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Optimal Taxation in Asset Markets with Adverse Selection

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Abstract

Constrained efficiency is characterized in an asset market, subject to search frictions, where sellers are privately informed about the type of their asset. The type determines the opportunity cost of the asset for sellers and the quality of the asset for buyers. The constrained efficient allocation can be implemented using a sales tax schedule. The role of these taxes is to redistribute resources between different types of sellers to relax incentive constraints. The optimal tax schedule strictly increases welfare compared with the laissez-faire equilibrium, can sometimes lead to an allocation that Pareto dominates the equilibrium, and can sometimes lead to the first-best allocation (i.e., taxation can correct all inefficiencies caused by adverse selection).

The shape of the optimal tax schedule is also investigated. If the quality of assets for buyers is a monotonic function of the sellers' opportunity cost (e.g., more distressed sellers have lower-quality assets), the schedule requires that the trading of low-quality assets be subsidized and trading of high-quality assets be taxed, although the schedule is not necessarily monotone in the quality or price of the assets. Otherwise, trading of some low-quality assets may be taxed and trading of some high-quality assets may be subsidized.

Topics: Economic models; Financial markets; Financial system regulation and policies; Market structure and pricing
JEL codes: D82, D83, E24, G10, J31, J64

1 Introduction

Adverse selection and search frictions are prevalent in the asset, insurance, labor and housing markets. For example, consider markets for assets traded over the counter such as mortgage-backed securities, collateralized debt obligations, structured credit products and corporate bonds. Sellers in these markets may have private information about the value of their assets, and they must incur search costs to find buyers for their assets. During the financial crisis, activity in some of these markets declined to close to zero (Gorton and Metrick (2012)), and the government/central bank undertook various policies to retrieve trading. Many policy questions have arisen ever since, one of which is whether government interventions are a good policy from a social point of view. Guerrieri and Shimer (2014a) study the effects of asset purchase and asset subsidy programs by an entity with deep pockets and show that these programs can increase the liquidity and price of assets, therefore saving the market from a liquidity crisis. I contribute to this discussion by studying the optimal policy that is budget balanced in a model of the asset market with adverse selection, which is a static version of Chang's (2018) model.[1]

The model economy is populated by a continuum of sellers of a fixed measure. Sellers have private information about their type, which determines the opportunity cost of the asset to them (their liquidity need) and the value of the asset to the buyer (the quality of the asset). A large number of buyers, whose population is determined through free entry, can enter the market and post prices to attract sellers. Sellers choose the price at which they want to trade. Sellers and buyers who want to trade with a given price form a submarket in which they match bilaterally, subject to a matching technology, and trade if matched successfully.

In the laissez-faire economy, if the quality of assets for buyers is a monotonic function of the sellers' opportunity cost (e.g., more distressed sellers have lower-quality assets), then there is a unique, separating equilibrium, where homogeneous buyers offer different prices to attract different types of sellers.[2] For example, for the case in which the quality of assets is an increasing function, buyers offering higher prices attract sellers with a higher opportunity cost. Those sellers are matched with a lower probability (i.e., sell more slowly) in the equi-

[1] The main ideas regarding equilibrium and constrained efficiency are captured in this static model.

[2] As a minor point, this result is an extension of Chang (2018) to the cases in which the quality of assets can be either increasing or decreasing in the sellers' opportunity cost. Chang's results are only for the increasing case.

librium with adverse selection compared with the equilibrium with complete information. If the quality of assets for buyers is not a monotonic function, then fully separating equilibrium does not exist, because there is a tension between respecting the incentive compatibility constraint of sellers and the free-entry condition of buyers. For example, the buyers would offer sellers with a higher opportunity cost a higher price, but the quality of those sellers' assets would not be high enough to compensate the buyers for their entry costs.

I study constrained efficiency in this environment. I characterize the problem of a planner who maximizes welfare, the ex-ante payoff to all types, subject to the matching technology and to the incentive compatibility constraints associated with the sellers' private information about their type. I find that the laissez-faire equilibrium is always constrained inefficient. That is, the constrained efficient allocation achieves strictly higher welfare compared with the equilibrium and can sometimes lead to allocations that Pareto dominate the equilibrium. For example, for the case in which the quality of assets is an increasing function of the sellers' opportunity cost, the probability of matching for sellers in the constrained-efficient allocation is higher compared with the laissez-faire equilibrium; i.e., sellers can trade more and higher surplus is created in the constrained efficient allocation.

The constrained inefficiency of the equilibrium can be explained in terms of externalities. In the equilibrium, entry of one more buyer to a submarket changes the payoff to sellers in that submarket and, through incentive compatibility constraints, changes the set of prices that the buyers in other submarkets can offer, and eventually changes the payoff to sellers in other submarkets. The planner takes this externality into account and is therefore able to increase welfare.

The constrained efficient allocation can be implemented using a submarket-specific tax schedule. The role of these taxes is to redistribute resources between different types of sellers to relax incentive constraints. I derive conditions under which the optimal tax schedule leads to the first-best allocation.[3] That is, using an appropriate set of submarket-specific taxes and subsidies can correct all inefficiencies caused by adverse selection. More generally, I characterize the optimal taxation in this environment even if the first-best cannot be achieved.

I study the shape of the optimal tax schedule. I explore trading of which assets should be taxed and trading of which assets should be subsidized, and whether the optimal tax

[3]The first-best allocation is the welfare-maximizing allocation under complete information given the matching technology.

schedule should be monotonic in the price or quality of assets.[4] First suppose the quality of assets for buyers is a monotonic function of the sellers' opportunity cost. The optimal taxation requires the trading of sufficiently low-quality assets to be subsidized and trading of other assets to be taxed. Interestingly, the optimal tax schedule is sometimes non-monotonic in the quality or price of the assets, e.g., under certain conditions, trading of some assets with a low quality is subject to a low subsidy, trading of some assets with an intermediate quality is subject to a high subsidy and trading of some assets with a high quality is subject to a positive tax.[5] If the quality is not a monotonic function of the opportunity cost, the optimal taxation sometimes requires trading of some low-quality assets to be taxed and trading of some high-quality assets to be subsidized.[6] These results are different from conventional wisdom suggesting that the higher the quality of the asset, the higher the corrective taxes should be.[7] Intuitively, in the present paper, an increasing tax schedule cannot respect the incentive constraints of sellers together with the free-entry condition of buyers.

Related Literature. From a theoretical point of view, this paper is closely related to a companion paper, Davoodalhosseini (2019), in which I show that the equilibrium is constrained inefficient in an environment with adverse selection and directed search using a discrete-type space. That environment was first introduced by Guerrieri et al. (2010). There are important differences between the present paper and Davoodalhosseini (2019). First, the focus of this paper is on the implementation of the constrained efficient allocation and the shape of the optimal tax schedule, while the focus of that paper is on the constrained efficient allocation, not on the tax schedule that implements it. Second, in this paper I extend the environment of that paper along two dimensions: (i) I extend the environment to a continuous-type space, which is easier to work with compared with a discrete-type space in Davoodalhosseini (2019). This is not a trivial extension, because several proofs must be revisited. Specifically, the proof for the constrained inefficiency of equilibrium is different

[4]The implementation of a monotonic tax schedule is less sensitive to the planner's knowledge of the distribution of types and is easier in practice, because it can be approximated by a linear function (a lump-sum transfer and a fixed marginal tax/subsidy).

[5]For the tax schedule to be monotonic, the quality of assets for buyers should be sufficiently increasing or sufficiently decreasing in the sellers' opportunity cost.

[6]I further show that if the planner can use entry tax in addition to sales tax, it is always possible to find monotonic tax schedules in the price and quality of assets.

[7]The argument for that is based on a two-type example of Akerlof's (1970) lemons market, which has been studied since the early 1970s. A version of a two-type example of a market with adverse selection with search frictions is studied in Section 4 of Davoodalhosseini (2019).

from that paper and implies a different allocation and cross-subsidization scheme. In the present paper, the planner uses taxation to increase liquidity for *all* assets in the market, while in that paper, the planner changes the allocation only for *lower* types. (ii) I extend the analysis to the cases in which the quality of assets for buyers can be not only monotonic but also a non-monotonic function of the sellers' opportunity cost. This important case is not covered in Davoodalhosseini (2019) nor in Guerrieri et al. (2010), and has new implications.[8] In those papers, it has been shown that when the equilibrium is separating, the equilibrium is dominated by a pooling or semi-pooling allocation in terms of efficiency. I show in this paper that, in the cases where the equilibrium is pooling—which is the case where the quality of assets for buyers is a non-monotonic function of the sellers' opportunity cost—the planner can design an optimal tax schedule to achieve higher welfare via a separating allocation. In other words, those papers show that sometimes the laissez-faire equilibrium is separating and a pooling allocation can improve efficiency, and I show here that sometimes the laissez-faire equilibrium is pooling and a separating allocation can improve efficiency.

From an applied point of view, this paper is related to Chang (2018) and Guerrieri and Shimer (2014a), who use similar models. None of them study optimal taxation with a balanced budget. Guerrieri and Shimer (2014a) study implications of asset subsidy and asset purchase programs and show that these programs can improve the liquidity of assets. I study the optimal policy under the frictions of the environment using a budget-balanced policy. Chang (2018) characterizes equilibrium and shows that if the quality of assets for buyers is an increasing function of the sellers' opportunity cost, then there exists a unique, separating equilibrium. If the quality is not a monotonic function of the opportunity cost, pooling or semi-pooling equilibrium with ironing or fire sales will arise. My results imply that there exists a tax schedule that improves welfare relative to the equilibrium and may even lead to the first-best allocation.

Other papers have used models with search and information frictions to study over-the-counter markets and to understand the effects of government interventions. For example, Chiu and Koeppl (2016) study the optimal design of government asset purchase programs and focus on the announcement effect of such programs. Fuchs and Skrzypacz (2015) also study the timing of optimal policy, albeit in a model without search frictions, and show that subsidizing early and taxing later is optimal. Camargo and Lester (2014) show that the

[8]In Assumption 1(i) of Davoodalhosseini (2019) or Assumption A1 of Guerrieri et al. (2010), the value of the asset to buyers should be a monotonic function of types.

effects of government interventions depend on their size and duration.

Philippon and Skreta (2012) study interventions aimed at stabilizing the financial markets affected by adverse selection. They focus, among other things, on the stigma that participants in government programs may face, because their participation may reveal information about their types. Tirole (2012) studies the interventions when the government's proposed mechanism affects the set of participants, which affects the market outcome, which in turn affects the reservation payoff of participants in the mechanism. In short, reservation payoffs of participants depend on the proposed mechanism itself.[9]

The paper is organized as follows. I introduce the environment of the model in Section 2 and define the planner's problem. In Section 3, I characterize the constrained efficient allocation and compare it with the equilibrium allocation. In Section 4, I study the shape of the optimal tax schedule along with several examples. I conclude in Section 5. All proofs appear in the Appendix.

2 Model

I introduce the environment of the model of an asset market with a continuous-type space first and then define the constrained efficient allocation (i.e., the planner's problem) for that. By revelation principle, I show in Lemma 1 that using a direct mechanism is equivalent to using submarket-specific taxes and subsidies. Therefore, I focus on the direct mechanism in most of the paper. Next, I characterize the optimal tax schedule that implements the constrained efficient allocation.

[9]More broadly, my paper is related to the literature on directed search and on adverse selection. For a non-exhaustive list of contributions on directed search, see Peters (1991), Moen (1997), Acemoglu and Shimer (1999), Shi (2001, 2002), Shimer (2005), Mangin and Julien (2017) and Wright et al. (2017) among many others. For some important contributions on adverse selection, see Rothschild and Stiglitz (1976), Wilson (1977), Miyazaki (1977), Spence (1978), Holmström and Myerson (1983) and Maskin and Tirole (1992). Golosov et al. (2013) study optimal taxation in a model of labor market with moral hazard where the planner cannot see whether the workers have searched or not, or toward which firms if they have. Davoodalhosseini (2020) studies an asset market with search frictions and adverse selection where some buyers are more informed than others about the quality of the assets.

2.1 Environment

There is a continuum of measure one of heterogeneous sellers indexed by $z \in Z \equiv [z_L, z_H] \subset \mathbb{R}$, with $F(z)$ denoting the measure of sellers with types below z. F is continuously differentiable and strictly increasing in z, and F' is its derivative. Type z is sellers' private information and determines the quality of the asset to buyers and the value of the asset to (or equivalently the liquidity needs of) sellers. Buyers' and sellers' payoffs are quasi-linear in a numeraire good. The payoff of a buyer who enters the market and matches with a type z is $h(z) - p - k$, where $h(z)$ denotes the value of the asset to the buyer, $p \in \mathbb{R}$ denotes the amount of the numeraire good that he produces and k is the entry cost, all in terms of the numeraire good. His payoff is $-k$ if unmatched. The payoff of a type z seller matched with a buyer is $p - c(z)$, where $c(z)$ denotes the value of the asset to the seller and $p \in \mathbb{R}$ denotes the amount of the numeraire good that he consumes in terms of the numeraire good. His payoff is 0 if unmatched. Functions $h : Z \to \mathbb{R}$ and $c : Z \to \mathbb{R}$ are twice continuously differentiable almost everywhere. I assume c is strictly increasing.[10] Matching function $m(.)$ is increasing, strictly concave and twice continuously differentiable.

As a benchmark, I characterize the complete information or first-best (FB) allocation: $U^{FB}(z) = \max_\theta\{m(\theta)(h(z) - c(z)) - k\theta\}$, and $\theta^{FB}(z) \equiv \arg\max_\theta\{m(\theta)(h(z) - c(z)) - k\theta\}$, so

$$m'(\theta^{FB}(z))(h(z) - c(z)) = k, \tag{1}$$

for both the planner's problem with complete information and the laissez-faire economy with complete information. I assume for simplicity that there are positive gains from trade for all types; i.e., $U^{FB}(z) > 0$ for all z.

2.2 Constrained Efficient Allocation

I first describe the laissez-faire economy in which the planner imposes no taxation. The buyers simply post prices and commit to them. Given the distribution of posted prices, the sellers decide which price (i.e., submarket) they want to visit. When contemplating posting a price, buyers take into account what type of sellers will be attracted to that submarket and what the market tightness at that submarket will be. Given the posted prices, the sellers

[10] One could assume without loss of generality that $c(z) = z$ throughout the paper and no insights would be lost.

choose to go to a submarket that maximizes their payoff. Buyers and sellers share the same beliefs about the market tightness in every submarket.

Now we can define the planner's problem. The planner uses two types of transfers: $\tilde{t}(p)$ is a sales tax levied on buyers who want to buy at price p, and $\tilde{t}_0$ is a lump-sum transfer paid to sellers. We call $\{\tilde{t}(.), \tilde{t}_0\}$ a policy.[11] Given a policy, buyers and sellers engage in the same game that they play in the laissez-faire economy. That is, for a given policy, buyers choose what price they want to post, and given the posted prices, the sellers choose what submarket they want to visit. I define the implementable allocation first and then the constrained efficient allocation, i.e., the welfare-maximizing implementable allocation. In the laissez-faire economy (i.e., under the policy of zero taxes where $\tilde{t}_0 = 0$ and $\tilde{t}(.) = 0$), the definition of implementable allocation reduces to the static version of equilibrium definition in Chang (2018).

An allocation is a four-tuple $\{G, \mathcal{P}, \Theta, \mu\}$ where $G(p)$ denotes the measure of buyers posting prices below p, and $\mathcal{P}$ denotes the support of G, so $\mathcal{P}$ encompasses all prices that are posted—i.e., all submarkets that attract some buyers and some sellers—in the allocation. Θ denotes the market tightness—i.e., the ratio of buyers to sellers—at every submarket p. Finally, μ denotes the distribution of types at every price p.

Definition 1. *An allocation $\{G, \mathcal{P}, \Theta, \mu\}$ is implementable through policy $\{\tilde{t}(.), \tilde{t}_0\}$ if the following conditions are satisfied:*

(i) ***Buyers' profit maximization and free entry***

For any $p \in \mathbb{P}$,

$$q(\Theta(p))\Big(\int h(z)\mu(z|p)dz - p - \tilde{t}(p)\Big) \leq k,$$

with equality if $p \in \mathcal{P}$.

(ii) ***Sellers' optimal search***

Let $U(z) = \max\left\{0, \max_{p' \in \mathcal{P}}\left\{m(\Theta(p'))(p' - c(z))\right\}\right\} + \tilde{t}_0$ and $U(z) = \tilde{t}_0$ if $\mathcal{P} = \emptyset$. Then, for any $p \in \mathbb{P}$ and z, $U(z) \geq m(\Theta(p))(p - c(z)) + \tilde{t}_0$ with equality if $\Theta(p) < \infty$ and $\mu(z|p) > 0$. Moreover, if $p - c(z) < 0$, either $\Theta(p) = \infty$ or $\mu(z|p) = 0$.

(iii) ***Feasibility or market clearing***

For all z, $\int_{\mathcal{P}} \frac{\mu(z|p)}{\Theta(p)} dG(p) \leq F'(z)$, with equality if $U(z) > \tilde{t}_0$.

[11] I assume in this paper that there are positive gains from trade for all types, so $\tilde{t}_0$ is redundant and set to 0 because it can be incorporated into $\tilde{t}(.)$. I included $\tilde{t}_0$ so that the model can be easily used even if there are no gains from trade for some types so it may be optimal that they do not participate in the market.

(iv) ***Planner's budget-balance condition***

$$\int_{\mathcal{P}} q(\Theta(p))\tilde{t}(p)dG(p) \geq \tilde{t}_0.$$

This definition describes, for a given policy, what the resulting allocation would be when agents play their optimal strategies, where the planner is subject to a budget-balance condition. Given a policy $\{\tilde{t}(.), \tilde{t}_0\}$, condition (i) states that buyers maximize their profit by posting a price (i.e., choosing a submarket to enter), and because of the free-entry condition, they get a zero profit if that price is actually posted in the market. Condition (ii) states that sellers choose a submarket that maximizes their payoff. To pin down the type of sellers and the market tightness at each submarket, I impose the same belief restrictions that Guerrieri et al. (2010) introduce. Since these restrictions have been extensively discussed in other papers, I do not discuss them here.[12] Condition (iii) simply states that the total measure of each type of sellers allocated to different submarkets should not exceed their measure in the population. Condition (iv) is simply a budget-balanced condition stating that all payments to sellers should be financed by the taxes levied on buyers not from external sources.

Definition 2. *A constrained efficient allocation is an implementable allocation that maximizes welfare among all implementable allocations. That is, a constrained efficient allocation solves the following problem:*

$$\max_{\{G,\mathcal{P},\Theta,\mu\},\{\tilde{t}(.),\tilde{t}_0\}} \int U(z)dF(z)$$

$$\textit{subject to } \{G, \mathcal{P}, \Theta, \mu\} \textit{ is implementable through } \{\tilde{t}(.), \tilde{t}_0\},$$

where $U(z)$ is defined in part (ii) of Definition 1.

Using the revelation principle, we can assume without loss of generality that sellers are allocated to different submarkets through a direct mechanism. Let $\{\theta(.), p(.), t(.), t_0\}$ be a direct mechanism. In a direct mechanism, the planner allocates each seller a market tightness, $\theta : Z \to \mathbb{R}_+$, and a transfer conditional on finding a match, $p : Z \to \mathbb{R}$, depending on the seller's self-reported type, and an unconditional transfer, $t_0 \in R_+$. Also, the planner charges each buyer who is matched $t : Z \to \mathbb{R}$ units of the numeraire good based on the type of the

[12] Guerrieri et al. (2010) elaborate on these restrictions and Chang (2018) uses them too. The Appendix of Davoodalhosseini (2019) includes the rationales behind these restrictions.

seller that the buyer is matched with.[13] A feasible mechanism and an optimal mechanism, which are counterparts of these definitions in the discrete-type space (Definitions 1 and 2 in Davoodalhosseini (2019)) are defined below.

Definition 3. *A feasible mechanism is a set* $\{(\theta(.), p(.), t(.), t_0)\}$ *such that the following conditions hold:*

*(i)**Incentive compatibility of sellers:** For all* z *and* $\hat{z}$*,*

$$U(z) \equiv m(\theta(z))(p(z) - c(z)) + t_0 \geq U(z, \hat{z}) \equiv m(\theta(\hat{z}))(p(\hat{z}) - c(z)) + t_0.$$

*(ii) **Participation constraint of sellers:** For all* z*,*

$$U(z) \geq 0.$$

*(iii) **Buyers' zero profit condition:** For all* z*,*

$$q(\theta(z))(h(z) - p(z) - t(z)) - k = 0.$$

*(iv) **Planner's budget-balance condition***

$$\int m(\theta(z))t(z)dF(z) \geq t_0.$$

Definition 4. *An optimal mechanism is a feasible mechanism that maximizes welfare among all feasible mechanisms.*

The lemma below states that we can use direct mechanisms without loss of generality. That is, we can characterize the optimal mechanism first, then find a policy and an allocation, which is implementable through that policy, in which all sellers receive the same payoff as in the direct mechanism.

Lemma 1. *Assume* $c'(z) > 0$ *for all* z*. Take any feasible mechanism in which all types receive a strictly positive payoff. Then there exists an associated implementable allocation under which all types receive exactly the same payoff as in the direct mechanism.*

Elements of the first-best, constrained efficient and equilibrium allocations are denoted, respectively, by superscript FB, CE and EQ.

[13] Notation-wise, when variable p is used as a function, the price function in a direct mechanism is concerned, but if it is used independently, an individual price in the market is concerned. Also, whenever $\tilde{}$ is used for a variable, it indicates that an allocation, not a direct mechanism, is concerned.

3 Characterization

To characterize the planner's problem, as common in the mechanism design literature, I first replace the incentive compatibility constraint of sellers by other constraints—a monotonicity constraint and an envelope condition. Next, I use buyers' profit maximization together with the budget-balance condition to obtain a simpler problem for characterization of the optimal mechanism. As mentioned earlier, we assume $t_0 = 0$.[14]

In a direct mechanism, if a type z seller reports $\hat{z}$, their payoff is given by:

$$U(z, \hat{z}) \equiv \{m(\theta(\hat{z}))(p(\hat{z}) - c(z))\}.$$

Therefore, the payoff to type z is given by:

$$U(z) = \max_{\hat{z}} U(z, \hat{z}). \tag{2}$$

The incentive compatibility (IC) constraint implies that $\hat{z} = z$.

Furthermore, $t(z)$ can be substituted from part (iii) of Definition 3 into the budget-balance condition. The planner's problem then turns into:

Problem 1.

$$\max_{\theta(z),p(z)} \int \Big[m(\theta(z))(h(z) - c(z)) - k\theta(z)\Big] dF(z)$$

$$\textit{subject to: } z \in \arg\max_{\hat{z}} U(z, \hat{z}) \ \textit{(Incentive Compatibility or IC)},$$

$$U(z) \geq 0 \ \textit{(Participation Constraint or PC)},$$

$$\int \Big[m(\theta(z))(h(z) - p(z)) - k\theta(z)\Big] dF(z) = 0 \ \textit{(Budget Balance or BB)}.$$

This problem states that, to obtain an optimal mechanism, the planner maximizes the total surplus created in this economy subject to IC, PC and BB. No transfers appear in the objective function, because agents have quasi-linear preferences, so the transfers are immaterial for the planner. I have also assumed that all types participate in the mechanism. This will be verified in the proofs. Finally, the planner does not want to keep any resources because distributing any remaining resources equally across all sellers can increase welfare, so the planner's budget-balance condition has been written with equality.

[14]This is without loss of generality if all types are active—i.e., $\theta(z) > 0$—because then $p(z)$ can be changed to $p(z) + \frac{t_0}{m(\theta(z))}$. It can be easily verified that all types should be active under the requirements of Theorem 1.

Lemma 2 (Necessary and sufficient condition for incentive compatibility)**.** *Assume $c'(.) > 0$.*

(i) Take any mechanism $\{(\theta(z), ., ., .)\}$ that satisfies IC. If $\theta(z)$ is a piecewise C^1 (continuous) function, then

$$\frac{d\theta(z)}{dz} \leq 0 \tag{3}$$

wherever $\theta(z)$ is differentiable at z.

(ii) Consider any piecewise C^1 function $\theta(z)$ satisfying $\frac{d\theta(z)}{dz} \leq 0$. Then there exists transfer schedules $p(.)$ such that the mechanism $\{(\theta(z), p(.), ., .)\}$ satisfies IC.

(iii) If mechanism $\{(\theta(z), ., ., .)\}$ satisfies IC, then

$$U(z) = U(z_H) + \int_z^{z_H} m(\theta(z_0))c'(z_0)dz_0. \tag{4}$$

This lemma states that in any direct mechanism that satisfies IC, higher types should be matched with a lower probability. Furthermore, with a decreasing schedule for market tightness, it is possible to find a transfer schedule that, together with the schedule for market tightness, forms an IC direct mechanism. Finally, (4) is simply the integral form of the envelope condition applied to (2).

From IC, $U(z) = m(\theta(z))(p(z) - c(z))$ for all z. I substitute $U(.)$ from (4) into $U(z) = m(\theta(z))(p(z) - c(z))$ to derive the transfer function:

$$p(z) = c(z) + \frac{U(z_H) + \int_z^{z_H} m(\theta(z_0))c'(z_0)dz_0}{m(\theta(z))}. \tag{5}$$

Given that the planner's BB holds with equality, the budget-balance condition can be used to derive $U(z_H)$:

$$U(z_H) = \int \left[m(\theta(z)) \left(h(z) - c(z) - c'(z)\frac{F(z)}{F'(z)} \right) - k\theta(z) \right] F'(z)dz \tag{6}$$

See the Appendix for derivation. The integral in (6) indicates the number of resources left for the highest type. According to (4) and because $c'(.) > 0$, if $U(z_H) \geq 0$, then $U(z) \geq 0$ for all z. Hence, if the expression in (6) is positive, BB and PC constraints of all types will be satisfied. Altogether, the planner's problem can now be written only in terms of $\theta(z)$:

Problem 2.

$$\max_{\theta(z)} \int \Big[m(\theta(z))[h(z) - c(z)] - k\theta(z) \Big] F'(z)dz$$

subject to $\dfrac{d\theta(z)}{dz} \leq 0$ *(monotonicity constraint or MC) and* $U(z_H) \geq 0$*.*

Then, $p(z)$ will be automatically given by (5) and $t(z)$ will be given by buyers' zero profit condition.

In the main results of the paper, we derive sufficient conditions under which the planner achieves the first-best and then compare the equilibrium allocation with the constrained efficient allocation. Before presenting the main results, I first characterize the equilibrium allocation briefly to make the comparison easier.

3.1 Equilibrium

As mentioned earlier, the environment here is basically the static version of Chang (2018), so I take the characterization of equilibrium from her paper.

Proposition 1 (Equivalent to Proposition 1 in Chang (2018))**.** *Suppose $c'(z) > 0$, $h'(z) > 0$ and $U^{FB}(z) > 0$ for all z. Given the policy of zero taxes (the laissez-faire economy), a unique implementable allocation (equilibrium) exists. The equilibrium is separating. The market tightness solves the differential equation (8). The initial condition and prices are given by $\theta^{EQ}(z_L) = \theta^{FB}(z_L)$ and $p^{EQ}(z) = h(z) - \frac{k}{q(\theta^{EQ}(z))}$.*

The IC constraints faced by agents in the laissez-faire economy are the same as those faced by the planner; therefore, (2) can be used to describe IC constraints in equilibrium too. However, the number of transfers that each type receives is different in equilibrium than the constrained efficient allocation, because they are pinned down by the free-entry condition absent of any cross-subsidization.

Following Guerrieri et al. (2010), Chang shows that the equilibrium under the assumptions $c'(.) > 0$ and $h'(.) > 0$ is separating, so free entry implies that $p^{EQ}(z) = h(z) - \frac{k}{q(\theta^{EQ}(z))}$ for all z. Therefore, the payoff to type z in the laissez-faire economy, denoted by $U^{EQ}(z)$, is calculated as follows:

$$U^{EQ}(z) = \max_{\hat{z}}\{m(\theta^{EQ}(\hat{z}))(p(\hat{z}) - c(z))\} = \max_{\hat{z}}\{m(\theta^{EQ}(\hat{z}))(h(\hat{z}) - c(z)) - k\theta^{EQ}(\hat{z})\}, \quad (7)$$

where the objective function is the payoff to type z if he reports type $\hat{z}$. FOC with respect to $\hat{z}$, together with the assumption of differentiability of $\theta(z)$ almost everywhere, yields

$$\left[m'(\theta^{EQ}(z))(h(z) - c(z)) - k\right] \frac{d\theta^{EQ}(z)}{dz} + m(\theta^{EQ}(z))h'(z) = 0, \quad (8)$$

where I used the fact that at the solution, $\hat{z} = z$ because of IC. With respect to the initial condition, roughly speaking, the market delivers the complete information payoff to the type

Condition	Necessary condition for IC	Initial conditions
$0 < c'(z)$ and $0 < h'(z)$	$\frac{d\theta}{dz} \leq 0$	$\theta(z_L) = \theta^{FB}(z_L)$
$0 < c'(z)$ and $0 > h'(z)$	$\frac{d\theta}{dz} \leq 0$	$\theta(z_H) = \theta^{FB}(z_H)$
$0 > c'(z)$ and $0 < h'(z)$	$\frac{d\theta}{dz} \geq 0$	$\theta(z_L) = \theta^{FB}(z_L)$
$0 > c'(z)$ and $0 > h'(z)$	$\frac{d\theta}{dz} \geq 0$	$\theta(z_H) = \theta^{FB}(z_H)$

Table 1: Equilibrium allocation in different cases

that has the most incentive to deviate. For example, when $h' > 0$, the lowest type has the most incentive to deviate, so the market tightness for this type is set to the complete information level; i.e., $\theta^{EQ}(z_L) = \theta^{FB}(z_L)$.

The necessary condition for IC and the initial conditions for the differential equation are depicted in Table 1 under different assumptions. I maintain the assumption that $c' > 0$. All results can be obtained similarly if $c' < 0$. Also note that Chang has not analyzed the case where buyers and sellers rank the assets in the opposite order ($c' > 0$ and $h' < 0$).[15]

For example and for future reference, if $c' > 0$ and $h' > 0$, the differential equation implies that $m'(h - c) - k$ is positive. Therefore, θ is distorted downward relative to the first-best. That is:

$$c' > 0 \text{ and } h' > 0 \Rightarrow \theta^{EQ}(z) < \theta^{FB}(z) \text{ for all } z \in (z_L, z_H].^{16} \tag{9}$$

Let's focus on the case with $c' > 0$. Table 1 has two messages. First, the probability of trade for high-type sellers is lower than low-type sellers, so that sellers with a low type, who value the asset less or are more willing to sell, do not want to pretend to be a high type. In both cases, low-type sellers want to get rid of their assets so they prefer a higher probability of match (i.e., lower wait) to a higher price. Second, under- or over-participation of buyers happens in equilibrium depending on whether buyers and sellers rank the assets in an identical or opposite order, respectively. When h is increasing, those sellers with higher-quality assets value their assets more. The participation of buyers is less than that in the first-best (under-participation) so that IC and free-entry conditions are both satisfied. When

[15]If the value of assets to buyers is an increasing (decreasing) function of the sellers' opportunity cost, then we say that the buyers' and sellers' ranking of the assets are identical (opposite). Hence, the rankings are identical if c' and h' are both positive, and are opposite if c' is positive but h' is negative. If the value of assets to buyers is not a monotonic function of the sellers' opportunity cost, then we say that the buyers' and sellers' rankings of the assets are different.

[16]With a similar argument, we can write: $c' > 0$ and $h' < 0 \Rightarrow \theta^{EQ}(z) > \theta^{FB}(z)$ for all $z \in [z_L, z_H)$.

h is decreasing, those sellers with higher-quality assets value their assets less. In this case, the participation of buyers is more than that in the first-best (over-participation).

3.2 Constrained Efficiency

To solve the planner's problem, I use a somewhat backward approach. I first guess that the planner can achieve the first-best. That is, the planner can maximize his objective function for each type separately without worrying about the IC constraints. I then find a set of transfers and taxes, $p(z)$ and $t(z)$, such that the sellers' maximization condition and buyers' zero profit condition are satisfied. Finally, I derive sufficient conditions under which the planner's budget-balance condition is satisfied. Define $H_0(z) \equiv \int_{z_L}^{z} m(\theta^{FB}(\hat{z}))h'(\hat{z})d\hat{z}$.[17]

Theorem 1. *Suppose $c'(z) > 0$ and $U^{FB}(z) > 0$ for all z.*

(i) Suppose $h'(z) \leq 0$ for all z, then the planner achieves the first-best.

(ii) Suppose $h'(z) \leq c'(z)$ for all z. The planner achieves the first-best if and only if

$$\bar{H}_0 \equiv \int H_0(z)dF(z) \geq \int m(\theta^{FB}(z))c'(z)dz - U^{FB}(z_L). \tag{10}$$

In Theorem 1 the planner achieves the first-best. That is, $\theta^{CE}(z)$ is given by:

$$\theta^{CE}(z) = \theta^{FB}(z) \text{ for all } z,$$

$p^{CE}(z)$ and $U^{CE}(z_H)$ are given by (5) and (6) with $\theta(z)$ being replaced by $\theta^{CE}(z)$, and

$$t^{CE}(z) = h(z) - p^{CE}(z) - \frac{k}{q(\theta^{CE}(z))} \text{ for all } z. \tag{11}$$

Some comments are in order regarding this result. First, technically speaking, part (i) of the proposition is redundant, as it is implied by part (ii). This is because if $h'(z) \leq 0$, then both $h'(z) \leq c'(z)$ and (10) are satisfied. I included part (i) to have a clear characterization. Second, the right-hand side (RHS) of condition (10) is independent of the distribution of types. Therefore, the average cost to the sellers (or any other moment of the cost distribution)

[17] Theorem 1 is an extension of Theorem 3 in Davoodalhosseini (2019) to an environment with a continuous-type space. Theorem 2 is an extended and modified version of Theorems 1 and 2 in Davoodalhosseini (2019). Proposition 2 is an extension of Theorem 4 in Davoodalhosseini (2019). The proof techniques are different, though, because the type space is continuous here in contrast to the discrete-type space in that paper. Even the idea of some proofs is different, as explained in the text.

is irrelevant as long as the average value of H_0, which I call effective value, over the population is sufficiently high. Third, (10) is equivalent to

$$\int \big[m(\theta^{FB}(z))\big(h(z) - c(z) - c'(z)\frac{F(z)}{F'(z)}\big) - k\theta^{FB}(z)\big]F'(z)dz \geq 0, \tag{12}$$

or

$$\int \Big(\frac{h(z) - c(z)}{c'(z)}\eta(\theta^{FB}(z)) - \frac{F(z)}{F'(z)}\Big)m(\theta^{FB}(z))c'(z)F'(z)dz \geq 0,$$

where $\eta(\theta) \equiv -\frac{\theta q'(\theta)}{q(\theta)}$. See the derivation in the proof of Theorem 1(ii) in the Appendix. For the latter inequality to hold, it is sufficient for the terms inside the brackets to be positive for all z. That is, if $h'(.) \leq c'(.)$, then $(h(.) - c(.))\eta(\theta^{FB}(.))/c'(.) \geq F(.)/F'(.)$ is a sufficient condition for the planner to achieve the first-best regardless of whether h is monotone or not. This condition is easy to verify in applications.

Given the formulation of the planner's problem in Problem 2, the proof is simple. In either case of $h'(.) \leq 0$ or $h'(.) \leq c'(.)$, $\theta^{FB}(z) \equiv m'^{-1}(\frac{k}{h(z)-c(z)})$ is decreasing in z, so the first constraint in Problem 2 is satisfied. For the second constraint, I show that if $h'(.) \leq 0$, then the planner has enough resources to distribute among agents regardless of the distribution; i.e., $U(z_H) \geq 0$ is not binding. If $h'(z) \leq 0$ does not hold for some z, as long as $h'(z) - c'(z) \leq 0$ for all z, I show in the proof that (10) is equivalent to $U(z_H) \geq 0$.

Intuitively, efficiency requires that the planner allocates fewer buyers (i.e., a lower θ) to the sellers with assets with a lower surplus (i.e., sellers with a lower $h - c$). Incentive compatibility requires that the planner allocates fewer buyers to the sellers with a higher opportunity cost (i.e., sellers with a higher c). This is because the sellers with a lower opportunity cost are willing to trade faster, so it is more costly for them to wait relative to sellers with a higher opportunity cost. When $h - c$ and c move in the opposite directions, the planner can achieve both objectives of efficiency and satisfying incentive compatibility. Notice that so far the value of the asset to the buyers (i.e., h) has not been discussed in the planner's motives, apart from $h - c$, so where does (10) come from? The above intuition suggests that the planner can find a schedule for θ to achieve both efficiency and IC. However, any allocation requires payments to support a given IC scheme. (10) requires that the average effective value of the asset to buyers should be sufficiently high that buyers are willing to participate and pay enough taxes to satisfy the planner's BB constraint.

3.3 Constrained Efficiency Versus Equilibrium

In this section I compare the constrained efficient and constrained Pareto efficient allocations with the equilibrium allocation. An allocation is constrained Pareto efficient if there does not exist another allocation that is implementable and gives all sellers a weakly higher payoff and a strictly positive measure of sellers a strictly higher payoff. Define $H_1(z) \equiv -\int_z^{z_H} m(\theta^{FB}(\hat{z}))h'(\hat{z})d\hat{z}$.

Theorem 2. *Suppose $c'(z) > 0$ and $U^{FB}(z) > 0$ for all z.*

(i) Suppose $h'(z) \leq 0$, then the welfare level in the constrained efficient allocation is strictly higher than that in equilibrium. Also, if

$$\bar{H}_1 \equiv \int H_1(z)dF(z) \geq \int (m(\theta^{EQ}(z)) - m(\theta^{FB}(z)))c'(z)dz, \tag{13}$$

then the constrained efficient allocation Pareto dominates the equilibrium allocation.

(ii) Suppose $0 < h'(z)$. Then the welfare level in the constrained efficient allocation is strictly higher than that in equilibrium.

(iii) Suppose $0 \leq h'(z) \leq c'(z)$ and

$$\bar{H}_0 \geq \int (m(\theta^{FB}(z)) - m(\theta^{EQ}(z)))c'(z)dz, \tag{14}$$

then the constrained efficient allocation Pareto dominates the equilibrium allocation.

Remember from Theorem 1 that when h is decreasing, the planner can achieve the first-best. The equilibrium, on the other hand, does not achieve the first-best because the market tightness is distorted relative to the first-best, as discussed in Section 3.1. Therefore, when h is decreasing, the equilibrium is never constrained efficient. Part (i) of this theorem gives us an even stronger result. It also establishes conditions under which equilibrium is not constrained Pareto efficient or, more precisely, conditions under which the constrained efficient allocation Pareto dominates the equilibrium allocation.

Part (ii) establishes the constrained inefficiency of equilibrium when h is increasing (even if (10) does not hold, in which case the planner does not achieve the first-best). This result is an extension of Theorem 1 in Davoodalhosseini (2019), although the proof is different because the construction method in that paper relies on the discrete-type space assumption and cannot be replicated here. The proof can be summarized below. In the equilibrium, the market tightness is distorted downward relative to the first-best allocation. In order to increase welfare, the planner allocates to all sellers a slightly higher market tightness, so

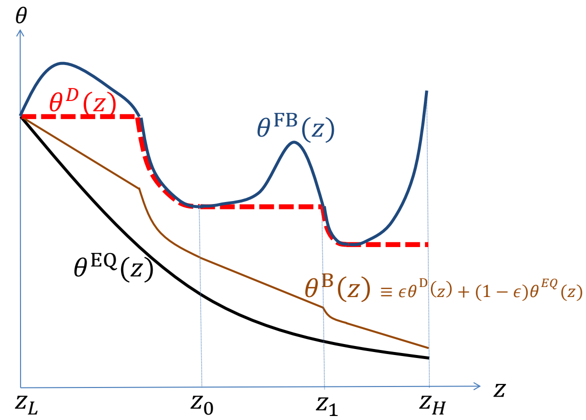

Figure 1: Schematic diagram demonstrating the construction of a better allocation with market tightness θ^B than equilibrium allocation under requirements of Theorem 2(ii).

that they can trade with a higher probability. Therefore, the aggregate surplus increases as more trades happen in the economy. The challenge here is that the market tightness should continue to be decreasing such that the IC continues to hold. In order to construct a market tightness function that satisfies IC, I first find the greatest weakly decreasing function that does not exceed $\theta^{FB}(.)$ and call it $\theta^D(.)$. See Figure 1. I then use a linear combination of $\theta^D(.)$ and $\theta^{EQ}(.)$ to obtain a decreasing market tightness function $\theta^B(.)$.[18] If $\theta^B(.)$ is sufficiently close to $\theta^{EQ}(.)$, then the change in the sellers' payoffs is small enough that the sellers continue to receive a payoff greater than zero.

Regarding part (iii), notice that if $0 \le h'(z) \le c'(z)$ and (14) both hold, then (10) holds, too. Therefore, the planner achieves the first-best according to Theorem 1(ii). I show in the proof that all sellers are better off in the constrained efficient allocation (where they trade with the first-best level of market tightness) relative to the equilibrium.

In Figure 2, the sellers' payoffs in different allocations are illustrated. The slopes of payoff functions in the constrained efficient allocation and equilibrium are equal to $m(\theta^{FB}(z))c'(z)$ and $m(\theta^{EQ}(z))c'(z)$, respectively. The market tightness in the equilibrium is distorted downward—i.e., $\theta^{EQ}(.) \le \theta^{CE}(.) \equiv \theta^{FB}(z)$ by (8)—so $U^{CE}(z)$ is steeper than $U^{EQ}(z)$. As a result, in order for the constrained efficient allocation to Pareto dominate the equilibrium

[18]Since I have not imposed any assumption on $h - c$, the market tightness at the first-best level can be non-monotone; therefore, we cannot just use a linear combination of $\theta^{FB}(.)$ and $\theta^{EQ}(.)$ as a proposal for increasing welfare.

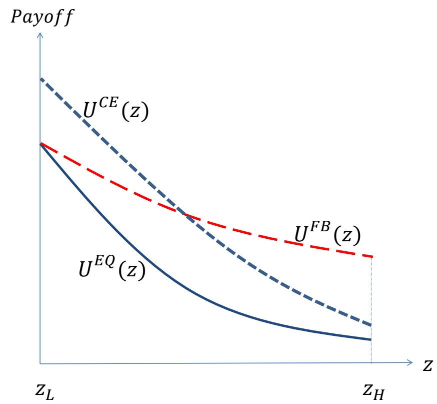

Figure 2: Schematic diagram for sellers' payoffs in different allocations under requirements of Theorem 2 (iii).

allocation, the necessary and sufficient condition is that the payoff to the highest type should be weakly higher in the former than the latter. That is, $U^{CE}(z_H) \geq U^{EQ}(z_H)$. It is easy to see that (14) is equivalent to this condition after some manipulation. (13) can be explained in a similar manner.

Finally, notice that (13) and (14) have been stated in terms of $\theta^{EQ}(.)$, for which the closed-form solution is not generally available, as it is described by (8). However, they are easy to check numerically. In Example 3, I assume identical gains from trade for all types, solve for $\theta^{EQ}(.)$ and give explicit conditions equivalent to (14) as well as (10). Similarly, in Example 4 in the Appendix, I adopt a specific matching function, solve for $\theta^{EQ}(.)$ and give explicit conditions equivalent to (13) and (14).

3.4 What If the First-Best Cannot Be Achieved?

In the next proposition, I keep the assumptions that $c'(.) > 0$ and $h'(.) - c'(.) \leq 0$, but I assume that the distribution of types is such that the planner cannot achieve the first-best. Therefore, the probability of matching should be distorted.

Proposition 2. *Assume $c'(z) > 0$, $h'(z) - c'(z) \leq 0$, and*

$$m'(0)\left[h(z) - c(z) - c'(z)\frac{F(z)}{F'(z)}\right] \quad > \quad k, \tag{15}$$

$$\frac{d}{dz}\left[h(z) - c(z) - c'(z)\frac{F(z)}{F'(z)}\right] \quad \leq \quad 0, \tag{16}$$

for all z. Then there exists a unique $\mu \geq 0$ such that the market tightness for the constrained efficient allocation, $\theta^{CE}(.)$, solves the following equations:

$$m'(\theta^{CE}(z))\left[h(z) - c(z) - \frac{\mu}{1+\mu}c'(z)\frac{F(z)}{F'(z)}\right] = k, \tag{17}$$

$$U(z_H) = \int \left[m(\theta^{CE}(z))\left[h(z) - c(z) - c'(z)\frac{F(z)}{F'(z)}\right] - k\theta^{CE}(z)\right] F'(z)dz \geq 0 \text{ with equality if } \mu > 0.$$

Moreover, $p^{CE}(z)$ and $t^{CE}(z)$ are given by (5) and (11).

Following Theorem 1(ii), it was shown that a sufficient condition to achieve the first-best is that $m(\theta(z))[h(z) - c(z) - c'(z)\frac{F(z)}{F'(z)}] - k\theta(z)$ is positive for all z. However, if that is negative for some types, or more generally if (10) is not satisfied, then Proposition 2 would be useful. It provides sufficient conditions for solving the planner's problem even if the first-best cannot be achieved. It requires that $h(z) - c(z) - c'(z)\frac{F(z)}{F'(z)}$, called virtual surplus, be positive and decreasing in z for all z. Chang (2018) shows that if $h(.)$ is not monotone, it is not possible for the market to separate types. However, this proposition states that, under its requirements, even if h is not monotone, the planner still wants to separate different types.[19]

If h is monotone but under other conditions (as in Example 3), the equilibrium is separating but the constrained efficient allocation is not. It is true that there is some cross-subsidization in the pooling allocation, which improves efficiency, but different types are allocated to the same θ, which hinders efficiency. Because of this trade-off, drawing conclusions about efficiency of an allocation by simply considering whether it is pooling or separating is misleading in a general case with more than two types.

If the expression in brackets in (17) is decreasing in z, even if $h - c$ is not, then Proposition 2 can be used. If that expression is not decreasing for some z, then we need to solve Problem 2 in a general form, which will lead to pooling of some types. This is despite the fact that if h is increasing, the market separates the types completely. For the sake of brevity, I delegate this part to the Appendix.

[19] Chang (2018) extends her analysis to allow for two-dimensional private information. See also Guerrieri and Shimer (2014b) for a closely related model. Following this extension, function h may have a strict local maximum and, therefore, full separation of types in the market is not possible (Proposition 4 in Chang (2018)). With a non-monotone h, I effectively capture the two-dimensional model in terms of equilibrium and efficiency properties.

4 Optimal Taxation

In this section I study the optimal tax schedule first and then present several examples.

The following proposition states that when the buyers' and sellers' rankings are identical or opposite, the low-quality assets should be subsidized and the high-quality assets should be taxed.

Proposition 3 (Single-crossing property of sales tax schedule)**.** *Assume* $0 < c'(.)$*. Also assume (i)* $h'(z) \leq 0$ *for all* z*, or (ii) both* (10) *and* $0 \leq h'(z) \leq c'(z)$ *for all* z *hold. Then*

$$\frac{dt^{CE}}{dz} = h'(z) + \frac{(m'(\theta^{FB}(z)))^2}{m(\theta^{FB}(z))m''(\theta^{FB}(z))} \frac{t^{CE}(z)}{h(z) - c(z)}(h'(z) - c'(z)). \tag{18}$$

Under case (i), there is a unique $\hat{z} \in (z_L, z_H)$ *where* $t(\hat{z}) = 0$*,* $t(z) > 0$ *for* $z < \hat{z}$ *and* $t(z) < 0$ *for* $z > \hat{z}$*. Similarly, under case (ii), there is a unique* $\hat{z} \in (z_L, z_H)$ *where* $t(\hat{z}) = 0$*,* $t(z) < 0$ *for* $z < \hat{z}$ *and* $t(z) > 0$ *for* $z > \hat{z}$*.*

Guerrieri et al. (2010) study the laissez-faire equilibrium of a similar example with only two types, which is an extension of the Akerlof (1970) lemons market to an environment with search frictions. Davoodalhosseini (2019) conducts welfare analysis for that example where the buyers' and sellers' rankings are identical and obtains the same result. Proposition 3 extends the intuition obtained from that simple example to environments (i) with a continuous-type space, and (ii) where the buyers' and sellers' rankings of assets are opposite. The intuition can be summarized below for the two-type example for the case where the rankings are identical. (The intuition is similar if the rankings are opposite.) In the laissez-faire equilibrium, there are two submarkets: a low-price one where low-quality assets are sold with a high probability, and a high-price one where the high-quality assets are sold with a low probability. The participation of buyers in the high-price submarket is endogenously limited so that the low-quality sellers do not want to go to that submarket. The planner subsidizes trade in the low-price submarket and taxes trade in the high-price submarket to discourage low-quality sellers from going to the high-price submarket. As a result, more buyers can enter the high-price submarket to trade with previously unmatched high-quality sellers.

I show in the following lemma that even if the buyers' and sellers' rankings are identical or opposite, still the tax schedule may not be monotone in the price or quality of assets.[20]

[20]By monotonicity of taxes I refer only to the taxes levied on on-the-equilibrium-path prices. For other prices (i.e., those prices that the planner does not want posted), the taxes should be prohibitively high.

This result is novel and cannot be obtained in a simple two-type example.

Lemma 3 (Conditions for non-monotonicity of sales tax schedule)**.** *Assume* $0 < c'(.)$*.*

[Identical rankings] Assume $h'(.) \leq 0$*. If* $h'(z_L) = 0$ *and* $h'(.) < 0$ *for a strictly positive measure of sellers, then the optimal sales tax schedule is not monotone in the price of assets. Specifically,* $\frac{dt^{CE}(p)}{dp}|_{p=p^{CE}(z_L)} > 0$ *and* $\frac{dt^{CE}(p)}{dp}|_{p=p^{CE}(z_0)} < 0$ *for some* $z_0 \in (z_L, z_H]$*.*

[Opposite rankings] Assume $0 \leq h'(.) < c'(.)$ *and* (10) *hold. If* $h'(z_L) = 0$ *and* $0 < h'(.)$ *for a strictly positive measure of sellers, then the optimal sales tax schedule is not monotone in the price of assets. Specifically,* $\frac{dt^{CE}(p)}{dp}|_{p=p^{CE}(z_L)} < 0$ *and* $\frac{dt^{CE}(p)}{dp}|_{p=p^{CE}(z_0)} > 0$ *for some* $z_0 \in (z_L, z_H]$*.*

This lemma states that the level of trading tax is not necessarily monotone in the quality of the assets. For example, for the case where buyers' and sellers' rankings are opposite, if the slope of value of the asset to buyers is zero at z_L, then the tax schedule becomes decreasing around z_L. That is, when the quality of assets are low, as the quality increases, the trading of those assets is subsidized even more.

Changes in $t^{CE}(z)$ are influenced not only by changes in $h(z)$ but also by another term that has the same sign as $t^{CE}(z)$. Therefore, when the first term above is sufficiently small, the second term may dominate the first term and $t^{CE}(z)$ may become decreasing. More intuitively, the free-entry condition can be written as follows, if the constrained efficient allocation is separating:

$$t^{CE}(z) = h(z) - \underbrace{\frac{k}{q(\theta^{CE}(z))}}_{decreasing} - \underbrace{p^{CE}(z)}_{increasing}.$$

The term $\frac{k}{q(\theta^{CE}(z))}$ is decreasing in z because $\theta^{CE}(z)$ is decreasing in z. Also, $p^{CE}(.)$ is increasing, as higher-type sellers should be compensated for selling with a lower probability (formally proved in Lemma 4 in the Appendix). Hence, $t^{CE}(z)$ may not be generally monotone in z. Put differently, efficiency is a force to make the tax schedule increasing and incentive compatibility is a force to make the tax schedule decreasing. The following proposition introduces conditions under which the optimal tax schedule is monotone. Intuitively, it states that if h changes sufficiently fast, then changes in h dominate the effects of the two forces above.

Proposition 4 (Conditions for monotonicity of sales tax schedule)**.** *Assume* $0 < c'(.)$*.*

Case (i): Suppose $h'(z) \leq 0$ for all z and

$$h'(z) \leq \Big(- \frac{m'^2(\theta^{FB}(z))}{m(\theta^{FB}(z))m''(\theta^{FB}(z))} \Big) \frac{h'(z) - c'(z)}{h(z) - c(z)} t_L \text{ for all } z.$$

Case (ii): Suppose (10), $0 \leq h'(z) \leq c'(z)$ for all z and

$$h'(z) \geq \Big(- \frac{m'^2(\theta^{FB}(z))}{m(\theta^{FB}(z))m''(\theta^{FB}(z))} \Big) \frac{h'(z) - c'(z)}{h(z) - c(z)} t_L \text{ for all } z \tag{19}$$

all hold.

In both cases, t_L is defined:

$$t_L \equiv t^{CE}(z_L) = h(z_L) - c(z_L) - \frac{k}{q(\theta^{CE}(z_L))} - \frac{U(z_H) + \int_{z_L}^{z_H} m(\theta^{CE}(z_0))c'(z_0)dz_0}{m(\theta^{CE}(z_L))}.$$

Under case (i) (case (ii)), t^{CE} is a weakly decreasing (increasing) function of z.

Cases (i) and (ii) are similar, so I focus on case (ii). Condition (19) states that h should be sufficiently increasing so that the taxation schedule becomes increasing. The distribution matters only through t_L, where t_L is affected by that only through $U(z_H)$. The rest is independent of the distribution, making this condition easy to verify in applications. One special case, studied in Example 3 below, is where the gains from trade in the match, $h - c$, are constant and independent of z. In this case, the condition is trivially satisfied because its RHS is zero.

4.1 Examples of Optimal Taxation

Three examples are presented in this section to compare the first-best, equilibrium and constrained efficient allocations, to identify the types that should be taxed and the types that should be subsidized, and to examine whether or not taxes are monotone in the type of sellers or in the price of assets.

Example 1. *Model parameters: $m(\theta) = 1 - e^{-\theta}$, $Z = [1, 2]$, $c(z) = z$, $h(z) = 0.5(z-1)^2 + 4$, $k = 1.85$, and $F(.)$ is uniform.*

Here, $c' > 0$, $h' \geq 0$ and $h' - c' \leq 0$. It is easy to check that Theorem 1(ii) applies, so the market tightness for the constrained efficient allocation is given by $\theta^{CE}(z) = \theta^{FB}(z) = m'^{-1}(\frac{k}{h(z)-c(z)}) = ln(\frac{h(z)-c(z)}{k})$, and $p^{CE}(z)$ and $t^{CE}(z)$ are given by (5) (in which θ is replaced by θ^{CE}) and (11). The net payment that buyers make in the constrained efficient allocation,

$p^{CE}(z)+t^{CE}(z)$, is equal to $p^{FB}(z) \equiv h(z) - \frac{k}{q(\theta^{FB}(z))}$, the price that buyers pay in the market with complete information. For the equilibrium allocation, $\theta^{EQ}(z)$ is given by (8):

$$\left[exp(-\theta^{EQ}(z))(h(z) - c(z)) - k\right] \frac{d\theta^{EQ}(z)}{dz} + (1 - exp(-\theta^{EQ}(z)))h'(z) = 0,$$

with the initial condition $\theta^{EQ}(1) = \theta^{FB}(1)$ and $\theta^{EQ}(z)$ being decreasing in z. The price that buyers pay in equilibrium is $p^{EQ}(z) = h(z) - \frac{k}{q(\theta^{EQ}(z))}$.

Figure 3 illustrates the first-best, equilibrium and constrained efficient allocations for Example 1. Here, $\theta^{EQ}(.) \leq \theta^{FB}(.)$. Market tightness is the tool that buyers in the laissez-faire economy use to screen high-type sellers. Low-type sellers prefer to sell their assets more quickly, because they do not want to be left with their "lemons." Consequently, $p^{EQ}(z)$ is generally greater than $p^{FB}(z) \equiv h(z) - \frac{k}{q(\theta^{FB}(z)}$. Also, $p^{CE}(.)$ is higher for lower types and lower for higher types compared with $p^{FB}(.)$. Since the market tightness is the same in the first-best and constrained efficient allocations, the net price that buyers should pay is the same in both cases so that buyers' zero profit condition is satisfied. The amount of tax that buyers should pay, $t^{CE}(.)$, is simply equal to the difference, $p^{FB}(.) - p^{CE}(.)$.

Next, I study an example in which h is not monotone and, therefore, separation of types in equilibrium is not possible, as explained in the last section.

Example 2. *Model parameters:* $m(\theta) = 1-e^{-\theta}$, $Z = [0, 2]$, $c(z) = z$, $h(z) = 0.5(z-1)^2+4$, $k = 1.85$, *and* $F(.)$ *is uniform.*

Theorem 1(ii) applies, so the market tightness at the constrained efficient allocation is similarly given by $\theta^{CE}(z) = \theta^{FB}(z) = m'^{-1}(\frac{h(z)-c(z)}{k}) = ln(\frac{h(z)-c(z)}{k})$. According to Proposition 3 in Chang (2018), I construct one semi-pooling equilibrium in which types $z \in [0, 1)$ trade in a pool with a low price but with high probability. Types $z \in (1, 2]$ trade in separating submarkets. Type $z = 1$ is indifferent between the pool and one of the separating submarkets. This equilibrium can be called a fire-sale equilibrium in that many sellers sell their assets in a pool with a low price but very quickly. Some sellers in the pool have low-quality assets (i.e., sellers with z close to 1) and some of them have high-quality assets but are in need of liquidity (i.e., sellers with z close to 0). Prices and taxes are calculated similarly as in Example 1.

Figure 4 is similar to Figure 3 for parameters in Example 2. Market tightness in the constrained efficient allocation is the same as that in the first-best. Market tightness in equilibrium is higher than that in the first-best for most types in the pool. In Figure 5, the payoff to sellers of different assets is depicted for both Examples 1 and 2. In the left (right) graph of Figure 6, $t^{CE}(z)$ is drawn in terms of z (in terms of $p^{CE}(z)$). An interesting fact here is that in both examples—even in Example 1, in which buyers' and sellers' rankings of assets are identical, as h and c are both increasing—the amount of tax levied on buyers is neither monotone in the type of sellers that buyers meet, nor in the price paid to the sellers.

Furthermore, in Example 1, since the rankings are identical, all low-quality assets are subsidized and high-quality assets are taxed although the tax schedule is not monotone. In contrast, in Example 2, the assets in the middle are subsidized although they are not the lowest-quality ones. (Note the asymmetry of the tax schedule around $z = 1$ in Example 2.) For example, the $z = 0.4$ assets are subsidized but the $z = 1.6$ assets are taxed although their quality is identical. In terms of surplus, the $z = 1.8$ assets (with a lower surplus) are taxed while the $z = 1$ assets (with a higher surplus) are subsidized.

Example 3. *Special case:* $h(z) = c(z) + x$, $x > 0$. *Since the gains from trade,* x, *are fixed,* $\theta^{FB}(z) = m'^{-1}(k/x)$ *does not depend on* z. *Denote* $\bar{\theta}^{FB} \equiv m'^{-1}(k/x)$ *and* $\bar{U}^{FB} \equiv m(\bar{\theta}^{FB})x - k\bar{\theta}^{FB}$. *Assume* $\bar{U}^{FB} > 0$. *The equilibrium market tightness is given by:*

$$\int_{\bar{\theta}^{FB}}^{\theta^{EQ}(z_L)} \frac{m'(\theta)x - k}{m(\theta)} d\theta = c(z) - c(z_L). \tag{20}$$

I show that if x *is sufficiently high and* m *has a decreasing elasticity (which is correct for most applications), then the planner can achieve the first-best regardless of the distribution of types. If the distribution satisfies the following:*

$$\int_{z_L}^{z_H} \frac{m(\theta^{EQ}(z))}{m(\bar{\theta}^{FB})} c'(z) dz \geq \int_{z_L}^{z_H} F(z) c'(z) dz, \tag{21}$$

then the constrained efficient allocation Pareto dominates the equilibrium allocation. The proofs are shown in the Appendix.

It is easy to see that $U(z) = \bar{U}^{FB} - m(\bar{\theta}^{FB}) \int_{z_L}^{z_H} F(z)c'(z)dz$, and $p(z) = \frac{\bar{U}^{FB}}{m(\bar{\theta}^{FB})} + \int_{z_L}^{z_H} c(z)F'(z)dz \equiv \bar{p}$. Notice that $p(z)$ is not a function of z. This is natural because the market tightness for all types is identical in the first-best allocation since all types produce identical surplus. A way that the planner can implement this direct mechanism is to have a pool with price $\bar{p}$ composed of all sellers without any explicit tax (and arbitrarily large taxes on other prices with which the planner does not want agents to trade). The buyers enter the pool until the market tightness becomes equal to θ^{FB}, at which point the buyers' zero profit condition is also satisfied.

4.2 Sales Tax and Entry Tax

Implementation of a non-monotone tax schedule is difficult in practice, as it requires the planner to have precise information about the distribution of types. Although it is usually assumed in the literature, including in this paper, that the planner has such information, one ideally wants to reduce the dependence of what the planner should do on the details of the environment.

Entry tax is introduced in this subsection, so buyers will be subject to two types of taxes: sales tax, which is conditional on trade as before, and entry tax, denoted by $\tilde{t}_e(p)$, which is a sales tax levied on buyers who want to buy at price p regardless of whether they trade. The definition of implementable allocation should be slightly modified to include both types of taxes. See Definition 5 in the Appendix for the details. I show in the following proposition that any feasible mechanism can be implemented by using a decreasing entry tax and an increasing sales tax in the price of assets.[21]

Proposition 5 (Implementation of the direct mechanism with monotone entry and sales tax schedules)**.** *Take any feasible mechanism in which all types receive a strictly positive payoff, and in which the market tightness allocated to different types is all different. Then, there exists an associated implementable allocation with monotone tax schedules in the price of assets, decreasing entry tax and increasing sales tax, such that all types receive the same payoff as their payoff in the feasible mechanism.*

In the language of direct mechanisms, to design a monotone $t^{CE}(z)$, we add an entry tax, ${t_e}^{CE}(z)$, for each submarket so the free-entry condition can be written as $t^{CE}(z) = h(z) - \frac{k+{t_e}^{CE}(z)}{q(\theta(z))} - p^{CE}(z)$. If ${t_e}^{CE}(z)$ is constructed to be decreasing sufficiently fast in z, then the effect of $\frac{k+{t_e}^{CE}(z)}{q(\theta(z))}$ dominates the effect of $p^{CE}(z)$, and $t^{CE}(z)$ becomes increasing in z. The following corollary is implied directly by Proposition 5.

Corollary 1. *Take an optimal mechanism. Under the requirements of Proposition 5, there exists an associated allocation that is constrained efficient and in which all types receive exactly the same payoff as in the optimal mechanism, and the associated entry tax and sales tax are, respectively, decreasing and increasing in the price of assets.*

[21] If the entry tax for a submarket is less than $-k$, then buyers pay this negative tax—i.e., receive a positive subsidy of $t_e + k$—and then do not participate in the matching stage that delivers them a strictly negative payoff. Therefore, another constraint that should be added to the definition of implementable allocation is $k + t_e(z) \geq 0$. See Definition 5 in the Appendix.

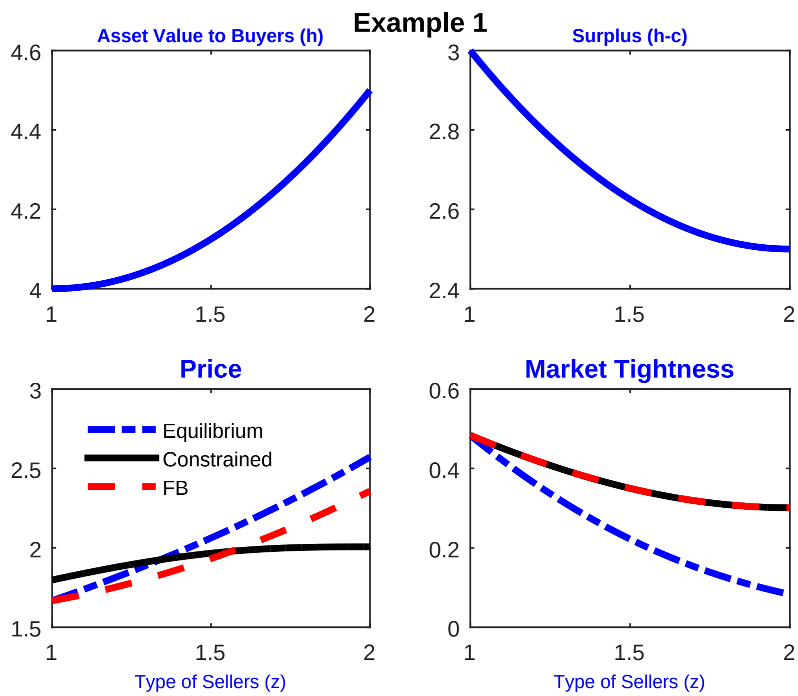

Figure 3: Model parameters are defined in Example 1. In the upper left (right) graph, the value of type z asset to buyers (surplus from the match) is depicted. In the lower left (right) graph, the price that sellers receive (the market tightness) in the first-best, constrained efficient and equilibrium allocation is depicted.

A schedule of monotone sales tax and entry tax, which implements the first-best allocation, is depicted for Examples 1 and 2 in Figure 7.

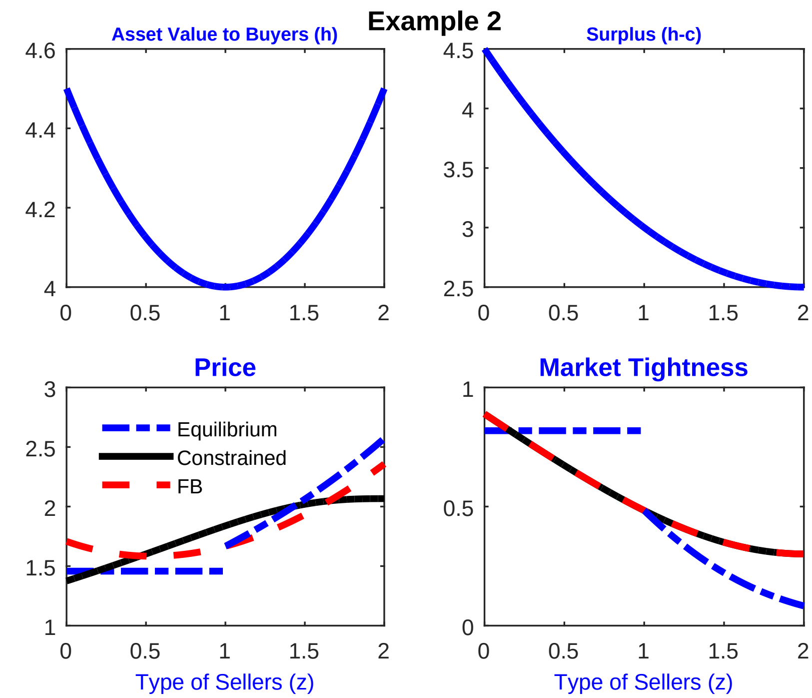

Figure 4: This figure is similar to Figure 3 but with model parameters defined in Example 2.

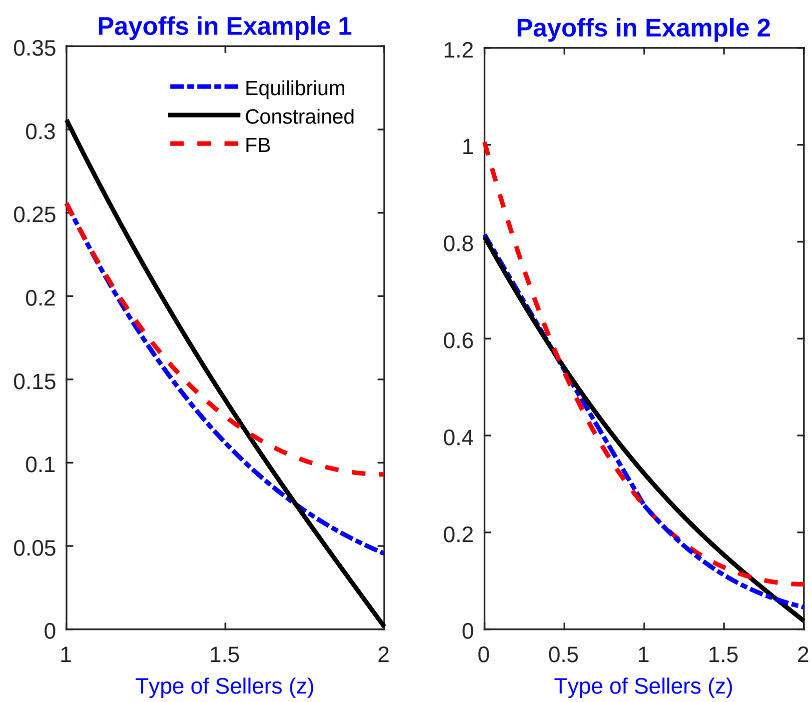

Figure 5: Model parameters are defined in Example 1 (2) for the left (right) graph. The expected payoff to sellers in the first-best, constrained efficient and equilibrium allocation is depicted.

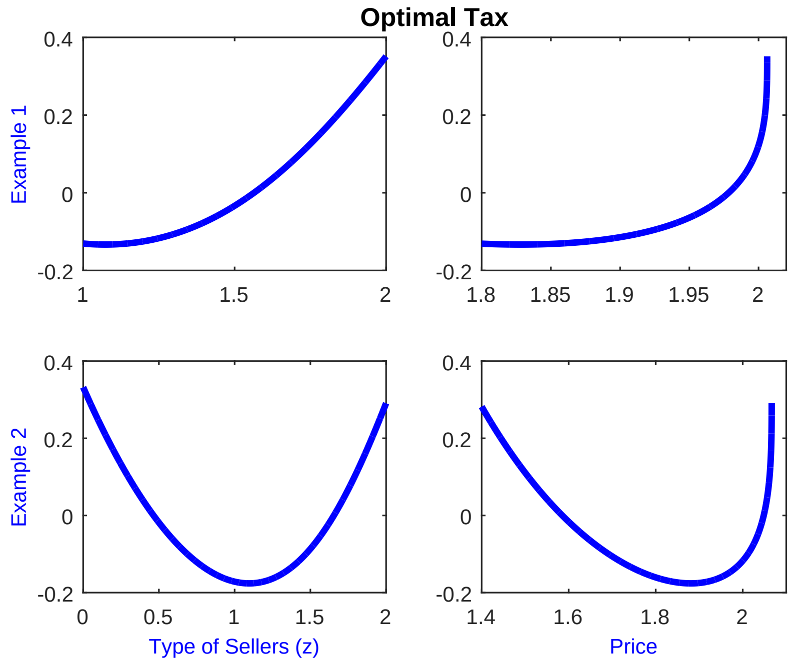

Figure 6: The upper (lower) graphs represent Example 1 (2). The optimal sales tax schedule is non-monotone in the type of sellers or price of assets in both examples. Specifically in Example 1, the optimal tax has a strict global minimum at $z = 1.07$.

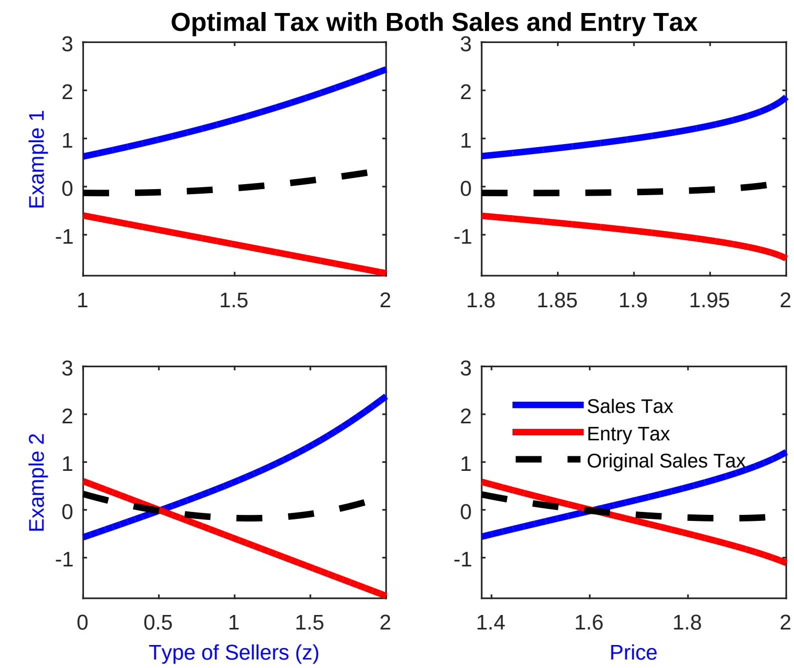

Figure 7: The upper (lower) graphs represent Example 1 (2). When entry tax is introduced, it can be designed so that both entry tax and sales tax become monotone in the type of sellers or price of assets in both examples.

5 Concluding Remarks

I characterized the optimal taxation in an asset market with search frictions and adverse selection. I derived conditions under which the planner can correct the inefficiencies caused by adverse selection. I also investigated the shape of the optimal tax schedule. What assets should be taxed and what assets should be subsidized, and is the tax schedule monotone in the price of assets? I showed that the optimal tax schedule is sometimes non-monotonic, then I derived sufficient conditions for its monotonicity. Finally, I showed that the laissez-faire economy is not constrained efficient. That is, the planner can always improve upon the market allocation.

Two main frictions that I focused on in this paper are search frictions and adverse selection. The presence of search frictions is less debatable because many assets, a few of them named in the Introduction, are traded over the counter; agents must engage in a time-consuming search to find a trading partner and there is no competitive price to clear the market. What about adverse selection? Is it relevant in the financial markets at all? Guerrieri and Shimer (2014a) admit that "[i]n practice, it is difficult to measure the extent of adverse selection in any market simply because the data demands are acute." Yet, they discuss in detail some empirical evidence on the relevance of adverse selection in financial markets, so I do not elaborate on that here.

In this paper, I considered bilateral (one-on-one) meetings. However, one could consider many-on-one meetings—i.e., allowing a buyer to meet several sellers so that sellers face some competition after meeting a buyer—which could be more realistic in some markets. Would that induce sellers to reveal their types in a less costly manner, and would the equilibrium remain constrained inefficient?

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Appendix of "Optimal Taxation in Asset Markets with Adverse Selection:" Definitions and Proofs

Proof of Lemma 1. Consider a feasible mechanism $\{\theta(.), \mathbf{p}(.), t(.), t_0\}$. Denote by $U(z)$ the payoff to type z in this mechanism. I construct an allocation $\{G, \mathcal{P}, \Theta, \mu\}$ as follows:

$$\mathcal{P} \equiv [p_L, p_H] \subseteq \mathbb{P} \equiv R_+ \text{ where } p_L \equiv \mathbf{p}(z_L) \text{ and } p_H \equiv \mathbf{p}(z_H)$$

and p is given by (5).[22] Define $\bar{m} = \lim_{\theta\to\infty} m(\theta)$. The market tightness for this allocation is given by:

$$\begin{cases} \Theta(p) = \infty & \text{for } p \leq c(z_L) \\ m(\Theta(p)) = \min\{\bar{m}, \frac{U(z_L)}{p-c(z_L)}\} & \text{for } p \in (c(z_L), p_L) \\ \Theta(p) = \theta(p^{-1}(p)) & \text{for } p \in [p_L, p_H] \\ m(\Theta(p)) = \min\{\bar{m}, \frac{U(z_H)}{p-c(z_H)}\} & \text{for } p > p_H \end{cases}$$

$$G(p) = \begin{cases} 0 & \text{for } p < p_L \\ \int_{p_L}^{p} \Theta(p) F'(\mathbf{p}^{-1}(p)) dp & \text{for } p \in [p_L, p_H] \\ 1 & \text{for } p > p_H \end{cases}$$

$$\int \mu(z|p) dz = 1 \text{ for all } p, \text{ and } \mu(z|p) = \begin{cases} 0 & \text{for } p < p_L \text{ and } z \neq z_L \\ 0 & \text{for } p \neq \mathbf{p}(z) \text{ and } p \in [p_L, p_H] \\ 0 & \text{for } p > p_H \text{ and } z \neq z_H \end{cases}.$$

The policy is given by:

$$t_0 = 0, t(p) = \begin{cases} h(z_L) - p & \text{for all } p < p_L \\ h(\mathbf{p}^{-1}(p)) - p - \frac{k}{q(\Theta(p))} & \text{and } p \in [p_L, p_H] \\ h(z_H) - p & \text{for } p > p_H \end{cases}.$$

[22] According to Lemma 4, $p(z)$ is weakly increasing in z. I assume in this proof that the price function $p(z)$ is *strictly* increasing. If it is not, which happens when the direct mechanism requires some pooling of types, then the proof can be extended, but I do not include that case in here for the sake of brevity. The treatment is discussed in the Appendix of Davoodalhosseini (2019) in the proof of Lemma 1.

Since $\mathbf{p}(z)$ is strictly increasing and continuous, the set of prices in the constructed implementable mechanism is $\mathcal{P} \equiv [p_L, p_H]$.

The construction is straightforward. We allocate to all types the same market tightness and transfer that they were given in the direct mechanism. For construction of off-the-equilibrium-path beliefs, if $p < p_L$, then the only type attracted to this post is z_L. Therefore, $\mu(z|p) = 0$ for all $z \neq z_L$, and $\mu(z|p)$ has a mass point at $z = z_L$. Similarly, if $p > p_H$, then the only type attracted to this price is z_H. Therefore, $\mu(z|p) = 0$ for all $z \neq z_H$. Given the above beliefs, the tax amount for all p is constructed such that buyers receive a net profit of exactly 0 for $p \in \mathcal{P}$ and $-k$ for $p \notin \mathcal{P}$. Note that the choice of t is not unique for $p \notin \mathcal{P}$. We could construct t differently such that buyers receive any non-positive amount of profit for $p \notin \mathcal{P}$. $G(p)$ is easily constructed given the construction of $\Theta(.)$.

The conditions for implementability should be verified now. It is easy to check that the buyers' zero profit condition is satisfied because of the construction of t. The feasibility or market-clearing condition is satisfied because of the construction of G. The budget-balance condition is satisfied because of the choice of $U(z_H)$.

Regarding the sellers' optimal search condition, first note that the restriction on off-the-equilibrium-path beliefs is equivalent to:

$$m(\Theta(p)) = \min\left\{\bar{m}, inf_{z\in\{z|c(z)<p\}}\frac{U(z)}{p - c(z)}\right\},$$

if $\{z|c(z) < p\}$ is non-empty. Otherwise, set $\Theta(p) = \infty$. See Chang (2018) for a more detailed discussion. Now it is easy to see that sellers' optimal search is satisfied because of the construction of $\Theta(p)$. The only thing worth explaining here is why only z_L is attracted to any price less than p_L (and similarly, why only z_H is attracted to any price greater than p_H). To see why, I begin by writing the incentive compatibility condition for any feasible mechanism: $m(\theta(z_L))(\mathbf{p}(z_L) - c(z)) \leq U(z)$ for all z. Remember that the payoff to type z is the same in the mechanism and in the proposed allocation. After using the fact that $U(z_L) = m(\theta(z_L))(\mathbf{p}(z_L) - c(z_L))$, one can write:

$$U(z_L) - U(z) \leq m(\theta(z_L))(c(z) - c(z_L)) \text{ for all } z.$$

$$\Rightarrow U(z_L) - U(z) \leq m(\theta(z_L))(c(z) - c(z_L)) = \frac{U(z_L)}{\mathbf{p}(z_L) - c(z_L)}(c(z) - c(z_L))$$

$$\leq \frac{U(z_L)}{p - c(z_L)}(c(z) - c(z_L)) \text{ for all } z \text{ and for } p \in (c(z_L), \mathbf{p}(z_L)),$$

or equivalently,

$$\frac{U(z_L)}{p - c(z_L)} \leq \frac{U(z)}{p - c(z)} \text{ for all } z \text{ and for } p \in (c(z_L), \mathbf{p}(z_L)).$$

Therefore, for the above choice of $m(\Theta(p))$, the restriction on off-the-equilibrium-path beliefs is satisfied. $\square$

Proof of Lemma 2. Define $V(W, R, z) \equiv Wc(z) + R$, $w(z) \equiv -m(\theta(z))$ and $r(z) \equiv m(\theta(z))p(z) + t_0$. Obviously, $U(z, \hat{z}) = V(w(\hat{z}), r(\hat{z}), z)$. A necessary condition for $w(.)$ to satisfy IC is $\frac{\partial}{\partial z}\big(\frac{\frac{\partial V}{\partial W}}{\frac{\partial V}{\partial R}}\big)\frac{dw}{dz} \geq 0$, whenever $w(.)$ is differentiable at z, according to Theorem 7.1, Fudenberg and Tirole (1991). But $\frac{\partial}{\partial z}\big(\frac{\frac{\partial V}{\partial W}}{\frac{\partial V}{\partial R}}\big)\frac{dw}{dz} = \frac{\partial}{\partial z}(\frac{c(z)}{1})(-m'(\theta(z)))\frac{d\theta(z)}{dz}$. Also $c'(.) > 0$ and $m'(.) \geq 0$; therefore, the necessary condition is equivalent to

$$c'(z)\frac{d\theta(z)}{dz} \leq 0. \tag{22}$$

According to Theorem 7.3 in Fudenberg and Tirole (1991), a sufficient condition for $w(.)$ to satisfy IC is that $\frac{dw(z)}{dz} \geq 0$, or equivalently, $c'(z)\frac{d\theta(z)}{dz} \leq 0$.

For the third part of the lemma, I use Corollary 1 in Milgrom and Segal (2002). Their result states that if $\theta(z)$ satisfies IC, then $U(.)$ can be written as follows:

$$U(z) = U(z_H) - \int_z^{z_H} \frac{\partial U(z_0, z_0)}{\partial z} dz_0 = U(z_H) + \int_z^{z_H} m(\theta(z_0))c'(z_0)dz_0. \tag{23}$$

This equation is derived from the envelope theorem and is standard in the mechanism design literature. The requirements of the result of Milgrom and Segal (2002) are as follows:

1. $U(z, \hat{z})$ is differentiable and absolutely continuous in z. This is satisfied because c is assumed to be twice differentiable.
2. $\sup_{\hat{z}} \left|\frac{\partial U(z,\hat{z})}{\partial z}\right|$ is integrable. This is satisfied because $\sup_{\hat{z}} \left|\frac{\partial U(z,\hat{z})}{\partial z}\right| \leq \left|c'(z)\right| < M$ for some $M \in \mathbb{R}_+$, because $c'(.)$ is continuous and is defined over a compact set $[z_L, z_H]$.
3. $\theta(z)$ is obviously non-empty. $\square$

Derivation of Equation (6). Begin from the BB condition:

$$0 = \int \left[m(\theta(z))[h(z) - p(z)] - k\theta(z)\right] F'(z)dz$$

$$= \int \left[m(\theta(z))[h(z) - c(z)] - k\theta(z) - m(\theta(z))(p(z) - c(z))\right] F'(z)dz$$

$$= \int \left[m(\theta(z))(h(z) - c(z)) - k\theta(z) - \int_z^{z_H} m(\theta(z_0))c'(z_0)dz_0 - U(z_H)\right] F'(z)dz.$$

The third equality follows from (5) and the fact that $U(z) = m(\theta(z))(p(z) - c(z))$. Using integration by parts, one yields (6) from the last equation. $\square$

Proof of Theorem 1(i). Theorem 1 is proved using a guess-and-verify approach. I guess that the first-best is achievable, and then I verify the conditions for feasibility.

I need to check that the two constraints of Problem 2 are satisfied for the proposed mechanism under the respective assumptions. The first-best level of market tightness, $\theta^{FB}(z)$, is given by $m'(\theta^{FB}(z))(h(z)-c(z))-k=0$. By differentiating it with respect to z, one yields $\frac{d\theta^{FB}(z)}{dz}=-\frac{k(h'(z)-c'(z))}{m''(\theta^{FB}(z))(h(z)-c(z))^2}\leq 0$, where the inequality is due to the fact that $h'(.)-c'(.)\leq 0$ and $m''(.)\leq 0$. Hence, MC in problem 2 is satisfied. Moreover,

$$U^{CE}(z_H)=\int\left[m(\theta(z))(h(z)-c(z)-c'(z)\frac{F(z)}{F'(z)})-k\theta(z)\right]F'(z)dz$$

$$=\int\left[\left(-\int_z^{z_H}m(\theta^{FB}(z))(h'(z)-c'(z))dz+U^{FB}(z_H)\right)-m(\theta^{FB}(z))c'(z)\frac{F(z)}{F'(z)}\right]F'(z)dz$$

$$=-\int\left[m(\theta^{FB}(z))(h'(z)-c'(z))+m(\theta^{FB}(z))c'(z)\right]F(z)dz+U^{FB}(z_H)$$

$$=-\int m(\theta^{FB}(z))h'(z)F(z)dz+U^{FB}(z_H)\geq 0. \tag{24}$$

The second equality uses the fact that $\theta(z)=\theta^{FB}(z)$ and also the fact that $\frac{dU^{FB}(z)}{dz}=\frac{d[\max_\theta\{m(\theta)(h(z)-c(z))-k\theta\}]}{dz}=m(\theta^{FB})(h'(z)-c'(z))$. The third equality is derived by using integration by parts. The inequality holds because $h'(z)<0$ by assumption, and $U^{FB}(z_H)\geq 0$ because there are positive gains from trade for all types. Both constraints in Problem 2 are satisfied. Finally, the proposed mechanism achieves the first-best, which is the highest possible welfare, so it is not needed to check whether any other allocation achieves higher welfare.

□

Proof of Theorem 1(ii) ("if" part). Again, I need to show that the proposed mechanism is feasible. But $h'(z)-c'(z)\leq 0$, so $\frac{d\theta^{FB}(z)}{dz}\leq 0$, thus the first constraint in Problem 2 is satisfied. Furthermore,

$$U^{CE}(z_H)=\int\left[m(\theta^{FB}(z))[h(z)-c(z)]-k\theta^{FB}(z)-m(\theta^{FB}(z))c'(z)\frac{F(z)}{F'(z)}\right]dF(z)$$

$$=\int\left[\int_{z_L}^{z}m(\theta^{FB}(z))(h'(z)-c'(z))dz+U^{FB}(z_L)-m(\theta^{FB}(z))c'(z)\frac{F(z)}{F'(z)}\right]dF(z)$$

$$=\int H_0(z)dF(z)-\int\int_{z_L}^{z}m(\theta^{FB}(z))c'(z)dzdF(z)-\int m(\theta^{FB}(z))c'(z)F(z)dz+U^{FB}(z_L)$$

$$=\int H_0(z)dF(z)-\left(\int m(\theta^{FB}(z))c'(z)dz-U^{FB}(z_L)\right)\geq 0,$$

where the last equality is derived using integration by parts and the last inequality is derived using (10).[23]

□

Proof of Theorem 1(ii) ("only if" part). The planner achieves the first-best, so $\theta^{FB}(z)$ must solve Problem 2; thus the second constraint, specifically, must hold. Exactly similar to the last part, one can manipulate the integral to show that condition (10) must be satisfied too. □

Proof of Theorem 2(i). By Theorem 1(i), the planner achieves the first-best and the equilibrium does not. Therefore, the welfare level in the former is strictly higher than the latter.

For Pareto inefficiency of equilibrium, let's start from the envelope conditions implied by IC: $U'^{CE}(z) = -m(\theta^{FB}(z))c'(z)$ and $U'^{EQ}(z) = -m(\theta^{EQ}(z))c'(z)$ almost everywhere. Since $h'(z) \leq 0 < c'(z)$, the boundary condition in the equilibrium is $\theta^{EQ}(z_H) = \theta^{FB}(z_H)$. Equilibrium condition (8) implies that $\theta^{EQ}(z) \geq \theta^{FB}(z)$ for all z. Therefore, $|U'^{CE}(z)| \leq |U'^{EQ}(z)|$ almost everywhere. To show that the equilibrium allocation is Pareto dominated by the constrained efficient allocation, it is necessary and sufficient to show that $U^{CE}(z_L) \geq U^{EQ}(z_L)$. But $U^{CE}(z_L) = U^{CE}(z_H) + \int_{z_L}^{z_H} m(\theta^{FB}(z))c'(z)dz = U^{FB}(z_H) - \int_{z_L}^{z_H} m(\theta^{FB}(z))h'(z)F(z)dz + \int_{z_L}^{z_H} m(\theta^{FB}(z))c'(z)dz$, where the second equality follows from (24). Moreover,

$$U^{EQ}(z_L) = U^{EQ}(z_H) + \int_{z_L}^{z_H} m(\theta^{EQ}(z))c'(z)dz = U^{FB}(z_H) + \int_{z_L}^{z_H} m(\theta^{EQ}(z))c'(z)dz.$$

Hence, it is necessary and sufficient to show that

$$-\int_{z_L}^{z_H} m(\theta^{FB}(z))h'(z)F(z)dz \geq \int (m(\theta^{EQ}(z)) - m(\theta^{FB}(z)))c'(z)dz.$$

The proof is now complete, because the left-hand side (LHS) is exactly the same as the LHS of (13) using integration by parts. □

Proof of Theorem 2 (ii). I construct a function for market tightness that satisfies the constraints of Problem 2 and achieves a higher value for the objective function. I proceed in three steps:

[23] To show that (10) and (12) are equivalent, begin from the integral in the first line above. That is positive if and only if (10) is satisfied. Now notice that the integrand is equal to $m(\theta^{FB}(z))[h(z) - c(z)] - k\theta^{FB}(z) - m(\theta^{FB}(z))c'(z)\frac{F(z)}{F'(z)} = -\frac{\theta^{FB}(z)q'(\theta^{FB}(z))}{q(\theta^{FB}(z))}(h(z) - c(z))m(\theta^{FB}(z)) - m(\theta^{FB}(z))c'(z)\frac{F(z)}{F'(z)}$, where (1) is used for the equality. But the latter expression is the same integrand as in (12).

Step 1. Denote the set of all weakly decreasing functions on $[z_L, z_H]$ by D. Define $\theta^D \in D$ as follows:

(i) $\theta^D(z) \leq \theta^{FB}(z)$ for all z.

(ii) $\theta^D(z) \geq \bar{\theta}(z)$ for all z and all $\bar{\theta} \in D$.

$\theta^D(z)$ is denoted in Figure 1. Intuitively, $\theta^D(z)$ is the greatest decreasing function not exceeding $\theta^{FB}(z)$.

Step 2. For any $\epsilon \in (0, 1)$, define

$$\theta^B(z) \equiv \epsilon\theta^D(z) + (1-\epsilon)\theta^{EQ}(z).$$

First, notice that $\theta^B(z)$ is a strictly decreasing function of z by definition of $\theta^D(z)$ and the fact that $\theta^{EQ}(z)$ is strictly decreasing.

Second,

$$\theta^B(z) = \epsilon\theta^D(z) + (1-\epsilon)\theta^{EQ}(z) \leq \epsilon\theta^{FB}(z) + (1-\epsilon)\theta^{FB}(z) = \theta^{FB}(z),$$

where the inequality holds by definition of $\theta^D(z)$ and the fact that $\theta^{EQ}(z)$ is distorted in equilibrium downward relative to the first-best. For this, see the differential equilibrium (8) that characterizes $\theta^{EQ}(z)$.

Third,

$$\theta^B(z) = \epsilon\theta^D(z) + (1-\epsilon)\theta^{EQ}(z) > \epsilon\theta^{EQ}(z) + (1-\epsilon)\theta^{EQ}(z) = \theta^{EQ}(z)$$

for all $z > z_L$. Notice that $\theta^D(z)$ is either strictly decreasing, in which case $\theta^D(z) = \theta^{FB}(z)$ and the inequality follows. Or, $\theta^D(z)$ is constant over an interval of $[z_0, z_1]$ where $\theta^D(z_0) = \theta^{FB}(z_0)$. Then, $\theta^D(z) = \theta^D(z_0) = \theta^{FB}(z_0) \geq \theta^{EQ}(z_0) > \theta^{EQ}(z)$, where the first inequality is due to the fact that $\theta^{EQ}(z)$ is distorted in equilibrium downward relative to the first-best, and the second inequality is due to the fact that $\theta^{EQ}(z)$ is strictly decreasing. See Figure 1 again.

The second and third points above together imply that $\theta^B(z) \in (\theta^{EQ}(z), \theta^{FB}(z)]$ for all $z \in (z_L, z_H]$. Therefore, the amount of surplus created with market tightness $\theta^B(z)$ is higher than that with market tightness $\theta^{EQ}(z)$ for all $z \in (z_L, z_H]$.

Step 3. Notice that $U(z_H) > 0$ in equilibrium. Now choose $\epsilon > 0$ sufficiently small such that the change in $U(z_H)$, calculated from (6), is sufficiently small so that $U(z_H) \geq 0$ continues to hold. For this ϵ, $\theta^B(z)$ is decreasing and leads to strictly higher welfare than equilibrium. This concludes the proof. $\square$

Proof of Theorem 2(iii). Since $h'(z) \geq 0$, it is easy to see that (14) is stronger than (10); therefore, the planner achieves the first-best in this case too, according to Theorem 1(ii). Similarly as above, by IC we have $U'^{CE}(z) = -m(\theta^{FB}(z))c'(z)$ and $U'^{EQ}(z) = -m(\theta^{EQ}(z))c'(z)$ almost everywhere. Since $h'(z) \geq 0$, then the boundary condition in the equilibrium is $\theta^{EQ}(z_L) = \theta^{FB}(z_L)$. Equilibrium condition (8) implies that $\theta^{EQ}(z) \leq \theta^{FB}(z)$ for all z. Therefore, $|U'^{CE}(z)| \geq |U'^{EQ}(z)|$ almost everywhere.

To show that the equilibrium allocation is Pareto dominated by the constrained efficient allocation, it is necessary and sufficient to show that $U^{CE}(z_H) \geq U^{EQ}(z_H)$. From (24), one yields $U^{CE}(z_H) = U^{FB}(z_H) - \int_{z_L}^{z_H} m(\theta^{FB}(z))h'(z)F(z)dz$. Moreover,

$$U^{EQ}(z_H) = U^{EQ}(z_L) - \int_{z_L}^{z_H} m(\theta^{EQ}(z))c'(z)dz = U^{FB}(z_L) - \int_{z_L}^{z_H} m(\theta^{EQ}(z))c'(z)dz$$

$$= U^{FB}(z_H) - \int_{z_L}^{z_H} m(\theta^{FB}(z))(h'(z) - c'(z))dz - \int_{z_L}^{z_H} m(\theta^{EQ}(z))c'(z)dz.$$

Hence, it is necessary and sufficient to show that

$$\int_{z_L}^{z_H} m(\theta^{FB}(z))h'(z)(1 - F(z))dz \geq \int (m(\theta^{FB}(z)) - m(\theta^{EQ}(z)))c'(z)dz.$$

The proof is now complete, because the LHS is exactly the same as the LHS of (14) using integrating by parts. $\square$

Proof of Proposition 2. I assume that the IC constraints are not binding, so I maximize the planner's objective function in Problem 2 given only the $U(z_H) \geq 0$ constraint. Next, I will check that the IC constraints are not binding under condition (16).

The Lagrangian can be written as:

$$\int \left[s(\theta(z), z) + \mu \Big(s(\theta(z), z) - m(\theta(z))c'(z)\frac{F(z)}{F'(z)} \Big) \right] F'(z)dz,$$

where $s(\theta, z) \equiv m(\theta)[h(z) - c(z)] - k\theta$, and μ is the Lagrangian multiplier associated with the $U(z_h) \geq 0$ constraint. The conditions for optimality imply:

$$\begin{aligned} \left(s_\theta(\theta(z), z) - \frac{\mu}{1+\mu} m'(\theta(z))c'(z)\frac{F(z)}{F'(z)} \right) F'(z)(1+\mu) &= 0, \\ U(z_H) = \int \left[m(\theta(z)) \Big[h(z) - c(z) - c'(z)\frac{F(z)}{F'(z)} \Big] - k\theta(z) \right] F'(z)dz &\geq 0 \text{ with equality if } \mu > 0. \end{aligned}$$

The first condition is equivalent to (17) and the second one is identical to that in the proposition. These conditions together are necessary and sufficient for optimality under the requirements of Proposition 2.

Combining (16) together with $h'(.) - c'(.) < 0$, one can see that θ obtained from (17) is decreasing in z, confirming the premise that the IC constraints are not binding. Also, because of (15), all types will be active under the planner's solution because their contribution to the Lagrangian, $(s(\theta(z), z) - \frac{\mu}{1+\mu} m(\theta(z))c'(z)\frac{F(z)}{F'(z)})F'(z)(1+\mu)$, is positive.

Finally, note that if the constraint is not binding, then $\mu = 0$ and the solution is obviously unique, because the first-best can be achieved. If the constraint is binding, $\theta(.)$ is uniquely determined for a given μ. As μ increases, $\theta(z)$ decreases, so does $U(z_H)$. As a result, there exists a unique μ such that the constraint becomes binding at $U(z_H) = 0$. This completes the proof. $\square$

Proof of Proposition 3. I suppress the superscript CE in this proof to reduce the notation when there is no danger of confusion. If $h' < 0$, the proof is similar, so I focus only on the case where both $0 < h' < c'$ and (10) hold.

Theorem 1(ii) applies, so $\theta(z) = \theta^{FB}(z)$ for all z. I calculate $m(\theta(z))t(z)$ and take its derivative with respect to z:

$$\begin{aligned} m(\theta(z))t(z) &= m(\theta^{FB}(z))(h(z) - c(z)) - k\theta^{FB}(z) - U(z_H) - \int_z^{z_H} m(\theta^{FB}(z_0))c'(z_0)dz_0 \\ &= U^{FB}(z) - U(z_H) - \int_z^{z_H} m(\theta^{FB}(z_0))c'(z_0)dz_0 \\ \Rightarrow \frac{\partial}{\partial z}[m(\theta(z))t(z)] &= \frac{dU^{FB}(z)}{\partial z} + m(\theta^{FB}(z))c'(z) = m(\theta^{FB}(z))h'(z). \end{aligned} \tag{25}$$

The second equality is derived by applying the envelope theorem to the following maximization problem: $U^{FB}(z) = \max_\theta\{m(\theta)(h(z) - c(z)) - k\theta\}$. Therefore,

$$\frac{dt}{dz} = h'(z) - \frac{m'(\theta(z))}{m(\theta(z))}\frac{d\theta(z)}{dz}t(z).$$

Also, $\frac{d\theta(z)}{dz}$ can be calculated from (1) to obtain $\frac{dt}{dz}$ as in Lemma 3.

Since $\int t(z)dF(z) = 0$, there exists a $\hat{z} \in (z_L, z_H)$ such that $t(\hat{z}) = 0$. At such a point, $\frac{dt^{CE}}{dz} = h'(z) > 0$ at $z = \hat{z}$. Since t^{CE} is continuous, there cannot be more than one such $\hat{z}$. If so, the slope of t^{CE} for at least one of such $\hat{z}$ should be negative. $\square$

Proof of Lemma 3. The proof for part (i) is similar, so I focus only on part (ii). $h(.) - c(.)$ is strictly decreasing, so $\theta^{FB}(.)$ is strictly decreasing; therefore, the associated implementable allocation must be separating. Hence, there is a one-to-one mapping from types to prices. I show below that $t(z)$ is decreasing in z at $z = z_L$. Furthermore, $\frac{dt}{dp} = \frac{dt/dz}{dp/dz}$, and as shown in Lemma 4 that the denominator is always positive, $\frac{dt}{dp}$ must be negative for $p = p_L$. Now consider (18) at $z = z_L$. Given the assumption that $h'(z_L) = 0$ and given the fact that $\theta'(z) < 0$ for all z, it is sufficient to show that $t(z_L) < 0$. It will follow that $t'(z_L) < 0$. To calculate $t(z)$, I use the planner's budget-balance condition to write: $\int m(\theta(z))t(z)dF(z) = 0$. Let $\chi(.) \equiv m(\theta(z))t(z)$, then $0 = \int \chi(z)dF(z) = -\chi(z)(1-F(z))\big|_{z_L}^{z_H} + \int \chi'(z)(1-F(z))dz$ by using integration by parts. Therefore, $\chi(z_L) = -\int \chi'(z)(1-F(z))dz < 0$. The inequality holds because $\chi'(z) = m(\theta(z))h'(z)$ from (25) and the fact that $h'(z) \geq 0$. Also, the inequality is strict because $h'(z) > 0$ for a positive measure of z. Hence, $t(z_L) = \frac{\chi(z_L)}{m(\theta(z_L))} < 0$ by definition of $\chi(.)$.

Finally, I show that $\frac{dt}{dp} > 0$ for some p. The facts that $\int m(\theta(z))t(z)dF(z) = 0$, $t(z_L) < 0$, $t(z)$ is continuous, and F has full support, together imply that $t(z)$ must be strictly positive for a strictly positive measure of z, therefore, $t'(z) > 0$ for for a strictly positive measure of z. Finally, $\frac{dt}{dp} = \frac{dt/dz}{dp/dz}$, so $\frac{dt}{dp}$ must be strictly positive for some p, using Lemma 4.

□

Lemma 4. *Assume $c'(.) > 0$. Take any feasible mechanism. The transfer function in this mechanism, $p(z)$, is increasing in z.*

Proof of Lemma 4. According to (5), $\frac{d[m(\theta(z))p(z)]}{dz} = m'(\theta(z))\frac{d\theta(z)}{dz}c(z)$, so

$$\frac{dp(z)}{dz} = -\frac{m'(\theta(z))}{m(\theta(z))}\frac{d\theta(z)}{dz}(p(z) - c(z)) \geq 0. \tag{26}$$

The inequality holds because $\theta(z)$ is decreasing in z following the fact that the mechanism satisfies the incentive compatibility constraint. Moreover, $p(z) - c(z)$ is positive for all types following the fact that the mechanism satisfies the participation constraint. Therefore, $p(z)$ is **weakly** increasing in z.

Finally, $p(z) - c(z) = \frac{U(z)}{m(\theta(z))} = \frac{U(z_H) + \int_z^{z_H} m(\theta(z_0))c'(z_0)dz_0}{m(\theta(z))} > \frac{U(z_H)}{m(\theta(z))} \geq 0$ for $z < z_H$. Therefore, if $\frac{d\theta(z)}{dz} > 0$, then $p(z)$ is **strictly** increasing in z. □

Proof of Proposition 4. Again, I focus only on case (ii), as the proof is similar for case (i).

From (18) and (19), one obtains $\frac{dt^{CE}}{dz} > 0$ at $z = z_L$. Now suppose t^{CE} is not increasing; then it must be the case that t^{CE} has a local maximum. Consider the first local maximum, say at $z = z_M$. Since t^{CE} is increasing at $z = z_L$, then $t^{CE}(z_M) > t^{CE}(z_L)$. Now we can write the following at $z = z_M$:

$$h'(z) = -\frac{m'^2}{mm''}\frac{h' - c'}{h - c}t^{CE}(z_M),$$

and

$$h'(z) \geq -\frac{m'^2}{mm''}\frac{h'-c'}{h-c}t_L$$

because of (19). The last two inequalities together imply that $t^{CE}(z_M) \leq t^{CE}(z_L)$, which is a contradiction. This completes the proof. $\square$

Proofs of the claims in Example 3. Theorem 1(ii) and 2(iii) apply, so we need to check conditions (10) and (14). H_0 and $\bar{H}_0$ can be calculated as follows:

$$H_0(z) = m(\bar{\theta}^{FB})\int_{z_L}^{z} h'(\hat{z})d\hat{z} = m(\bar{\theta}^{FB})(c(z) - c(z_L)).$$

$$\bar{H}_0 = \int H_2(z)dF(z) = m(\bar{\theta}^{FB})\int_{z_L}^{z_H}(c(z) - c(z_L))dz.$$

To verify (10), we need to verify that:

$$m(\bar{\theta}^{FB})\int_{z_L}^{z_H}(1-F(z))c'(z)dz \geq m(\bar{\theta}^{FB})\int c'(z)dz - U^{FB},$$

$$\Leftrightarrow \frac{\bar{U}^{FB}}{m(\bar{\theta}^{FB})} \geq \int_{z_L}^{z_H} F(z)c'(z)dz.$$

I argue that if the gains from trade, x, are sufficiently high, then this inequality holds, so the planner can always achieve the first-best. The RHS is independent of x, so it is enough to show that the LHS is increasing in x, and when x goes to ∞, the LHS is greater than the RHS. The LHS can be written as $\frac{U^{FB}}{m(\theta^{FB})} = x - k/q(\theta^{FB})$. Therefore,

$$\frac{\partial \frac{U^{FB}}{m(\theta^{FB})}}{\partial x} = 1 + \frac{kq'\frac{\partial \theta^{FB}}{\partial x}}{q^2} = 1 - \frac{k^2 q'}{q^2 s^2 m''} = 1 - \frac{q'm'}{qm''}\frac{m'}{q} \geq 1/2 > 0,$$

where the second and third equalities follow from the definition of θ^{FB}. The inequality follows from the fact that m has a decreasing elasticity (See Eeckhout and Kircher (2010) or Davoodalhosseini (2015)). This completes the proof for verifying (10).

It is easy to see that (14) is equivalent to (21) in this example. Here, I characterize only the equilibrium allocation, which is given by the following differential equation:

$$\left[m'(\theta^{EQ}(z))x - k\right]\frac{d\theta^{EQ}(z)}{dz} + m(\theta^{EQ}(z))c'(z) = 0.$$

Rewrite the equation in a separable form in z and θ:

$$-\frac{m'(\theta^{EQ})x - k}{m(\theta^{EQ})}d\theta^{EQ} = c'(z)dz.$$

Integrate the LHS over θ and the RHS over z to obtain (20):

$$\int_{\theta^{EQ}(z)}^{\theta^{EQ}(z_L)} \frac{m'(\theta)x-k}{m(\theta)}d\theta = c(z)-c(z_L).$$

Notice that the upper bound of the integral is equal to θ^{FB}. This equation gives us $\theta^{EQ}(z)$ (which is less than θ^{FB}) in terms of z and can be easily solved. □

Example 4 (Characterizing equilibrium and constrained efficient allocation for $m(\theta)=\mu\min\{\theta,1\}$). *Assume $0<h'(z)\le c'(z)$ and $\mu(h(z)-c(z))>k$ for all z. According to the differential equation (8), $\theta^{EQ}(z)$ solves: $\theta^{EQ}(z)exp\left(\int_{z_L}^{z}\frac{q(\theta^{EQ}(\hat{z}))h'(\hat{z})}{m'(\theta^{EQ}(\hat{z}))(h(\hat{z})-c(\hat{z}))-k}d\hat{z}\right)=const$. But $0<h'(.)\le c'(.)$, so $\theta^{EQ}(z_L)=\theta^{FB}(z_L)$, thus $const=\theta^{FB}(z_L)$. The equilibrium market tightness is distorted downward for all $z>z_L$, so $q(.)$ and $m'(.)$, given by $q(\theta)=m'(\theta)=\mu$ for all $\theta<1$, are both differentiable almost everywhere. Therefore, $\theta^{EQ}(z)=\theta^{FB}(z_L)exp\left(-\int_{z_L}^{z}\frac{h'(\hat{z})}{h(\hat{z})-c(\hat{z})-k/\mu}d\hat{z}\right)$. Moreover, $\theta^{FB}(z)=1$ for all z, so $H_0=h(z)-h(z_L)$ and $\bar{H}_0=\int h(z)dF(z)-h(z_L)\equiv\mathbb{E}(h(z))-h(z_L)$. Therefore, (14) in this example reduces to:*

$$\mathbb{E}(h(z))-h(z_L)\ge\int\left(1-exp\left(-\int_{z_L}^{z}\frac{h'(\hat{z})}{h(\hat{z})-c(\hat{z})-k/\mu}d\hat{z}\right)\right)c'(\hat{z})dz.$$

This condition is explicitly on the fundamentals of the model. (13) can be given in a similar fashion.

With this special matching function, $\theta^{FB}(z)=1$. To achieve the first-best, we need

$$\int\left[\mu\left(h(z)-c(z)-c'(z)\frac{F(z)}{F'(z)}\right)-k\right]F'(z)dz\ge 0, \tag{27}$$

which is equivalent to (12) or (10).

Planner's problem in a general form

If the expression in brackets in (17) is decreasing in z, even if $h-c$ is not, then Proposition 2 can be used. If that expression is not decreasing for some z, then we need to solve Problem 2 in a general form, which is described below. We continue to assume that $c'(z)>0$.

Problem 3.

$$\max_{\theta(z)}\int\Big[m(\theta(z))[h(z)-c(z)]-k\theta(z)\Big]F'(z)dz$$

subject to:

$$\begin{aligned}
\phi(z): \quad \tfrac{d\theta(z)}{dz} = &\quad \chi(z) \\
\eta(z): \quad -\chi(z) \geq &\quad 0 \\
\mu: \quad U(z_H) \geq &\quad 0.
\end{aligned}$$

with complementary slackness for $\eta(z)$ and μ.

The second constraint is to ensure that the IC is satisfied. Functions $\phi(z)$ and $\eta(z)$ are the associated multipliers for the first two constraints, and μ, as in Proposition 2, is the multiplier for the last constraint. Also, $U(z_H)$ is calculated from (6).

This is an optimal control problem with state variable θ and control variable χ, so we form the Hamiltonian to solve the problem:

$$H = s(\theta(z), z)F'(z) + \mu\Big(s(\theta(z), z) - m(\theta(z))c'(z)\frac{F(z)}{F'(z)}\Big)F'(z) + \phi(z)\chi(z) - \eta(z)\chi(z),$$

where $s(\theta, z) \equiv m(\theta)[h(z) - c(z)] - k\theta$. Hence:

$$\frac{\partial H}{\partial \theta(z)} = \Big(s_\theta(\theta(z), z) - \frac{\mu}{1+\mu}m'(\theta(z))c'(z)\frac{F(z)}{F'(z)}\Big)F'(z)(1+\mu) = -\frac{d\phi(z)}{dz} \tag{28}$$

$$\frac{\partial H}{\partial \chi(z)} = \phi(z) - \eta(z) = 0 \tag{29}$$

If IC constraints are not binding, then $\eta(z) = 0$. Hence $\phi(z) = 0$ by (29). Then, (28) implies that $s_\theta(\theta(z), z) - \frac{\mu}{1+\mu}m'(\theta(z))c'(z)\frac{F(z)}{F'(z)} = 0$, which is identical to (17). Under the requirements of Proposition 2, θ obtained by (17) is decreasing, confirming the premise that the IC constraints are not binding.

Now suppose IC constraints are binding for some z. More precisely, define

$$\theta^{SB}(z) \equiv \max_\theta \left\{ s(\theta, z) - \frac{\mu}{1+\mu}m'(\theta)c'(z)\frac{F(z)}{F'(z)} \right\}.$$

Suppose $\theta^{SB}(z)$ is increasing between z_1 and z_2. See Figure 8. This means that the constraint efficient allocation would entail pooling of some types. We want to find the pooling region. Suppose $\theta^{SB}(z)$ has only one local minimum at $z_1 > z_L$ and one local maximum at $z_2 < z_H$. At the solution:

$$\theta^{CE}(z) = \begin{cases} \theta^{SB}(z) & z_L \leq z \leq z_a \\ \bar{\theta} & z_a < z \leq z_b \\ \theta^{SB}(z) & z_b < z \leq z_H \end{cases},$$

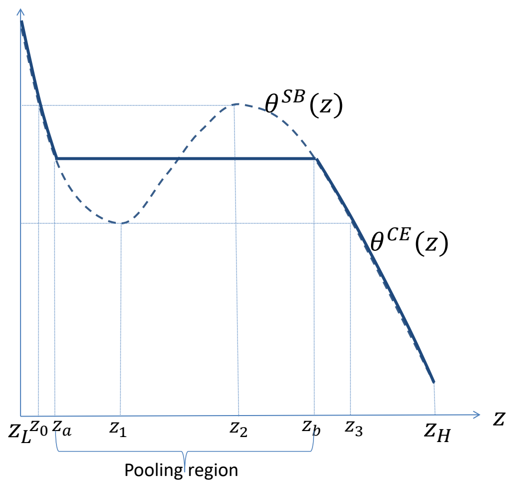

Figure 8: Schematic diagram for the pooling region in the planner's problem.

where $\bar{\theta} \equiv \theta^{SB}(z_a)$. The unknowns are z_a and z_b.[24] Since the constraint is not binding for z below z_a and above z_b, then $\phi(z_a) = \phi(z_b) = 0$. Integrating (28) over $[z_a, z_b]$ leads to:

$$\int_{z_a}^{z_b} \left(s_\theta(\bar{\theta}, z) - \frac{\mu}{1+\mu} m'(\bar{\theta}) c'(z) \frac{F(z)}{F'(z)} \right) F'(z)(1+\mu) dz = -\phi(z_b) + \phi(z_a) = 0.$$

Define $I(\bar{\theta})$ to be $z \in [z_0, z_1]$ that solves $\theta^{SB}(z) = \bar{\theta}$. Similarly, define $J(\bar{\theta})$ to be $z \in [z_2, z_3]$ that solves $\theta^{SB}(z) = \bar{\theta}$. See Figure 8 again. Now define

$$\Delta(\bar{\theta}) \equiv \int_{I(\bar{\theta})}^{J(\bar{\theta})} \left(s_\theta(\bar{\theta}, z) - \frac{\mu}{1+\mu} m'(\bar{\theta}) c'(z) \frac{F(z)}{F'(z)} \right) F'(z)(1+\mu) dz.$$

Note that at $\bar{\theta} = \theta^{SB}(z_0)$, $\Delta(\bar{\theta}) < 0$ and at $\bar{\theta} = \theta^{SB}(z_1)$, $\Delta(\bar{\theta}) > 0$. Note also that m is a concave function, (15) holds and $I(\bar{\theta})$ and $J(\bar{\theta})$ are the optimal choices, so $\Delta(\bar{\theta})$ is decreasing in $\bar{\theta}$. Hence, there exists a unique $\bar{\theta}$ that solves $\Delta(\bar{\theta}) = 0$. Then, $z_a = I(\bar{\theta})$ and $z_b = J(\bar{\theta})$. This completes the characterization.

The main message is that, if maximization of the virtual surplus gives rise to a function that is not decreasing in z, then the planner wants to pool some types. This is despite the fact that if h is increasing, the types will be separated completely in the market.

[24] The technique to solve the problem in the case where there is more than one local minimum or maximum has been discussed in Fudenberg and Tirole (1991), Appendix of Chapter 7, for example.

Entry Tax

The implementable allocation with two types of taxes on buyers, sales tax and entry tax, is defined here similarly to the definition of implementable allocation in Definition 1. The main difference is that the policy here includes $\tilde{t}_e(p)$, which is an entry tax schedule, the tax levied on buyers when they enter a submarket whether or not they find a match.

Definition 5 (Implementable allocation for continuous-type space with sales and entry tax)**.** *An allocation, $\{G, \mathcal{P}, \theta, \mu\}$, is implementable through policy $\{\tilde{t}, \tilde{t}_e, \tilde{t}_0\}$ if the following conditions are satisfied:*

(i) ***Buyers' profit maximization, free entry and no commitment***

For any $p \in \mathbb{P}$,

$$q(\theta(p))[\int h(z)\mu(z|p)dz - \tilde{t}(p)] \leq k + \tilde{t}_e(p),$$

with equality if $p \in \mathcal{P}$. Also, $0 \leq k + \tilde{t}_e(p)$ for any $p \in \mathcal{P}$.

(ii) ***Sellers' optimal search****: the same as Definition 1(ii).*

(iii) ***Feasibility or market clearing****: the same as Definition and 1(iii),*

(iv) ***Planner's budget-balance condition***

$$\int_{\mathcal{P}} [q(\theta(p))\tilde{t}(p) + \tilde{t}_e(p)]dG(p) \geq \tilde{t}_0.$$

Proof of Proposition 5. I focus on the cases in which c and h are both increasing. Analyzing other cases are similar. The set of admissible prices, $\mathbb{P}$, is assumed to be $[c(z_L), \infty)$ as opposed to $(0, \infty)$. This assumption is not restrictive (and made only to avoid some technical difficulties), as no seller would have incentives to apply to $p < c(z_L)$.

Consider again a feasible mechanism $\{\theta(.), \mathbf{p}(.), t(.), t_0\}$. I construct the allocation $\{G, \mathcal{P}, \Theta, \mu\}$ and policy $\{\tilde{t}, \tilde{t}_e, \tilde{t}_0\}$ and show that if $M \in \mathbb{R}_+$ and $M' \in \mathbb{R}_+$, defined below, are chosen sufficiently large, then this allocation is implementable and $\tilde{t}_e(p)$ is strictly decreasing and $\tilde{t}(p)$ is strictly increasing in p. The allocation is constructed as follows:

$$\mathcal{P} \equiv [p_L, p_H] \subseteq \mathbb{P} \equiv R_+, \text{ where } p_L \equiv \mathbf{p}(z_L) \text{ and } p_H \equiv \mathbf{p}(z_H),$$

and p^{CE} is given by (5) (where θ should replace θ^{CE}). Moreover,

$$\begin{cases} \Theta(p) = 1 & \text{for } p = c(z_L) \\ m(\Theta(p)) = \min\{\bar{m}, \frac{U(z_L)}{p - c(z_L)}\} & \text{for } p \in (c(z_L), p_L) \\ \Theta(p) = \theta(\mathbf{p}^{-1}(p)) & \text{for } p \in [p_L, p_H] \\ m(\Theta(p)) = \min\{\bar{m}, \frac{U(z_H)}{p - c(z_H)}\} & \text{for } p > p_H \end{cases}$$

$$G(p)=\begin{cases}0 & \text{for } p\in[c(z_L),p_L)\\ \int_{p_L}^{p}\Theta(p)F'(\mathbf{p}^{-1}(p))dp & \text{for } p\in[p_L,p_H]\\ \int_{p_L}^{p_H}\Theta(p)F'(\mathbf{p}^{-1}(p))dp & \text{for } p>p_H\end{cases}$$

$$\int\mu(z|p)dz=1 \text{ for all } p, \text{ and } \mu(z|p)=\begin{cases}0 & \text{for } p<p_L \text{ and } z\neq z_L\\ 0 & \text{for } p\neq p(z) \text{ and } p\in[p_L,p_H]\\ 0 & \text{for } p>p_H \text{ and } z\neq z_H\end{cases}.$$

The policy is given by:

$$\tilde{t}_0=0,\tilde{t}_e(p)=\begin{cases}-k+M(p_H-p) & \text{for } p\in[c(z_L),p_H]\\ -k & \text{for } p>p_H\end{cases}$$

$$\tilde{t}(p)=\begin{cases}h(z_L)-p-\frac{k+\tilde{t}_e(p)}{q(\Theta(p))} & \text{for all } p\in[c(z_L),p_L)\\ h(p^{-1}(p))-p-\frac{k+\tilde{t}_e(p)}{q(\Theta(p))} & \text{and } p\in[p_L,p_H]\\ \tilde{t}(p_H)+(p-p_H)M' & \text{for } p>p_H\end{cases}.$$

The conditions for implementability can be verified easily, so I do not repeat them here.

Regarding monotonicity of taxes, it is obvious that $\tilde{t}_e(p)$ is decreasing in p for all $p\in[p_L,p_H]$ if $M>0$. It is just left to show that $\tilde{t}(p)$ is increasing in p for $p\in[p_L,p_H]$. I take a derivative of $\tilde{t}(p)$ with respect to p:

$$\tilde{t}'(p)=h'(\mathbf{p}^{-1}(p))\frac{d(\mathbf{p}^{-1}(p))}{dp}-1+M\frac{q(\Theta(p))+q'(\Theta(p))\Theta'(p)(p_H-p)}{q(\Theta(p))^2}.$$

Now, define

$$M_1\equiv\max\left\{1,\sup_{p\in[p_L,p_H]}\frac{(1-h'(\mathbf{p}^{-1}(p))\frac{d(\mathbf{p}^{-1}(p))}{dp})q(\Theta(p))^2}{q(\Theta(p))+q'(\Theta(p))\Theta'(p)(p_H-p)},\sup_{p\in[c(z_L),p_L]}\frac{q(\Theta(p))^2}{q(\Theta(p))+q'(\Theta(p))\Theta'(p)(p_H-p)}\right\}.$$

M_1 is a lower bound for M and 1 is merely an arbitrary positive number. I want to show that $M_1<\infty$, so the second and third expressions in the max have to be less than ∞. Consider the second one. If $q(\Theta(p))\to 0$, then the expression goes to 0; therefore, I simply need to show that $\frac{d(\mathbf{p}^{-1}(p))}{dp}>-\infty$. But $\frac{d\mathbf{p}}{dz}$ has been already calculated in Lemma 4, so $\frac{d(\mathbf{p}^{-1}(p))}{dp}$, which is merely the inverse of $\frac{d\mathbf{p}}{dz}$, is always positive too. Since z lies in a compact interval, $1-h'(.)\frac{d\mathbf{p}}{dz}$ is not greater than 1 and the proof in this part is complete. The same argument applies to the third expression but for $p\in[c(z_L),p_L]$.

Therefore, if $M > M_1$ and $M' > 0$, then $\tilde{t}(p)$ is strictly increasing over each separate interval. Since $\tilde{t}(p)$ is continuous by construction, it is therefore increasing over the entire domain. $\square$