixed premium (of 225 basis points). Unlike $\frac{m_M}{DI}$, tests of structural breaks imply that the HELOC debt service ratio $\frac{m_{LOC}}{DI} = R_{LOC} \frac{d_{LOC}}{DI}$ shows evidence of permanent upward shifts around 1998 and 2004. As a result, the assumed time series of $\overline{\frac{m_{LOC}}{DI}}$ allows for smoothed versions of those breaks. It is then straightforward to calculate $\left(\overline{\frac{d_{LOC}}{DI}} \right)_t = \left(\overline{\frac{m_{LOC}}{DI}} \right)_t / \overline{R_{LOC,t}}$.

Appendix C

Adaptive Metropolis algorithm

We use the following steps in the adaptive Metropolis algorithm to estimate the domestic block of ToTEM III.

Optimization: Similar to the strategy in Dorich et al. (2013), an approximate maximum for the posterior probability density function (posterior mode) was found using a Covariance Matrix Adaptation Evolution Strategy (CMA-ES) algorithm using a diagonal covariance matrix (Ros and Hansen 2008).

Initializing covariance matrix: The initial covariance matrix was calculated using output from an adaptive simulated annealing algorithm based on that of Corana et al. (1987) and Goffe, Ferrier and Rogers (1994). We ran 20 rounds of 20 jumps per parameter starting at the obtained posterior mode, with temperature set at 1 throughout (ensuring that draws are accepted or rejected on the same basis as they would be in a Metropolis-Hastings algorithm). The step length for jumps of each parameter were gradually adjusted so as to guarantee acceptance of a jump 50 percent of the time. The vector of step lengths was then used to construct an initial, diagonal parameter covariance matrix.

Adaptive Metropolis-Hastings sampling: The adaptive Metropolis algorithm ran as follows. As a first pass, the Metropolis-Hastings algorithm was run for an initial N draws (in the results reported in Chapter 2, $N = 5000$) using a proposal distribution with covariance matrix $c_0^2 C_0$, with the scaling parameter c_0 set to yield an acceptance rate of about one-quarter. For the second round of N draws, the proposal distribution was re-set to $c_1^2 C_1$, where C_1 is the empirical covariance matrix of the initial N_0 draws and the scaling factor $c_1 = 2.38/\sqrt{k} \approx 0.2087$, given $k = 130$ estimated parameters. Afterward, we use the empirical covariance matrix for all $2N = 10000$ draws to obtain the proposal distribution for the third round of N draws.

Each subsequent round was obtained as follows. In round n, N draws were generated using a proposal distribution proportional to the empirical covariance matrix of the previous $(n-1)N$ draws. The empirical covariance matrix was then recalculated with all nN extant draws to obtain the proposal distribution for round $n + 1$. In other words, in each round, the proposal distribution used to generate draw t equals $\Sigma_{prop,n} = c_1^2 C_n$, where C_n satisfies:

$$C_n = cov(X_1, X_2, ...X_{(n-1)N}), (n-1)N + 1 < t \leq nN, n \geq 2, \tag{C.1}$$

and X_t is the vector of parameters of draw t. In all, 100 rounds of 5,000 draws were run, for a total of 500,000 draws.