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Allocative Efficiency and Aggregate Productivity Growth in Canada and the United States

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Abstract

This paper evaluates the contribution of allocative efficiency to the aggregate productivity growth in Canada and the US. In particular, we are interested in explaining two puzzling facts: 1) the slowdown in productivity growth during the 1970s and the 2000s in the US, and 2) the widening Canada-US productivity gap since the middle of the 1980s. We extend the framework of Oberfield (2013) to derive sufficient statistics for allocative efficiency and decompose aggregate productivity in an input-output economy à la Jones (2013). The lack of improvement in allocative efficiency can explain two-thirds of the US's productivity slowdown and more than one-third of the widening Canada-US productivity gap. The allocation of capital, rather than labor, was the main driver behind the overall movement in allocative efficiency. Resources allocated to service sectors were significantly lower than the optimal level. It improved markedly over time, especially in the US before the 2000s.

Bank topics: Economic models, Productivity
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1 Introduction

This paper addresses empirically the role of allocative efficiency in aggregate labor productivity growth. Figure 1 shows the two motivating facts of this paper. Panel A shows that real output per worker in the US slowed down significantly in the 1970s and the 2000s. The post-2000 slowdown in productivity growth is often discussed in the context of the "secular stagnation" debate and has attracted much attention from both academic and policy circles. At the same time, the labor productivity growth in Canada has fallen behind the US over the past three decades, resulting in a widening Canada-US productivity gap. In 1985, Canadian workers earned 83 percent of their American counterparts' earnings; in 2015, that number fell below 75 percent (Panel B).

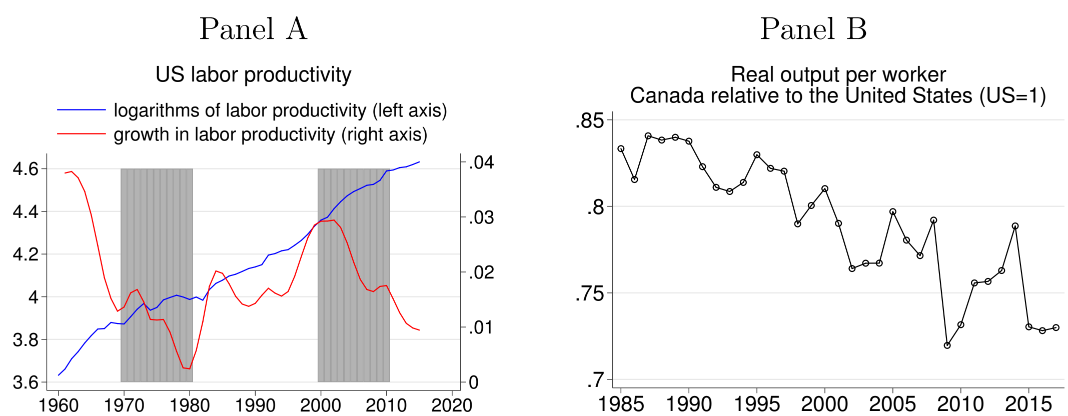


Figure 1: **Aggregate productivity in the US and Canada**

Source: BLS, FRED and PWT 9.1.

Notes: Panel A plots the logarithms and growth of real output per worker in the United States business sector. Panel B plots the real labor productivity of Canada relative to the US, where labor productivity is measured as real GDP per worker.

It is not surprising that we are not the first paper that documents and explains these facts about aggregate productivity (we discuss this literature in section 1.1). The first fact concerns the slowdown of productivity growth in the world productivity frontier (the US). The second fact is about whether a country is moving towards the frontier, and at what rate. These are two aspects of the global economic growth that determine our well-being in the long-run and form the basis of many policy proposals (Startz, 2020). Thus, understanding the forces behind these patterns is of first order importance. It also helps guide economic policies by distinguishing between the theories of the current and the future state of the macroeconomy. In this paper, we show that allocative efficiency—or more precisely, the lack of improvement in allocative efficiency—is the common factor behind all episodes of slow productivity growth presented in Figure 1.

Our paper performs an exercise that decomposes aggregate productivity growth into changes in allocative efficiency and a residual term that we interpret as changes in fundamental technology. The measure of allocative efficiency closely follows the notion and approach in the misallocation literature (Hsieh and Klenow, 2009). Using the language of Baqaee and Farhi (2019), our notion measures the *changes in allocative efficiency relative to the (production possibility) frontier* and differs from that in Basu and Fernald (2002) and Baqaee and Farhi (2019).

Our empirical exercises rely on the sector-level data in KLEMS and the national input-output tables collected by World Input-Output Table (WIOT), both the 2013 version. KLEMS and WIOT are harmonized datasets that, to a certain extent, allow us to make cross-country comparisons. In particular, similar to Oberfield (2013), our analysis relies on the assumption that the measurement errors in these datasets do not change systematically over time. We recognize the caveat of using these datasets. Namely, we restrict our study to the cross-sector allocations, and thus, our results are silent on the movement of within-sector allocative efficiency. However, without access to high-quality firm-level datasets that span the entire economy, the detailed sector-level data is the only data source that allows us to study allocation beyond a few sectors, such as manufacturing. As our results show, the allocative efficiency of the manufacturing sectors is significantly different from that of the entire economy. It would have been misleading to assume otherwise (see section 5.2).

Our theoretical framework builds on Oberfield (2013) and Jones (2013) and features a multi-sector value-added economy and an input-output economy à la Jones (2013). We compare the results in these two economies to evaluate the role of input-output linkages in measuring the *changes* in allocative efficiency. We characterize a planner's optimal allocation problem and derive sufficient statistics for measuring allocative efficiency. Intuitively, the sufficient statistics of allocative efficiency capture the deviation of the cross-sector allocation of production factors—capital, labor, and intermediate inputs—in the data from that under the optimal allocation. The approach we take is closely related to the decentralized approach used in Hsieh and Klenow (2009) and Jones (2013).

The main finding of this paper is that allocative efficiency played a quantitatively important role in explaining the productivity slowdown in the US and the widening Canada-US productivity gap.

In the US, allocative efficiency improved gradually over time. From 1960 to 2007, allocative efficiency grew by approximately 18 percent and contributed to approximately 20 percent of the aggregate productivity growth. The two decades of slow productivity growth—the 1970s and the 2000s—were also the two decades with no improvement of allocative efficiency (2000s) or even a deterioration of allocative efficiency (1970s). We compare the productivity growth in the data and that under the optimal allocation to evaluate the quantitative role of allocative efficiency in causing the slowdown of productivity growth. The lack of improvement in allocative efficiency can explain approximately two-thirds of the slowdown in productivity growth.

Compared with the US, allocative efficiency had been stagnant in Canada. It stayed almost unchanged from 1985 to the early 2000s and decreased afterward. Depending on the specification, the lack of improvement in allocative efficiency compared to the US can account for 35 to 62.5 percent of the widening productivity gap between the two countries.

We also study which sectors and production factors contributed the most to the movement in allocative efficiency. Capital, not labor or intermediate inputs, was the main driver behind the movements in allocative efficiency in both countries. On average, service sector allocation had been further away from the optimal level than the rest of the economy. The resources allocated in service sectors were significantly below the optimal level in the early years of the sample. They are much closer to the optimal level now. The magnitude of the improvement was larger in the US compared with Canada, especially before the 2000s, which contributed to the aggregate difference between the two countries.

Measurement of allocative efficiency is a challenging task, and it is often carried out under certain explicit or implicit assumptions. In the following paragraphs, we discuss two key measurement issues in our paper and how they affect our results.

First, one of the biggest challenges in comparing allocative efficiency across countries and over time is specification errors. In the main text, we employ two specifications to obtain output elasticities in the production functions. Specification 1 assumes that the factor shares are undistorted, corresponding to the form of sectoral level wedges in Jones (2013). In specification 2, we allow distorted factor shares each year, but we assume that the factor shares are, on average, undistorted over time (Oberfield, 2013). As a robustness check, we consider two more specifications. We find our results are robust to the different specifications.

Second, we explore how the specifications affect the measurement of allocative efficiency. Recall that allocative efficiency captures the deviation of cross-sector allocation of production factors in the data from the optimal allocation. Under specification 1, we show that the crosssector allocation in the data, when measured using the expenditure of the factors, is always optimal. Therefore, only when the cross-sector factor allocation measured using quantity differs from that measured using expenditure can we identify misallocation in the data. In other words, the type of misallocation we can uncover under specification 1 comes from the dispersion of implied prices across sectors. This finding, in a very extreme way, reinforces the common understanding in the misallocation literature that we need both quantity and expenditure to measure misallocation properly. In the data, intermediate inputs are only measured using expenditure (nominal value). Hence, we cannot evaluate their allocative efficiency under specification 1. Under specification 2, the allocation of expenditure in the data is not optimal. However, we show that the misallocation of intermediate inputs under specification 2 is quantitatively very small.

Third, we introduced input-output linkages into the model based on the prior that they might change the measured allocative efficiency. Our empirical result shows that the *level* of allocative efficiency is indeed lower in the input-output economy than the value-added economy. However, the *changes* of allocative efficiency are almost identical in these two economies for the period when the input-output information is available. As a result, input-output linkages do not alter the decomposition of the aggregate productivity growth by much because it is the change, not the level, that matters for growth. In section 2.3, we provide some intuitions on why and how the input-output structure matters for measurement.

1.1 Literature review

The literature has studied the US productivity slowdown in the 1970s extensively. Vandenbroucke (2019) summarizes that the arguments in the literature include the rise in oil price, measurement errors, information technology, and demography. We show in this paper that allocative efficiency played an important role in explaining the 1970 productivity slowdown, which, to our knowledge, has not been discussed in the context of the productivity slowdown in the 1970s.

The post-2000 slowdown in productivity growth in advanced economies is a topic that attracted much attention from the academic and policy circles (see Jones, 2017 for a summary). Byrne et al. (2016) and Syverson (2017) explore whether the increasing difficulty in measurement can explain the productivity slowdown. Byrne et al. (2016) find that the slowdown in TFP is not much affected after several adjustments to IT-related hardware, software, and services. Syverson (2017) also finds that the mismeasurement hypothesis cannot explain a substantial part of the productivity slowdown. Cette et al. (2016) and Aum et al. (2018) study the role played by IT technology. Cette et al. (2016) show that the easing in utilization and adaption of IT technology contributed to the slowdown in productivity growth before the Great Recession in the US. Aum et al. (2018) instead argue when productivity grows at different rates across sectors or occupations, there will be a slowdown in aggregate productivity due to the complementarity between the sectors and occupations. They show that routinization and computerization together explain the aggregate productivity slowdown since the 2000s. Finally, the last strand of literature explores the role played by misallocation. Thus they are the closest to our paper. Several papers that study southern European countries such as Spain and Italy emphasize the impact of an increase in misallocation on the sluggish productivity growth (Gopinath et al., 2017, Cette et al., 2016 and Calligaris et al., 2018). Using the Longitudinal Business Dynamics (LBD) dataset, Decker et al. (2017) show that impaired growth in allocative efficiency can account for the bulk of the productivity slowdown from the late 1990s to the mid-2000s in the US. Like us, Cette et al. (2016) use the KLEMS data, but they do not directly measure allocative efficiency, nor do they take into account input-output linkages. Gopinath et al. (2017) use the Orbis firm-level data and follow a similar framework as Hsieh and Klenow (2009) to estimate the within-sector misallocation. Calligaris et al. (2018) decompose misallocation into a within- and a between-sector component. They show that it is the within-sector component that contributed the most to the increase in misallocation. Unlike these two papers, our analysis is based on sector-level data and, therefore, identifies between-sector misallocation. The LBD data used in Decker et al. (2017) is firm-level data covering all sectors in the US economy, and they follow the dynamic Olley Pakes method of Melitz and Polanec (2015) to decompose aggregate labor productivity, which differs from our framework. The data we use also allow us to evaluate the allocation of all production factors, whereas Decker et al. (2017) focus on labor. One of our paper's main findings, which points to the slowdown in the improvement of allocation in the service sectors, is also new to the literature.

The productivity gap between Canada and the US is also a well-known and well-studied fact (see Sharpe, 2003 for a review). Leung et al. (2008) and Tang (2014) both point to the fact that the share of small-firm workers in Canada is larger than in the US and that small firms are less productive than large ones. Ranasinghe (2017) uses a quantitative model to show that differences in innovation costs account for the majority of firm size and productivity differences between Canada and the US. However, the widening productivity gap—the focus of our paper—is much less explored in the literature. Bernstein et al. (2002) and Tang (2014) are the notable exceptions. However, none of these papers explored the role played by allocative efficiency.

In addition to the papers already mentioned, several other papers of the misallocation literature are closely related to ours. Caliendo et al. (2018) build a model of the world economy with both domestic and international input-output linkages and derive sufficient statistics for the loss from both internal and external distortions. Osotimehin and Popov (2018) study how the measured allocative efficiency is affected by the substitution of elasticity between intermediate inputs in an input-output economy. The production system and how we treat markups are the two notable differences between these two papers and our paper. Both Caliendo et al. (2018) and Osotimehin and Popov (2018) feature a general CES production function, whereas we opt for a Cobb-Douglas production system as the benchmark. Epifani and Gancia (2011) and Osotimehin and Popov (2018) show that the level of allocative efficiency decreases with the degree of substitution between inputs. In section 5.1, we consider a CES production system and confirm the findings of these two papers. Also, a higher degree of substitutability leads to larger changes in measured allocative efficiency. Second, since KLEMS data do not separately report profit from capital income, our benchmark model assumes perfect competition and abstracts from markups. This is different from both Caliendo et al. (2018) and Osotimehin and Popov (2018), where they make assumptions to separate markups from capital income in the data.

Lastly, our paper is closely related to Basu and Fernald (2002) and Baqaee and Farhi (2019) in that both decompose aggregate productivity growth into technology and allocation. These papers have different notions of changes in allocative efficiency, as pointed out in Baqaee and Farhi (2019). Our notion measures the distance from the production possibility frontier. It differs from Baqaee and Farhi (2019) when there are changes in technology and factor supplies. Frameworks in Baqaee and Farhi (2019) and Basu and Fernald (2002) allow the flexibility to apply the theory without knowing the production functions and the form of the distortions, but they rely on linearization to obtain the results. Our approach requires the specification of the production system, but the result does not rely on linearization.

The structure of the paper is as follows: in section 2, we characterize the optimal allocation and the decomposition framework. Sections 3 and 4 apply the framework to Canada and the US. Section 5 presents several robustness checks. Section 6 concludes.

2 Theoretical framework

This section presents the theoretical framework in three steps. First, we characterize the optimal allocation across sectors as a planner's problem. Second, we derive sufficient statistics to measure allocative efficiency using the optimal allocation from step 1. Last, we show a simple framework that decomposes the aggregate productivity growth using the measured allocative efficiency from step 2. In section 2.1 we consider an economy without input-output linkages. We then turn to the economy with input-output linkages in section 2.2. In the literature, these are also often referred to as the value-added economy and the input-output economy, respectively.

2.1 Value-added economy

There are N sectors in the economy ($i = \{1, ..., N\}$). In year t, each sector produces a good $Y_{i,t}$ using capital, labor and a Cobb-Douglas production function

$$Y_{i,t} = A_{i,t} K_{i,t}^{\alpha_{i,t}} L_{i,t}^{1-\alpha_{i,t}},$$

where $A_{i,t}$ is the sectoral productivity.

There is one final good Y_t, which is produced by aggregating all sectoral goods, such that

$$Y_t = \prod_{i=1}^{N} Y_{i,t}^{\theta_{i,t}},$$

in which $\sum_i \theta_{i,t} = 1$.

2.1.1 Planner's problem

The planner's problem is to allocate aggregate capital K_t and labor L_t into the N sectors to maximize the output of final good Y_t, such that,

$$\max \ Y_t = \prod_{i=1}^{N} Y_{i,t}^{\theta_{i,t}}, \text{s.t. } Y_{i,t} = A_{i,t} K_{i,t}^{\alpha_{i,t}} L_{i,t}^{1-\alpha_{i,t}}, \ \sum_i K_{i,t} = K_t, \ \sum_i L_{i,t} = L_t$$

The following proposition characterizes the optimal allocation across sectors and the optimal output.

Proposition 1. *The optimal allocation of capital and labor in this economy is such that* $K_{i,t} = \chi_{i,t}^{k*} K_t$ *and* $L_{i,t} = \chi_{i,t}^{l*} L_t$*, where* $\chi_{i,t}^{k*} = \frac{\theta_{i,t}\alpha_{i,t}}{\sum_i \theta_{i,t}\alpha_{i,t}}$ *and* $\chi_{i,t}^{l*} = \frac{\theta_{i,t}(1-\alpha_{i.t})}{\sum_i \theta_{i,t}(1-\alpha_{i,t})}$.

Proof. See Appendix C.1. □

Under the optimal allocation, aggregate capital and labor are allocated to each sector according to the optimal sectoral shares $\chi_{i,t}^{k*}$ and $\chi_{i,t}^{l*}$. Intuitively, the optimal sectoral shares reflect the relative importance of sector i's capital and labor in the production of the final good ($\alpha_i\theta_i$ and $(1-\alpha_i)\theta_i$, respectively).

Allocative efficiency We define allocative efficiency $\mathbf{E}_t$ as the ratio between output in the data (Y_t) and output under the optimal allocation (Y_t^*),

$$\mathbf{E}_t = \frac{Y_t}{Y_t^*}.$$

It can be shown, using proposition 1, that

$$\mathbf{E}_t = \prod_{i=1}^{N} [(\frac{\chi_{i,t}^{k}}{\chi_{i,t}^{k*}})^{\alpha_{i,t}} (\frac{\chi_{i,t}^{l}}{\chi_{i,t}^{l*}})^{1-\alpha_{i,t}}]^{\theta_{i,t}}, \tag{1}$$

where $\chi_{i,t}^{k} = \frac{K_{i,t}}{K_t}$ and $\chi_{i,t}^{l} = \frac{L_{i,t}}{L_t}$ are the sector i's capital and labor as a share of aggregate K_t and L_t in the data, respectively. Intuitively, $(\frac{\chi_{i,t}^{k}}{\chi_{i,t}^{k*}})^{\alpha_{i,t}} (\frac{\chi_{i,t}^{l}}{\chi_{i,t}^{l*}})^{1-\alpha_{i,t}}$ measures sector i's allocative efficiency, which is the deviations of data allocation from the optimal allocation in sector i. The aggregate allocative efficiency $\mathbf{E}_t$ is then simply the weighted geometric average of the sectoral allocative efficiency with the weights being θ_i—the share of good i in the final good production.

2.2 Input-output economy

Similar to the previous section, each sector produces good $Q_{i,t}$ using capital, labor, domestic and imported intermediate goods, such that

$$Q_{i,t} = A_{i,t}(K_{i,t}^{\alpha_{i,t}} L_{i,t}^{1-\alpha_{i,t}})^{1-\sigma_{i,t}-\lambda_{i,t}} (\prod_{j=1}^{N} d_{ij,t}^{\sigma_{ij,t}}) (\prod_{j=1}^{N} m_{ij,t}^{\lambda_{ij,t}}),$$

where $d_{ij,t}$ is the domestic intermediate good j used by sector i, $m_{ij,t}$ is the imported intermediate good j used by sector i, $\sigma_{i,t} = \sum_{j=1}^{N} \sigma_{ij,t}$, and $\lambda_{i,t} = \sum_{j=1}^{N} \lambda_{ij,t}$.

There is a final good in the economy, produced by aggregating over the sectoral goods,

$$Y_t = \prod_{i} Y_{i,t}^{\theta_{i,t}},$$

where $\sum_{i=1}^{N} \theta_{i,t} = 1$.

The resource constraint on the sectoral good i therefore can be written as

$$Q_{i,t} = Y_{i,t} + \sum_{j=1}^{N} d_{ji,t},$$

and the total expenditure on imported goods is

$$X_t = \sum_{i=1}^{N}\sum_{j=1}^{N} \bar{P}_{j,t} m_{ij,t},$$

where $\bar{P}_{j,t}$ is the price of imported intermediate good j relative to the final good.[1]

2.2.1 Planner's problem

The planner's problem is to allocate aggregate capital K_t, aggregate labor L_t, intermediate goods $Y_{i,t}$, $d_{ij,t}$ and $m_{ij,t}$ such that the aggregate output net of imports $(Y - X)$ is maximized,

$$\begin{aligned}
\max_{\{K_{i,t},L_{i,t},d_{ij,t},m_{ij,t}\}_{i,j=1}^{N}} \quad & Y_t - X_t = \prod_i Y_{i,t}^{\theta_{i,t}} - \sum_{i=1}^{N}\sum_{j=1}^{N} \bar{P}_{j,t} m_{ij,t} \\
\text{s.t.} \quad & Q_{i,t} = A_{i,t}(K_{i,t}^{\alpha_{i,t}} L_{i,t}^{1-\alpha_{i,t}})^{1-\sigma_{i,t}-\lambda_{i,t}} (\prod_{j=1}^{N} d_{ij,t}^{\sigma_{ij,t}})(\prod_{j=1}^{N} m_{ij,t}^{\lambda_{ij,t}}), \\
& Q_{i,t} = Y_{i,t} + \sum_{j=1}^{N} d_{ji,t}, \ \sum_i K_{i,t} = K_t, \ \sum_i L_{i,t} = L_t.
\end{aligned}$$

The optimal allocation is characterized by the following proposition:

Proposition 2. *The optimal allocation of capital, labor and intermediate goods in the economy can be characterized using optimal sectoral shares* ($\chi_{i,t}^{k*}$, $\chi_{i,t}^{l*}$, $\gamma_{ij,t}^{*}$, $\chi_{i,t}^{y*}$), *such that* $K_{i,t}^* = \chi_{i,t}^{k*} K_t$, $L_{i,t}^* = \chi_{i,t}^{l*} L_t$, $d_{ij,t}^* = \gamma_{ij,t}^* Q_{j,t}^*$, $Y_{j,t}^* = \chi_{j,t}^{y*} Q_{j,t}^*$, *and* $m_{ij,t}^* = (\frac{\theta_{i,t}\lambda_{ij,t}}{\chi_{i,t}^{y*}}) \frac{Y_t^*}{\bar{P}_{j,t}}$ *such that*

1. $\chi_{i,t}^{k*} = \frac{\theta_{i,t}\alpha_{i,t}(1-\sigma_{i,t}-\lambda_{i,t})}{1-\sum_j \gamma_{ji,t}^*} / \sum_s \frac{\theta_{s,t}\alpha_{s,t}(1-\sigma_{s,t}-\lambda_{s,t})}{1-\sum_j \gamma_{js,t}^*}$, $\forall i \in \{1, ..., N\}$.

2. $\chi_{i,t}^{l*} = \frac{\theta_{i,t}(1-\alpha_{i,t})(1-\sigma_{i,t}-\lambda_{i,t})}{1-\sum_{j,t} \gamma_{ji,t}^*} / \sum_s \frac{\theta_{s,t}(1-\alpha_{s,t})(1-\sigma_{s,t}-\lambda_{s,t})}{1-\sum_j \gamma_{js,t}^*}$, $\forall i \in \{1, ..., N\}$.

[1] As shown in the next section, our measure of allocative efficiency relies only on the expenditure on imported intermediate goods, not their prices.

3. $\{\chi^{y*}_{i,t}\}_{i=1}^{N}$ *solve the system of equations*

$$\frac{1}{\chi^{y}_{i,t}} = 1 + \frac{1}{\theta_{i,t}} \sum_{s} (\frac{\theta_{s,t}}{\chi^{y}_{s,t}} \sigma_{si,t}), i \in \{1, ..., N\}, \tag{2}$$

and

$$\gamma^{*}_{ij,t} = \frac{\theta_{i,t}\chi^{y*}_{j,t}}{\theta_{j,t}\chi^{y*}_{i,t}} \sigma_{ij,t}. \tag{3}$$

4. $\{Q^{*}_{i,t}\}_{i=1}^{N}$ *solve for the system of equations*

$$Q_{i,t} = \chi_{Qi,t}(\prod_{s=1}^{N} Q_{s,t}^{\sigma_{is,t}+\lambda_{i,t}\theta_{s,t}}), i \in \{1, ..., N\},$$

where $\chi_{Qi,t} = A_{i,t}[(\chi^{k*}_{i,t}K_t)^{\alpha_{i,t}}(\chi^{l*}_{i,t}L_t)^{1-\alpha_{i,t}}]^{1-\sigma_{i,t}-\lambda_{i,t}}(\prod_{j=1}^{N} \gamma_{ij,t}^{*\sigma_{ij,t}})[\theta_{i,t}\prod_{s}(\frac{\chi^{y*}_{s,t}}{\chi^{y*}_{i,t}})^{\theta_{s,t}}]^{\lambda_{i,t}}$ $\prod_{j=1}^{N}(\frac{\lambda_{ij,t}}{\bar{P}_{j,t}})^{\lambda_{ij,t}}$.

Proof. See Appendix C.2. Note that the optimal shares only depend on the output elasticities in the production functions. □

Allocative efficiency We define the allocative efficiency as the ratio between the output net of imports in the data and that under the optimal allocation, such that

$$\mathbf{E}_t = \frac{Y_t - X_t}{Y^{*}_t - X^{*}_t}.$$

It can be shown using proposition 2 that $\mathbf{E}_t$ can be written as a product of allocative efficiency of capital, labor, domestic and imported intermediate goods, and intermediate goods used for final good production, such that

$$\mathbf{E}_t = E^{kl}_t \cdot E^{d}_t \cdot E^{m}_t \cdot E^{y}_t, \tag{4}$$

- $E^{kl}_t = \prod_{i=1}^{N}(((\frac{\chi^{k}_{i,t}}{\chi^{k*}_{i,t}})^{\alpha_{i,t}}(\frac{\chi^{l}_{i,t}}{\chi^{l*}_{i,t}})^{1-\alpha_{i,t}})^{1-\sigma_{i,t}-\lambda_{i,t}})^{\sum_n \theta_{n,t}C_{ni,t}}$ is the allocative efficiency of capital and labor.

- $E^d_t = \prod_{i=1}^N (\prod_{j=1}^N (\frac{\gamma_{ij,t}}{\gamma^*_{ij,t}})^{\sigma_{ij,t}})^{\sum_n \theta_{n,t} C_{ni,t}}$ is the allocative efficiency of domestic intermediate goods.
- $E^m_t = \frac{1-\sum_{n=1}^N \frac{\theta_{n,t}\lambda_{n,t}}{\chi^y_{n,t}}}{1-\sum_{n=1}^N \frac{\theta_{n,t}\lambda_{n,t}}{\chi^{y*}_{n,t}}}$ is the allocative efficiency of imported intermediate goods.
- $E^y_t = \prod_{n=1}^N (\frac{\chi^y_{n,t}}{\chi^{y*}_{n,t}})^{\theta_{n,t}} \prod_{i=1}^N (\frac{\prod_s (\frac{\chi^y_{s,t}}{\chi^y_{i,t}})^{\theta_{s,t}}}{\prod_s (\frac{\chi^{y*}_{s,t}}{\chi^{y*}_{i,t}})^{\theta_{s,t}}})^{\lambda_{i,t} \sum_n (\theta_{n,t} C_{ni,t})}$ is the allocative efficiency of intermediate goods used in the final goods production.

where C_t is the $N \times N$ Leontif inverse matrix, such that $C_t = (I - \Omega_t)^{-1}$ and $\Omega_t(i,j) = \sigma_{ij,t} + \lambda_{i,t}\theta_{j,t}$. In the above equation, $(\chi^{k*}_{i,t}, \chi^{l*}_{i,t}, \gamma^*_{ij,t}, \chi^{y*}_{i,t})$ are the sectoral shares under optimal allocation, and $(\chi^k_{i,t}, \chi^l_{i,t}, \gamma_{ij,t}, \chi^y_{i,t})$ are the data analog of these shares.

2.3 Value-added vs. input-output economy

We study both the value-added and the input-output economy based on the prior idea that input-output linkages might affect the measure of allocative efficiency. In this section, we provide some intuition of why and how input-output linkages matter to the measurement. We do this through a simple comparison between equation 1 and 4, the sufficient statistics of allocative efficiency in the value-added and the input-output economy, respectively.

First, not surprisingly, equation 1 does not contain terms that measure the allocation of intermediate inputs. This is simply because the value-added economy only concerns the allocation of capital and labor since they are the only production factors.

Second, we turn to the terms that measure capital and labor allocation—equation 1 and the E_{kl} term in equation 4. Both terms are a weighted geometric average of sectoral level allocative efficiency, but they have different sets of sectoral weights. The weights in the value-added economy are θ_i, the share of sector i in the final good production, or final good consumption. The weights become $(1 - \sigma_i - \lambda_i) \sum_n \theta_n C_{ni}$ in the input-output economy. In this term, $\sum_n \theta_n C_{ni}$ is equal to the Domar weight—sector i's sales to GDP—in an undistorted economy. Multiplying it by $(1 - \sigma_i - \lambda_i)$ yields the value-added share of sector

i.[2] As Leal (2015) pointed out, the value-added share and θ_i could potentially differ from each other, in which case the value-added economy is not equivalent to the input-output economy.

Third, the optimal allocation of capital and labor—$\chi^k_{j,t}, \chi^l_{j,t}$—differ as well. Recall that the optimal allocation reflects the relative importance of a sector's capital and labor in producing the final good. In the value-added economy, the relative importance of a sector's capital and labor is, again, θ_i, whereas in the input-output economy, it is $\frac{\theta_i(1-\sigma_i-\lambda_i)}{1-\sum_j \gamma^*_{ji}}$. It can be shown that this term is also equal to sector i's value-added share.[3]

Taking stock, adding input-output linkages alters the measurement of allocative efficiency in two significant ways: 1) it accounts for the allocation of intermediate inputs, and 2) the set of sectoral weights to measure capital and labor allocation changes from θ_i to value-added shares. Intuitively, the *level* of allocative efficiency would be underestimated in the value-added economy if θ_i is smaller than the sectoral value-added share in the highly misallocated sectors. Similarly, the value-added economy would underestimate the *improvement* in allocative efficiency if θ_i is smaller than the sectoral value-added share in the sectors that experienced an improvement in allocation.

2.4 Decomposition of aggregate productivity in the data

This section uses the theoretical results in the previous two sections and shows the decomposition of the aggregate labor productivity in the data.[4]

Proposition 3. *Aggregate labor productivity measured in the data* LP_t *can be decomposed into 1) allocative efficiency* $\mathbf{E}_t$ *and 2) aggregate labor productivity under optimal allocation*

[2]Note that the concept of Domar weights is one related to a decentralized economy because it involves sales (value of the sectoral gross output) and GDP (value of the final good). See Appendix A.3 for a decentralized version of our model.

[3]To see this, we only need to show that $\frac{\theta_i}{1-\sum_j \gamma^*_{ji}}$ is equal to the Domar weight: $\frac{\theta_i}{1-\sum_j \gamma^*_{ji}} = \frac{\theta_i}{\chi_i} = \frac{P_i\theta_i Q_i}{P_i Y_i} = \frac{P_i Q_i}{Y}$. The second equality holds because of the definition of χ_i, such that $\chi_i = Y_i/Q_i$. The last equality holds because $\theta_i Y = P_i Y_i$.

[4]In Appendix A.2, we provide a decomposition framework for aggregate TFP and discuss the difficulties with the exercise of TFP decomposition.

LP_t^*, *such that*

$$\mathsf{LP}_t = \mathsf{LP}_t^* \mathbf{E}_t \tag{5}$$

$$\Delta\mathsf{log}\mathsf{LP}_t = \Delta\mathsf{log}\mathsf{LP}_t^* + \Delta\mathsf{log}\mathbf{E}_t. \tag{6}$$

Proof. The proof can be found in Appendix C.3. □

Equations 5 and 6 are the decomposition of the level and the growth rate of labor productivity. The focus of our paper is to study the contribution of changes in allocative efficiency ($\Delta\mathsf{log}\mathbf{E}_t$) to aggregate productivity growth ($\Delta\mathsf{log}\mathsf{LP}_t$). In comparing the allocative efficiency across countries and over time, one key issue is the measurement errors (Hsieh and Klenow, 2009 and Bils et al., 2020). Similar to Oberfield (2013), our exercises rely on the assumption that the extent of measurement error does not systematically change over time in both countries.

3 Application to Canadian and US data

In this section, we discuss the datasets used in the paper (section 3.1) as well as the empirical strategies to back out the cross-sector allocation in the data (section 3.2) and the output elasticities in the production functions (section 3.3).

3.1 Data description

We use the 2013 version of the KLEMS dataset and the 2013 version of the world input-output table (WIOT) for Canada and the US.[5] The 2013 versions of KLEMS and WIOT are both based on the ISIC Rev. 3 classification, thus allowing us to perform a straightforward mapping between the two datasets. To the best of our knowledge, the 2013 version of KLEMS and WIOT is the latest version of the datasets that have the same industry classification

[5]See http://www.wiod.org/database/wiots13 and http://www.worldklems.net/data.htm.

and are available for both Canada and the US. For both countries, we restrict our analysis to $\mathsf{N} = 28$ private sectors in the economy, as shown in Table 1.

The US KLEMS dataset covers the period 1947–2010, while the input-output table covers 1995–2011, thus restricting the analysis with input-output linkages to the period of 1995–2010. The analysis without input-output linkages spans the longer period of 1947–2010, which allows us to study the slowdown in US productivity growth during the 1970s. To study the widening gap between Canada and the US since the middle of 1985, we consider the model without input-output linkages with the KLEMS datasets in Canada and the US over the period 1985–2007.

Table 1: List of sectors

Sectors
AtB Agriculture, hunting, forestry and fishing
C Mining and quarrying
D Manufacturing
15t16 Food products, beverages and tobacco
17t19 Textiles, textile products, leather and footwear
20 Wood and products of wood and cork
21t22 Pulp paper, paper products, printing and publishing
23 Coke refined petroleum products and nuclear fuel
24 Chemicals and chemical products
25 Rubber and plastics products
26 Other non-metallic mineral products
27t28 Basic metals and fabricated metal products
29 Machinery nec (not elsewhere classified)
30t33 Electrical and optical equipment
34t35 Transport equipment
36t37 Manufacturing nec (not elsewhere classified); recycling
E Electricity gas and water supply
F Construction
G Wholesale and retail trade
50 Wholesale trade and commission trade except of motor vehicles and motorcycles
51 Sale, maintenance and repair of motor vehicles and motorcycles; retail sale of fuel
52 Retail trade except of motor vehicles and motorcycles; repair of household goods
H Hotels and restaurants
I Transport and storage and communication
60t63 Transport and storage
64 Post and telecommunications
J Financial intermediation
K Real estate, renting and business activities
70 Real estate activities
71t74 Renting of m&eq and other business activities
M Education
N Health and social work

In the benchmark analysis, we use real capital stock and the number of workers to measure capital and labor inputs. This choice is based on the fact that real capital stock and the number workers are taken from national accounts. We are less worried about measurement errors when using these variables. The construction of these variables does not rely on survey data as do hours and number of workers of different skill types. Nor do they rely on optimization conditions as do the measures of compensation of different types of capital.

KLEMS also does not distinguish between profit and capital income, i.e., capital compensation + labor compensation = value added. Since assumptions are needed to split profit from capital income, we do not try to measure sector-level markups in the data. We note, however, that our frame can be extended to include markups since markups can be written as distortions to the allocation of intermediate inputs.

Below we list all the variables used in the empirical exercise. For each variable, we distinguish whether it is expenditure ($) or quantity. Note that, of all the inputs, the data provide measures of both expenditure and quantity for capital and labor at the sector level, but only provide measures of expenditure for intermediate inputs.

KLEMS (1) Sector-level value-added and gross output ($), (2) sector-level capital and labor compensation, and cost of intermediate goods ($), (3) sector-level real capital stock, and the number of workers (quantity).

WIOT (1) Sector i's use of domestic sector j good ($), (2) sector i's use of foreign sector j good ($), (3) sector i's output used in final good production ($).

3.2 Cross-sector allocation in the data

To calculate $\mathbf{E}_t$, we first need to compute the allocation of capital, labor and intermediate inputs across sectors in the data $(\chi_i^k, \chi_i^l, \chi_i^y, \gamma_{ij})$. Ideally, we would like to use inputs measured in quantities to calculate the allocation across sectors. We are able to do so for capital and labor, such that $\chi_{i,t}^K = \frac{K_{i,t}}{\sum_i K_{i,t}}$ and $\chi_{i,t}^L = \frac{L_{i,t}}{\sum_i L_{i,t}}$, where $K_{i,t}$ is the real capital stock and

$L_{i,t}$ is the number of workers in sector i. For $\gamma_{ij,t}$ and $\chi^{y}_{i,t}$, we use expenditure to compute these sectoral shares in the data, such that $\gamma_{ij,t} = \frac{\$d_{ij,t}}{\$Q_{j,t}}$, where $\$d_{ij,t}$ is the expenditure of sector i's use of sector j good and $\$Q_{j,t}$ is the nominal value of sector j's gross output. We show in the next section that the lack of measurement for the quantity of intermediate goods has important implications for the measurement of allocative efficiency.

3.3 Output elasticities in the production functions

We employ two specifications to back out the set of sector-level output elasticities ($\alpha_{i,t}$, $\sigma_{ij,t}$, $\lambda_{ij,t}$) from the data. In specification 1, we assume that the factor shares are undistorted for both countries in the data. Namely, the distortions are in the form of sector-level taxes/subsidies as in Jones (2013), and they are not input specific. Although this is a very strong assumption, we think that this is a good benchmark. In specification 2, we relax this assumption. Following Oberfield (2013), we assume that sectors might face input-specific distortions each year, but on average the factor shares are not distorted. More formally, we take a rolling window of three years ($t-1, t, t+1$ for the year t) and use the average expenditure share to back out the output elasticities.[6] Hsieh and Klenow (2009) assume that the US factor shares are undistorted and apply them to China and India. As a robustness check, we follow this approach and apply the US shares to study the allocative efficiency in Canada and find that our results are robust to the alternative specifications (section 5.3). In the following, we provide details of the two specifications and analyze the implications of each specification on the measurement of allocative efficiency.

3.3.1 Specification 1: year-by-year shares

Since the factor shares are undistorted, the output elasticities are equal to the expenditure shares in the data for each year. We label this specification "year-by-year shares" in the

[6] In the long-run, the deviation of expenditure share from the average level in the rolling window could come from both misallocation and technological differences (see discussions in Appendix A.1). As a result, we construct the factor shares using a relatively short rolling window.

empirical results.

We find that for this specification, the allocation of the *expenditure* of the factors across sectors in the data is always equal to the optimal allocation. Only when the cross-sector factor allocation measured using quantity differs from that measured using expenditure can we identify misallocation in the data.

As a corollary of this result, since intermediate goods allocation in the data can only be computed using their expenditure, not quantity, the allocation of intermediate goods is always optimal under this specification.

More formally:

Proposition 4. *If we assume that the expenditure factor shares are undistorted for all* t *and* $(\gamma_{ij,t},\ \chi^{y}_{i,t})$\$ *are computed using expenditure data,* $E^{d}_{t} = E^{m}_{t} = E^{y}_{t} = 1$ *holds. The measured allocative efficiency with input-output linkages reduces to*

$$\mathbf{E}_t = E^{kl}_t = \prod_{i=1}^{N}(((\frac{\chi_{Ki,t}}{\chi^{*}_{Ki,t}})^{\alpha_{i,t}}(\frac{\chi_{Li,t}}{\chi^{*}_{Li,t}})^{1-\alpha_{i,t}})^{1-\sigma_{i,t}-\lambda_{i,t}})^{\sum_n \theta_{n,t} C_{ni,t}}.$$

Proof. See Appendix C.4.

□

This proposition highlights the fact that both the expenditure and quantity of the factors of production are needed to measure allocative efficiency. Any misallocation uncovered in the data comes from the dispersion in implied prices across sectors, i.e., when the cross-sector allocation of expenditure differs from that of quantities.[7]

3.3.2 Specification 2: average shares

In this specification, we take a rolling window of three years ($t-1, t, t+1$ for the year t) and use the average expenditure share to back out the output elasticities. Under this specification,

[7] We call this "implied" prices because we do not observe price directly in the data. However, we can infer that there is dispersion in price across sectors if the allocation of expenditure differs from that of quantity.

the misallocation we uncovered in the data comes from 1) dispersion in implied prices, and 2) the deviation of expenditure shares from the average level of that rolling window.[8]

3.4 Slowdown of productivity growth in the US

In this section, we study the evolution of allocative efficiency in the US and show how it contributed to aggregate productivity growth.

3.4.1 Allocative efficiency over time

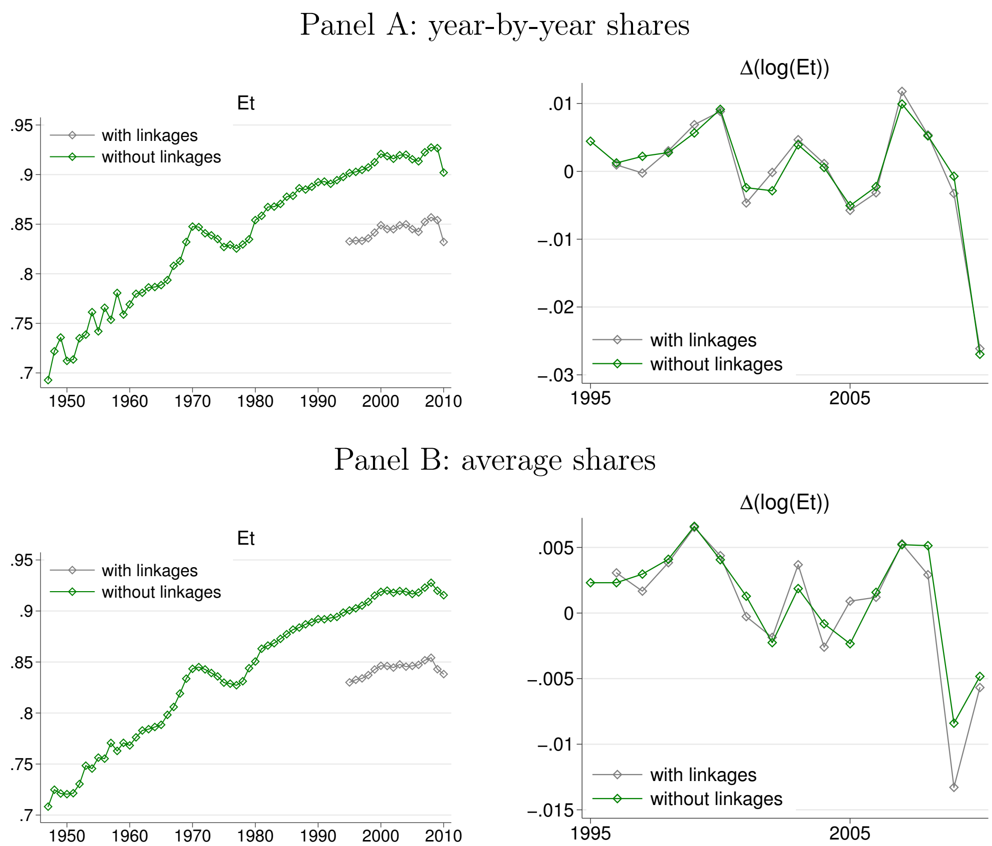


Figure 2: **Allocative efficiency in the US over time**

Note: In this figure, capital K is measured using real capital stock, and L is measured using the number of workers. The black line corresponds to the model with the input-output linkage, and the green line is the one without the linkage.

[8] In the long-run, the deviation of expenditure share from the average level could come from both misallocation and technological differences (see discussions in Appendix A.1). As a result, we construct the factor shares using a relatively short rolling window.

Figure 2 displays the allocative efficiency with and without input-output linkages over time. Since solving the allocation problem without input-output linkages does not require information about the input-output structure, the result goes back to 1947. Measured allocative efficiency is higher without input-output linkages, which suggests that missing the linkages leads to an underestimate of the loss from misallocation. Despite the difference in level, the two lines show very similar trends for the period 1995–2010. The results are robust across the two specifications in Panel A and Panel B. In the year-by-year specification, $\mathsf{E}^{\mathsf{m}} = \mathsf{E}^{\mathsf{d}} = 1$, as discussed in the previous section. We find that E^{m} and E^{d} in the average shares specification are very close to one. However, it is important to note that E^{m} and E^{d} might play a more important role if the intermediate goods can be measured in quantity in addition to expenditure.

Table 2 shows the contribution of allocative efficiency to the aggregate productivity growth in the US. Except for the 1970s and the 2000s, changes in allocative efficiency contributed significantly to the aggregate productivity growth, ranging from 12 to 32 percent when using the year-by-year shares (column 4) and 13 to 34 percent when using the average shares (column 6). Over the period 1960–2007, changes in allocative efficiency account for 18 to 20 percent of aggregate productivity growth. In the next section, we examine how much the slowdown in productivity growth in the 1970s and the 2000s can be attributed to allocative efficiency.

3.4.2 Slowdown of productivity growth in the 1970s and 2000s

As shown in the third column of Table 3, these two decades (marked red) are characterized by a slowdown in productivity compared with their previous decades. In the data, the growth rate of the 1970s is 12 percentage points lower than that of the 1960s. Under optimal allocation, however, the slowdown in labor productivity of the 1970s compared with the 1960s is only 3 percentage points (year-by-year shares) or 4 percentage points (average shares). In other words, the slowdown in the improvement in allocation contributes to 2/3 to 3/4 of

Table 2: **Contribution of allocative efficiency to productivity growth**

	Data	year-by-year shares		average shares	
	$\Delta\log(\mathrm{LP}_t)$	$\Delta\log\mathbf{E}_t$	$\frac{\Delta\log\mathbf{E}_t}{\Delta\log(\mathrm{LP}_t)}$	$\Delta\log\mathbf{E}_t$	$\frac{\Delta\log\mathbf{E}_t}{\Delta\log(\mathrm{LP}_t)}$
(1)	(2)	(3)	(4)	(5)	(6)
1960–1969	+0.24	+0.08	+0.32	+0.08	+0.34
1970–1979	+0.13	-0.02	-0.12	+0.00	+0.01
1980–1989	+0.15	+0.04	+0.27	+0.04	+0.30
1990–1999	+0.19	+0.02	+0.12	+0.03	+0.13
2000–2007	+0.16	+0.00	+0.01	+0.00	+0.03
1960–2007	+0.89	+0.18	+0.20	+0.16	+0.18

Source: BLS, FRED, authors' own calculations.

Note: This table shows the growth rate and the changes in the growth rate of labor productivity for different periods, both in the data and under optimal allocation. dy/y is the growth in labor productivity, measured in log differences, and $\Delta dy/y$ is the change in growth compared to the previous period. Labor productivity is computed as real output per worker.

the slowdown in productivity growth in the 1970s. Similarly, during the 2000s, productivity growth slowed down by 3 percentage points compared to the 1990s. Under the optimal allocation, however, the growth rate differential between the 2000s and the 1990s is only -1 percentage point.

3.5 The widening Canada-US productivity gap

There exists a sizable labor productivity gap between Canada and the US. This fact is well-documented and studied in the literature. In this paper, we focus on explaining why the gap has been widening since the middle of the 1980s, as documented in Panel B of Figure 1. In 1985, Canadian labor productivity is around 83 percent that of the US; the number is now at approximately 75 percent. Over the period of 1985–2007, Canadian labor productivity relative to the US declined at approximately 0.4 percent per year.

Figure 3 compares the changes in allocative efficiency in the two countries. Allocative efficiency in Canada had remained relatively stable before 2000 and declined significantly afterward. On the other hand, allocative efficiency in the US increased significantly before the 2000s and started to stabilize afterward. There are two notable patterns. First, allocative

Table 3: **Growth in labor productivity in the data and under optimal allocation**

	Data		Optimal year-by-year shares		Optimal average shares	
	dy/y	$\Delta dy/y$	dy/y	$\Delta dy/y$	dy/y	$\Delta dy/y$
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1960–1969	+0.24		+0.16		+0.16	
1970–1979	+0.14	-0.12	+0.13	-0.03	+0.13	-0.04
1980–1989	+0.15	+0.02	+0.11	-0.02	+0.10	-0.03
1990–1999	+0.19	+0.04	+0.17	+0.06	+0.17	+0.07
2000–2007	+0.16	-0.03	+0.16	-0.01	+0.16	-0.01

Source: BLS, FRED, authors' own calculations
Note: This table shows the growth rate and the changes in the growth rate of labor productivity for different periods, both in the data and under optimal allocation. dy/y is the growth in labor productivity, measured in log differences, and $\Delta dy/y$ is the change in growth compared to the previous period. Labor productivity is computed as real output per worker.

efficiency has improved faster in the US than in Canada. Second, there is a trend break at the beginning of the 2000s in both countries.

Figure 4 shows that under the optimal allocation, the productivity gap between Canada and the United States has been relatively stable over the same period. In the data, Canadian labor productivity relative to the US has been declining at 0.4 percent per year, compared with 0.15 percent per year under the optimal allocation in both specifications. In other words, 62.5 percent ($\frac{0.25}{0.4}$) of the widening productivity gap can be accounted for by the lack of improvement of allocative efficiency in Canada relative to the US.

4 Capital, labor and sector allocation

Next, we explore which sectors and factors have contributed to changes in allocative efficiency. To avoid repeating the exercise, we study the period 1985–2007 using the model without input-output linkages and in the context of the Canada-US productivity gap.

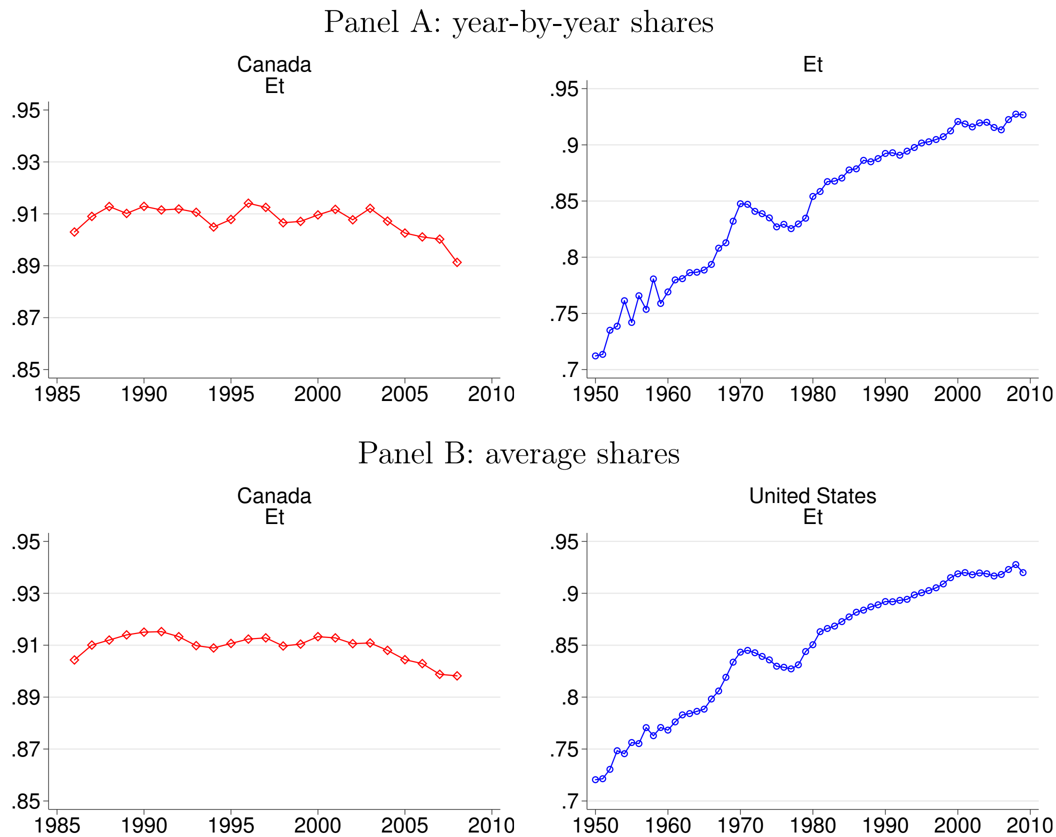


Figure 3: $\mathsf{E_t}$ **in Canada and the US over time**

Source: KLEMS, authors' own calculations.
Note: Allocative efficiency is calculated using KLEMS data without input-output linkages. K is measured using real capital stock and L as number of workers.

4.1 Capital and labor

The allocative efficiency can be decomposed into capital and labor allocative efficiency, such that

$$\mathbf{E}_t = E^{k,t} \cdot E^{l,t},$$

where $E^{k,t} = \prod_{i=1}^{N}[(\frac{x^{k}_{i,t}}{x^{k*}_{i,t}})^{\alpha_{i,t}\theta_{i,t}}$ and $E^{l,t} = \prod_{i=1}^{N}(\frac{x^{l}_{i,t}}{x^{l*}_{i,t}})^{(1-\alpha_{i,t})\theta_{i,t}}$ are the allocative efficiency of capital and labor, respectively.

As shown in Figure 5, in the United States capital allocative efficiency improved drastically over the period 1985 to 2000, whereas labor allocative efficiency stayed stable. In

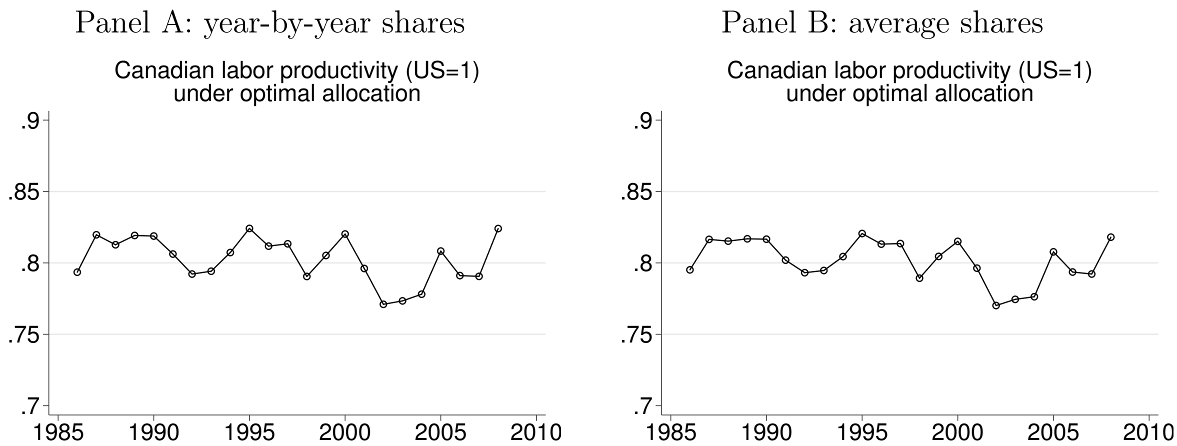


Figure 4: **Canada-US labor productivity gap under optimal allocation**

Source: KLEMS, BLS, FRED, authors' own calculations.
Note: This figure plots Canadian labor productivity under the optimal allocation relative to US labor productivity under optimal allocation.

contrast, there is no evidence of improvement in either capital or labor allocative efficiency in Canada. In fact, the allocative efficiency of capital was in decline throughout the mid-to-late 2000s.

4.2 Sectors

Similarly, the aggregate allocative efficiency can be decomposed into sectoral allocative efficiency $E_{i,t}$, such that,

$$\mathbf{E}_t = \prod_{i=1}^{N} E_{i,t}^{\theta_{i,t}},$$

where $E_{i,t} = (\frac{\chi^k_{i,t}}{\chi^{k*}_{i,t}})^{\alpha_{i,t}}(\frac{\chi^l_{i,t}}{\chi^{l*}_{i,t}})^{1-\alpha_{i,t}}$.

Figure 6 plots the distribution of $E_{i,t}$ over time, where different shades of colors represent different percentiles of the $E_{i,t}$ distribution in year t. Under the optimal allocation, sector-level allocative efficiency $E_{i,t} = 1$ for all sectors, and the distribution collapses into one point at 1. If $E_{i,t} < 1$ for sector i, it means that capital and labor allocated to this sector is lower

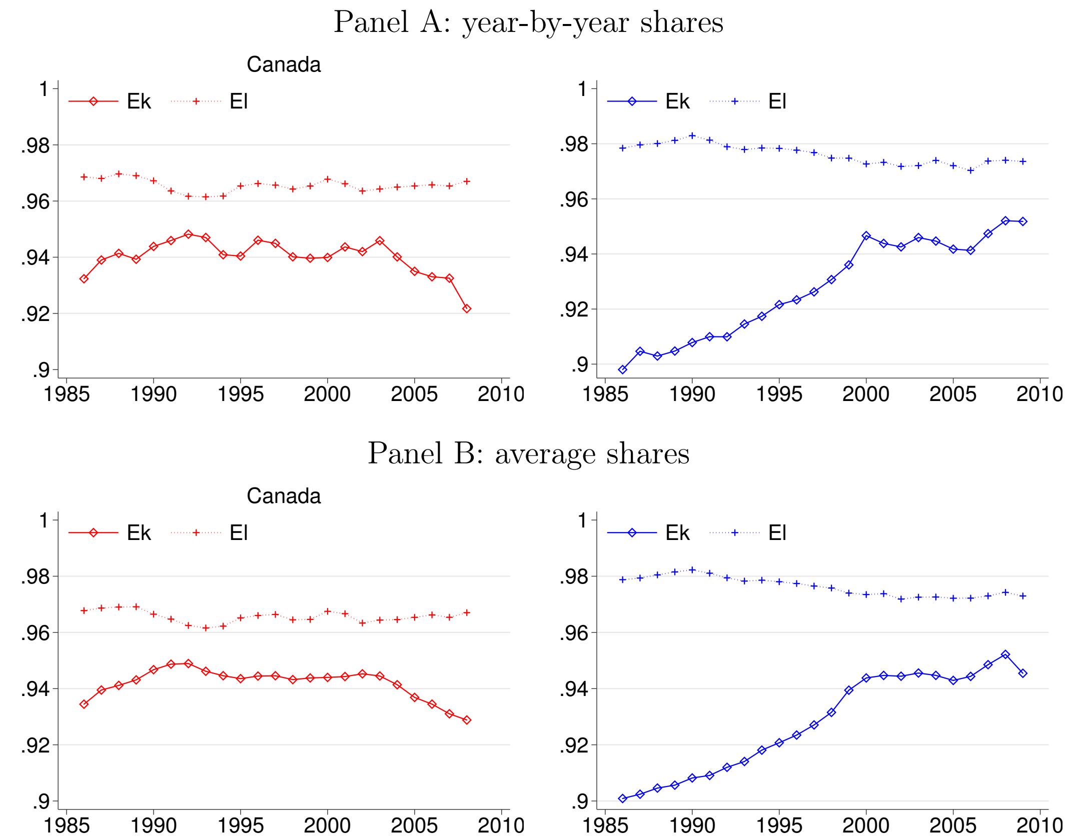


Figure 5: E^k **and** E^l **over time**

Source: KLEMS, authors' own calculations.
Note: This figure plots the evolution of E_k and E_l in Canada and the US in a model without input-output linkages.

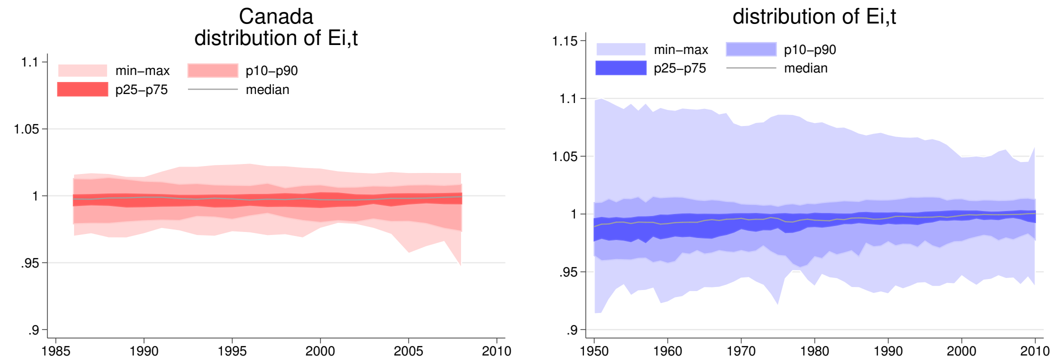
than the optimal level and vice versa.

There is a significant narrowing of the distribution in the US between the middle of 1985 and 2000, which is most visible among the industries that belong to the top and bottom tenth percentile of the E_i distribution. In contrast, there are no significant changes to the distribution in Canada at the same time.

After 2000, the distribution in the US stays relatively stable, whereas it becomes more dispersed in Canada. Notably, in Canada, the industries that belong to the bottom tenth percentile in the distribution have moved further away from the optimal level.

Next, we examine why both countries experienced a slowdown (or decline) in allocative efficiency after 2000. Figure 7 plots the changes in E_i by sector in these two countries for

Panel A: year-by-year shares



Panel B: average shares

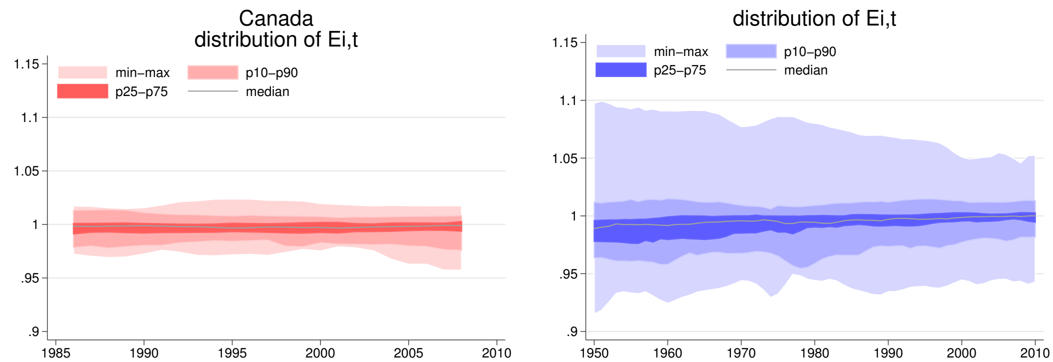


Figure 6: **Distribution of $\mathtt{E}_i$ over time**

Source: KLEMS, authors' own calculations.
Note: This figure plots the distribution of $\mathtt{E}_i$ in Canada and the US in a model without input-output linkages.

the period 1986–2000 and 2000–2007 (the beginning of the financial crisis). The circle and cross represent the beginning and end of each period, respectively. Therefore, the distance between the circle and cross is the magnitude of the change. We mark the sectors green/black if their allocative efficiency has improved/deteriorated during this period ($\mathtt{E}_i$ moved closer to/further away from 1).

In the US, several service sectors experienced significant improvements in $\mathtt{E}_i$ during 1986–2000, most notably the sector of renting machinery and equipment and other business activities (K71t74), whereas there is very little change in the manufacturing sectors (Panel A). As shown in Panel B, during 2000–2007, the service sectors' allocative efficiency continued to improve, but with a much smaller magnitude. In addition, the sector coke-refined petroleum

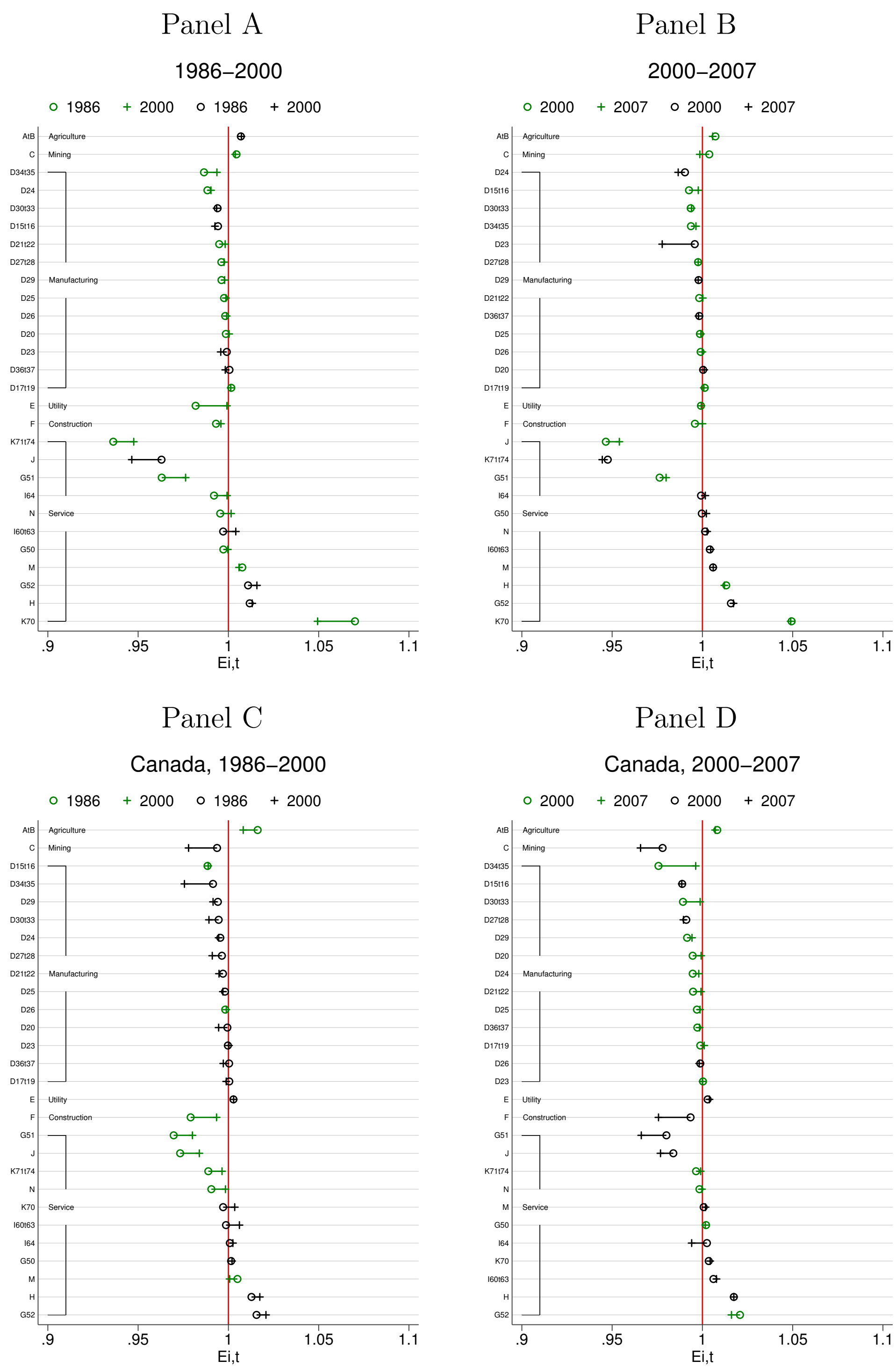


Figure 7: **Changes in E_i over 1986–2000 and 2000–2007, year-by-year shares**

Source: KLEMS, authors' own calculations.

Note: This figure plots the evolution of E_k and E_l in Canada and the US in a model without input-output linkages. Appendix Figure B.3 plots the same statistics under the average shares.

products and nuclear fuel (D23) experienced a significant decline in allocative efficiency. The E_i was close to 1 in 2000 but decreased significantly in 2007, which indicates that the capital and labor allocated to this sector were significantly lower than the optimal level.

In contrast, in Canada during 1986–2000, the service sectors also experienced some improvements in E_i (Panel C). However, the magnitude is smaller than in the US (Panel C). At the same time, the E_i of the mining sector (C) and the transport equipment sector (D34t35) deteriorated, which is perhaps the reason why aggregate allocative efficiency did not change significantly during this period. As shown in Panel D, during 2000–2007, a majority of the Canadian manufacturing sectors' E_i improved. However, several sectors, including mining (C), construction (F), sale, maintenance and repair of motor vehicles, retail sale of fuel (G51), and financial intermediation (J), experienced a significant decline in E_i. All five sectors' E_i were significantly below 1 in 2000, and they became even lower in 2007, which is consistent with the pattern in Figure 6 that the increase in the dispersion of the E_i distribution is driven by sectors in the bottom tenth percentiles moving further away from the optimal level.

5 Robustness

In this section, we carry out several robustness checks. Section 5.1 studies the impact of the elasticity of substitution on our analysis in a CES production system. In section 5.2, we restrict our analysis to the manufacturing sectors only. Following Hsieh and Klenow (2009), we consider two more specifications in section 5.3.

5.1 Elasticity of substitution

The literature shows that the elasticity of substitution between production factors has an important implication on the measurement of allocative efficiency (see Osotimehin and Popov, 2018 and Epifani and Gancia, 2011). On the other hand, however, the elasticities of substitution between inputs are notoriously difficult to measure from data.[9] Our analysis so far is based on a Cobb-Douglas production system.

In this section, we extend the model into a CES production system and show how changes in the elasticity could affect our result. Since we are unable to measure profit separately from capital income at the sector level, we assume that the expenditure share of good i in the final consumption is not distorted by sectoral markups.[10]

The final good is a CES aggregation of the N intermediate goods, such that

$$Y = (\sum_i \omega_i Y_i^{1-\frac{1}{\rho}})^{\frac{\rho}{\rho-1}}, \tag{7}$$

where ρ measures the elasticity of substitution and ω_i is the weight of good Y_i in the final good production.

The production function of the intermediate good Y_i is the Cobb-Douglas form, such that

$$Y_i = A_i K_i^{\alpha_i} L_i^{1-\alpha_i},$$

and the planner solves the following optimization problem:

$$max\ Y, s.t \sum_i K_i = K, \sum_i L_i = L.$$

The following proposition characterizes the solution to the problem and the measured allocative efficiency.

Proposition 5. *The allocative efficiency* $\mathbf{E}_t$ *can be written as*

$$\mathbf{E}_t = \frac{Y_t^*}{Y_t} = \{\sum_j \{(\frac{P_j Y_j}{PY})^{\frac{\rho}{\rho-1}} (\frac{\alpha_j / K_j}{\bar{\alpha}^* / K})^{\alpha_j} [\frac{(1-\alpha_j)/L_j}{(1-\bar{\alpha}^*)/L}]^{1-\alpha_j}\}^{\rho-1}\}^{\frac{1}{1-\rho}}, \tag{8}$$

[9]Recent development of the literature includes Oberfield and Raval (Forthcoming) and Ruane and Peter (2020).

[10]As shown in Epifani and Gancia (2011), markups only lead to misallocation if they are heterogeneous across sectors. This framework can be extended to incorporate markups with the availability of sector-level data.

The optimal allocation is characterized by $\{\bar{\alpha}^*, \chi_i^{k*}, \chi_i^{l*}\}$*, such that*

$$\chi_i^{k*} = \frac{K_i^*}{K} = \frac{\frac{P_i Y_i}{PY}\alpha_i}{\bar{\alpha}^*} \quad \chi_i^{l*} = \frac{L_i^*}{L} = \frac{\frac{P_i Y_i}{PY}(1-\alpha_i)}{1-\bar{\alpha}^*}, \quad \bar{\alpha}^* = \sum_i \frac{P_i Y_i}{PY}\alpha_i,$$

where $\frac{P_j Y_j}{PY}$ *is the expenditure share of good* j *in the final good consumption in the data.*

Proof. See Appendix C.5. □

We show the impact of ρ on measured allocative efficiency in Figure 8, where lower ρ means the goods are more complementary to each other. The measured allocative efficiency is higher when the goods are more complementary to each other (Panel A). This result is broadly consistent with the findings in Epifani and Gancia (2011) and Osotimehin and Popov (2018).

Besides, we see from Panel A that the growth rate of $\mathbf{E}_t$ increases with ρ (the line becomes steeper). In Panel B, a higher ρ is also associated with higher volatility in measured $\mathbf{E}_t$. This means that our benchmark model underestimates (overestimates) the role of allocative efficiency in explaining aggregate growth and volatility if the goods are more substitutable (complementary) than the Cobb-Douglas case.

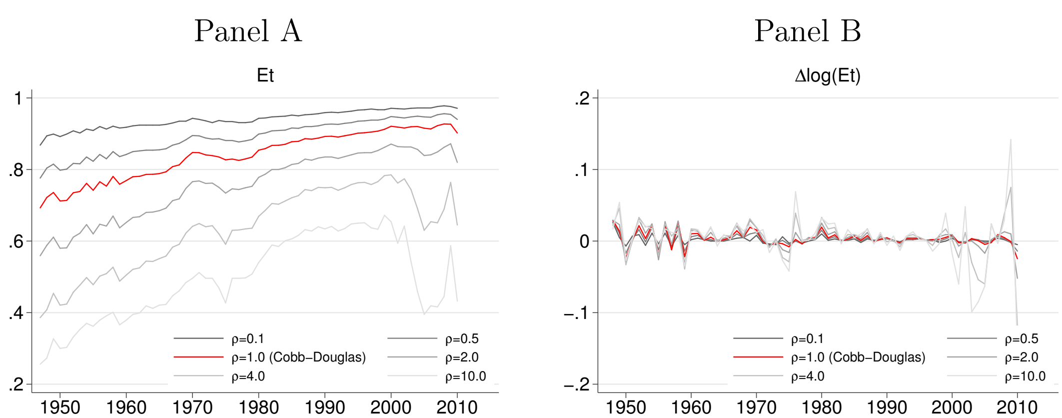


Figure 8: **$\mathbf{E}_t$ and $\Delta\log(\mathbf{E}_t)$, different ρ**

Source: KLEMS, authors' own calculations.
Note: This figure plots $\mathbf{E}_t$ and $\Delta\log(\mathbf{E}_t)$ under different values for ρ and without input-output linkages. Capital is measured using real capital stock, and labor is measured using the number of employers.

5.2 Manufacturing sectors

The literature has focused mostly on studying the allocation of the manufacturing sectors, which arguably suffers less from measurement errors than the other sectors but has the caveat of not covering the whole economy. In this section, we study allocation across the 13 manufacturing industries (see Table 1).

In Figure 9, we plot the allocative efficiency of the manufacturing industries in the US over time. The pattern of E_t of the manufacturing sector is significantly different than that of the entire economy. As shown by the black line (the model without input-output linkages), E_i of the manufacturing sectors was very stable before 1980; it increased slightly during the 1980s, stabilized again in the 1990s, and declined significantly after the beginning of the 2000s. The green line (model with input-output linkages) has a much shorter time series and shows a significant decline during the 2000s, especially during the 2007–2009 financial crisis.

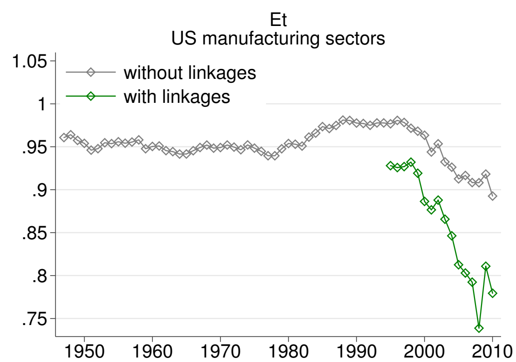


Figure 9: E_i **in the US manufacturing sectors over time, year-by-year shares**

Source: KLEMS, WIOT, authors' own calculations.
Note: This figure plots the allocative efficiency across the 13 manufacturing sectors in the US. Capital K is measured using real capital stock, and L is measured using the number of workers. The black line corresponds to the model without input-output linkages, and the green line is the one with input-output linkages.

As shown in Panel A of Figure B.4, before 2010, the labor productivity of the US manufacturing sector has been growing at a relatively constant rate of 10–11 percent per year. Interestingly, it has slowed down significantly after 2010, the last year of our data series.

Unfortunately, we cannot speak to the post-2010 slowdown in productivity growth, given the current data availability.

5.3 US factor shares

In the literature, the commonly adopted assumption is that the US factor shares are undistorted and thus can be applied to other countries (Hsieh and Klenow, 2009). In this section, we consider two additional specifications inspired by this approach.

First, we assume that the US economy is relatively undistorted in the later years. Then we apply the factor shares of the later years to study the allocative efficiency over time. When applied to a long-term horizon, this specification runs into the issue of compounding the impacts of distortions and technological change on expenditure shares (see Appendix A.1).

Second, we assume that the US factor shares are undistorted and apply them to evaluate allocative efficiency in Canada. We find that our results are robust under this alternative specification. The allocative efficiency of Canada experienced a small increase before 2000 (2 percentage points) and declined after 2000 (Panel A of Figure 10). The Canada-US productivity gap under optimal allocation widens at 0.26 percent per year, which is faster than under the two benchmark specifications, but it is significantly slower than the 0.4 percent per year in the data. Under this specification, approximately 35 percent (0.14/0.4) of the widening productivity gap can be explained by allocative efficiency.

6 Conclusion

In this paper, we showed the importance of allocation in explaining aggregate productivity growth. Applying the theory to the US and Canada using the KLEMS and WIOT datasets, we showed that allocative efficiency could go a long way in explaining both the slowdown in productivity growth in the US and the widening Canada-US productivity gap. Furthermore,

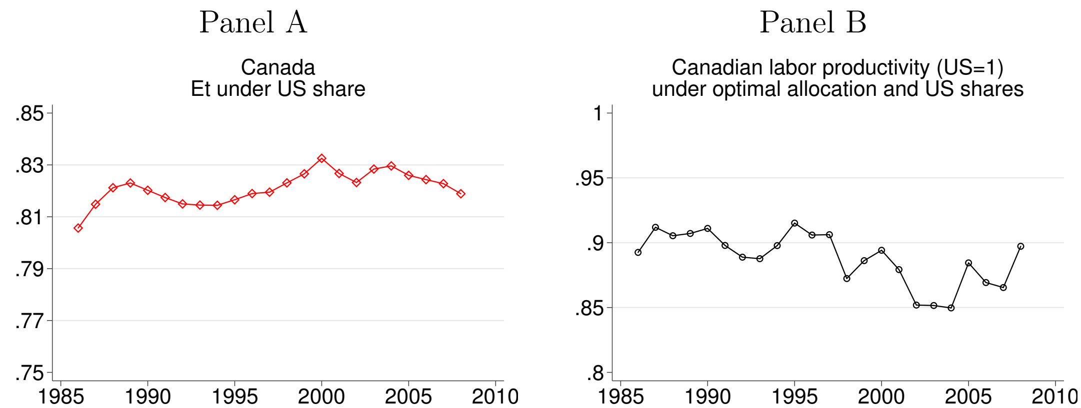


Figure 10: **Canada $\mathsf{E_i}$ and the Canada-US productivity gap using the US shares**

Source: KLEMS, authors' own calculations.

Note: Panel A plots the $\mathsf{E_i}$ of the Canadian economy over time by applying the US factor shares to the Canadian economy. Panel B plots the productivity gap under optimal allocation.

we found that capital allocation is the main driver behind the trends in allocative efficiency in both countries, whereas the allocation of labor stayed relatively unchanged. Most notably, several service sectors in the US have gained resources over time. They are much closer to the optimal level now than several decades ago, which contributes significantly to the overall improvement in allocative efficiency.

References

Aum, S., S. Y. T. Lee, and Y. Shin (2018): "Computerizing Industries and Routinizing Jobs: Explaining Trends in Aggregate Productivity," *Journal of Monetary Economics*, 97, 1–21.

Baqaee, D. R. and E. Farhi (2019): "Productivity and Misallocation in General Equilibrium," Working paper.

Basu, S. and J. G. Fernald (2002): "Aggregate Productivity and Aggregate Technology," *European Economic Review*, 46, 963–991.

Bernstein, J. I., R. G. Harris, and A. Sharpe (2002): "The Widening Canada-US Productivity Gap in Manufaturing," *International Productivity Monitor*, 5, 3–22.

Bils, M., P. J. Klenow, and C. Ruane (2020): "Misallocation or Mismeasurement?" Working paper.

Byrne, D. M., J. G. Fernald, and M. B. Reinsdorf (2016): "Does the United States Have a Productivity Slowdown or a Measurement Problem?" *Brookings Papers on Economic Activity*, 2016, 109–182.

Caliendo, L., F. Parro, and A. Tsyvinski (2018): "Distortions and the Structure of the World Economy," Working paper.

Calligaris, S., M. D. Gatto, F. Hassan, G. Ottaviano, and F. Schivardi (2018): "The Productivity Puzzle and Misallocation: An Italian Perspective," Working paper.

Cette, G., J. Fernald, and B. Mojon (2016): "The Pre-Great Recession Slowdown in Productivity," *European Economic Review*, 88, 3–20.

Decker, R. A., J. Haltiwanger, R. S. Jarmin, and J. Miranda (2017): "Declining Dynamism, Allocative Efficiency, and the Productivity Slowdown," *American Economic Review*, 107, 322–26.

Epifani, P. and G. Gancia (2011): "Trade, Markup Heterogeneity and Misallocations," *Journal of International Economics*, 83, 1–13.

Gopinath, G., Ş. Kalemli-Özcan, L. Karabarbounis, and C. Villegas-Sanchez (2017): "Capital Allocation and Productivity in South Europe," *The Quarterly Journal of Economics*, 132, 1915–1967.

Hsieh, C.-T. and P. J. Klenow (2009): "Misallocation and Manufacturing TFP in China and India," *The Quarterly Journal of Economics*, 124, 1403–1448.

Jones, C. I. (2013): *Misallocation, Economic Growth, and Input-Output Economics*, Cambridge University Press, vol. 2 of *Econometric Society Monographs*, 419–456.

——— (2017): "The Productivity Growth Slowdown in Advanced Economies," ECB Forumon Central Banking.

Leal, J. (2015): "Equivalence Between Input-Output and Value-Added Economies," Working paper.

Leung, D., C. Meh, and Y. Terajima (2008): "Firm Size and Productivity," Bank of Canada Staff Working Papers.

Melitz, M. J. and S. Polanec (2015): "Dynamic Olley-Pakes Productivity Decomposition with Entry and Exit," *RAND Journal of Economics*, 46, 362–375.

Oberfield, E. (2013): "Productivity and Misallocation During a Crisis: Evidence from the Chilean Crisis of 1982," *Review of Economic Dynamics*, 16, 100–119.

Oberfield, E. and D. Raval (Forthcoming): "Micro Data and Macro Technology," *Econometrica*.

Osotimehin, S. and L. Popov (2018): "Misallocation and Intersectoral Linkages," Working Paper.

Ranasinghe, A. (2017): "Innovation, Firm Size and the Canada-U.S. Productivity Gap," *Journal of Economic Dynamics and Control*, 85, 46–58.

Ruane, C. and A. Peter (2020): "The Aggregate Importance of Intermediate Input Substitutability," Working paper.

Sharpe, A. (2003): "Why Are Americans More Productive than Canadians?" *International Productivity Monitor*, 6, 19–37.

Startz, R. (2020): "The Next Hundred Years of Growth and Convergence," *Journal of Applied Econometrics*, 35, 99–113.

Syverson, C. (2017): "Challenges to Mismeasurement Explanations for the US Productivity Slowdown," *Journal of Economic Perspectives*, 31, 165–86.

Tang, J. (2014): "Are Small or Large Producers Driving the Canada-U.S. Labour Productivity Gap? Recent Evidence from Manufacturing," *Canadian Journal of Economics*, 47, 517–539.

Vandenbroucke, G. (2019): "The Baby Boomers and the Productivity Slowdown," Working paper.

Online Appendix

Not for Publication

A Extensions

A.1 Technologcial change vs. distortions

One commonly-used assumption in the literature is that the US factor shares are undistorted and thus reflect the fundamental technology and can be applied to evaluate allocation in the other countries. In this section, we consider a slightly revised specification by assuming that the factor shares are undistorted in the US in a *base year*.

It is plausible that the factor shares are more likely to be undistorted in later years compared with earlier years. We pick 2005 to be the base year to minimize the impact of the Great Recession. Figure A.1 compares the measured allocative efficiency between specification 1 in the main text (blue) and the current specification with 2005 as the base year (grey). It is not surprising that the two lines intersected in 2005. Compared with specification 1, measured allocative efficiency is in general much lower and smoother under the current specification. The gap between these two lines gets larger the further one moves away from 2005. Applying the factor shares of 2005 to the earlier decades seems to run into the issue of compounding technology change and distortions.

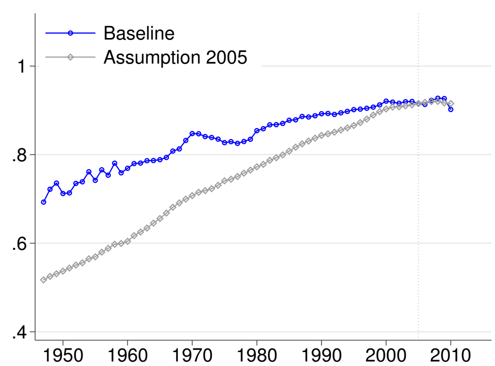


Figure A.1: E_i **in the US, assuming undistorted factor shares in 2005**

Source: KLEMS, authors' own calculations.

Note: This figure plots the allocative efficiency (without input-output linkages) in the US under specification 1 (baseline) and the assumption of undistorted factor shares in 2005 (assumption 2005). Capital K is measured using real capital stock, and L is measured using the number of workers.

Generally speaking, we are concerned with applying factor shares of one year over a sample of several decades. This concern can be seen more clearly in Figure A.2, where we plot $\mathbf{E_t}$ by varying the base year in the specification. Panel A plots the lines with 1960, 1970 and 1980 as the base year, and Panel B plots lines with 1990, 2000 and 2010 as the base year. For example, as shown in Panel A, if we apply the 1960 factor shares to the whole sample, measured allocative efficiency $\mathbf{E}_t$ is inverted-U shaped and peaked at the beginning of the 1980s.

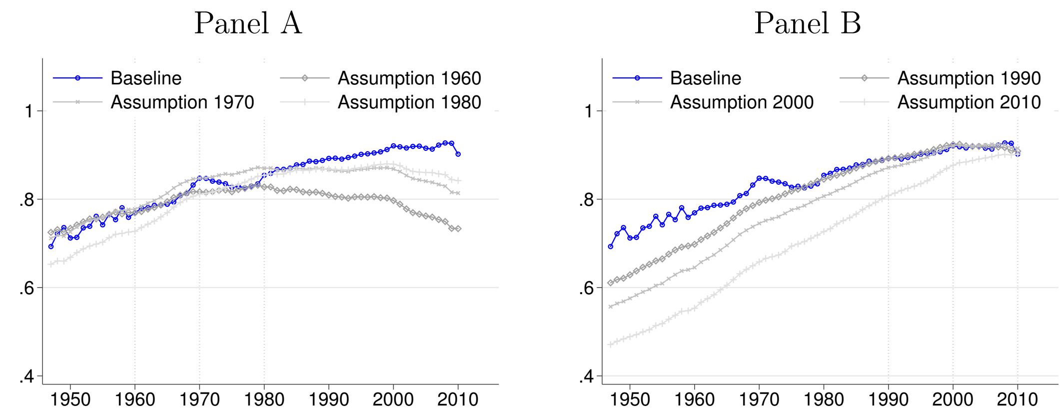


Figure A.2: E_i **in the US, assuming undistorted factor shares in various years**

Source: KLEMS, authors' own calculations.

Note: This figure plots the allocative efficiency (without input-output linkages) in the US under specification 1 (baseline) and the assumption of undistorted factor shares in year 1960/1970/1980/1990/2000/2005/2010 (assumption 1960/1970/1980/1990/2000/2005/2010). Capital K is measured using real capital stock, and L is measured using the number of workers.

A.2 The decomposition of aggregate TFP in the data

Proposition 6. *Aggregate TFP in the data A_t can be decomposed into three components: 1) allocative efficiency $\mathbf{E}_t$, 2) TFP under optimal allocation A_t^*, and 3) a mismeaurement term $(\frac{K_t}{L_t})^{\alpha_t^*-\alpha_t}$, such that*

$$A_t = A_t^* \mathbf{E}_t (\frac{K_t}{L_t})^{\alpha_t^*-\alpha_t}. \tag{9}$$

For the economy without input-output linkages, $\alpha_t^* = \sum_i \alpha_i \theta_i$; for the economy with input-output linkages, $\alpha_t^* = \sum_n (\sum_i (\alpha_{i,t}(1-\sigma_{i,t}-\lambda_{i,t})C_{ni,t})\theta_{n,t})$.

Proof. The proof can be found in Appendix C.6. □

Compared with proposition 3, the decomposition of aggregate TFP has one additional component $(\frac{K_t}{L_t})^{\alpha_t{}^*-\alpha_t}$, where α_t is the capital income share in the data and α^* is the capital output elasticity in the aggregate production function under the optimal allocation.[11] We show in proposition 7 that under optimal allocation $\alpha = \alpha^*$.

Proposition 7. *If there is no distortion in the economy, capital income share in the data is equal to α^*, the capital output elasticity in the aggregate production function under optimal allocation.*

[11]Notice that TFP in the data is computed as $A_t = \frac{Y_t}{K_t^{\alpha_t} L_t^{1-\alpha_t}}$.

Proof. See Appendix C.7 □

If the economy deviates from optimal allocation, α_t measured in the data is different than α_t^*, which leads to a bias in the measurement of the aggregate TFP. We label the term $(\frac{K_t}{L_t})^{\alpha_t{}^*-\alpha_t}$ as "mismeasurement" component in the decomposition. Therefore, any measurement error in the real capital-labor ratio $\frac{K_t}{L_t}$, such as the price of the capital stock, would lead to changes in this mismeasurement component, making the TFP decomposition exercise vulnerable to measurement errors.

A.3 Decentralized problem

In this section, we characterize the decentralized problem with inputs-specific distortions.

Firm in sector i will solve the following profit maximization problem:

$$\max_{K_i,L_i,\{d_{ij},m_{ij}\}} P_i A_i (K_i^{\alpha_i} L_i^{1-\alpha_i})^{1-\sigma_i-\lambda_i} d_{i1}^{\sigma_{i1}} \dots d_{iN}^{\sigma_{iN}} m_{i1}^{\lambda_{i1}} \dots m_{iN}^{\lambda_{iN}}$$
$$- (1+\tau_i^k) R K_i - (1+\tau_i^l) w L_i - \sum_{j=1}^{N} (1+\tau_{ij}^d) P_j d_{ij} - \sum_{j=1}^{N} (1+\tau_{ij}^m) \bar{P}_j m_{ij}$$

First order conditions:

$$\alpha_i (1-\sigma_i-\lambda_i) \frac{P_i Q_i}{K_i} = R(1+\tau_i^k) \tag{10}$$

$$(1-\alpha_i)(1-\sigma_i-\lambda_i) \frac{P_i Q_i}{L_i} = w(1+\tau_i^l) \tag{11}$$

$$\sigma_{ij} \frac{P_i Q_i}{d_{ij}} = P_j(1+\tau_{ij}^d), j = 1, \cdots, N \tag{12}$$

$$\lambda_{ij} \frac{P_i Q_i}{m_{ij}} = \bar{P}_j(1+\tau_{ij}^m), j = 1, \cdots, N \tag{13}$$

Market clearing condition for sector j

$$Y_j + \sum_{i=1}^{N} d_{ij} = Q_j$$

or

$$P_j Y_j + \sum_{i=1}^{N} P_j d_{ij} = P_j Q_j$$

Plugging in equation (12)

$$P_j Y_j + \sum_{i=1}^{N} \sigma_{ij} \frac{P_i Q_i}{(1+\tau_{ij}^d)} = P_j Q_j$$

From final goods problem

$$P_j = \theta_j \frac{Y}{Y_j}$$

Substitute for P_j and cancel for Y

$$\theta_j + \sum_{i=1}^{N} \sigma_{ij} \frac{Q_i}{(1+\tau_{ij}^d)} \frac{\theta_i}{Y_i} = \frac{\theta_j}{Y_j} Q_j$$

Following Jones (2013), we define $\gamma_j = \frac{\theta_j Q_j}{Y_j}$, then the following equation will solve $\{\nu_j\}$

$$\theta_j + \sum_{i=1}^{N} \frac{\sigma_{ij}}{(1+\tau_{ij}^d)} \gamma_i = \gamma_j, j = 1, \cdots, N$$

Denote $\theta_=[\theta_1, \cdots, \theta_N]_{N\times 1}$, $\gamma = [\gamma_1, \cdots, \gamma_N]_{N\times 1}$, $\bar{B}(i,j) = \frac{\sigma_{ij}}{1+\tau_{ij}^d}$, then

$$\gamma = (I - \bar{B})^{-1}\theta$$

where γ is the distorted Domar weights.

Given γ, by the first order condition in equation (12), we can solve d_{ij} as

$$d_{ij} = \sigma_{ij} \frac{P_i Q_i}{P_j(1+\tau_{ij}^d)} = \sigma_{ij} \frac{\gamma_i}{\gamma_j(1+\tau_{ij}^d)} Q_j$$

From the first order condition in equation (13),

$$\frac{m_{ij}}{d_{ij}} = \frac{1+\tau_{ij}^d}{1+\tau_{ij}^m} \frac{\lambda_{ij}}{\sigma_{ij}} \frac{P_j}{\bar{P}_j}$$

then

$$m_{ij} = \frac{\lambda_{ij}}{1+\tau_{ij}^m} \frac{P_j}{\bar{P}_j} \frac{\gamma_i}{\gamma_j} Q_j = \frac{\lambda_{ij}}{1+\tau_{ij}^m} \frac{\gamma_i}{\bar{P}_j} Y$$

From the first order condition in equation (10),

$$K_i = \alpha_i (1 - \sigma_i - \lambda_i) \frac{P_i Q_i}{R(1+\tau_i^k)} = \alpha_i (1 - \sigma_i - \lambda_i) \frac{\gamma_j Y}{R(1+\tau_i^k)}$$

Denote

$$\delta_i^k = \alpha_i (1 - \sigma_i - \lambda_i) \frac{\gamma_i}{(1+\tau_i^k)}$$

then $K_i = \delta_i^k \frac{Y}{R}$, define $\tilde{\delta}_i^k = \frac{\delta_i^k}{\sum_{i=1}^N \delta_i^k}$

$$K_i = \tilde{\delta}_i^k K.$$

Similarly, define

$$\delta_i^l = (1 - \alpha_i)(1 - \sigma_i - \lambda_i) \frac{\gamma_i}{(1+\tau_i^l)}$$

and $\tilde{\delta}_i^l = \frac{\delta_i^l}{\sum_{i=1}^N \delta_i^l}$, then

$$L_i = \tilde{\delta}_i^l L$$

define $\delta^d_{ij} = \frac{\sigma_{ij}}{1+\tau^d_{ij}}$ and $\delta^m_{ij} = \frac{\lambda_{ij}}{1+\tau^m_{ij}}$, then

$$d_{ij} = \delta^d_{ij} \frac{\gamma_i}{\gamma_j} Q_j$$

$$m_{ij} = \delta^m_{ij} \frac{\gamma_i}{\bar{P}_j} Y$$

Given $\{K_i, L_i, d_{ij}, m_{ij}\}$, the gross output Q_i can be computed as

$$Q_i = P_i A_i [(\tilde{\delta}^k_i K)^{\alpha_i} (\tilde{\delta}^l_i L)^{1-\alpha_i}]^{1-\sigma_i-\lambda_i} \prod_{j=1}^{N} (\delta^d_{ij} \frac{\gamma_i}{\gamma_j} Q_j)^{\sigma_{ij}} \prod_{j=1}^{N} (\delta^m_{ij} \frac{\gamma_i}{\bar{P}_j} Y)^{\lambda_{ij}}.$$

Take the log of both sides

$$\log(Q_i) = \log\{P_i A_i [(\tilde{\delta}^k_i K)^{\alpha_i} (\tilde{\delta}^l_i L)^{1-\alpha_i}]^{1-\sigma_i-\lambda_i}\} + \sum_{j=1}^{N} \sigma_{ij} \log(\delta^d_{ij} \frac{\gamma_i}{\gamma_j}) + \sum_{j=1}^{N} \lambda_{ij} \log(\delta^m_{ij} \frac{\gamma_i}{\bar{P}_j})$$
$$+ \sum_{j=1}^{N} \sigma_{ij} \log(Q_j) + \sum_{j=1}^{N} \lambda_{ij} \log(Y)$$

and write in matrix form

$$q = D + \sigma q + \lambda \log(Y)$$

where q is N by 1 matrix, $q(i) = \log(Q_i)$, D is N by 1 matrix,

$$D(i) = \log\{P_i A_i [(\tilde{\delta}^k_i K)^{\alpha_i} (\tilde{\delta}^l_i L)^{1-\alpha_i}]^{1-\sigma_i-\lambda_i}\} + \sum_{j=1}^{N} \sigma_{ij} \log(\delta^d_{ij} \frac{\gamma_i}{\gamma_j}) + \sum_{j=1}^{N} \lambda_{ij} \log(\delta^m_{ij} \frac{\gamma_i}{\bar{P}_j}).$$

σ is N by N matrix, $\sigma(i,j) = \sigma_{ij}$ and λ is N by 1 matrix, $\lambda(i) = \lambda_i$. Then

$$q = (I - \sigma)^{-1} [D + \lambda \log(Y)]$$

Since $Y = \prod_i Y_i^{\theta_i}$ and $\gamma_j = \frac{\theta_j Q_j}{Y_j} = \frac{P_j Q_j}{Y}$, then $Y = \prod_i (\frac{\theta_i Q_i}{\gamma_i})^{\theta_i}$. Taking the log we have

$$\log(Y) = \sum_i \theta_i \log(\frac{\theta_i}{\gamma_i}) + \sum_i \theta_i \log(Q_i)$$

then

$$\log(Y) = \sum_i \theta_i \log(\frac{\theta_i}{\gamma_i}) + \theta' (I - \sigma)^{-1} [D + \lambda \log(Y)]$$

where θ is N by 1 matrix, $\theta(i) = \theta_i$, then

$$\log(Y) = \frac{1}{1 - \theta' (I - \sigma)^{-1} \lambda} [\sum_i \theta_i \log(\frac{\theta_i}{\gamma_i}) + \theta' (I - \sigma)^{-1} D]$$

Q.E.D.

B Additional tables and figures

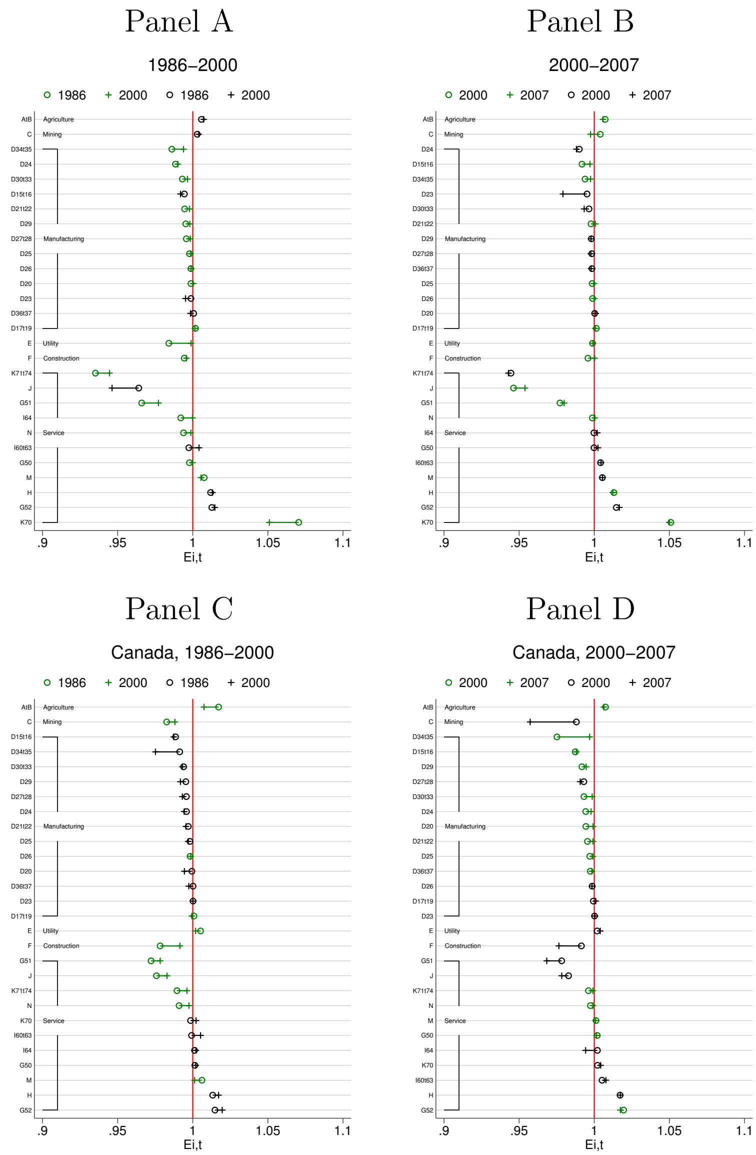


Figure B.3: **Changes in $\mathtt{E}_i$ over 1986–2000 and 2000–2007, average shares**

Source: KLEMS, authors' own calculations.

Note: This figure plots the evolution of $\mathtt{E}_k$ and $\mathtt{E}_l$ in Canada and the US in a model without input-output linkages.

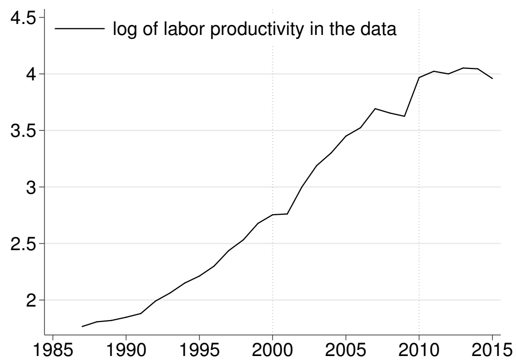


Figure B.4: **Labor productivity of the manufacturing sector**

Source: BLS.

Note: This figure plots the logarithms of real labor productivity in the data for the manufacturing sector.

C Proofs

In the proofs, we drop the time subscript t to simplify notation.

C.1 Proof of proposition 1

The solution to the planner's problem requires the equalization of MPK and MPL across sectors, such that

$$\frac{\partial \log Y}{\partial K_i} = \lambda$$
$$\frac{\partial \log Y}{\partial L_i} = \eta$$

They can be written as,

$$K_i = \frac{\theta_i \alpha_i}{\lambda}$$
$$L_i = \frac{\theta_i (1-\alpha_i)}{\eta}$$

Given the resource constraint, we get

$$K_i = \frac{\theta_i \alpha_i}{\sum_i \theta_i \alpha_i} K$$

$$L_i = \frac{\theta_i (1-\alpha_i)}{\sum_i \theta_i (1-\alpha_i)} L$$

The final good output can be written as

$$\begin{aligned} Y &= \prod_i Y_i^{\theta_i} \\ &= \Pi_i (A_i K_i^{\alpha_i} L_i^{1-\alpha_i})^{\theta_i} \\ &= \Pi_i (A_i (\frac{\theta_i \alpha_i}{\sum_i \theta_i \alpha_i} K)^{\alpha_i} (\frac{\theta_i (1-\alpha_i)}{\sum_i \theta_i (1-\alpha_i)} L)^{1-\alpha_i})^{\theta_i} \\ &= \bar{A} K^{\sum_i \alpha_i \theta_i} L^{\sum_i (1-\alpha_i)\theta_i}, \end{aligned}$$

where $\bar{A} = \Pi_i (A_i (\frac{\theta_i \alpha_i}{\sum_i \theta_i \alpha_i})^{\alpha_i} (\frac{\theta_i (1-\alpha_i)}{\sum_i \theta_i (1-\alpha_i)})^{1-\alpha_i})^{\theta_i}$. *Q.E.D.*

C.2 Proof of proposition 2

The planner's problem is

$$C = \prod_{i=1}^{N} (Q_i - \sum_{j=1}^{N} d_{ji})^{\theta_i} - \sum_i \sum_j \bar{P}_j m_{ij}.$$

FOCs The first order conditions for K_i, L_i, d_{ij}, m_{ij} are

$$\begin{aligned}
\frac{\partial C}{\partial K_i} &= \theta_i \frac{Y}{Y_i}\frac{Q_i}{K_i}\alpha_i(1-\sigma_i-\lambda_i) \\
\frac{\partial C}{\partial L_i} &= \theta_i \frac{Y}{Y_i}\frac{Q_i}{L_i}(1-\alpha_i)(1-\sigma_i-\lambda_i) \\
\frac{\partial C}{\partial d_{ij}} &= \theta_i \frac{Y}{Y_i}[\frac{Q_i}{d_{ij}}\sigma_{ij} - I_{\{i=j\}}] + \theta_j \frac{Y}{Y_j}[\frac{Q_j}{d_{jj}}\sigma_{jj} I_{\{i=j\}} - 1] \\
\frac{\partial C}{\partial m_{ij}} &= \theta_i \frac{Y}{Y_i}\frac{Q_i}{m_{ij}}\lambda_{ij} - \bar{P}_j
\end{aligned}$$

The FOC $\frac{\partial C}{\partial d_{ij}} = 0$ implies

$$d_{ij} = \frac{\theta_i Y_j}{\theta_j Y_i}\sigma_{ij} Q_i, \tag{14}$$

therefore

$$\begin{aligned}
Y_j = Q_j - \sum_{i=1}^{N} d_{ij} &= Q_j - \sum_{i=1}^{N} \frac{\theta_i Y_j}{\theta_j Y_i}\sigma_{ij} Q_i, \\
Y_j[1 + \frac{1}{\theta_j}\sum_i (\frac{\theta_i Q_i}{Y_i}\sigma_{ij})] &= Q_j.
\end{aligned}$$

Let $\chi_j = \frac{Y_i}{Q_i}$, $\{\chi_i\}_{i=1}^N$ solve the following equations:

$$\frac{1}{\chi_i} = 1 + \frac{1}{\theta_i}\sum_s (\frac{\theta_s}{\chi_s}\sigma_{si}) \tag{15}$$

or

$$1 - \chi_j = \sum_i \sigma_{ij}\frac{\theta_i \chi_j}{\theta_j \chi_i}.$$

Let $\gamma_{ij} = \frac{\theta_i \chi_j}{\theta_j \chi_i}\sigma_{ij}$ in equation 14, then $d_{ij} = \gamma_{ij} Q_j$. The market clearing condition for Q_i implies

$$\chi_i = 1 - \sum_s \gamma_{si}.$$

The FOC $\frac{\partial C}{\partial m_{ij}} = 0$ implies

$$m_{ij} = \theta_i \frac{Y}{Y_i} Q_i \frac{\lambda_{ij}}{\bar{P}_j}. \tag{16}$$

Since

$$Y = \prod_i Y_i^{\theta_i} = \prod_i (\chi_i Q_i)^{\theta_i}$$

we have

$$m_{ij} = \theta_i \prod_s (\frac{\chi_s}{\chi_i})^{\theta_s} \prod_s (Q_s)^{\theta_s} \frac{\lambda_{ij}}{\bar{P}_j}. \tag{17}$$

The FOC $\frac{\partial C}{\partial K_i} = 0$ and $\frac{\partial C}{\partial L_i} = 0$ lead to

$$K_i = \chi_{Ki} K \tag{18}$$

$$L_i = \chi_{Li} L \tag{19}$$

where

$$\chi_{Ki} = \frac{\frac{\theta_i \alpha_i (1-\sigma_i-\lambda_i)}{(1-\sum_j \gamma_{ji})}}{\sum_s \frac{\theta_s \alpha_s (1-\sigma_s-\lambda_s)}{(1-\sum_j \gamma_{js})}}, \chi_{Li} = \frac{\frac{\theta_i (1-\alpha_i)(1-\sigma_i-\lambda_i)}{(1-\sum_j \gamma_{ji})}}{\sum_s \frac{\theta_s (1-\alpha_s)(1-\sigma_s-\lambda_s)}{(1-\sum_j \gamma_{js})}}. \tag{20}$$

Solve for Q_i Rewrite production function of Q_i as

$$\begin{aligned} Q_i &= A_i (K_i^{\alpha_i} L_i^{1-\alpha_i})^{1-\sigma_i-\lambda_i} (\gamma_{i1} Q_1)^{\sigma_{i1}} \cdots (\gamma_{iN} Q_N)^{\sigma_{iN}} \prod_{j=1}^{N} \{\theta_i \prod_s (\frac{\chi_s}{\chi_i})^{\theta_s} \prod_s (Q_s)^{\theta_s} \frac{\lambda_{ij}}{\bar{P}_j}\}^{\lambda_{ij}} \\ &= A_i (K_i^{\alpha_i} L_i^{1-\alpha_i})^{1-\sigma_i-\lambda_i} (\prod_{j=1}^{N} \gamma_{ij}^{\sigma_{ij}}) (\prod_{j=1}^{N} Q_j^{\sigma_{ij}}) [\prod_s (Q_s)^{\theta_s}]^{\lambda_i} [\theta_i \prod_s (\frac{\chi_s}{\chi_i})^{\theta_s}]^{\lambda_i} \prod_{j=1}^{N} (\frac{\lambda_{ij}}{\bar{P}_j})^{\lambda_{ij}} \\ &= A_i [(\chi_{Ki} K)^{\alpha_i} (\chi_{Li} L)^{1-\alpha_i}]^{1-\sigma_i-\lambda_i} (\prod_{j=1}^{N} \gamma_{ij}^{\sigma_{ij}}) [\theta_i \prod_s (\frac{\chi_s}{\chi_i})^{\theta_s}]^{\lambda_i} \prod_{j=1}^{N} (\frac{\lambda_{ij}}{\bar{P}_j})^{\lambda_{ij}} (\prod_{s=1}^{N} Q_s^{\sigma_{is}+\lambda_i \theta_s}) \end{aligned}$$

Define

$$\chi_{Qi} = A_i [(\chi_{Ki} K)^{\alpha_i} (\chi_{Li} L)^{1-\alpha_i}]^{1-\sigma_i-\lambda_i} (\prod_{j=1}^{N} \gamma_{ij}^{\sigma_{ij}}) [\theta_i \prod_s (\frac{\chi_s}{\chi_i})^{\theta_s}]^{\lambda_i} \prod_{j=1}^{N} (\frac{\lambda_{ij}}{\bar{P}_j})^{\lambda_{ij}} \tag{21}$$

The above equation can be written as

$$Q_i = \chi_{Qi} (\prod_{s=1}^{N} Q_s^{\sigma_{is}+\lambda_i \theta_s}). \tag{22}$$

Taking log of equation 22 gives $\log Q_i = \log \chi_{Qi} + \sum_{j=1}^{N} [(\sigma_{ij} + \lambda_i \theta_j) \log(Q_j)]$. Letting $x = [\log(Q_1), \ldots, \log(Q_N)]'_{N\times 1}$, equation 22 can be written as

$$x_{N\times 1} = b_{N\times 1} + \Omega_{N\times N} x_{N\times 1},$$

where $b(i) = \log \chi_{Qi}$ and $\Omega(i,j) = \sigma_{ij} + \lambda_i \theta_j$. Therefore x can be solved as $x = Cb$, where $C_{N\times N} = (I - \Omega)^{-1}$ and $Q_n = \prod_{i=1}^{N} (\chi_{Qi}^{C_{ni}})$.

Rewrite χ_{Qi} as

$$\chi_{Qi} = z_i K^{\alpha_i (1-\sigma_i-\lambda_i)} L^{(1-\alpha_i)(1-\sigma_i-\lambda_i)}$$

where $z_i = A_i ({\chi_{Ki}}^{\alpha_i} {\chi_{Li}}^{1-\alpha_i})^{1-\sigma_i-\lambda_i} (\prod_{j=1}^{N} \gamma_{ij}^{\sigma_{ij}}) [\theta_i \prod_s (\frac{\chi_s}{\chi_i})^{\theta_s}]^{\lambda_i} \prod_{j=1}^{N} (\frac{\lambda_{ij}}{\bar{P}_j})^{\lambda_{ij}}$.

Then Q_n can be rewritten as

$$Q_n = \tilde{A}_n K^{\tilde{\alpha}_n} L^{\tilde{\beta}_n} \tag{23}$$

where $\tilde{A}_n = \{\prod_{i=1}^N z_i{}^{C_{ni}}\}$, and $\tilde{\alpha}_n = \sum_i(\alpha_i(1-\sigma_i-\lambda_i)C_{ni})$, $\tilde{\beta}_n = \sum_i((1-\alpha_i)(1-\sigma_i-\lambda_i)C_{ni})$.

Next we show that $\tilde{\alpha}_n + \tilde{\beta}_n = 1$. Let $B = I - \Omega$, following the definition of Ω

$$\sum_j B(i,j) = 1 - (\sigma_i + \lambda_i).$$

Since

$$\tilde{\alpha}_n + \tilde{\beta}_n = \sum_i (C_{ni}(1-\sigma_i-\lambda_i)) = \sum_i \sum_j C(n,i)B(i,j).$$

By definition, $BC = CB = I$, then $\sum_j \sum_i C(n,i)B(i,j) = 1$, for any n. Therefore $(\tilde{\alpha}_n+\tilde{\beta}_n) = 1$.

Aggregate output under optimal allocation can be written as a function of aggregate capital K and aggregate labor L

$$Y = \bar{A}K^{\bar{\alpha}}L^{\bar{\beta}},$$

where $\bar{A} = \prod_{i=1}^N (\chi_i \tilde{A}_i)^{\theta_i}$ is the aggregate TFP under optimal allocation, and $\bar{\alpha} = \sum_n(\tilde{\alpha}_n\theta_n)$, $\bar{\beta} = \sum_n(\tilde{\beta}_n\theta_n)$. It can be shown easily that $\bar{\alpha} + \bar{\beta} = 1$.

The expenditure on imported goods is

$$\bar{P}_j m_{ij} = [\prod_s (\chi_s \tilde{A}_s)^{\theta_s}]\{\frac{\theta_i}{\chi_i} K^{\sum_s \theta_s \tilde{\alpha}_s} L^{\sum_s \theta_s \tilde{\beta}_s}\}\lambda_{ij} = (\frac{\theta_i \lambda_{ij}}{\chi_i})Y.$$

The total expenditure on imported goods is

$$X = [\sum_{i=1}^N (\frac{\theta_i \lambda_i}{\chi_i})]Y.$$

The output net of imported goods is

$$Y - X = Y[1 - \sum_{i=1}^N (\frac{\theta_i \lambda_i}{\chi_i})].$$

Q.E.D.

C.3 Proof of Proposition 3

According to the definition of E_t, the following equation holds: $Y_t = Y_t^* \mathbf{E}_t$. Dividing both sides by the aggregate labor inputs yields $LP_t = LP_t^* \mathbf{E}_t$. Taking a log difference on both sides yields $\Delta \log(LP_t) = \Delta \log(LP_t^*) + \Delta \log \mathbf{E}_t$. *Q.E.D.*

C.4 Proof of Proposition 4

To prove this proposition, we need to show that $E_t^m = E_t^d = E_t^y = 1$ in equation 4. It is sufficient to show that $\chi_{i,t}^y = \chi_{i,t}^{y*}$, where $\chi_{i,t}^{y*}$ is the optimal allocation and $\chi_{i,t}^y$ is its data analog.

To simplify notations, we drop the time subscript. Note that x_i^{y*} is a solution to the system of equation 2. We intend to show that χ_i^y also satisfies equation 2, which we have reproduced here: $\frac{1}{\chi_i^y} = 1 + \frac{1}{\theta_i}\sum_s(\frac{\theta_s}{\chi_s^y}\sigma_{si})$. Letting $\eta_i = \frac{1}{\chi_i^y}$, the above system of equations can be written as,

$$\eta_i = 1 + \sum_s (\eta_s \frac{\theta_s}{\theta_i}\sigma_{si}), \forall i \in \{1, ..., N\},$$

equivalently,

$$\begin{pmatrix} \eta_1 - 1 \\ \vdots \\ \eta_N - 1 \end{pmatrix} = \underbrace{\begin{pmatrix} \sigma_{11} & \cdots & \frac{\theta_N}{\theta_1}\sigma_{N1} \\ \vdots & \ddots & \vdots \\ \frac{\theta_1}{\theta_N}\sigma_{1N} & \cdots & \sigma_{NN} \end{pmatrix}}_{\Pi} \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_N \end{pmatrix} \tag{24}$$

in which $\Pi(i, j) = \frac{\theta_j}{\theta_i}\sigma_{ji}$.

We compute the data allocation of intermediate inputs as

$$\eta_i \;=\; \frac{1}{\chi_i^y} = \frac{\$Q_i}{\$Y_i},$$

where the dollar sign \$ indicates a measure of expenditure (nominal value).

Under specification 1, we assume that the expenditure shares are undistorted in each year and thus they are equal to the output elasticities in the production functions. More specifically, under this specification, the elasticities θ_i and σ_{ij} are calculated as

$$\sigma_{ij}\theta_i = \frac{\$Y_i}{\sum_s \$Y_s}, \quad \sigma_{ij} = \frac{\$d_{ij}}{\$Q_i}.$$

Taking the data η_i, σ_{ij} and θ_i back to equation 24, the RHS can be written as

$$\begin{pmatrix} \frac{\$d_{11}}{\$Q_1} & \cdots & \frac{\$Y_N}{\$Q_N}\frac{\$d_{N1}}{\$Y_1} \\ \vdots & \ddots & \vdots \\ \frac{\$Y_1}{\$Q_1}\frac{\$d_{1N}}{\$Y_N} & \cdots & \frac{\$d_{NN}}{\$Q_N} \end{pmatrix} \begin{pmatrix} \frac{\$Q_1}{\$Y_1} \\ \vdots \\ \frac{\$Q_N}{\$Y_N} \end{pmatrix} = \begin{pmatrix} \frac{\sum_{s=1}^N \$d_{s1}}{\$Y_1} \\ \vdots \\ \frac{\sum_{s=1}^N \$d_{sN}}{\$Y_N} \end{pmatrix} = \begin{pmatrix} \frac{\$Q_1 - \chi_1\$Q_1}{\chi_1\$Q_1} \\ \vdots \\ \frac{\$Q_N - \chi_N\$Q_N}{\chi_N\$Q_N} \end{pmatrix} = \begin{pmatrix} \eta_1 - 1 \\ \vdots \\ \eta_N - 1 \end{pmatrix},$$

which is equal to the LHS of the equation. The first equality is simple algebra, the second equality holds because of the market clear condition for Q_i, and the third equality is because of the definition of η_i.[12] *Q.E.D.*

[12]Note that if intermediate inputs are measured using their quantity, i.e., $\eta_i = \frac{Q_i}{Y_i}$, the first equality no longer holds.

C.5 Proof of Proposition 5

The FOCs of the planner's problem give

$$\omega_i(\frac{Y_i}{Y})^{1-\frac{1}{\rho}} = \frac{K_i/\alpha_i}{\sum_i(K_i/\alpha_i)},$$
$$\omega_i(\frac{Y_i}{Y})^{1-\frac{1}{\rho}} = \frac{L_i/(1-\alpha_i)}{\sum_i[L_i/(1-\alpha_i)]}.$$

To simplify notations, we denote $\tilde{K}_i = K_i/\alpha_i$, $\tilde{L}_i = L_i/(1-\alpha_i)$ and define $\tilde{K} = \sum_i \tilde{K}_i$ and $\tilde{L} = \sum_i \tilde{L}_i$. It is clear that, from the FOCs, $\frac{\tilde{K}_i}{\tilde{L}_i} = \frac{\sum_i \tilde{K}_i}{\sum_i \tilde{L}_i} = \frac{\tilde{K}}{\tilde{L}}$. We can rewrite K_i and L_i using the production functions as

$$K_i = (\alpha_i\tilde{K})\omega_i^{\rho}[\frac{A_i(\alpha_i\tilde{K})^{\alpha_i}[(1-\alpha_i)\tilde{L}]^{1-\alpha_i}}{Y}]^{\rho-1},$$
$$L_i = (1-\alpha_i)\tilde{L}\omega_i^{\rho}[\frac{A_i(\alpha_i\tilde{K})^{\alpha_i}[(1-\alpha_i)\tilde{L}]^{1-\alpha_i}}{Y}]^{\rho-1}.$$

Given $\sum_i K_i = K$,$\sum_i L_i = L$, we can solve $Y, \tilde{K}, \tilde{L}$ with the system of three equations

$$K = \sum_i\{(\alpha_i\tilde{K})\omega_i^{\rho}[\frac{A_i(\alpha_i\tilde{K})^{\alpha_i}[(1-\alpha_i)\tilde{L}]^{1-\alpha_i}}{Y}]^{\rho-1}\},$$
$$L = \sum_i\{(1-\alpha_i)\tilde{L}\omega_i^{\rho}[\frac{A_i(\alpha_i\tilde{K})^{\alpha_i}[(1-\alpha_i)\tilde{L}]^{1-\alpha_i}}{Y}]^{\rho-1}\},$$
$$Y^{\rho-1} = \sum_i \omega_i^{\rho}\{A_i(\alpha_i\tilde{K})^{\alpha_i}[(1-\alpha_i)\tilde{L}]^{1-\alpha_i}\}^{\rho-1}.$$

In particular,

$$\frac{K}{\tilde{K}} = \sum_i\{\alpha_i\frac{\omega_i^{\rho}\{A_i(\alpha_i\tilde{K})^{\alpha_i}[(1-\alpha_i)\tilde{L}]^{1-\alpha_i}\}^{\rho-1}}{\sum_j \omega_j^{\rho}\{A_j(\alpha_j\tilde{K})^{\alpha_j}[(1-\alpha_j)\tilde{L}]^{1-\alpha_j}\}^{\rho-1}}\},$$
$$\frac{L}{\tilde{L}} = \sum_i\{(1-\alpha_i)\frac{\omega_i^{\rho}\{A_i(\alpha_i\tilde{K})^{\alpha_i}[(1-\alpha_i)\tilde{L}]^{1-\alpha_i}\}^{\rho-1}}{\sum_j \omega_j^{\rho}\{A_j(\alpha_j\tilde{K})^{\alpha_j}[(1-\alpha_j)\tilde{L}]^{1-\alpha_j}\}^{\rho-1}}\},$$

and $\frac{K}{\tilde{K}} + \frac{L}{\tilde{L}} = 1$.

Denote $\bar{\alpha} = \frac{K}{\tilde{K}}$, then $\frac{L}{\tilde{L}} = 1-\bar{\alpha}$, and $\bar{\alpha}$ solves the following equation:

$$\bar{\alpha} = \sum_i\{\alpha_i\frac{\omega_i^{\rho}\{A_i(\frac{\alpha_i}{\bar{\alpha}}K)^{\alpha_i}[\frac{(1-\alpha_i)}{(1-\bar{\alpha})}L]^{1-\alpha_i}\}^{\rho-1}}{\sum_j \omega_j^{\rho}\{A_j(\frac{\alpha_j}{\bar{\alpha}}K)^{\alpha_j}[\frac{(1-\alpha_j)}{(1-\bar{\alpha})}L]^{1-\alpha_j}\}^{\rho-1}}\}, \tag{25}$$

and the output under optimal allocation is

$$Y^* = \{\sum_j \omega_j^\rho \{A_j(\frac{\alpha_j}{\bar{\alpha}}K)^{\alpha_j}[\frac{(1-\alpha_j)}{(1-\bar{\alpha})}L]^{1-\alpha_j}\}^{\rho-1}\}^{\frac{1}{\rho-1}}.$$

If we replace, in the previous equation, A_i with $\frac{Y_i}{K_i^{\alpha_i}L_i^{1-\alpha_i}}$ and Y_i with $(\frac{P_iY_i/\omega_j}{PY})^{\frac{\rho}{\rho-1}}Y$, we can rewrite the allocative efficiency as

$$\begin{aligned}
\mathbf{E} = \frac{Y}{Y^*} &= \{\sum_j \omega_j^\rho \{\frac{(\frac{P_jY_j/\omega_j}{PY})^{\frac{\rho}{\rho-1}}}{K_j^{\alpha_j}L_j^{1-\alpha_j}}(\frac{\alpha_j}{\bar{\alpha}}K)^{\alpha_j}[\frac{(1-\alpha_j)}{(1-\bar{\alpha})}L]^{1-\alpha_j}\}^{\rho-1}\}^{\frac{1}{1-\rho}}, \\
&= \{\sum_j \{(\frac{P_jY_j}{PY})^{\frac{\rho}{\rho-1}}(\frac{\alpha_j/K_j}{\bar{\alpha}/K})^{\alpha_j}[\frac{(1-\alpha_j)/L_j}{(1-\bar{\alpha})/L}]^{1-\alpha_j}\}^{\rho-1}\}^{\frac{1}{1-\rho}},
\end{aligned}$$

which means that E is a function of $\bar{\alpha}$, the expenditure share $\frac{P_jY_j}{PY}$ in the data, capital allocation $\frac{K_j}{K}$ and labor allocation $\frac{L_j}{L}$ in the data. All the other measures, except for $\bar{\alpha}$, are clearly unitless. We show next that so is $\bar{\alpha}$. By replacing A_i in equation 25, we can write the equation as follows:

$$\begin{aligned}
\bar{\alpha} &= \sum_i \{\alpha_i \frac{\omega_i^\rho\{A_i(\frac{\alpha_i}{\bar{\alpha}}K)^{\alpha_i}[\frac{(1-\alpha_i)}{(1-\bar{\alpha})}L]^{1-\alpha_i}\}^{\rho-1}}{\sum_j \omega_j^\rho\{A_j(\frac{\alpha_j}{\bar{\alpha}}K)^{\alpha_j}[\frac{(1-\alpha_j)}{(1-\bar{\alpha})}L]^{1-\alpha_j}\}^{\rho-1}}\} \\
&= \sum_i \{\alpha_i \frac{\omega_i^\rho\{\frac{Y_i}{K_i^{\alpha_i}L_i^{1-\alpha_i}}(\frac{\alpha_i}{\bar{\alpha}}K)^{\alpha_i}[\frac{(1-\alpha_i)}{(1-\bar{\alpha})}L]^{1-\alpha_i}\}^{\rho-1}}{\sum_j \omega_j^\rho\{\frac{Y_j}{K_j^{\alpha_j}L_j^{1-\alpha_j}}(\frac{\alpha_j}{\bar{\alpha}}K)^{\alpha_j}[\frac{(1-\alpha_j)}{(1-\bar{\alpha})}L]^{1-\alpha_j}\}^{\rho-1}}\} \\
&= \sum_i \{\alpha_i \frac{\omega_i^\rho\{(\frac{P_iY_i/\omega_j}{PY})^{\frac{\rho}{\rho-1}}Y(\frac{\alpha_i}{\bar{\alpha}}K/K_i)^{\alpha_i}[\frac{(1-\alpha_i)}{(1-\bar{\alpha})}L/L_i]^{1-\alpha_i}\}^{\rho-1}}{\sum_j \omega_j^\rho\{(\frac{P_jY_j/\omega_j}{PY})^{\frac{\rho}{\rho-1}}Y(\frac{\alpha_j}{\bar{\alpha}}K/K_j)^{\alpha_j}[\frac{(1-\alpha_j)}{(1-\bar{\alpha})}L/L_j]^{1-\alpha_j}\}^{\rho-1}}\} \\
&= \sum_i \{\alpha_i \frac{Y^{\rho-1}\{(\frac{P_iY_i}{PY})^{\frac{\rho}{\rho-1}}(\frac{\alpha_i}{\bar{\alpha}}K/K_i)^{\alpha_i}[\frac{(1-\alpha_i)}{(1-\bar{\alpha})}L/L_i]^{1-\alpha_i}\}^{\rho-1}}{\sum_j Y^{\rho-1}\{(\frac{P_jY_j/\omega_j}{PY})^{\frac{\rho}{\rho-1}}(\frac{\alpha_j}{\bar{\alpha}}K/K_j)^{\alpha_j}[\frac{(1-\alpha_j)}{(1-\bar{\alpha})}L/L_j]^{1-\alpha_j}\}^{\rho-1}}\} \\
&= \sum_i \{\alpha_i \frac{\{(\frac{P_iY_i}{PY})^{\frac{\rho}{\rho-1}}(\frac{\alpha_i}{\bar{\alpha}}\frac{K}{K_i})^{\alpha_i}(\frac{1-\alpha_i}{1-\bar{\alpha}}\frac{L}{L_i})^{1-\alpha_i}\}^{\rho-1}}{\sum_j\{(\frac{P_jY_j}{PY})^{\frac{\rho}{\rho-1}}(\frac{\alpha_j}{\bar{\alpha}}\frac{K}{K_j})^{\alpha_j}[\frac{(1-\alpha_j)}{(1-\bar{\alpha})}\frac{L}{L_j}]^{1-\alpha_j}\}^{\rho-1}}\},
\end{aligned} \tag{26}$$

which makes it clear that $\bar{\alpha}$ only depends on the expenditure share $\frac{P_jY_j}{PY}$ in the data, capital allocation $\frac{K_j}{K}$ and labor allocation $\frac{L_j}{L}$ in the data. Note that $\bar{\alpha}$ is unitless.

In addition, one can easily verify that $\bar{\alpha} = \sum_i \frac{P_iY_i}{PY}\alpha_i$ and the following allocation of capital and labor

$$\frac{K_i}{K} = \frac{\frac{P_iY_i}{PY}\alpha_i}{\sum_i \frac{P_iY_i}{PY}\alpha_i} = \frac{\frac{P_iY_i}{PY}\alpha_i}{\bar{\alpha}} \text{ and } \frac{L_i}{L} = \frac{\frac{P_iY_i}{PY}(1-\alpha_i)}{\sum_i \frac{P_iY_i}{PY}(1-\alpha_i)} = \frac{\frac{P_iY_i}{PY}(1-\alpha_i)}{1-\bar{\alpha}}$$

solve equations 8 and 26 and therefore are the optimal allocation. We denote $\alpha^* = \sum_i \frac{P_iY_i}{PY}\alpha_i$,

$\chi_i^{k*} = \frac{\frac{P_i Y_i}{PY}\alpha_i}{\sum_i \frac{P_i Y_i}{PY}\alpha_i}$ and $\chi^{l*} = \frac{\frac{P_i Y_i}{PY}(1-\alpha_i)}{\sum_i \frac{P_i Y_i}{PY}(1-\alpha_i)}$. Note that the optimal allocation of capital and labor does not depend on the elasticity of substitution ρ. We can rewrite the allocative efficiency as

$$\begin{aligned}
\mathbf{E} &= \{\sum_j \{(\frac{P_j Y_j}{PY})^{\frac{\rho}{\rho-1}}(\frac{\alpha_j/K_j}{\bar{\alpha}/K})^{\alpha_j}[\frac{(1-\alpha_j)/L_j}{(1-\bar{\alpha})/L}]^{1-\alpha_j}\}^{\rho-1}\}^{\frac{1}{1-\rho}} \\
&= \{\sum_j \{(\frac{P_j Y_j}{PY})^{\frac{\rho}{\rho-1}}(\frac{\alpha_j/K_j}{\bar{\alpha}/K})^{\alpha_j}[\frac{(1-\alpha_j)/L_j}{(1-\bar{\alpha})/L}]^{1-\alpha_j}\}^{\rho-1}\}^{\frac{1}{1-\rho}}
\end{aligned}$$

C.6 Proof of Proposition 6

Following proposition 2 and the decentralized problem in section C.2, we know that

$$A^* = \frac{Y^*}{K^{\alpha^*}L^{1-\alpha^*}}.$$

Since in the data,

$$A = \frac{Y}{K^{\alpha}L^{1-\alpha}},$$

we can write the optimal TFP A^* as

$$A^* = A\frac{Y^*}{Y}K^{\alpha-\alpha^*}L^{\alpha^*-\alpha} = A\frac{1}{\mathbf{E}}(\frac{K}{L})^{\alpha-\alpha^*}.$$

C.7 Proof of Proposition 7

In this section, we show that $\alpha = \alpha^*$ when the economy has no distortions. From the FOCs of the decentralized problem, we know that $RK_i = p_i Q_i \alpha_i (1-\sigma_i)$ if there are no distortions in the economy. Therefore, the measured capital income share in the data α_t can be written as

$$\alpha = \frac{\sum_i RK_i}{Y} = \frac{\sum_i p_i Q_i \alpha_i (1-\sigma_i)}{Y}.$$

Following the notations of the decentralized problem (see Appendix A.3), denote $\gamma_i = \frac{p_i Q_i}{Y}$ as the Domar weight for sector i. We can rewrite the above equation as

$$\alpha = \sum_i \gamma_i \alpha_i (1-\sigma_i).$$

On the other hand, α^* can be written as

$$\begin{aligned}
\alpha^* &= \sum_n \theta_n \sum_i (\alpha_i (1-\sigma_i) C_{ni}) \\
&= \sum_i \alpha_i (1-\sigma_i) \sum_n \theta_n C_{ni},
\end{aligned}$$

where $C = (I-\Omega)^{-1}$. We've shown in the decentralization problem that $\gamma_i = \sum_n \theta_n C_{ni}$, which implies $\alpha = \alpha^*$.