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by

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Abstract

Many central banks are contemplating whether to issue a central bank digital currency (CBDC). CBDC has certain potential benefits, including the possibility that it can bear interest. However, using CBDC is costly for agents, perhaps because they lose their anonymity when using CBDC instead of cash. I study optimal monetary policy when only cash, only CBDC, or both cash and CBDC are available to agents. If the cost of using CBDC is not too high, more efficient allocations can be implemented by using CBDC than with cash, and the first best can be achieved. Having both cash and CBDC available may result in lower welfare than in cases where only cash or only CBDC is available. The welfare gains of introducing CBDC are estimated as up to 0.64% for Canada.

Bank topics: Digital currencies; Monetary policy
JEL codes : E42, E50

Résumé

Nombreuses sont les banques centrales qui se demandent si elles devraient émettre une monnaie électronique. Une monnaie électronique de banque centrale (MEBC) présente des avantages potentiels, dont la possibilité qu'elle puisse porter intérêt. Son utilisation entraîne cependant un coût pour les agents, sans doute en raison de la perte de l'anonymat que leur offre l'argent liquide. Je cherche à déterminer la politique monétaire optimale selon que les moyens de paiement à la disposition des agents sont les espèces seules, la MEBC seule, ou les espèces et la MEBC. Si le coût lié à l'usage de la MEBC n'est pas trop élevé, des formes de répartition plus efficientes que ne le permettent les espèces peuvent être mises en œuvre, et l'optimum de premier rang peut être atteint. La coexistence des espèces et de la MEBC peut se traduire par un niveau de bien-être économique moindre que dans les cas où l'argent liquide (ou la MEBC) constitue l'unique moyen de paiement. Au Canada, les gains de bien-être issus de l'émission d'une MEBC pourraient atteindre 0,64 %.

Sujets : Monnaies numériques; Politique monétaire
Codes JEL : E42, E50

Non-technical summary

Many central banks are contemplating whether to issue central bank digital currency (CBDC). If they do, CBDC will co-exist with other means of payment, including cash. On one hand, CBDC has certain potential benefits, including the possibility that it can bear interest. On the other hand, cash may be preferred by agents over CBDC, perhaps because they can remain anonymous in transactions by using cash.

Interactions between cash and CBDC have not been well understood in the literature. I put together a model in which cash and CBDC co-exist, and agents with heterogeneous transaction needs can choose their portfolios with varying mixtures of each. Using this model, I investigate how monetary policy is affected by the introduction of CBDC, and study the circumstances under which its use is desirable. I show that CBDC provides more flexibility for the central bank to conduct monetary policy. This is because the central bank can monitor agents' portfolios of CBDC and can cross-subsidize between different types of agents, but these actions are not possible if agents use cash.

Having both cash and CBDC available to agents sometimes results in lower welfare than in cases where only cash or only CBDC is available. This fact suggests that removing cash from circulation may be a welfare-enhancing policy if the motivation to introduce CBDC is to improve monetary policy effectiveness. When the availability of both cash and CBDC results in higher welfare than in the above-mentioned cases, optimal cash inflation is strictly positive, and agents endogenously use cash in small-value transactions and CBDC in large-value transactions.

The welfare gains of introducing CBDC are estimated as up to 0.64% for Canada.

"[P]hasing out paper currency is arguably the simplest and most elegant approach to clearing the path for central banks to invoke unfettered negative interest rate policies should they bump up against the 'zero lower bound' on interest rates."

Ken Rogoff (2016) in *The Curse of Cash*

"Some economists advocate that the central bank should replace cash with a digital currency that can be given a negative interest rate. ... This reasoning is based on the central bank being prevented from setting a negative interest rate to the extent considered necessary to stimulate economic activity. Personally, I am not convinced that this problem would arise in Sweden and I would once again like to say that the Riksbank has a statutory requirement to issue banknotes and coins. I see e-krona primarily as a complement to cash."

Skingsley (2016), Deputy Governor of the Bank of Sweden

1 Introduction

There has been a great deal of discussion in recent years about the effects of introducing central bank digital currency (CBDC) into economies and whether cash should be eliminated, as the quotes above indicate. Some central banks have already started the decision-making process on whether to introduce CBDC into their respective economies. For example, the central bank of Sweden wants to decide soon on whether to issue CBDC (what they call "e-krona"), and if yes, what type of CBDC.[1] Also, some officials at the central bank of China have expressed their desire to issue their own digital currency as a way to support their digital economy.[2] If central banks issue CBDC, important questions arise, some of which are as follows: Should central banks eliminate cash from circulation? What would be the optimal (i.e., welfare-maximizing) monetary policy if agents can choose between cash and CBDC? And quantitatively, what are the welfare gains of introducing CBDC into the economy?[3]

[1] See the first interim report on the Riksbank's E-krona Projects (Sveriges Riksbank (2017)). Also note that CBDC can be of different types. Several authors have suggested taxonomies to understand various types of electronic money, some forms of which are CBDC. See Bech and Garratt (2017), Bjerg (2017) or the Committee on Payments and Market Infrastructures (2015) report on digital currencies.

[2] See here: https://goo.gl/kEpHhV.

[3] It may seem that these questions are about the payment systems only, but the endogenous choice of means of payment by agents affects the optimal monetary policy that the central bank can adopt and, consequently, welfare.

To address these and similar questions, I use the framework of Lagos and Wright (2005) to build a model in which two means of payment could be available to agents: cash and CBDC. What I mean by CBDC in this paper is the money issued by the central bank in electronic format and universally accessible; i.e., all agents in the economy can use it to purchase goods and services.[4] I study the optimal monetary policy when only one or both means of payment are available to agents. Cash and CBDC are different along two dimensions in this paper. First, the ability of the central bank to implement monetary policy is different across these means of payment. The central bank can allocate transfers to agents based on their CBDC balances but the central bank cannot do so based on their cash balances because the central bank cannot see agents' cash balances. Therefore, the only policy that the central bank can implement with cash is to distribute the newly created cash evenly across all agents.[5] Second, carrying CBDC is more costly relative to cash. This cost is perhaps due to the fact that agents lose their anonymity if using CBDC. This cost creates a sensible tradeoff for the central bank regarding the means of payment that the central bank would like agents to use. While CBDC is a more flexible policy instrument, it is more costly than cash.

There are two main results of the paper. First, given that the cost of carrying CBDC is sufficiently small, the fact that CBDC is interest bearing in a non-linear fashion allows the central bank to achieve better allocations than with cash. In particular, it is possible to achieve the first-best level of production by using CBDC if the agents are patient enough and if the bargaining power of buyers is sufficiently high, while it is never possible to achieve the first best by using cash. Second, when cash and CBDC are both available to agents and valued in equilibrium, the monetary policy may be more constrained (i.e., welfare may be lower) compared with the case in which only one means of payment is available.

To elaborate on these results, consider three different schemes: only cash is available to the agents (cash-only scheme), only CBDC is available to the agents (CBDC-only scheme), and both cash and CBDC are available (co-existence scheme).[6] If only cash is available, then the optimal inflation in this economy is zero. It would not be possible to implement a negative inflation as the central bank cannot force agents to pay taxes, and a positive

[4] A similar form of money was proposed by Tobin (1987). In his own words: "I think the government should make available to the public a medium with the convenience of deposits and the safety of currency, essentially currency on deposit, transferable in any amount by check or other order."

[5] It is important to note that taxing cash balances is not feasible here; otherwise, the central bank can simply run the Friedman rule to achieve the first best, and adding CBDC or replacing cash by CBDC would not offer any potential improvements.

[6] The optimal monetary policy under co-existence is defined to be one that maximizes welfare among all policies under which both cash and CBDC are valued in equilibrium and used as means of payment.

inflation would lead agents to economize on their real balances relative to the first best, so the production level is distorted. If only CBDC is available, then the set of implementable allocations is larger, because the balance-contingent transfers are allowed with CBDC, not with cash, and even the first-best level of production can be achieved. However, there is welfare loss resulting from the cost of carrying CBDC. Comparing cash-only and CBDC-only schemes, we find that the tradeoff for the central bank is simply between distorting the allocation relative to the first best under the cash-only scheme or having the agents incur the cost of carrying CBDC under the CBDC-only scheme.

Under the co-existence scheme, agents with lower transaction needs endogenously choose to use cash, and agents with higher transaction needs choose to use CBDC. In this case, the central bank faces a constraint stemming from the endogenous choice of means of payment. Because cash is available, agents who would have used CBDC if cash had not been available can now use cash as a way to evade the taxation that CBDC users are subjected to. To discourage these agents from using CBDC, the central bank could set the cash inflation too high, but it would hurt cash users. Therefore, the availability of cash in the presence of CBDC imposes a constraint for the central bank's maximization problem. Whether or not the co-existence scheme is optimal (i.e., leading to higher welfare) relative to cash-only or CBDC-only schemes depends on how tight this constraint is. If the constraint is too tight, then the central bank would prefer to have only one means of payment used by agents. In this case, if the cost of carrying CBDC is not too high, then central bank eliminates cash, and if the cost is too high, then the central bank eliminates CBDC. On the other hand, if the constraint is relatively relaxed, then the central bank would have both cash and CBDC circulating in the economy.

For both cash and CBDC to be used by agents, the cash inflation must be strictly positive. This result is obtained despite the fact that it is feasible to implement a negative cash inflation rate when both are available (through open market operations where cash is traded for CBDC), but it would induce CBDC users to use cash instead. Therefore, CBDC would not be adopted under a negative cash inflation rate.

Rogoff (2016) has argued in favor of eliminating cash from circulation, except perhaps for small-denomination notes, as the first quote above indicates. One of his main arguments is that by eliminating cash, central banks can stimulate the economy in downturns via setting negative nominal interest rates. If cash is available, since cash guarantees the nominal interest rate of zero for agents, the ability of central banks to stimulate the economy will be restricted. The reason that co-existence of cash and CBDC may not be optimal in my model is similar to Rogoff's argument. In both, cash provides an outside option for agents, restricting the set of feasible allocations that the central bank can achieve. However, it is important to note that the effectiveness of CBDC is not only due to the fact that it allows for the possibility of achieving negative interest rates, but it also allows for implementation of non-linear transfer schemes, the feature that I use to show that the first-best level of production can be achieved using CBDC. Furthermore, CBDC provides more information to the central bank, such as whether the agent is a buyer or seller or the size of transaction. Altogether, even if cash is not eliminated, CBDC can still positively affect monetary policy, although its effectiveness is sometimes enhanced if cash is eliminated.[7]

To give a sense of the welfare gains of introducing CBDC, I calibrate the model to the Canadian and US data. I show that introducing CBDC can lead to an increase of up to 0.64% in consumption for Canada and up to 1.6% for the US, compared with their respective economies if only cash is used. Assuming that there are only two sizes of transactions (large-value and small-value transactions), I calculate the welfare gains of introducing CBDC for different values of the cost of carrying CBDC and for various values of the relative size of large- to small-value transactions. As an example, if the monetary cost of carrying CBDC relative to cash is 0.25% of the transaction value and the average value of large transactions is around six times the size of small-value transactions, then introducing CBDC will lead to an increase in consumption of 0.16% for Canada.

In an extension of the benchmark model, I assume that CBDC is not a perfect substitute for cash, in that CBDC cannot be used in a fraction of transactions in which cash can be used. In this case, if the cost of carrying CBDC is low, then co-existence might be optimal. Also, the first best can be achieved as long as the fraction of meetings in which only CBDC can be used and also the discount factor are sufficiently high. This result shows, interestingly, that CBDC helps to achieve the first best even for the meetings in which only cash can be used.

One may argue that the type of CBDC addressed in this paper is difficult to implement in practice because the interest payments suggested here are traditionally in the realm of fiscal policy, not monetary policy. This argument ignores two facts: First, the central banks in most advanced economies already make interest payments on reserves, but only to some financial institutions that have exclusive access to the central bank facilities. Second, those interest payments are non-linear in that the interest rate paid on reserves is different from

[7]Another point is that in Rogoff's argument, the policy of negative nominal interest rates is needed for short-run stabilization. In contrast, I analyze the steady state of the model, and the interests paid on CBDC balances are aimed at maximizing the long-term welfare of the population, not stabilizing the economy in the short run.

the rate charged to borrowers. Central banks have recognized that the payments on reserves in the current system can serve their policy objectives, so why not extend access to all agents if economic efficiency requires that?[8]

The rest of the paper is organized as follows. After briefly discussing the related literature, I lay out the model in Section 2. I assume in all sections except Section 7 that cash and CBDC are perfect substitutes; i.e., both can be used in all transactions. In Section 3 and as a benchmark, I assume cash and CBDC are both costless to carry. In Section 4, I assume CBDC is more costly to carry relative to cash. I show, among other results, that if both cash and CBDC are used by agents under the optimal policy, cash is used in small-value transactions and CBDC is used in large-value transactions. In Section 5, I focus on a special case in which there are only two sizes of transactions—large-value and small-value transactions—and characterize conditions under which cash and CBDC are both used by agents under the optimal policy. In Section 6, I calibrate the model to the Canadian and US data to estimate the welfare gains of introducing CBDC into these economies. In Section 7, I assume that cash and CBDC are not perfect substitutes in that in a fraction of meetings, only cash can be used, and in a fraction of meetings, only CBDC can be used. I show here that co-existence may be welfare enhancing relative to cash-only or CBDC-only schemes. Section 8 concludes.

Related Literature. This paper is related to the monetary theory literature, especially the models that emphasize the micro-foundations of money. The model is built on the framework developed by Lagos and Wright (2005) and Rocheteau and Wright (2005), and has the same structure of a centralized market (CM) and a decentralized market (DM). The CBDC in my paper is similar to the interest-bearing money in Andolfatto (2010). However, he does not study the endogenous choice of means of payment when (non-interest-bearing) cash and interest-bearing money are both available to agents; nor does he have idiosyncratic preference shocks that lead to endogenous adoption of different means of payment by agents with different transaction needs in my paper.

A closely related paper is Chiu and Wong (2015). They show that electronic money allows the first-best allocation to be implemented under a broader set of parameter values relative to cash.[9] Another related paper is Gomis-Porqueras and Sanches (2013), in which there are two payment systems—fiat money and credit—and there is a cost effectively incurred by buyers

[8] Furthermore, I take the implementation of monetary policy more seriously than most papers in monetary economics where the creation of new money is assumed to be done through helicopter drop (lump-sum transfers). In my model, the implementation is done either through open market operations (exchange of cash with CBD) or through direct transfers to CBDC accounts.

[9] I elaborate in the appendix on the differences between my paper and some closely related papers.

to access the credit system. The credit system in their paper is similar to the CBDC in my paper. Dong and Jiang (2010) show that two monies can expand the set of parameters for which the first best is achievable in an environment in which agents have private information about their preference types. Zhu and Hendry (2017) study currency competition between cash and privately issued digital currency. They show that if the private issuer is not welfare maximizing, there will be coordination problems between the central bank and the private issuer and welfare will be lower relative to the case in which the central bank has full control of monetary policy.[10] In other models in the literature, money and credit are studied in the same model (like Gu et al. (2016) and Chiu et al. (2012)). Using credit is not possible in my model, as the central bank cannot keep track of the agents' actions in the DM. The central bank observes only the agents' balances at the end of the CM. My model is also related to Rocheteau et al. (2014) in that in both papers, an open market operation (OMO) is used. OMO is used in my model as a cross-subsidization device between cash and CBDC users. Finally, on estimating the costs and benefits of issuing CBDC, Barrdear and Kumhof (2016) estimate that CBDC issuance could increase GDP by as much as 3%, mostly through lowering the real interest rates.

2 Model

The model is based on Lagos and Wright (2005), LW hereafter, with two means of payment: cash and CBDC. I use index c to refer to cash and index e (for **e**lectronic money) to refer to CBDC. Time is discrete: $t = 0, 1, 2, \ldots$. Each period consists of two subperiods: DM and CM. In the DM, a decentralized market, and in the CM, a decentralized market, is active. There is a continuum of buyers and continuum of sellers, each with a unit mass. Both have discount factor $\beta \in (0, 1)$ from CM to DM. In the CM, both can consume and produce. In the DM, sellers can only produce and buyers can only consume. In the CM, one unit of labor supply produces one unit of perishable consumption good. In the DM, a buyer and seller meet randomly with probability σ and split the gains from trade based on proportional

[10] There is a growing body of literature studying CBDC and its implications for the payment systems, monetary policy implementation and financial stability. I cannot do justice to all the papers in this literature, but to mention only some examples, Fung and Halaburda (2016) study a framework to assess why a central bank should issue digital currency. Kahn et al. (2017) study different schemes of CBDC and discuss how these schemes can meet the central bank's objectives. Finally, Berentsen and Schar (2018) argue in favor of central banks issuing CBDC. In particular, they argue that implementing monetary policy using CBDC is more transparent than the current way of implementing monetary policy.

bargaining. The buyer's utility function is given by:

$$E_0 \sum_{t=0}^{\infty} \beta^t (w_t u(q_t) + X_t),$$

where buyer's preference shock, w_t, is an i.i.d. draw across time and agents from CDF $F(w)$ and $w \in [w_{min}, w_{max}]$, $u(q)$ is the utility of consuming q units of the DM good, and X_t is the consumption of numeraire in the CM. Introducing this preference shock is the first departure from the standard LW model. The seller's utility function is given by:

$$E_0 \sum_{t=0}^{\infty} \beta^t (-c(q_t) + X_t).$$

where $c(q)$ is the cost of producing q units of the DM good. Sellers do not receive a preference shock. All interesting actions come from the buyers' decisions in this paper. I assume that $u'' < 0 < u', u(0) = 0$ and $0 < c'', 0 < c', c(0) = 0$. Clearly, the first-best production level for type w is given by:

$$q_w^* = \arg\max_q \{wu(q) - c(q)\}.$$

Another departure from the standard LW model is that there are two means of payment in this economy—cash and CBDC. Denote by $c_e(z_e) : \mathbb{R}_+ \to \mathbb{R}_+$ the cost of carrying z_e units of real balances (in terms of the CM good) in the form of CBDC from CM to DM. It is assumed that the buyer incurs this cost. The cost of carrying real balances in the form of cash is assumed to be zero.

The timing of actions and realization of shocks in period t are specified as follows. In the DM, agents are randomly matched and trade according to the proportional bargaining protocol, with $\theta \in [0, 1]$ being the share of the buyer. In the bilateral meeting, w is known both to the buyer and seller, so there is no problem regarding private information. After agents trade in the DM and get separated from the match, the buyers learn their w for the next period. Next, agents trade in the CM. They work and choose the amount of CBDC and cash they want to carry to the next DM. At the end of the CM, new cash and CBDC are transferred to agents as will be described below. Denote by $z_c \in \mathbb{R}_+$ the number of pre-transfer real balances in the form of cash. Similarly, denote by $z_e \in \mathbb{R}_+$ the number of pre-transfer real balances in the form of CBDC. Denote by $t_c \in \mathbb{R}_+$ the helicopter drop of cash in real terms (units of the CM good) to all buyers. Denote by $t_e(z_e, w) : \mathbb{R}_+ \to \mathbb{R}_+$ the number of CBDC transfers in real terms to type w buyers that have brought z_e from the CM. It is assumed that the planner has complete information about the buyer's type if the buyer uses CBDC. Yet, the only feasible policy using cash is a helicopter drop. This assumption

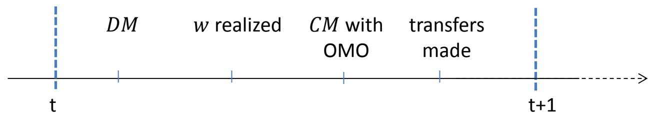

Figure 1: Timing

requires that the preference shock that buyers receive becomes known to the planner when CBDC is used, but the planner cannot identify people when cash is used.[11] Given the policy described above, post-transfer cash and CBDC balances are given by $z_c(w) + t_c$ and $z_e(w) + t_e(z_e(w), w)$. The following notation will be used in the rest of the paper: $x \equiv x_t$ and $x_+ \equiv x_{t+1}$.

The growth rates for cash and CBDC supply are denoted by $\gamma_c > 0$ and $\gamma_e > 0$, respectively, so

$$M_{t+1} = \gamma_c M_t, \quad E_{t+1} = \gamma_e E_t, \tag{1}$$

where M_t and E_t denote the cash and CBDC stock, respectively, at the beginning of the CM at time t. Each buyer is endowed with the steady state level of cash and CBDC in the DM of $t = 0$.

There is a rationale for both fixed and flexible exchange rates. Under the fixed exchange rate, the inflation rates for cash and CBDC are the same, while they can be different under the flexible exchange rate. On one hand, one dollar issued by the central bank has traditionally had the same value regardless of whether it is in the agent's pocket in the form of cash or with their account in electronic form. On the other hand, there is no reason why this should be the case. As a fixed exchange rate for domestic versus foreign currencies was a dominant paradigm at some point and then partially or completely abandoned, so why not let the exchange rate between cash and CBDC be flexible too, should efficiency require? For now, I allow for a flexible exchange rate. It is shown in the proofs that the CBDC inflation rate is irrelevant as long as it is higher than a threshold, in which case the fixed exchange rate is not binding. This is because the planner can redistribute CBDC balances to the agents in an efficient way, since the planner can see both the type of the buyers and their CBDC balances if they use CBDC. However, if the optimal cash inflation is lower than that

[11] The preference shock is known to the planner when buyers use CBDC. This assumption can be motivated by the fact that the number of CBDC balances brought to the bilateral meetings can be used at the end of the DM to verify the agent's preference shock. As shown later, even if w is not observable to the planner, the main insights go through as indicated in Proposition 7, but the planner's problem would become harder to solve.

threshold, then imposing a fixed exchange rate will be a binding restriction for the planner's problem, leading to a less efficient allocation.[12]

We focus on the cases where total real cash and CBDC balances are constant over time: $\phi_t M_t = \phi_{t+1} M_{t+1}$ and $\psi_t E_t = \psi_{t+1} E_{t+1}$. This implies that:

$$\frac{\phi_t}{\phi_{t+1}} = \gamma_c, \quad \frac{\psi_t}{\psi_{t+1}} = \gamma_e. \tag{2}$$

I allow the planner to use OMO to change the relative supply of cash and CBDC. By OMO, I mean that the government trades CBDC for cash in the CM with the price $\frac{\psi}{\phi}$. In that case, the equilibrium conditions can be written as follows:

$$\bar{M} - M = -\frac{\psi}{\phi}(\bar{E} - E), \tag{3}$$

$$t_c = \phi_+(M_+ - \bar{M}), \tag{4}$$

$$\int t_e(z_e(w), w) dF(w) = \psi_+(E_+ - \bar{E}), \tag{5}$$

$$\int z_c(w) dF(w) = \phi_+ \bar{M}, \tag{6}$$

$$\int z_e(w) dF(w) = \psi_+ \bar{E}, \tag{7}$$

where $\bar{M}$ and $\bar{E}$ are the cash and CBDC supply after the OMO and before transfers are made to the agents, and $z_c(w)$ and $z_e(w)$ are real balances of cash and CBDC that a buyer of type w holds in the steady state.

Equation (3) states that the *net* number of real balances supplied to the CM in the form of cash and CBDC is equal to 0. Equations (4) and (5) simply pin down the value of transfers in the form of cash and CBDC, respectively, available to be distributed across agents. For example, t_c is the real value of balances in CM_{t+1} given to buyers in the transfer stage of period t. Equations (6) and (7) are market-clearing conditions for cash and CBDC.

Two points about OMO are worth mentioning. First, (3) shows that OMO is a cross-subsidization tool between cash and CBDC users. If there is no cross-subsidization, then $\bar{M} = M$ and $\bar{E} = E$, so t_c will be pinned down by the inflation rate of cash. However, OMO allows the amount of cash distributed among agents to be less than the amount of newly created cash, providing a tool for the planner to achieve better allocations. Second,

[12] Agarwal and Kimball (2015) argue that if cash and CBDC co-exist, allowing for the exchange rate to be different from par makes it possible to implement a negative interest rate policy. My paper and theirs share the feature that if a flexible exchange rate between cash and CBDC is allowed, the outcome is more efficient compared to that with a fixed exchange rate, at least under some parameters.

OMO can adjust the imbalances between the supplies of these two assets. Specifically, OMO can be used for short-run stabilization in this model, as in many standard macroeconomic models. I do not study short run-stabilization policies here, however.

Lemma 1. *With OMO, the following constraint should hold in the stationary equilibrium:*[13]

$$t_c + \int t_e(z_e(w), w)dF(w) = (\gamma_c - 1)\int z_c(w)dF(w) + (\gamma_e - 1)\int z_e(w)dF(w). \tag{8}$$

2.1 Agents' Problems in the CM

Buyer's problem in the CM:

$$W_w^B(z) = \max_{X,Y,z_c,z_e} \Big\{ X - Y - c_e(z_e + t_e(z_e, w)) + \beta V_w^B(z_c + t_c, z_e + t_e(z_e, w)) \Big\}$$

$$\text{s.t. } X + \frac{\phi}{\phi_+} z_c + \frac{\psi}{\psi_+} z_e = Y + z,$$

where z denotes the real balances that the buyer has at the beginning of the CM. Also, $V_w^B(z_c, z_e)$ is the value function in the DM of the buyer of type w with z_c real balances in cash and z_e real balances in CBDC. Incorporating the constraint into the objective function, one can write:

$$W_w^B(z) = z + \max_{z_c,z_e}\{-\frac{\phi}{\phi_+} z_c - \frac{\psi}{\psi_+} z_e - c_e(z_e + t_e(z_e, w)) + \beta V_w^B(z_c + t_c, z_e + t_e(z_e, w))\}. \tag{9}$$

The sellers' value function in the CM can be written similarly.

2.2 Agents' Problems in the DM

Buyers receive:

$$\begin{aligned} V_w^B(z_c, z_e) &= \mathbb{E}W_w^B(z_c + z_e) \\ &+\sigma\Big(wu(q_w(z_c, z_e)) + \mathbb{E}W_w^B(z_c + z_e - d_{c,w}(z_c, z_e) - d_{e,w}(z_c, z_e)) - \mathbb{E}W_w^B(z_c + z_e)\Big) \\ &= \mathbb{E}W_w^B(z_c + z_e) + \sigma\Big(wu(q_w(z_c, z_e)) - d_{c,w}(z_c, z_e) - d_{e,w}(z_c, z_e)\Big), \end{aligned}$$

where $q_w(z_c, z_e)$, $d_{c,w}(z_c, z_e)$, $d_{e,w}(z_c, z_e)$ denote, respectively, the production amount and the real transfer of cash and CBDC balances in the DM meetings in which the buyer has brought

[13] This constraint is consolidated for both cash and CBDC. Without OMO, this condition should be replaced by the following two constraints: $t_c = (\gamma_c - 1)\int z_c(w)dF(w)$ and $\int t_e(z_e(w), w)dF(w) = (\gamma_e - 1)\int z_e(w)dF(w)$. In this case, the gains of CBDC would be more limited.

z_c real balances in cash and z_e real balances in CBDC. Also, the expectation is taken over realizations of buyer types in the next period. Similarly, sellers receive:

$$V_w^S(z_c, z_e) = W^S(z_c + z_e) + \sigma\Bigg(- c(q_w(z_c, z_e)) + d_{c,w}(z_c, z_e) + d_{e,w}(z_c, z_e)\Bigg).$$

Superscript S represents the seller's associated variable. The linearity of W^S and W^B were used to simplify the DM value functions. Sellers do not need to bring balances to the DM because carrying balances is costly and the sellers do not use them until the next CM. Therefore, we focus only on the buyer's balances, determined from the bargaining protocol.

2.3 Proportional Bargaining in the DM

Terms of trade are determined from the following maximization problem:

$$\max_{q, d_c \in [-z_c^S, z_c], d_e \in [-z_e^S, d_e \leq z_e]} \Delta^B + \Delta^S$$
$$\text{subject to: } \Delta^B = \theta(\Delta^B + \Delta^S),$$

where Δ^B and Δ^S denote buyer's and seller's surplus, respectively, and are given by:

$$\Delta^B \equiv V_w^B(z_c - d_c, z_e - d_e) - V_w^B(z_c, z_e) + wu(q),$$

$$\Delta^S \equiv V_w^S(z_c^S + d_c, z_e^S + d_e) - V_w^S(z_c^S, z_e^S) - c(q).$$

The solution to the bargaining problem is given by:[14]

$$d_{c,w}(z_c, z_e) + d_{e,w}(z_c, z_e) = \min\Bigg\{ z_c + z_e, D_w(q_w^*), \Bigg\}$$

$$q_w(z_c, z_e) = D_w^{-1}\Bigg(d_{c,w}(z_c, z_e) + d_{e,w}(z_c, z_e), \Bigg) \tag{10}$$

where $D_w(.)$ is defined as follows:

$$D_w(q) \equiv \theta c(q) + (1-\theta) wu(q).$$

Equivalently, the solution is given by:

$$(q_w, d_{c,w} + d_{e,w}) = \begin{cases} (q_w^*, D_w(q_w^*)) & \text{if } z_c + z_e \geq D_w(q_w^*) \\ (D_w^{-1}(z_c + z_e), z_c + z_e) & \text{otherwise} \end{cases}.$$

[14]This problem is basically the same as follows: $\max_{x, d \in [-z', z]}[u(x) - x]$ subject to $u(x) - d = \theta(u(x) - x)$.

In words, $D_w(q)$ denotes the number of real balances that a type w buyer needs for buying q units of the DM good. If the buyer brings at least $D_w(q_w^*)$, then the first best is achievable; i.e., the first-best level of production, q_w^*, can be produced. Otherwise, the buyer spends the entire balances, and then the terms of trade are given by the second line above. Finally, the value function for buyers and sellers at the beginning of the DM can be written as follows:

$$V_w^B(z_c, z_e) = \mathbb{E}W_w^B(z_c + z_e) + \sigma\theta\Big(wu(q_w(z_c, z_e)) - c(q_w(z_c, z_e))\Big),$$

$$V_w^S(z_c, z_e) = W^S(z_c + z_e) + \sigma(1-\theta)\Big(wu(q_w(z_c, z_e)) - c(q_w(z_c, z_e))\Big).$$

2.4 CM and DM Value Functions Together

The buyer's problem turns into:

$$W_w^B(z) = z + \mathbb{E}W_w^B(0)$$
$$+\max_{z_c, z_e}\left\{-\frac{\phi}{\phi_+}z_c - \frac{\psi}{\psi_+}z_e - c_e(z_e + t_e(z_e, w)) + \beta(z_c + t_c) + \beta(z_e + t_e(z_e, w)) + \beta\sigma\theta(wu(q) - c(q))\right\}, \tag{11}$$

where q is implicitly given by $D_w(q) = \min\{D_w(q_w^*), z_c + t_c + z_e + t_e(z_e)\}$.

It is standard to show that sellers do not bring any balances to the DM. In the DM, if sellers get matched, they work to produce the DM good and sell it to the buyer, and then bring their balances to the CM and use them to purchase the CM good and consume it. Buyers work to acquire money (cash or CBDC) in the CM, and receive transfers from the planner. Then, they enter the DM with the entire money stock, and exchange it all for goods produced by sellers.

2.5 Equilibrium Definition

The equilibrium definition can now be written as follows.

Definition 1 (Stationary Equilibrium)**.** *Stationary equilibrium is a price system* $\{\phi_t\}$, $\{\psi_t\}$, *an allocation* $\{(q(w), z_c(w), z_e(w))\}_w$ *and a policy* $\{\gamma_c, \gamma_e, t_c, \{t_e(z_e, w)\}_w\}$ *such that the following conditions hold:*

(i) Buyer's maximization in CM: Given $z_{c,0}$, $z_{e,0}$, $\{\psi_t\}_{t=0}^\infty$ *and* $\{\phi_t\}_{t=0}^\infty$, $z_c(w)$ *and* $z_e(w)$ *solve* (9) *(where* $z_{c,0}$ *and* $z_{e,0}$ *denote initial values of* z_c *and* z_e*).*

(ii) Market clearing for cash and CBDC, planner's budget constraint and OMO: (8) *should hold.*

(iii) Proportional bargaining: $q(w)$ solves (10).

(iv) Growth equation (2) for cash and CBDC.

2.6 Planner's Problem

The planner's problem is to maximize welfare, calculated at the beginning of the CM, by choosing a policy:

Problem 1 (Planner's Problem)**.**

$$\max_{\{\gamma_c,\gamma_e,t_c,\{t_e(w)\}_w\}} \int \Big[\beta\sigma\big(wu(q(w)) - c(q(w))\big) - c_e(z_e(w))\Big] dF(w)$$

subject to: $\{(q(w), z_c(w), z_e(w))\}_w$ form an equilibrium together with some prices $\{\phi_t\}$, $\{\psi_t\}$ and the policy.

I make the following assumption on the functional form of cash and CBDC costs throughout the paper.

Assumption 1 (Cost Functions)**.** *CBDC costs $K \geq 0$ in terms of the CM good and is to be incurred in the CM. That is, $c_e(z) = K\mathbb{I}\{z > 0\}$.*

As will be shown later, if both cash and CBDC are costless, cash is redundant, because CBDC is a more powerful instrument for the planner to implement monetary policy. For the planner to have a non-trivial problem regarding which means of payment should be available to agents, CBDC needs to be disadvantageous to cash in some ways. Considering a fixed cost of using CBDC relative to cash, as assumed here, is one way to do so. This disadvantage is motivated by the fact that agents in the economy may value anonymity while doing transactions, and they may lose it if they use CBDC. Also, electronic means of payment including CBDC usually require some devices to process the payments, while cash does not, so this cost can summarize the costs of using such devices. This disadvantage is modeled for simplicity as a flat cost $K \geq 0$ for using CBDC. The flat cost of using CBDC is also consistent with the digital format of CBDC in that the dis-utility of losing anonymity for the agent may be independent of the number of balances that the agent holds.

2.6.1 Simplified Planner's Problem

The main constraint for the planner's problem, equation (11), can be written as:

$$(q(w), z_c(w), z_e(w)) \in \arg \max_{q\in[0,q^*_w],z_c,z_e} \Bigg\{ - (\frac{\phi}{\phi_+} - \beta)(z_c + t_c)$$

$$-(\frac{\psi}{\psi_+} - \beta)(z_e + t_e(z_e, w)) - c_e(z_e + t_e(z_e, w)) + \gamma_c t_c + \gamma_e t_e(z_e, w) + \beta\sigma\theta(wu(q) - c(q))\Big\}$$

$$\text{s.t. } D_w(q) = \min\{D_w(q_w^*), z_c + t_c + z_e + t_e(z_e, w)\}.$$

Given the assumption on the cost functions, the planner's problem can be simplified as follows:

Problem 2.

$$\max_{\{\gamma_c, \gamma_e, t_c, \{t_e(z_e, w)\}\}} \int \Big[\beta\sigma\Big(wu(q(w)) - c(q(w))\Big) - K\mathbb{I}(z_e(w) > 0)\Big] dF(w)$$

$$s.t.\ (q(w), z_c(w), z_e(w)) \in \arg \max_{q \in [0, q_w^*], z_c + z_e + t_c + t_e(z_e, w) = D_w(q)} \Big\{$$

$$-(\gamma_e - \beta)(z_e + t_e(z_e, w)) - (\gamma_c - \beta)(z_c + t_c) + \beta\sigma\theta\big(wu(q) - c(q)\big) - K\mathbb{I}(z_e > 0) + \gamma_c t_c + \gamma_e t_e(z_e, w)\Big\},$$

$$\textit{and } t_c + \int (t_e(z_e(w), w) - (\gamma_c - 1)z_c(w) - (\gamma_e - 1)z_e(w)) dF(w) = 0.$$

Proposition 1. *In the solution to the planner's problem, we can assume without loss of generality that $t_e(z, w)$ is a step function in z. That is,*

$$t_e(z, w) = \begin{cases} t_{0,w} & z \geq z_{0,w} \\ 0 & z < z_{0,w} \end{cases}$$

for some $t_{0,w} \in \mathbb{R}_+$, $z_{0,w} \in \mathbb{R}_+$.

This lemma states that we can restrict our attention to the CBDC transfer schemes that are step functions. That is, if an agent of type w brings at least $z_e(w)$, then he receives some transfers, but bringing any lower real balances in CBDC does not yield him any transfers. This is the most severe punishment of the agents by the planner.[15]

In the next section, I study the case in which cash and CBDC are costless as a benchmark ($K = 0$). Next, I study the case in which CBDC is more costly than cash ($K > 0$).

[15] This transfer scheme can easily be implemented by a fixed fee and interest payment on balances. Following Andolfatto (2010), assume agents are charged some fixed cost $\tilde{f}$ in the CM if they want to hold CBDC and are paid interest on their CBDC balances with $\tilde{i}$ interest rate in the transfer stage. This scheme with appropriate values for $\tilde{f}$ and $\tilde{i}$ can implement the same allocation as the transfer scheme here.

3 Costless Cash and CBDC

I show that if $K = 0$, cash is redundant. I also study conditions under which first best is achievable with CBDC. It is impossible to achieve the first best with only cash, because it is not possible to tax cash holdings nor to make transfers to agents based on their cash holdings.

Proposition 2 (Redundancy of cash)**.** *If both cash and CBDC are costless—i.e., $K = 0$—then cash is redundant. That is, any allocation that is achieved by using cash and CBDC can be achieved by using only CBDC.*

The idea is that if both cash and CBDC are costless, CBDC has a clear advantage for the planner, as the planner can provide incentive for buyers to bring enough balances to the DM by checking their CBDC balances. The planner can then punish agents who do not bring enough balances from the CM by making zero transfers to them. The cash growth rate is set sufficiently high that the gains from using cash and consequently the demand for cash become zero.

3.1 Homogeneous Buyers

The planner sees w and can make transfers to buyers who use CBDC contingent on their types. Cash can be distributed across agents only evenly (helicopter drop). In this section, there is only one type so only one means of payment is generally used and a flexible exchange rate is irrelevant for analyzing the steady state.

Proposition 3. *Suppose both cash and CBDC are costless; i.e., $K = 0$. Suppose also the distribution of types is degenerate at w; i.e., there is only one type. The first best is achievable if and only if:*

$$\beta\sigma\theta(wu(q_w^*) - c(q_w^*)) \geq (1 - \beta)D_w(q_w^*).$$

The left-hand side (LHS) of the condition is the buyer's gains from bringing the number of balances that the planner asks for. The right-hand side (RHS) is the real cost of holding balances. This is the inevitable cost of holding real balances with CBDC: Carrying CBDC for buying q units of the DM good imposes $(\gamma_e - \beta)D_w(q_w^*)$ cost of real balances on the buyer, but the newly created CBDC will be distributed across buyers such that they receive $(\gamma_e - 1)D_w(q_w^*)$ real balances if they have brought enough balances. Therefore, they have to incur the cost $(1 - \beta)D_w(q_w^*)$. Noticeably, the inflation rate of CBDC does not affect the incentives.

The condition required by this proposition is equivalent to:

$$\theta \geq \overline{\theta}(w) \equiv \frac{1-\beta}{(1-\beta(1-\sigma))(1-\frac{c(q_w^*)}{wu(q_w^*)})}.$$

For $\overline{\theta}(w) \leq 1$, we must have:

$$\beta \geq \left(1+\sigma\frac{wu(q_w^*)-c(q_w^*)}{c(q_w^*)}\right)^{-1}.$$

To achieve the first best, the proposition requires θ and also β to be sufficiently high. Even in the case of $\theta = 1$, in which the buyer takes the entire surplus, the buyer still needs to work in the CM_t to earn $c(q)$ real balances in CBDC. The benefits of CBDC will be realized in the DM_{t+1} with probability σ, in the DM_{t+2} with probability $(1-\sigma)\sigma$, and so on. For the benefits to dominate the costs, one needs: $wu(q)(\beta\sigma+\beta^2(1-\sigma)\sigma+\beta^3(1-\sigma)^2\sigma+...) \geq c(q)$, which is equivalent to the above condition.[16]

The following proposition implies that for a given set of model parameters, there exists a threshold for w below which the first best cannot be achieved and above which the first best can be achieved. The condition required in this proposition is satisfied, for example, when $c(q)$ is linear and $u(q) = \frac{(q+b)^{1-\eta}-b^{1-\eta}}{1-\eta}$ where $\eta \in (0,1)$ and $b > 0$.

Proposition 4. *$\bar{\theta}(w)$ is decreasing in w if $\frac{c'(q)u(q)}{c(q)u'(q)}$ is increasing in q.*

3.2 Heterogeneous Buyers

Now suppose that the distribution of types is not degenerate. The condition in the proposition may be binding for some types and not for others. The following proposition provides sufficient conditions to achieve the first best.

Proposition 5. *Suppose both cash and CBDC are costless; i.e., $K = 0$. With heterogeneous types, the first best is achievable if and only if:*

$$\beta\sigma\theta\int(wu(q_w^*)-c(q_w^*))dF(w) \geq (1-\beta)\int D_w(q_w^*)dF(w).$$

Compared with the case with homogeneous buyers, cross-subsidization is possible here. This can be seen clearly from comparing the conditions in Propositions 3 and 5. The idea is that if the condition in Proposition 3 is slack for some types, say w_2, and does not hold for other types, say w_1, the planner can charge type w_2 buyers to subsidize type w_1 buyers so

[16] Chiu and Wong (2015) provide this intuitive explanation for a related discussion.

that they bring enough balances to the DM. It is not possible to cross-subsidize with cash, because it is not possible to see types or the amount of balances.

I emphasize the fact that types are observable in the CBDC system, so cross-subsidization is possible. Without observability of types, the higher types would like to pretend to be of a lower type. Therefore, the planner faces a more constrained problem. The following proposition provides a condition for the planner to achieve the first best when the type of agents are not observable to the planner.

Proposition 6 (First best with no type-contingent transfers). *Suppose cash and CBDC are costless, i.e., $K = 0$. Assume that the CBDC transfers cannot be contingent on the type of buyers.*[17] *The first best is achievable if the following condition holds:*

$$\beta\sigma\theta(wu(q_w^*) - c(q_w^*))|_{w=w_{min}} \geq (1-\beta)\int D_w(q_w^*)dF(w).$$

The LHS of the condition in this proposition is associated with the surplus of the lowest type. The RHS of the condition is the same term as that in Proposition 5. This difference implies that private information, unsurprisingly, restricts the amount of cross-subsidization possible. Suppose we begin by having only one type, w_{min}. When higher types are added to the population, the RHS becomes larger while the LHS is kept constant, implying that the range of β's and θ's under which the first best can be achieved becomes smaller. Remember that in the complete information case, when higher types are added, both the RHS and LHS increase, which may result in achieving the first best for a broader range of parameters. When type-contingent transfers are not allowed, the minimum balances that the agents need to bring cannot depend on their type, so all buyers would be subject to the same minimum balances. As a result, high-type buyers would have less incentive to bring the same number of balances that they would bring under complete information, and consequently, it would be harder to achieve the first best.

4 Costless Cash and Costly CBDC

In this section, consider the case in which cash is still costless but CBDC requires flat cost $K > 0$ in real balances to carry from the CM to the DM. CBDC is costly, so cash may not be redundant anymore and the planner may want some types to use cash. An important task is to characterize the types who use cash and the types who use CBDC. Cash inflation

[17] In this case, buyers have private information regarding w relative to the planner, but the seller still can see w.

is costly for those agents who carry cash. However, it may still be optimal for them to bring cash because there is a direct cost associated with carrying CBDC. Thus, the interesting tradeoff here is whether the planner should increase cash inflation so as to encourage more buyers to use CBDC and achieve better allocations, or decrease cash inflation to have less distorted allocation for cash users and to save on CBDC carrying costs. Since the CBDC cost is independent of the amount of CBDC that buyers carry, if the planner wants a buyer to carry some CBDC, the planner wants the buyer to carry his entire balances in the form of CBDC.

I introduce the following notation, which will prove useful in the rest of the paper:

$$f(w,q) \equiv wu(q) - c(q), \tag{12}$$

$$s(w,q) \equiv -(1-\beta)D_w(q) + \beta\sigma\theta(wu(q) - c(q)), \tag{13}$$

$$O(w,\gamma) \equiv \max_q\{-(\gamma-\beta)D_w(q) + \beta\sigma\theta(wu(q) - c(q))\}, \tag{14}$$

$$\bar{q}(w,\gamma) \equiv \arg\max_q\{-(\gamma-\beta)D_w(q) + \beta\sigma\theta(wu(q) - c(q))\}, \tag{15}$$

$$e(w,\gamma) = \begin{cases} 1 & \text{type } w \text{ uses CBDC} \\ 0 & \text{otherwise} \end{cases}. \tag{16}$$

Function $f(w,q)$ is the surplus created in a match of a buyer of type w who consumes q units of the DM good. Function $s(w,q)$ is the present value of the payoff that a buyer of type w receives in the CM from working for (and holding) $D_w(q)$ units of real balances that can be used to buy q units of the DM good when the inflation rate is zero, assuming that only cash is available. Function $O(w,\gamma)$ is the maximum present value of the payoff that a buyer of type w can receive when the inflation rate is $\gamma-1$ and $\bar{q}(w,\gamma)$ is the consumption of the DM good when the inflation rate is $\gamma-1$, again assuming that only cash is available. Finally, $e(w,\gamma)$ is simply an indicator function for a buyer of type w when the cash inflation rate is $\gamma-1$ (and CBDC inflation rate is sufficiently high). It takes the value of 1 if the buyer uses CBDC and takes the value of 0 otherwise.

4.1 Homogeneous Buyers

Similar to the last section, I begin by analyzing the case for homogeneous buyers. Since there is no heterogeneity and the cost of using CBDC is flat, either all buyers use cash or all use CBDC. As a result, it suffices to calculate the highest possible welfare under cash and under CBDC separately and then compare them. Define $\tilde{e}(w)$ as follows. If only CBDC is used, then $\tilde{e}(w)=1$, and if only cash is used, then $\tilde{e}(w)=0$.

First, suppose buyers use CBDC. From the constraints in the planner's problem, we have $t_e = (\gamma_e - 1)z_e$ and $t_e + z_e = D_w(\tilde{q})$ where $\tilde{q}$ is the DM production under CBDC. Therefore, $t_e = (\gamma_e - 1)/\gamma_e D_w(\tilde{q})$. Note that t_c is set to 0 because distributing cash would only distort the allocation. The planner's problem can be written as follows:

$$\max_{\tilde{q}} \left\{ \beta\sigma f(w, \tilde{q}) - K \right\}$$

$$\text{s.t.} \quad -(1-\beta)D_w(\tilde{q}) + \beta\sigma\theta(wu(\tilde{q}) - c(\tilde{q})) - K \geq \max_{\tilde{q}}\{-(\gamma_c - \beta)D_w(q) + \beta\sigma\theta(wu(q) - c(q))\}.$$

The cash inflation rate, $\gamma_c - 1$, is chosen to be sufficiently high that the RHS of the constraint becomes 0. Also, $\gamma_e - 1$ must be sufficiently large so that CBDC transfers become positive. (Here, it suffices to have $\gamma_e - 1 > 0$.) Denote by $\tilde{q}(w)$ the solution to this problem. It is easy to see that if $K \leq s(w, q_w^*)$, then $\tilde{q}(w) = q_w^*$. If $K > s(w, q_w^*)$, then $\tilde{q}(w)$ is implicitly given by $K = -(1-\beta)D_w(\tilde{q}) + \beta\sigma\theta(wu(\tilde{q}) - c(\tilde{q}))$. In this case, obviously, $\tilde{q}(w) < q_w^*$.

Second, suppose buyers use cash, then it is optimal to set the cash inflation to the lowest possible level; i.e., $\gamma_c = 1$. The value of the objective function then equals $\beta\sigma f(w, \bar{q}(w, 1))$.

Therefore, it is optimal to use CBDC if and only if $\beta\sigma f(w, \bar{q}(w, 1)) < \beta\sigma f(w, \tilde{q}(w)) - K$. The following proposition summarizes this discussion.

Proposition 7. *The production level, $\tilde{q}(w)$, and the optimal choice of means of payment for the case in which all buyers are of type w, $\tilde{e}(w)$, can be summarized as follows:*

$$\textit{if } K_1(w) \leq K_2(w): \begin{cases} \tilde{q}(w) = q_w^*, \tilde{e}(w) = 1 & K \leq K^*(w) \equiv K_1(w) \\ \tilde{q}(w) = \bar{q}(w, 1), \tilde{e}(w) = 0 & K > K^*(w) \end{cases},$$

$$\textit{if } K_1(w) > K_2(w): \begin{cases} \tilde{q}(w) = q_w^*, \tilde{e}(w) = 1 & K \leq K_2(w) \\ \tilde{q}(w) < q_w^*, \tilde{e}(w) = 1 & K_2(w) < K \leq K^*(w) \\ \tilde{q}(w) = \bar{q}(w, 1), \tilde{e}(w) = 0 & K > K^*(w) \end{cases},$$

where $K_1(w) \equiv \beta\sigma f(w, q^*(w)) - \beta\sigma f(w, \bar{q}(w, 1))$, $K_2(w) \equiv s(w, q_w^*)$, and $K^*(w)$ denotes the cost threshold at which the planner is indifferent between the schemes in which only cash is used by everyone or only CBDC is used by everyone.

When $\tilde{e}(w) = 0$, the cash inflation rate is 0—i.e., $\gamma_c = 1$—and transfers are given by $t_e(z, w) = 0$ and $t_c = 0$. When $\tilde{e}(w) = 1$, the transfers are given by:

$$t_e(z, w) = \begin{cases} \frac{(\gamma_e - 1)D_w(\tilde{q}(w))}{\gamma_e} & z \geq \frac{D_w(\tilde{q}(w))}{\gamma_e} \\ 0 & z < \frac{D_w(\tilde{q}(w))}{\gamma_e} \end{cases}.$$

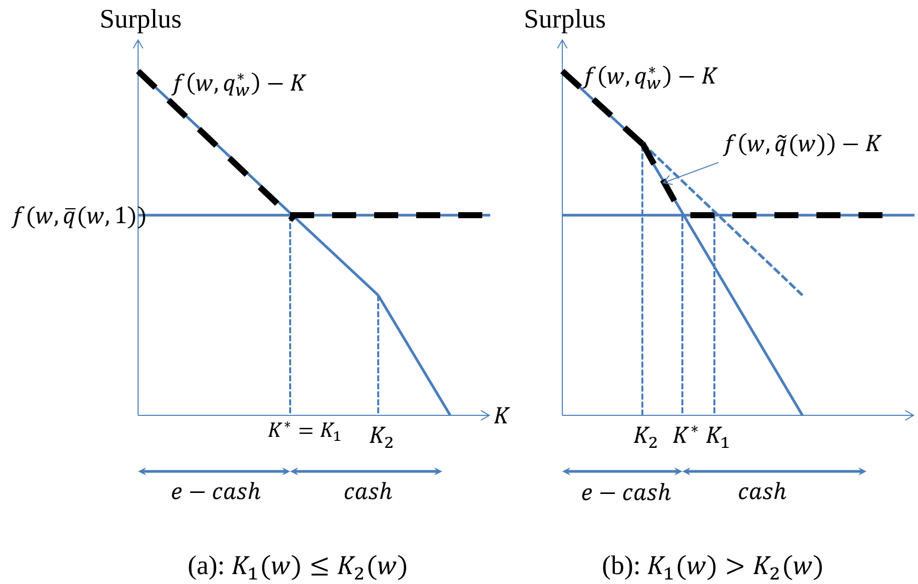

Figure 2: Usage of cash versus CBDC with fixed cost of using CBDC

Proposition 8. *There exists* $(\hat{\beta}, \hat{\theta}) \in (0,1)^2$ *such that if* $\beta > \hat{\beta}$ *and* $\theta > \hat{\theta}$*, then* $K_2(w) \geq K_1(w)$.

This proposition states that $K_1(w) \leq K_2(w)$ for sufficiently large values of β and θ. In this case, if CBDC is used, then the first best will be achieved. See Figure 2 for illustration. CBDC is used when K is small. Cash is used when K is large. If $K_1(w) > K_2(w)$, CBDC is optimally used for $K \in (K_2(w), K^*(w))$, but the first best cannot be achieved.

4.2 Heterogeneous Buyers

We can now compare between homogeneous and heterogeneous cases. As in the costless case, cross-subsidization is possible with multiple types, so if the incentive constraint is binding for some types and not for others, cross-subsidization can help to achieve better allocations. The incentive constraint states that CBDC users should gain a weakly higher payoff from using CBDC compared with cash.[18]

With heterogeneity, the planner is more restricted. For those types with K close to but

[18]We do not need to consider the constraint that cash users should gain a higher payoff from using cash relative to CBDC, because if they switch to using CBDC, their type would be immediately revealed and they would receive the net payoff of 0.

less than $K^*(w)$, use of CBDC is optimal when the population is homogeneously composed of w, because the cash inflation can be set very high. However, if there is a sufficiently high measure of agents who want to use cash, then setting a high cash inflation rate amounts to a significant loss in social welfare, because it affects all cash users. As a result, cash inflation cannot be too high. Therefore, such a type w may switch to cash (inefficiently compared with the homogeneous case) because the punishment for using cash cannot be severe enough to induce him to use CBDC.

4.2.1 Cash Is Used in Small-Value Transactions

In the following result, we establish that when co-existence is optimal, low-type buyers use cash and high-type buyers use CBDC. That is, cash is used for small-value transactions and CBDC is used for large-value transactions under the optimal policy. Define:

$$Q(r) \equiv \arg\max_q \{ru(q) - c(q)\}.$$

Proposition 9. *Assume* $\frac{rQ''(r)}{Q'(r)} \geq -1$. *Then there exists a threshold* $w^t > 0$ *such that agents with* $w < w^t$ *use cash and agents with* $w \geq w^t$ *use CBDC under the optimal policy.*

The only requirement of this result is that the coefficient of relative risk aversion of Q should be less than 1. This result is not trivial. Cash inflation is not ∞, so some agents can receive a strictly positive payoff by using cash. Since the size of the surplus is higher for higher types, they receive a higher payoff for a given cash inflation. If their payoff from holding cash increases very fast with their type, it may not be worth it for the planner to have these types use CBDC. It is shown that if $\frac{rQ''(r)}{Q'(r)} \geq -1$, this does not happen.[19] This condition is satisfied for the production and cost functions, u and c, usually used in economics. As an example, let $u(q) = q^{1-1/c_0}$ with $c_0 > 1$ and $c(q) = c_1 q$. Hence, $Q(r) = (\frac{(1-1/c_0)}{c_1})^{c_0} r^{c_0}$, so $\frac{rQ''(r)}{Q'(r)} = c_0(c_0 - 1)/c_0 = c_0 - 1 > 0 \geq -1$.[20]

[19]This finding is consistent with facts from a survey in Canada reported by Fung et al. (2015) that cash is used mainly for small-value transactions. They add that the share of cash usage relative to the usage of other means of payment has decreased. Interestingly, they report that the respondents to the survey attribute their cash usage mostly to its lower cost relative to other means of payment. Other factors, such as security concerns, acceptance by the merchants, and ease of use, come after the cost.

[20]More generally, consider a constant relative risk averse $Q(r)$ in which $-\frac{rQ''(r)}{Q'(r)} = 1 - c_0$ where $c_0 \geq 0$. This implies that $Q(r) = k_1 r^{c_0} + k_0$. Therefore, if c and u are such that $\frac{c'(q)}{u'(q)} = (\frac{q-k_0}{k_1})^{1/c_0}$ for some k_0, k_1 and c_0, then the required condition is satisfied.

5 Co-Existence of Cash and CBDC in a Two-Type Example

I focus in this section on the two-type example. Studying this case, as opposed to many types or a continuum of types, is relatively easy and captures the main tradeoff. Suppose there are two types w_1 and w_2 with $K^*(w_1) < K < K^*(w_2)$. If the population is homogeneous, type w_1 uses cash and type w_2 uses CBDC under the optimal policy. What should the planner do when there is more than one type and the optimal means of payment for some of them is different from others? Denote the measure of type w_2 buyers by $\pi_2 \equiv \pi$ and measure of type w_1 buyers by $\pi_1 \equiv 1 - \pi$. When all buyers are of type w_2, the planner could increase cash inflation so that no buyer uses cash. In contrast, when all the buyers are of type w_1, the planner has to use cash for w_1 and set the cash inflation rate to the lowest possible level, $\gamma_c = 1$.

Denote by E the optimal welfare level if both types use CBDC, denote by C the optimal welfare level if both types use cash, and finally denote by B the welfare level if only w_1 uses cash:

$$E = (1-\pi_2)\beta\sigma f(w_1, q_1^*) + \pi_2\beta\sigma f(w_2, q_2^*) - K,$$

$$C = (1-\pi_2)\beta\sigma f(w_1, \bar{q}_1(1)) + \pi_2\beta\sigma f(w_2, \bar{q}_2(1)),$$

$$B = (1-\pi_2)\beta\sigma f(w_1, \bar{q}_1(\gamma_c)) + \pi_2\beta\sigma f(w_2, q_2) - \pi_2 K.$$

We know from the results in the previous section that it is not optimal that only type w_2 use cash. Therefore, the planner's problem can be written as $\max\{E, C, \overline{B}\}$ where $\overline{B}$ denotes the optimal welfare level if only w_1 uses cash, and it is obtained from:

$$\overline{B} \equiv \max_{t_c, t_{e2}, z_{c1}, z_{e2}, \gamma_c, \gamma_e, q_2} B$$

$$\text{s.t. } t_c + \pi_2(t_{e2} - (\gamma_e - 1)z_{e2}) = (1-\pi_2)(\gamma_c - 1)z_{c1} \text{ (equivalent to (8))},$$

$$t_c + t_{e2} + z_{e2} = \max\{D_{w_2}(q_{w2}^*), D_{w_2}(q_2)\} \text{ (w_2's payment when using CBDC)},$$

$$t_c + z_{c1} = \max\{D_{w_1}(q_{w1}^*), D_{w_1}(q_1)\} \text{ (w_1's payment when using cash)},$$

$$O(w_2, \gamma_c) \leq -(\gamma_e - \beta)(t_{e2} + z_{e2}) - (\gamma_c - \beta)t_c + \beta\sigma\theta(w_2 u(q_2) - c(q_2)) - K + \gamma_e t_{e2} \text{(incentive constraint)}.$$

I assume that β and θ are sufficiently large that $\tilde{q}(w) = q_w^*$ under CBDC (according to Proposition 8). Agents do not want to bring more balances than the number of balances needed to buy the first-best level of production. The incentive constraint for the maximization problem can then be simplified to:

$$\max_q \{-(\gamma_c - \beta)D_{w_2}(q) + \beta\sigma\theta(w_2 u(q) - c(q))\}$$

$$\leq -(1-\beta)D_{w_2}(q_2)+\beta\sigma\theta(w_2u(q_2)-c(q_2))-K+(1-\pi_2)/\pi_2(\gamma_c-1)D_{w_1}(q_1)-\gamma_c t_c/\pi_2. \quad (17)$$

See the appendix for the derivation. A positive t_c makes the constraint only tighter. Since t_c does not appear in the objective function, it is optimal to set it to the lowest possible value; i.e., $t_c = 0$. Now, we obtain the following result.

Proposition 10 (Optimality of positive inflation)**.** *Suppose $K > 0$. In any equilibrium in which both means of payment are used, the cash inflation rate must be strictly positive; i.e., $\gamma_c > 1$.*

If $\gamma_c \leq 1$, then cash should be withdrawn from the CM, requiring CBDC to be injected into the CM using OMO. These CBDC balances should be financed from CBDC users. Moreover, as shown earlier, there is an opportunity cost of using CBDC, as the transfers made to buyers can be used to purchase the DM good only in the next period. This is as if the inflation for CBDC cannot be less than 1. Finally, CBDC users should incur cost K. Altogether, if $\gamma_c \leq 1$, then CBDC is a strictly dominated choice of payment for buyers, so co-existence is not possible. Finally, note that this result implies that when co-existence is optimal, the cash inflation must be strictly positive, although it is feasible to run a negative cash inflation rate.

5.1 Sufficient Conditions for Non-Optimality of Co-Existence

Assumption 2 (Condition for Non-Optimality of Co-Existence)**.** *Assume $\gamma_0 < \gamma_1$, where γ_0 and γ_1 are implicitly defined by the following equations:*

$$\beta\sigma f(w_1, \bar{q}(w_1, \gamma_0)) \equiv \beta\sigma f(w_1, q_1^*) - K,$$

$$\max_q\{-(\gamma_1-\beta)D_{w_2}(q)+\beta\sigma\theta(w_2u(q)-c(q))\} \equiv -(1-\beta)D_{w_2}(q_2^*)+\beta\sigma\theta(w_2u(q_2^*)-c(q_2^*))-K.$$

Proposition 11 (Non-Optimality of Co-Existence)**.** *Suppose buyers are of only two types: w_1 with probability $1-\pi$ and w_2 with probability π. Under Assumption 2, the co-existence is not optimal if π is sufficiently close to 1.*

The schematic diagram for this result can be found in Figure 3. A similar result can be obtained if π is close to 0. It is evident that when π is close to 1, the welfare level under co-existence is lower than that under the scheme in which both types use CBDC, which is in turn lower than that under the first best (i.e., the weighted average of the welfare level under the optimal scheme for each type).

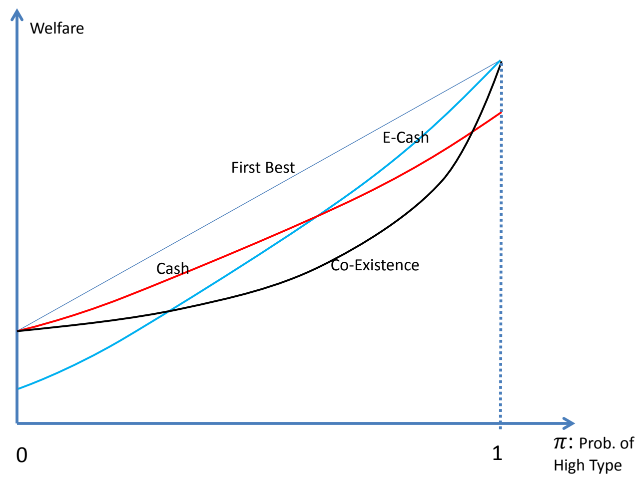

Figure 3: Co-existence is not optimal under conditions specified in Prop. 11.

It is not easy to characterize whether co-existence is optimal for all values of π. In several numerical examples, many of which are not reported here to save space, co-existence of cash and CBDC are shown to be unlikely. Theoretically speaking, the constraint that CBDC users should not have incentives to use cash imposes a restriction on the welfare level that the planner can achieve.

5.2 Numerical Example

I present three examples here for the two-type case to study the optimal choice of means of payment. The parameter values are given in Table 1. See Figure 4 for various graphs associated with Examples 1 (top row), 2 (middle row) and 3 (bottom row). On the horizontal axes, the measure of type w_2 buyers in the population is indicated. For each example, the left graph shows welfare under different schemes. In the first scheme, called "both-cash," both types use cash, in the second scheme, called "both-CBDC," both types use CBDC, and in the third scheme, called "co-existence," type w_1 uses cash and type w_2 uses CBDC. In the fourth scheme, type w_1 uses cash and type w_2 uses CBDC, but OMO is not allowed. The fifth scheme is the first-best level of welfare. Note that under the both-CBDC scheme, the first-best level of production can be achieved. In the right graph, optimal cash inflation is shown for various schemes.

In Example 1, co-existence is not optimal. When π is small, cash is used and when π is large, CBDC is used. When π is small, the incentive constraint is binding, and the cost of inclusion of type w_2 in the CBDC system would impose a relatively large inflation

Example	$u(q)$	$c(q)$	θ	β	σ	K	w_1	w_2
1	$q^{0.4}$	$0.1q$	0.9	0.8	0.5	0.06	1	1.15
2	$q^{0.5}$	$0.5q$	0.9	0.84	0.3	0.33	1	4
3	$q^{0.5}$	$0.5q$	0.5	0.85	0.8	0.06	1	4

Table 1: Parameters of the numerical examples

on cash users. When π is large, if cash is valued, then it will serve only a small fraction of the population but is a way for a large fraction of the population to evade the taxation scheme, and in order to discourage type w_2 buyers, their allocation should be distorted. This distortion is too costly, so it is optimal that cash is valued at 0. Example 3 is similar to Example 1. The only difference is that cash is used only for very small values of π, and for most values of π, CBDC should be used for both types of buyers. In Example 2, co-existence is optimal for sufficiently large values of π. Although the constraint is binding in this example, the cost of allocating cash to type w_1 and CBDC to w_2 is relatively low.

6 Calibration

The goal of this section is to calibrate the model to the Canadian and US data to estimate the potential welfare gains of introducing CBDC. I proceed in two steps. First, I estimate the parameters of the model and calculate the welfare costs of inflation. The main parameters to be estimated are the buyer's bargaining power, θ, the elasticity of the utility function, $1 - \eta$, where $u(q) = q^{1-\eta}$, and the level of production in the CM, A, as described below. Calculating the welfare costs of inflation is independently interesting, because it allows us to compare the results of the present paper with the existing literature using US data. Second, I use the parameters to find potential gains of introducing CBDC.

The data used in the analysis are M1, gross domestic product (GDP) in market prices and the three-month interest rate. For Canada, the data are from CANSIM for the period 1967-2008. For the US, the data are the same as those used by Lucas (2001) for the period 1900-2000. For the sake of brevity, details of the estimation and parameter restrictions are explained in the appendix.

The estimation method used here is mostly based on Lucas (2001). The theory shows that money demand, M/P, is proportional to real output, Y. Their ratio, $M/(PY)$, is a decreasing function of the opportunity cost of holding cash; i.e., the nominal interest rate. One way to calculate the welfare costs of inflation is to estimate this function from the data

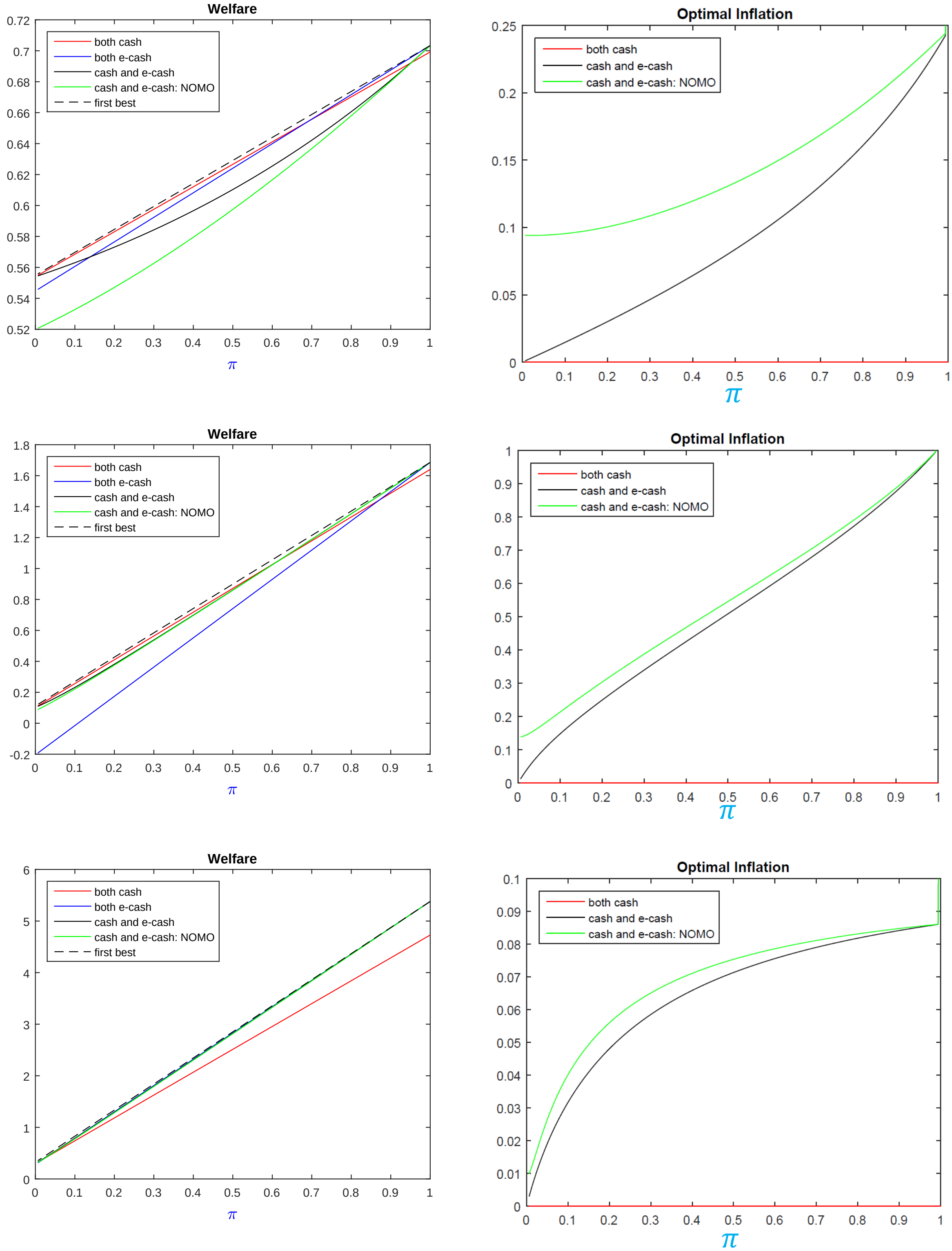

Figure 4: Welfare and optimal cash inflation in Examples 1 (top row), 2 (middle row) and 3 (bottom row).

by finding the best fit, then calculate the area under the demand curve from the inflation level of π_0 to $\pi_0 + 0.10$ to estimate the welfare costs of 10% inflation. Another way is to have a model that generates a money demand function. One should then try to find the parameters of the model such that the money demand function generated by the model fits the data as much as possible. Finally, one can calculate the welfare costs of inflation implied by the model. Many papers, including Lucas (2001), Lagos and Wright (2005) and Craig and Rocheteau (2008), also use the latter approach to estimate welfare costs of inflation for the US data. I follow this methodology.

In the first step, I estimate parameters (θ, η, A), taking as given the discount factor, β, and the probability of matching in the DM, σ. Throughout this section, I set $\beta = 0.97$; that is, the real interest rate is set to 3%. I assume that $c(q) = q$; that is, the cost of producing q units of the DM good for sellers is q. In the benchmark exercise, I set $\sigma = 0.5$, although I estimate parameters assuming other values for σ as well.[21] For estimating parameters, I assume agents are homogeneous; i.e., $w_1 = w_2$.[22] For the results to be comparable with the aforementioned papers, I normalize $w_1 = 1/(1-\eta)$; that is, the buyer's utility from consuming q units of the DM good is $q^{1-\eta}/(1-\eta)$. With a linear production function in the CM, the level of production in equilibrium is indeterminate. Following the literature, I assume the production function in the CM is $U(X) = A\ ln(X)$. This implies that the level of production in the CM is $X^* = A$.

I estimate (θ, η, A) by minimizing the distance between the data and the model-generated real balances–income ratio, $M/(PY)$, subject to the constraint that the markup (price over marginal cost minus 1) in the DM is $\mu = 10\%$ under the 2% inflation rate in the benchmark estimation.[23] Fixing the markup imposes a constraint on variables. The difference between my approach and that typically used in the literature is that I place this constraint explicitly in the minimization problem. In the estimation, I assume that only cash is available. Since I use $M1$ to represent cash, and most elements of $M1$ are not interest bearing, this assumption is consistent with the main presumption of the model that cash is not interest bearing. Introducing CBDC, then, is equivalent to introducing interest-bearing money into

[21] In the estimation, I fix σ. If I include σ in the optimization parameters, the estimates of welfare costs of inflation do not change substantially, but the estimates of parameters (θ, η) would be very sensitive, in that various pairs of (θ, η) lead to almost the same fit. Yet, I conduct various robustness checks depicted in Tables 2 and 3.

[22] In some exercises not reported here, I estimate the model allowing w_2 to be different from w_1, but w_2 is not identified, in that various combinations of parameters yield to almost the same fit.

[23] For the robustness check, I consider $\mu = 20\%$. Also, I consider the average markup of both DM and CM and estimate parameters, with this average markup being 10%.

the economy.

The estimates for Canada and the US are reported in Tables 2 and 3, respectively. The benchmark estimates are shown in the top row of the tables in bold. The ratio of real balances to income is plotted against the nominal interest rate for both data points and model-generated points based on the benchmark estimates in Figures 5 and 7 for Canada and the US, respectively. The welfare costs of inflation based on the benchmark estimates are shown in Figures 6 and 8. Following Lagos and Wright (2005), the welfare costs of a given level of inflation are calculated as the fraction of consumption that agents are willing to forgo to be in equilibrium with a 0 inflation rate (or 3% interest rate).

The welfare cost of 10% inflation in the benchmark estimation is 0.96% for Canada and 1.88% for the US. The range of welfare costs for different parameter values is from 0.8% to 2.8% for Canada and 1.19% to 5.81% for the US. My estimates of welfare costs of 10% inflation for the US are close to estimates in the literature. Lagos and Wright (2005) estimate these welfare costs as ranging from 1% to 5% and Craig and Rocheteau (2008) estimate these costs as ranging from 0.5% to 5%. Also, the lower bound in my estimates is close to the upper bound in the estimate of Lucas (2001) (less than 1%).

In the second part, I estimate the welfare gains of introducing CBDC into the economy. This estimate crucially depends on the choice of K, the cost of using CBDC relative to cash in a transaction. I calculate a range for possible gains of introducing CBDC when the cost of using CBDC in transactions relative to cash ranges from 0% to 0.5% of the transaction value. The welfare gains of introducing CBDC are calculated relative to 0% inflation. That is, I calculate the welfare at 0% inflation when only cash is used, and then compare it with the optimal level of welfare when CBDC can be used too. More precisely, I calculate the level of additional consumption that makes the agents indifferent between being in equilibrium with 0% inflation with only cash circulated in the economy, and being in equilibrium under the optimal policy with both cash and CBDC.

Results regarding the welfare gains of introducing CBDC are summarized in Tables 4 and 5 for Canada and the US, respectively.[24] The benchmark estimations are depicted in the top rows of the tables. When the cost of using CBDC relative to cash is 0%, the welfare gains of introducing CBDC are 0.11% for Canada and 0.26% for the US. With other parameter specifications, the welfare gains range from 0.11% to 64% for Canada and from 0.15% to 1.6% for the US.

I also allow buyers to be homogeneous ($w_1 = w_2$) or heterogeneous ($w_2 < w_1$). When w_2

[24]The optimal scheme requires co-existence when only w_2 uses CBDC. See the last column of Tables 4 and 5.

is increased relative to w_1, the importance of the DM relative to the CM indeed increases if A is kept fixed. In some specifications, when I increase w_2, I increase A as well, such that $\frac{\pi_1 w_1 + \pi_2 w_2}{A}$ remains constant.

μ (only DM)	σ	θ	η	A	Welfare loss of 10% inflation
0.1	**0.5**	**0.82**	**0.27**	**3.5**	**0.96**
0.1	0.8	0.72	0.2	3.41	1.15
0.1	0.2	0.94	0.52	3.69	0.8
0.2	0.5	0.7	0.31	3.82	1.17
0.2	0.8	0.57	0.24	3.57	1.5
μ (CM & DM)					
0.1	0.8	0.32	0.4	4.86	2.8
0.1	0.2	0.49	0.73	8.98	1.5
0.1	0.5	0.36	0.5	5.67	2.38

Table 2: Estimated parameters for Canada.

μ (only DM)	σ	θ	η	A	Welfare loss of 10% inflation
0.1	**0.5**	**0.76**	**0.2**	**2.02**	**1.88**
0.1	0.2	0.91	0.4	2.26	1.41
0.1	1	0.60	0.13	1.67	2.49
0.2	0.5	0.61	0.22	2.05	2.51
0.2	0.2	0.83	0.4	2.31	1.62
0.2	1	0.40	0.15	1.56	4.05
μ (CM & DM)					
0.1	0.5	0.29	0.23	1.87	5.81
0.1*	0.5	0.5	0.51	3.83	1.85
0.1*	0.4	1	0.18	1.90	1.44
0.1*	1	1	0.11	1.92	1.19

Table 3: Estimated parameters for the US. In the rows with an *, θ has been fixed, not estimated.

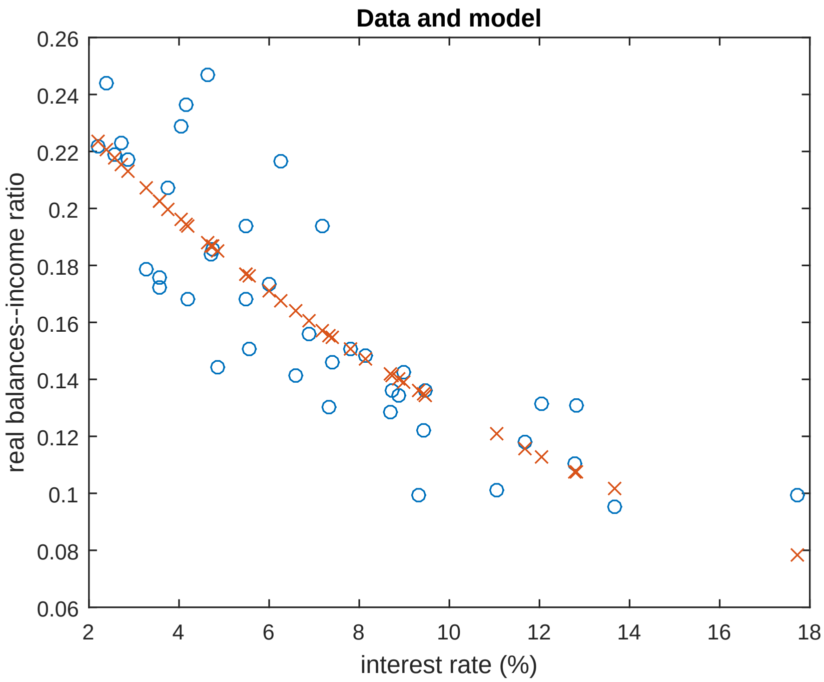

Figure 5: Data and model for Canada

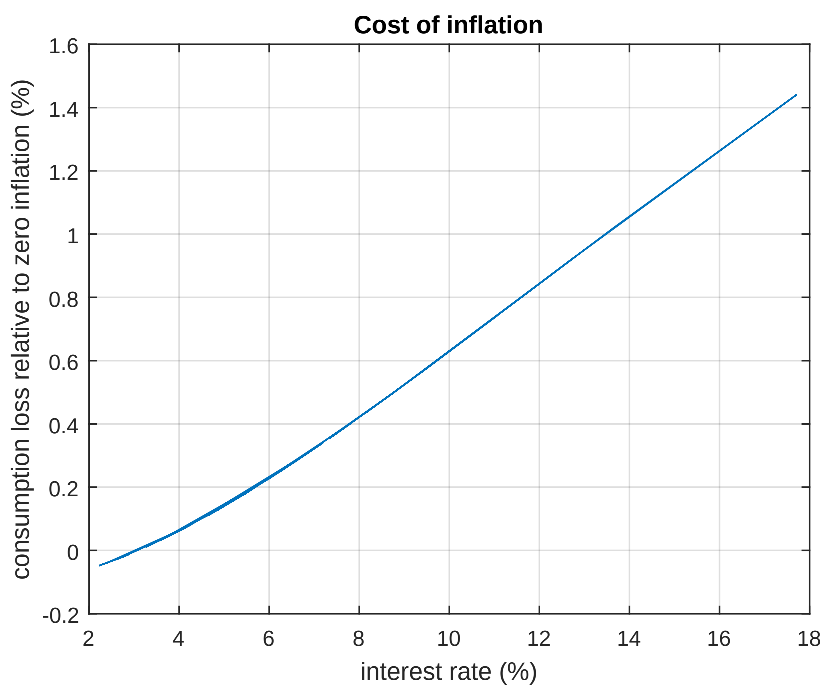

Figure 6: Welfare cost of inflation for Canada

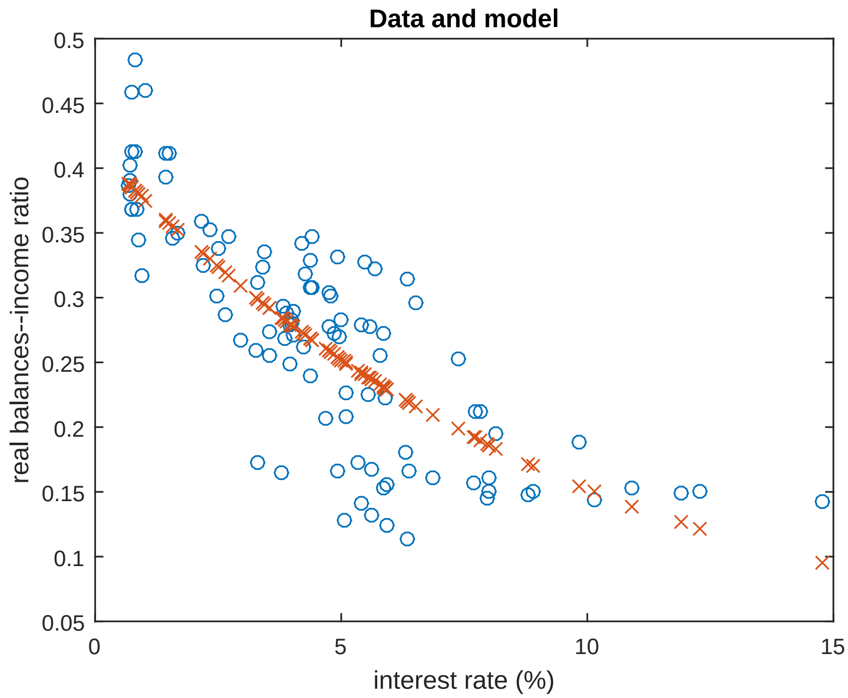

Figure 7: Data and model for the US

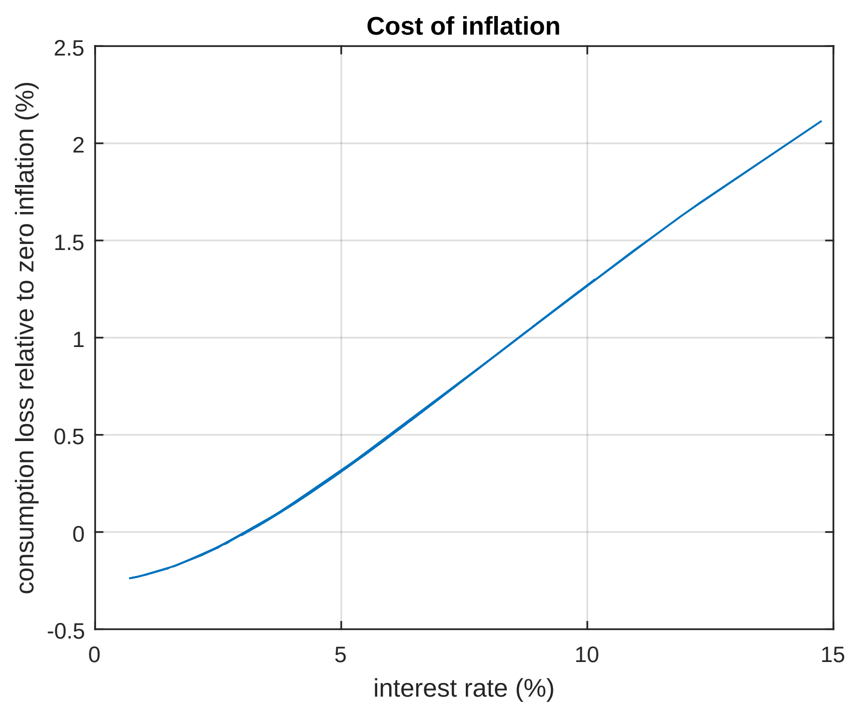

Figure 8: Welfare cost of inflation for the US

w_1	w_2	η	σ	θ	A	z_2/z_1	CBDC transaction cost (%)	Welfare gains of CBDC (%)	Who uses CBDC?
							0	**0.11**	w_1 & w_2
1.37	**1.37**	**0.27**	**0.5**	**0.82**	**3.5**	**1**	**0.25**	**0.06**	w_1 & w_2
							0.50	**0.00**	w_1 & w_2
							0	0.20	w_1 & w_2
1.37	2	0.27	0.5	0.82	4.31	4.06	0.27	0.06	w_2
							0.45	0.00	w_2
							0	0.15—0.27*	w_1 & w_2
1.37	2	0.27	0.5	0.82	4.31	4.06	0.27	0.02—0.11*	w_2
							0.45	-0.01—0.02*	w_2
							0	0.44	w_1 & w_2
1.37	3	0.27	0.5	0.82	5.58	18.23	0.25	0.20	w_2
							0.50	0.00	w_2
							0	0.15	w_1 & w_2
1.32	1.32	0.24	0.8	0.57	3.57	1.00	0.25	0.10	w_1 & w_2
							0.51	0.04	w_1 & w_2
							0	0.33	w_1 & w_2
1.32	2	0.24	0.8	0.57	4.50	5.72	0.25	0.16	w_2
							0.49	0.06	w_2
							0	0.64	w_1 & w_2
1.32	3	0.24	0.8	0.57	5.85	31.00	0.25	0.39	w_2
							0.51	0.16	w_2

Table 4: Estimates of the welfare gains of introducing CBDC for Canada. The gains are calculated relative to an economy with only cash under 0 % inflation.

(*): The lower and upper bounds in the range are associated with the cases where $\pi_1 = 0.2$ and $\pi_1 = 0.8$, respectively.

w_1	w_2	σ	η	θ	A	z_2/z_1	CBDC transaction cost (%)	Welfare gains of CBDC (%)	Who uses CBDC?
							0.00	**0.26**	w_1 & w_2
1.25	**1.25**	**0.5**	**0.2**	**0.76**	**2.02**	**1**	**0.25**	**0.18**	w_1 & w_2
							0.49	**0.11**	w_1 & w_2
							0.00	0.86	w_1 & w_2
1.25	2	0.5	0.2	0.76	2.02	10.49	0.26	0.51	w_2
							0.52	0.24	w_2
							0.00	0.77	w_1 & w_2
1.25	2	0.5	0.2	0.76	2.63	10.49	0.26	0.45	w_2
							0.50	0.20	w_2
							0	0.50—0.94*	w_1 & w_2
1.25	2	0.5	0.2	0.76	2.63	10.49	0.26	0.20—0.68*	w_2
							0.50	0.08—0.34*	w_2
							0.00	0.75	w_1 & w_2
1.30	1.30	1	0.23	0.29	1.87	1	0.25	0.67	w_1 & w_2
							0.49	0.60	w_1 & w_2
							0.00	1.60	w_1 & w_2
1.30	2.1	1	0.23	0.29	2.45	8.08	0.25	1.43	w_1 & w_2
							0.50	1.27	w_1 & w_2
							0.00	0.15	w_1 & w_2
2.04	2.04	0.5	0.51	0.5	3.83	1	0.25	0.08	w_1 & w_2
							0.50	0.00	w_1 & w_2
							0.00	0.32	w_1 & w_2
2.04	6.00	0.5	0.51	0.5	7.55	8.29	0.25	0.13	w_2
							0.46	0.01	w_2

Table 5: Estimates of the welfare gains of introducing CBDC for the US. The gains are calculated relative to an economy with only cash under 0% inflation.

(*): The lower and upper bounds in the range are associated with the cases where $\pi_1 = 0.2$ and $\pi_1 = 0.8$, respectively.

7 Extension

I assume in this section that there is no direct cost of carrying CBDC—i.e., $K = 0$—but CBDC cannot be used in a fraction of transactions. For example, the seller may not be able to accept CBDC because no internet or electricity connection is available, or because the seller does not have access to CBDC technology (such as un-banked agents in remote locations). Another reason might be that the buyers do not want to use CBDC in some transactions so that they can keep their anonymity. Formally, assume that in $\alpha_c \in [0, 1]$ fraction of transactions, only cash can be used (c-meetings); in $\alpha_e \in [0, 1]$ fraction of transactions, only CBDC can be used (e-meetings); and in $\alpha_b \in [0, 1]$ fraction of transactions, both cash and CBDC can be used (b-meetings).[25] Of course, $\alpha_c + \alpha_e + \alpha_b \leq 1$.[26] Note that the buyers learn the type of the meeting only after they match with a seller. Therefore, and in contrast to the previous sections, a given agent may want to hold both means of payment as they do not know which means of payment can be used in the DM. The case of $\alpha_c = 0$ is similar to the benchmark model in the previous sections with $K = 0$, in that using CBDC is costless, so cash is redundant. In the case of $\alpha_c > 0$, CBDC users may effectively incur a cost for using CBDC, as their means of payment cannot be used in some transactions.

I assume without loss of generality that the planner is not allowed to make cash transfers, i.e., $t_c = 0$.[27] Furthermore, I do not study the heterogeneity of types in this section; type w is fixed and the same for all buyers.[28] Therefore, the index w is removed from the notation. The rest of the environment is exactly the same as in the benchmark model. In particular, I continue to assume that the transfer to agents, whether in cash or CBDC form, cannot be negative. That is, the planner does not have the power of taxation.

For a given policy, the agent's problem is given as follows:

$$\max_{z_c, z_e} \left\{ -(\gamma_c - \beta)z_c - (\gamma_e - \beta)z_e + \beta t_e(z_e) + \beta\sigma\theta\Big(\alpha_c f(q_c) + \alpha_e f(q_e) + \alpha_b f(q_b)\Big) \right\} \tag{18}$$

[25] In making this assumption, I follow Rocheteau et al. (2014) and Zhu and Hendry (2017). In the former, fiat money and bond, and in the latter, money and private e-money are means of payment.

[26] Especially when doing comparative statics, I allow their sum to be less than 1. The interpretation is that in some transactions, the buyer and seller may not trade with any means of payment.

[27] To show this claim, start from an allocation with a strictly positive t_c. Change t_c to 0 but conduct OMO such that the same amount of cash is injected into the economy in exchange for CBDC. Next, transfer CBDC to the agents (and possibly adjust the CBDC inflation) so that the DM production levels in all three types of meetings remain the same.

[28] I had heterogeneity in the benchmark model to generate demand for both means of payment endogenously. In this section, since one means of payment may not be useful in some transactions, agents may want to bring both means of payment from the CM. Therefore, I do not need to have heterogeneity of types anymore.

where

$$q_c = \min\{q^*, D^{-1}(z_c)\}, \tag{19}$$

$$q_e = \min\{q^*, D^{-1}(z_e + t_e)\}, \tag{20}$$

$$q_b = \min\{q^*, D^{-1}(z_c + z_e + t_e)\}. \tag{21}$$

The planner's problem is to choose a policy that maximizes welfare, and can be written as follows:

$$\max_{z_c, z_e, t_e(.), \gamma_c, \gamma_e} \{\alpha_c f(q_c) + \alpha_e f(q_e) + \alpha_b f(q_b)\}$$
$$\text{s.t. } t_e(z_e) = (\gamma_c - 1)z_c + (\gamma_e - 1)z_e, \tag{22}$$

where (q_c, q_e, q_b) is obtained from the agent's maximization problem given above.

Since the punishment in the case that the agent does not bring enough CBDC balances is severe (i.e., receiving no transfers), we can assume without loss of generality that if an agent brings CBDC, then the agent will bring the exact amount of CBDC that the planner wants. Similar to the benchmark model, to support this punishment, the inflation rate for CBDC should be sufficiently high that the CBDC transfers for agents who bring enough balances become positive. Therefore, the only relevant constraint is that agents should not gain if they bring all their portfolio in the form of cash. This constraint is summarized by:

$$\max_{z_c} \big\{ -(\gamma_c - \beta)z_c + \beta\sigma\theta\big(\alpha_c f(q_c) + \alpha_b f(q_b)\big)\big\} - (\gamma_e - \beta)z_e + \beta t_e(z_e) + \beta\sigma\theta\alpha_e f(q_e)$$
$$\geq \max_{q} \big\{ -(\gamma_c - \beta)D(q) + \beta\sigma\theta(\alpha_c + \alpha_b)f(q)\big\}. \tag{23}$$

Also, define

$$z_c \in \arg\max_{z_c'} \big\{ -(\gamma_c - \beta)z_c' + \beta\sigma\theta\big(\alpha_c f(\min\{q^*, D^{-1}(z_c')\}) + \alpha_b f(\min\{q^*, D^{-1}(z_c' + z_e + t_e)\})\big)\big\}. \tag{24}$$

From now on, I summarize the optimal policy by z_c and q_b. From (19), z_c pins down q_c. One then obtains γ_c from (24) consistent with z_c, q_b and q_c, and z_e and t_e from (21) and (22) for a given γ_e. Finally, q_e is given by (20).

7.1 Sufficient Conditions to Achieve the First Best

Remember that in Section 3, the first-best level of welfare cannot be attained when $K > 0$, because even if β and θ are sufficiently high, the buyers still have to incur the direct cost of carrying CBDC. Here, in contrast, the first-best level of welfare can be achieved if β and θ are sufficiently high even when $\alpha_c > 0$. To achieve the first best, the planner runs deflation on cash at the rate of time preferences—i.e., $\gamma_c = \beta$—and then runs inflation on CBDC to finance the deflation by withdrawing cash and injecting CBDC in the CM through OMO. OMO is crucial for this result. Also, similar to the transfer scheme used previously, the planner makes CBDC transfers to agents only if they bring enough CBDC balances from the CM. If they do not, then the planner does not give them the subsidy.

Proposition 12. *Assume $\alpha_c = 0$. Then the first best is achievable if and only if:*

$$\beta\sigma(\alpha_e + \alpha_b)\theta(wu(q^*) - c(q^*)) \geq (1-\beta)D(q^*). \tag{25}$$

Now assume that $\alpha_c > 0$. Then the first best is achievable if:

$$\beta\sigma\alpha_e\theta(wu(q^*) - c(q^*)) \geq 2(1-\beta)D(q^*). \tag{26}$$

Some points are worth mentioning. If $\alpha_c = 0$, the condition needed to achieve the first best is equivalent to that in Proposition 3. In the general case of $\alpha_c > 0$, to achieve the first best, the planner needs to ensure that buyers have enough balances in all types of meetings to achieve the first best. Regarding the c-meetings, the planner needs to run a cash growth rate equal to the discount factor to reduce the cost of holding real balances to 0. Regarding the e-meetings, the planner simply needs to ensure that the number of balances in CBDC that the agents carry, together with what they receive as transfers, is enough to buy the first-best level of production. The planner would not worry about the b-meetings because agents have enough balances to buy the first-best level of consumption by using only one means of payment.

7.2 Optimal Scheme When the First Best Cannot Be Achieved

Now assume that the first best is achievable when there are no c-meetings—i.e., when $\alpha_c = 0$—but it is not achievable otherwise. That is, (25) holds but (26) does not. We are interested in knowing whether both means of payment are valued in the optimal allocation (co-existence) or only one means of payment is valued. Particularly, how is the use of means of payment changed when α_c increases from 0? Do buyers choose to bring a small amount of cash from the CM? Or, is there a threshold for α_c below which they do not use cash even though cash is useful in some meetings?

The tradeoff that the planner faces is similar to that in the benchmark model. If cash is valued in the equilibrium, then agents can substitute cash for CBDC in the b-meetings. Therefore, agents have less incentive to bring CBDC. This, in turn, implies that welfare gains from CBDC, due to its flexibility, are less likely to be realized. If cash is not valued, there will be no trade in the c-meetings. The optimal policy trades off these two forces.

Proposition 13. *Suppose $\alpha_c + \alpha_e + \alpha_b = 1$ and that $\alpha_c > 0$. Assume that (25) holds but (26) does not. That is, the first best can be achieved when $\alpha_c = 0$ but not when $\alpha_c > 0$. Then the following results are obtained:*

(a) Assume $\alpha_b = 0$. Suppose (25) holds with strict inequality. Then cash is valued in the optimal allocation.

(b) Assume $\alpha_e = 0$. If both means of payment are valued and the first best cannot be achieved in any meetings, then cash inflation must be strictly positive.

Part (a) states that when $\alpha_c > 0$, it is optimal to use cash. Notice that this result holds even without Inada conditions. There are two countervailing forces. When cash is valued, it is effectively a leeway for agents to evade the taxation policy to which the CBDC users are subjected. Since there are no b-meetings, the choice of cash is independent of the choice of CBDC. The planner would have cash circulated as long as there are some meetings for which cash is useful. Therefore, cash is absolutely essential in this economy. This result is in contrast to the result in the benchmark model in which under certain conditions cash is not used in the optimal allocation. Finally, when α_c is small, the planner needs to run deflation to implement this allocation. This means that using OMO (to inject CBDC and withdraw cash from the CM) is crucial here.

Part (b) simply states that cash inflation is necessary to provide incentive to agents to hold CBDC. If cash inflation is too low, agents would have incentive to use cash in both c-meetings and b-meetings, and therefore, the welfare gains associated with using CBDC would not be realized. This result is similar to the benchmark model.

Proposition 14. *(a) Assume $\alpha_e = 0$. There exists $\bar{\alpha}_c < 1$ such that if $\alpha_c \geq \bar{\alpha}_c$, then only cash is valued in the optimal allocation.*

(b) Assume $\alpha_b = 0$ and $u'(0) < \infty$. There exists $\tilde{\alpha}_c < 1$ such that if $\alpha_c \geq \tilde{\alpha}_c$, then only cash is valued in the optimal allocation.

I summarize the analytical results so far in the space of α_c and α_e in Figure 9. There are three schemes for means of payment: cash-only (in which all agents use only cash), CBDC-only (in which all agents use only CBDC), and both cash and CBDC (in which agents use both means of payment). Ideally, one would like to identify the regions under which each scheme is optimal, but analytically it is not easy to do that for the entire space. This is why I identify the optimal schemes on only some boundaries of the region. On the horizontal axes, α_e is depicted, and on the vertical axes, α_c is depicted.

When $\alpha_c = 0$, cash is redundant, and if (25) holds, even the first best is achievable. If $\alpha_e = 0$, then Proposition 14(a) implies that the cash-only scheme is optimal. Similarly,

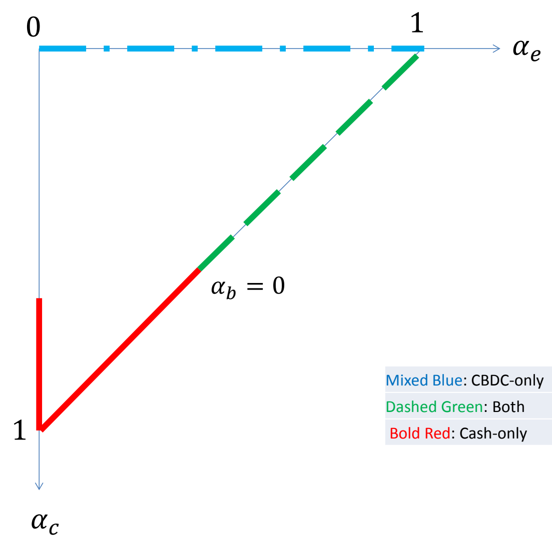

Figure 9: Demonstration of analytical results regarding the optimal means of payment. Here, $\alpha_b = 1 - \alpha_c - \alpha_e$. The blue line corresponds to Proposition 12. The green line corresponds to Proposition 13(a). The red lines correspond to Proposition 14(a) and (b).

if $\alpha_b = 0$, then the cash-only scheme is optimal from Proposition 14(b). If $\alpha_b = 0$, then Proposition 13(a) implies that cash should be valued. Also, Proposition 12 implies that for a sufficiently large α_e, CBDC should be valued, too.

The interesting result here is the asymmetry between usage of cash and CBDC. Assume $\alpha_b = 0$. If α_c is sufficiently close to 1, then using CBDC is not optimal (Proposition 14(b)), but if α_c is sufficiently close to 0, using cash is still optimal (Proposition 13(a)), because the taxation policy associated with CBDC can help achieve better allocations even in the c-meetings.

8 Concluding Remarks

In this paper, I study a model in which agents can hold two central bank-issued monies: cash and CBDC. CBDC is interest bearing potentially in a non-linear fashion, but at the same time, there is a cost associated with carrying it, perhaps due to the agents' anonymity concerns. I characterize the welfare-maximizing monetary policy with respect to these two means of payment. Assume first that cash and CBDC are perfect substitutes in the transactions. If the cost of carrying CBDC is small enough, then only CBDC is used under the optimal policy, and even the first best can be achieved. If CBDC is too costly, only cash is used. If the cost of carrying CBDC is intermediate, welfare under co-existence may be dominated by the scheme in which only one means of payment is used. This is because under co-existence, agents use cash as a way to evade the taxation scheme that CBDC users are subjected to; therefore, co-existence leads to under-utilization of gains of CBDC rising from its interest-bearing feature. When cash and CBDC are not perfect substitutes, then co-existence is more likely to be optimal. Furthermore, CBDC can help achieve better allocations even for the meetings in which only cash can be used. This is possible through OMO, where the planner exchanges cash for CBDC to run the Friedman rule for cash.

A system that can implement the CBDC discussed in this paper is a debit card system that is owned and monitored by the central bank (although its operations can be outsourced to other institutions such as "FinTech" companies, if they can do it with low operational costs). Each individual can have an account with the central bank, and individuals can use these balances for purchases of goods and services. Such a system provides access to the central bank balance sheet in electronic format for all agents in the population and allows them to earn interest on their balances. Currently, only some financial institutions have this privilege. With such a system, monetary policy directly affects agents' decisions to carry balances, rather than through the financial system, making the implementation of monetary policy more transparent. (See Berentsen and Schar (2018).)

Introducing CBDC would have non-trivial effects on the banking system. If CBDC is available, it provides direct access for agents to a perfectly safe deposit account. Such an account would be safer than the deposits at commercial banks (at least for the deposits exceeding the amount insured by, say, the Canada Deposit Insurance Corporation), so those agents who want a perfectly safe deposit vehicle do not deposit at commercial banks. Therefore, CBDC acts as a competitor for bank deposits. On one hand, this elevated level of competition may lead banks to improve their services by offering cheaper and/or better products. On the other hand, an increased level of competition may lead banks to invest in (or give loan to) riskier projects, resulting in a less stable financial system. Furthermore, banks may increase the interest rate on loans, which in turn can affect the real economy. The effects of CBDC on the banking system, both in the steady state and also in times of crisis, are yet to be explored.

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Appendix

In this appendix, I first discuss the relationship between my paper and two closely related papers. Second, I provide proofs for the results. Finally, I elaborate on the calibration exercise.

Relation with Chiu and Wong (2015)

My paper is among the first to study the payment choice between cash and CBDC, and how this choice affects the optimal monetary policy. The only paper with something of a similar objective is Chiu and Wong (2015). Perhaps the most important difference between my model and theirs is that the features of cash and CBDC emphasized in my model are different from theirs. In particular, I do not allow for communication between the planner and the agents if they use cash. Also, I do not allow for the planner's direct interventions in the bilateral meetings. I believe these two features are more realistic regarding cash and the versions of CBDC typically discussed in policy circles.

To elaborate, the first difference between my paper and theirs is with respect to how cash transfers are made to agents. In their model, the agents can communicate with the planner or mechanism designer to report their cash holdings, and the planner proposes an allocation that can depend on their report in a very general way; i.e., there is no requirement on the functional form of this dependence. In my model, the ability of the planner to implement monetary policy around cash is very limited. No communication between cash users and the planner is allowed, and the only possible form of implementing monetary policy regarding cash is the helicopter drop; i.e., the number of cash transfers to agents cannot depend on the agent's characteristics (such as transaction amount). As a result, the first best is never achievable with cash in my model. Roughly speaking, the CBDC in my paper is equivalent to cash, together with some form of communication in their paper.

The second difference is that in my model, the planner cannot intervene in the bilateral meetings. All interventions in my model are done either in the CM (using OMO) or in the transfer stage before bilateral meetings occur. In contrast, in their model, the planner has the power to either restrict access to CBDC balances in the DM or make transfers to the agents in the DM. Therefore, the set of interventions available to the planner in their paper is different from my paper.[29]

[29] There are other differences between the two papers, too. In my paper, the agents are heterogeneous in their transaction needs, while agents are homogeneous in their paper. I study the cases under which both means of payment co-exist and are used by different agents. In their paper, co-existence is not studied.

Relation with Gomis-Porqueras and Sanches (2013)

I do not assume that the planner has access to the history of transactions, but the fact that the planner can see the current CBDC balances somewhat summarizes all available information for the planner whether the agent has worked in the previous CM or not. In this sense, CBDC in my model acts like credit, which is the focus of their paper. Unlike in my benchmark model, the credit is not available in their paper in a fraction of meetings, so their setting is similar to the setting in Section 7 of my paper. Moreover, they do not have heterogeneity of buyers; therefore, the endogenous adoption of different means of payment for different agents does not happen in their equilibrium. Finally, they do not have a policy choice regarding credit. In contrast, in my model, the policy choice of the planner is to choose how CBDC transfers should be distributed to the agents to induce them to use the means of payment that leads to the socially optimal outcome.

Proofs

Proof of Lemma 1. One can write

$$t_c + \int t_e(\hat{z}_e(w), w)dF(w) = \phi_+ M_+ + \psi_+ E_+ - \phi_+ \bar{M} - \psi_+ \bar{E} = \phi_+(M_+ + \frac{\psi_+}{\phi_+}E_+) - \phi_+(\bar{M} + \frac{\psi}{\phi}\bar{E})$$

$$= \phi_+(\bar{M}_+ + \frac{\psi_+}{\phi_+}\bar{E}_+) - \phi_+(\bar{M} + \frac{\psi}{\phi}\bar{E}) = \frac{\phi_+}{\phi_{++}}\phi_{++}\bar{M}_+ - \phi_+\bar{M} + \frac{\psi_+}{\psi_{++}}\psi_{++}\bar{E}_+ - \psi_+\bar{E}$$

$$= (\gamma_c - 1)\int \hat{z}_c(w)dF(w) + (\gamma_e - 1)\int \hat{z}_e(w)dF(w).$$

For the first equality, (4) and (5) were used. For the third one, (3) was used for $t+1$. Equations (6) and (7) together with (2) were used for the last equality. □

Proof of Proposition 1. Consider problem 2. Take the optimal policy $\{\gamma_c, \gamma_e, t_c, \{t_e(z, w)\}_w\}$ and denote the equilibrium allocation by $\{(q(w), \hat{z}_c(w), \hat{z}_e(w))\}_w$. Consider another policy in which $t_e(z, w)$ is replaced with

$$t'_e(z, w) = \begin{cases} t_e(\hat{z}_e(w), w) & z \geq z_e(w) \\ 0 & z < z_e(w) \end{cases}.$$

It is easy to see that the same equilibrium objects form an equilibrium under this new policy, because the constraints of the planner's problem continue to hold. Especially note that type w buyers do not want to bring CBDC any less than in the equilibrium, as they would get weakly fewer transfers, and they do not want to bring any more either, as they cannot get any transfers more than $t_e(\hat{z}_e(w), w)$. Finally, their decision regarding cash does not change either, as cash payments to them are the same as before. $\square$

Proof of Proposition 2. For any optimal policy and the associated equilibrium allocation, another policy and equilibrium will be constructed in which cash is not used and the welfare remains the same. Consider problem 2. Take the optimal policy $\{\gamma_c, \gamma_e, t_c, \{t_e(z, w)\}_w\}$ and denote the equilibrium allocation by $\{(q(w), z_c(w), z_e(w))\}_w$. Consider another policy with $t'_c = 0$, $M'_0 = M_0$, $E'_0 = E_0$ inflation rate γ'_c, γ'_e, and

$$t'_c = 0, t'_e(z, w) = \begin{cases} D_w(q(w)) - \frac{D_w(q(w)) + \Lambda_c(w) + \Lambda_e(w)}{\gamma'_e} & z \geq \frac{D_w(q(w)) + \Lambda_c(w) + \Lambda_e(w)}{\gamma'_e} \\ 0 & \text{otherwise} \end{cases}$$

where

$$\Lambda_c(w) \equiv (\gamma_c - 1)(t_c + z_c(w)) - \gamma_c t_c,$$

$$\Lambda_e(w) \equiv (\gamma_e - 1)(t_e(z_e(w), w) + z_e(w)) - \gamma_e t_e(z_e(w), w).$$

Set γ'_c sufficiently high such that buyers do not want to bring any balances from the CM. Then it is easy to check that with this policy, agents will choose the same $q(w)$ as in the original equilibrium.

To give an idea of how the new policy was constructed, note that the buyer's payoff can be written as

$$-(1 - \beta)(\underbrace{z_c + z_e(w) + t_c + t_e(z_e(w), w)}_{D_w(q)}) + \beta\sigma\theta(wu(q) - c(q)) - \Lambda_c(w) - \Lambda_e(w)$$

following the first constraint in problem 2. I construct γ'_e and $t'_e(z, w)$ such that the real post-transfer balances of buyers remain the same as in the original allocation—i.e., $t'_e(z'_e(w), w) + z'_e(w) = D_w(q(w))$—and also their payoff remains the same: $-\Lambda_c(w) - \Lambda_e(w) = -(\gamma'_c - 1)(t'_c + z'_c(w)) + \gamma'_c t'_c - (\gamma'_e - 1)(t'_e(.) + z'_e(w)) + \gamma'_e t'_e(.)$.

Under this policy, if any buyer wants to carry cash, the buyer gets at most $\max_q\{-(\gamma'_c - \beta)D_w(q) + \beta\sigma\theta(wu(q) - c(q))\}$, which is 0 if γ'_c is set sufficiently high. If any buyer chooses to bring balances in CBDC but lower than $z'_e(w)$, then the buyer can get no more than his payoff if he uses cash, in which case his payoff is 0. If he brings $z'_e(w)$, then he gets a positive payoff as in the original equilibrium allocation.

Finally, I show below that the last constraint in the problem is also satisfied.

$$t'_c + \int \left[(t'_e(z'_e(w), w) - (\gamma'_c - 1)z'_c(w) - (\gamma'_e - 1)z'_e(w))\right] dF(w) = -\int \left[\Lambda_c(w) + \Lambda_e(w)\right] dF(w) = 0.$$

For the first equality, the construction of a new policy and a new equilibrium allocation was used. The second one can be simply derived after some algebra and using the fact that the corresponding constraint in the original equilibrium must hold. $\square$

Proof of Proposition 3. First, assume that the condition holds.
Consider the policy $\{\gamma_c, \gamma_e, t_c, \{t_e(z,w)\}_w\}$, the equilibrium allocation $\{(q(w), \hat{z}_c(w), \hat{z}_e(w))\}_w$, and CBDC transfer function $t_e(z,w)$:

$$t_e(z,w) = \begin{cases} \frac{(\gamma_e - 1)D_w(q_w^*)}{\gamma_e} & z \geq \frac{D_w(q_w^*)}{\gamma_e} \\ 0 & z < z_e(w) \end{cases}.$$

$t_c = 0$, $\gamma_e > 1$, and $\gamma_c > 1$ is set sufficiently high that $\max_q\{-(\gamma - \beta)D_w(q) + \beta\sigma\theta(wu(q) - c(q))\} = 0$. Under this policy, no buyer wants to carry any cash, as their payoff would be 0. Moreover, the buyer would take exactly $z_e(w) = \frac{D_w(q_w^*)}{\gamma_e}$ real balances from the CM. The constraint of the planner's problem is satisfied by the construction of t_e.

Second, assume that the first best is achievable. By Proposition 2, cash is not used without loss of generality, so we can assume that $t_c = 0$. Following the second constraint of the planner's problem, one obtains $t_e(z,w) = (\gamma_e - 1)z_e(w)$. Since first best is achievable, the balances that the agent takes from the CM plus the transfers that he receives, $z_e(w) + t_e(z_e(w), w)$, must be equal to $D_w(q_w^*)$. Therefore, $z_e(w) = \frac{D_w(q_w^*)}{\gamma_e}$, and consequently, $t_e(z,w) = (\gamma_e - 1)z_e(w) = \frac{(\gamma_e - 1)D_w(q_w^*)}{\gamma_e}$. The buyer must get a positive payoff, so

$$-(\gamma_e - \beta)D_w(q_w^*) + \beta\sigma\theta(wu(q_w^*) - c(q_w^*)) + \gamma_e t_e(z_e(w), w) \geq 0.$$

Otherwise, the buyer would skip the CM and DM; i.e., there would be no trade. Simple algebra ensures that the condition must hold using the values derived for $z_e(w)$ and $t_e(z_e(w), w)$. $\square$

Proof of Proposition 4. Define $r(q)$ as follows:

$$r(q) \equiv \frac{c'(q)}{u'(q)} \Rightarrow r' = \frac{c'}{u'}\left(\frac{c''}{c'} - \frac{u''}{u'}\right) \text{ and } q_w^* = r^{-1}(w),$$

where the arguments of the functions are eliminated when there is no danger of confusion. Therefore,

$$\bar{\theta}(w) \text{ is decreasing } \Leftrightarrow \frac{wu(q_w^*)}{c(q_w^*)} \text{ is increasing } \Leftrightarrow \frac{\partial}{\partial w}\left(\frac{wu(r^{-1}(w))}{c(r^{-1}(w))}\right) > 0.$$

Now,

$$c^2 \frac{\partial}{\partial w}\left(\frac{wu(r^{-1}(w))}{c(r^{-1}(w))}\right) = (u + wu'/r')c - wuc'/r'$$

$$= uc\Big(1 + w/r'(u'/u - c'/c)\Big) = uc\Big(1 + \frac{wu'/(wu) - c'/c}{c''/c' - u''/u'}\Big),$$

where in the last step, the fact that $u'(q_w^*) = c'(q_w^*)$ was used. But

$$1 + \frac{wu'/(wu) - c'/c}{c''/c' - u''/u'} > 0 \Leftrightarrow \frac{cc'' - c'^2}{cc'} > \frac{uu'' - u'^2}{uu'} \Leftrightarrow \frac{\partial}{\partial q} ln\Big(\frac{c'(q)u(q)}{c(q)u'(q)}\Big) > 0.$$

□

Proof of Proposition 5. Both means are costless, so we can assume without loss of generality that inflation is sufficiently high so that cash is not used and also t_e is characterized as a step function. Since the payoff from using cash is 0, the following must hold at any equilibrium:

$$-(\gamma_e - \beta)D_w(q(w)) + \beta\sigma\theta\Big(wu(q(w)) - c(q(w))\Big) + \gamma_e t_e(z(w), w) \geq 0 \forall w,$$

$$\int_w (t_e(z(w), w) - (\gamma_e - 1)z(w))dF(w) \leq 0.$$

One can write from the first constraint that $-\gamma_e t_e(z(w), w) = -(\gamma_e - \beta)D_w(q(w)) + \beta\sigma\theta(wu(q(w)) - c(q(w))) - \epsilon(w)$ for some $\epsilon(w) \geq 0$. Since $t_e(z(w), w) + z(w) = D_w(q(w))$, the budget constraint can be written as

$$0 = 1/\gamma_e\Big(\int(\gamma_e t_e(z(w), w) - \gamma(\gamma_e - 1)z(w))dF(w)\Big)$$

$$= 1/\gamma_e\Big((\gamma_e - \beta)D_w(q(w)) - \beta\sigma\theta(wu(q(w)) - c(q(w))) + \epsilon(w)$$

$$-(\gamma_e - 1)(\beta D_w(q(w)) + \beta\sigma\theta(wu(q(w)) - c(q(w)) - \epsilon(w))\Big)dF(w)$$

$$= \int((1 - \beta)D_w(q(w)) + \beta\sigma\theta(wu(q(w)) - c(q(w))) + \epsilon(w))dF(w). \tag{27}$$

"only if" part: Now suppose FB is achievable. Since $\epsilon(w) \geq 0$ for all w, the condition in the proposition will follow.

"if" part: If the condition is satisfied, then it is easy to verify that the first best is achievable with the following choice of $t_e(z, w)$:

$$t_e(z, w) = \begin{cases} \frac{(\gamma_e - \beta)D_w(q^*(w)) - \beta\sigma\theta(wu(q^*(w)) - c(q^*(w))) + v}{\gamma_e} & z \geq \frac{\beta D_w(q^*(w)) + \beta\sigma\theta(wu(q^*(w)) - c(q^*(w))) - v}{\gamma_e} \\ 0 & \text{otherwise} \end{cases},$$

where

$$v \equiv \int -(1 - \beta)D_w(q^*(w)) + \beta\sigma\theta(wu(q^*(w)) - c(q^*(w)))dF(w).$$

Also, set $t_c = 0$, and again $\gamma_c > 1$ is set sufficiently high that $\max_q\{-(\gamma_c - \beta)D_w(q) + \beta\sigma\theta(wu(q)-c(q))\} = 0$, and γ_e is set sufficiently high that $(\gamma_e-\beta)D_w(q^*(w))-\beta\sigma\theta(wu(q^*(w))-c(q^*(w))) \geq 0$. Note that $v \geq 0$ following the assumption in the proposition. Notice that I have assumed the distribution is such that $v \leq \frac{\beta D_w(q^*(w))+\beta\sigma\theta(wu(q^*(w))-c(q^*(w)))-v}{\gamma_e}$ for all types. If it does not hold for some types, then those types do not need to bring any balances. In that case, the transfer scheme needs to be modified, but I skip it for the sake of brevity. $\square$

Proof of Proposition 6. The buyers of type w want to bring $D_w(q_w^*)$—enough balances to be able to buy q_w^*—if

$$t_c = 0, t_e(z, w) = \begin{cases} \frac{\gamma_e-\beta}{\beta} z - \frac{\beta\sigma\theta(w_{min}u(q^*_{w_{min}})-c(q^*_{w_{min}}))}{\beta} + \Delta & z \geq \frac{D_{w_{min}}(q^*_{w_{min}})}{\gamma_e} \\ 0 & otherwise \end{cases},$$

where

$$\Delta \equiv \left(\beta\sigma\theta(wu(q_w^*) - c(q_w^*))|_{w=w_{min}} - (1-\beta)\int D_w(q_w^*)dF(w)\right)/\beta,$$

and γ_c is set sufficiently high that buyers get a 0 payoff by using cash, and γ_e is set sufficiently high too as in previous proofs. Therefore, they are willing to bring as much as $D_w(q_w^*)$ real balances. That is, $\frac{\gamma_e-\beta}{\beta}z_e - \frac{\beta\sigma\theta(w_{min}u(q^*_{w_{min}})-c(q^*_{w_{min}}))}{\beta} + \Delta + z_e = D_w(q_w^*)$. Subsequently, $z_e = \frac{\beta}{\gamma_e}(D_w(q_w^*) - \Delta + \sigma\theta(w_{min}u(q^*_{w_{min}}) - c(q^*_{w_{min}})))$, and $\gamma_e t_e(.) = (\gamma_e - \beta)D_w(q_w^*) + \beta\Delta - \beta\sigma\theta(w_{min}u(q^*_{w_{min}}) - c(q^*_{w_{min}}))$.[30] I check now that the incentive constraint is satisfied:

$$-(\gamma_e - \beta)(\hat{z}_e + t_e(\hat{z}_e, w)) - (\gamma_c - \beta)(\hat{z}_c + t_c) + \beta\sigma\theta(wu(q) - c(q)) + \gamma_c t_c + \gamma_e t_e(\hat{z}_e, w)$$

$$= -(\gamma_e - \beta)D_w(q_w^*) + \beta\sigma\theta(wu(q_w^*) - c(q_w^*)) + \gamma_e t_e(\hat{z}_e, w)$$

$$= \beta\sigma\theta(wu(q_w^*) - c(q_w^*)) + \beta\Delta - \beta\sigma\theta(w_{min}u(q^*_{w_{min}}) - c(q^*_{w_{min}})) \geq 0.$$

The last inequality holds true because $wu(q_w^*) - c(q_w^*)$ is increasing in w, $w \geq w_{min}$ and $\Delta \geq 0$ by assumption. Finally, I verify the planner's budget constraint:

$$\int (t_e + z_e - \gamma_e z_e)dF = \int \big((1-\beta)D^* - \beta\sigma\theta(w_{min}u(q^*_{w_{min}}) - c(q^*_{w_{min}})) + \beta\Delta\big)dF = 0,$$

which holds true by the choice of Δ.[31]

$\square$

[30] It is easy to check that $D_w(q_w^*) + \sigma\theta(w_{min}u(q^*_{w_{min}}) - c(q^*_{w_{min}})) \geq \Delta$, so all types need to bring positive balances from the CM.

[31] This argument will continue to work if only linear transfers (fixed cost and some interest rate) are allowed.

Proof of Proposition 8.

$$K_2(w) \equiv s(w, q_w^*) \geq K_1(w) \equiv \beta\sigma f(w, q_w^*) - \beta\sigma f(w, \bar{q}(w, 1))$$

$$\Leftrightarrow \beta\sigma f(w, \bar{q}(w, 1)) \geq \beta\sigma f(w, q_w^*) - s(w, q_w^*)$$

$$\beta\sigma(wu(\bar{q}(w, 1)) - c(\bar{q}(w, 1))) \geq (1-\beta)D_w(q_w^*) + \beta\sigma(1-\theta)(wu(q_w^*) - c(q_w^*))$$

The RHS is decreasing and the LHS is increasing in θ. As $\theta \to 1$, the LHS is equal to $\{\max_q -(1-\beta)c(q) + \beta\sigma(wu(q) - c(q))\} + (1-\beta)c(\bar{q}(w, 1))$ which is greater than the RHS, $(1-\beta)c(q_w^*)$, for β sufficiently close to 1. This concludes the proof. □

Proof of Proposition 9. I prove this result using three lemmas. First, I show that a necessary condition for co-existence is that agents using CBDC must be indifferent between cash and CBDC (Lemma 2). I then rewrite the planner's problem in a simpler form (Lemma 3). Finally, I write the necessary conditions for optimality and prove the result.

Lemma 2. *Suppose co-existence is optimal; i.e., it is optimal that both cash and CBDC are used. Then it can be assumed without loss of generality that all types who use CBDC are indifferent between using cash and CBDC.*

Proof of Lemma 2. Suppose the constraint is slack in Problem 2 for a strictly positive measure of types, so they strictly prefer to use CBDC. Then, the planner requires these types to bring more balances but the planner keeps $z_e + t_e$ constant for each type. The planner then uses the remaining balances to transfer to other types who use CBDC but do not get to the first best. If all types who use CBDC get their first-best level of consumption, the planner will distribute the remaining balances in the form of cash across everybody. This policy does not reduce welfare, as it does not change the incentives of buyers to bring enough balances. Therefore, we can assume without loss of generality that CBDC users are indifferent between cash and CBDC. □

Lemma 3. *Given γ_c, the planner's problem can be written as follows:*

$$\max_{\tilde{q}(w,\gamma_c), e(w,\gamma_c)} \int \Big[(\beta\sigma f(w, \tilde{q}(w, \gamma_c)) - K)e(w, \gamma_c) + \beta\sigma f(w, \bar{q}(w, \gamma_c))(1 - e(w, \gamma_c))\Big] dF$$

$$s.t. \int \Big(s(w, \tilde{q}(w, \gamma_c)) - K - s(w, \bar{q}(w, \gamma_c))\Big) e(w, \gamma_c) dF + \int (\gamma_c - 1) D_w(\bar{q}(w, \gamma_c)) dF \geq 0,$$

where $\tilde{q}(w, \gamma_c)$ is the buyer of type w's consumption in the DM when the buyer uses CBDC and the cash inflation rate is γ_c.

Proof of Lemma 3. If a type w buyer wants to use CBDC, his payoff from using CBDC must be higher than that from cash. Therefore, for any type who uses CBDC under the inflation rates γ_c and γ_e, there should exist $\epsilon(w) \geq 0$ such that:

$$\underbrace{-(\gamma_e - \beta)(\hat{z}_e + t_e(\hat{z}_e, w)) + \beta\sigma\theta(wu(\tilde{q}) - c(\tilde{q}))) - K + \gamma_e t_e(\hat{z}_e, w) + \gamma_c t_c}_{\text{gains from using CBDC}}$$

$$= \underbrace{-(\gamma_c - \beta)(\hat{z}_c + t_c) + \beta\sigma\theta(wu(\bar{q}) - c(\bar{q})) + \gamma_c t_c}_{\text{gains from using cash}} + \epsilon(w),$$

where $D_w(\bar{q}) = \hat{z}_c + t_c$ and $D_w(\tilde{q}) = \hat{z}_e + t_e(\hat{z}_e) + t_c$.[32] After some algebra, this constraint reduces to:

$$s(w, \tilde{q}) - K - (\gamma_e - 1)\hat{z}_e + t_e(\hat{z}_e) + (1 - \beta)t_c = s(w, \bar{q}) - (\gamma_c - 1)(\hat{z}_c + t_c) + \epsilon(w).$$

Now, consider the constraint of problem 2. Using the fact that each type carries either cash or CBDC, one can write:

$$t_c + \int \Big[\big(t_e(z_e(w), w) - (\gamma_e - 1)z_e(w)\big)e(w, \gamma) - (\gamma_c - 1)z_c(w)(1 - e(w, \gamma)) \Big] dF(w) = 0.$$

Combining the last two equations, one yields

$$\int (s(w, \tilde{q}(w, \gamma_c)) - K - s(w, \bar{q}(w, \gamma_c)))e(w, \gamma_c)dF + \int (\gamma_c - 1)D_w(\bar{q}(w, \gamma))dF$$

$$= \gamma_c t_c + \int \big(-(\gamma_c - \beta)t_c + \epsilon(w)\big)e(w, \gamma)dF.$$

Note that $\epsilon(w) \geq 0$, and that t_c does not show up anywhere in the objective function and in the LHS of this constraint; therefore, we can simply replace this constraint with another one where the LHS is greater than or equal to 0, as $t_c \geq 0$. $\square$

Following Lemma 3, the Lagrangian for the planner's problem can be written as follows:

$$\mathcal{L} = \int \Bigg[\big(\beta\sigma f(w, \tilde{q}(w, \gamma_c)) - K - \beta\sigma f(w, \bar{q}(w, \gamma_c))\big)e(w, \gamma) + \beta\sigma f(w, \bar{q}(w, \gamma_c))$$

$$+\lambda(\gamma_c)\big[\big(s(w, \tilde{q}(w, \gamma_c)) - K - s(w, \bar{q}(w, \gamma_c))\big)e(w, \gamma_c) + (\gamma_c - 1)D_w(\bar{q}(w, \gamma_c))\big]$$

$$+\mu_1(w)e(w, \gamma_c) + \mu_2(w)(1 - e(w, \gamma_c)) \Bigg] dF,$$

[32] The payoff from using CBDC has been written given the fact that if type w wants to use CBDC, he will use only CBDC. This is true due to the following: First, the cost of using CBDC is independent of how many balances the buyer carries. Second, the transfers in CBDC are in the form of a step function, so if a buyer already incurs the cost of CBDC, it is not worth it to have some balances in cash.

where the Lagrangian multipliers are denoted by λ, $\lambda\mu_1$ and $\lambda\mu_2$. Note that CBDC inflation is irrelevant as long as it is sufficiently high. Necessary conditions for optimality imply that:

$$\tilde{q}(w,\gamma) = \arg\max\{\beta\sigma f(w,q) + \lambda(\gamma)s(w,q)\}, \tag{28}$$

resulting in $\tilde{q}(w,1) \leq q_w^*$. Moreover, if a given type uses CBDC with an amount of production less than that if he uses cash, the planner can make zero transfers to this type, distribute the extra balances to CBDC users and have this type use cash. This would increase welfare. Therefore, the production level for type w under CBDC is always greater than that under cash; i.e., $\bar{q}(w,\gamma) \leq \tilde{q}(w,\gamma)$. As a result,

$$\tilde{q}(w,\gamma) \in [\bar{q}(w,\gamma), q_w^*].$$

Other necessary conditions for optimality:

$$e(w,\gamma)=1 \to \beta\sigma f(w,\tilde{q}(w,\gamma)) - K - \beta\sigma f(w,\bar{q}(w,\gamma)) + \lambda(\gamma)\Big[s(w,\tilde{q}(w,\gamma)) - K - s(w,\bar{q}(w,\gamma))\Big] = \mu_2 \geq 0, \tag{29}$$

$$e(w,\gamma)=0 \to \beta\sigma f(w,\tilde{q}(w,\gamma)) - K - \beta\sigma f(w,\bar{q}(w,\gamma)) + \lambda(\gamma)\Big[s(w,\tilde{q}(w,\gamma)) - K - s(w,\bar{q}(w,\gamma))\Big] = -\mu_1 \leq 0 \tag{30}$$

Now, by way of contradiction, assume that $e(w_1,\gamma_c)=1$, $e(w_2,\gamma_c)=0$, for $w_1 < w_2$. I show below that the following is increasing in w:

$$\beta\sigma f(w,\tilde{q}(w,\gamma)) - K - \beta\sigma f(w,\bar{q}(w,\gamma)) + \lambda(\gamma)\big(s(w,\tilde{q}(w,\gamma)) - K - s(w,\bar{q}(w,\gamma))\big).$$

But this is in contradiction with the necessary conditions for optimality. The above term is equal to:

$$\int_{\bar{q}(w,\gamma)}^{\tilde{q}(w,\gamma)} \big[f_q(w,q) + \lambda(\gamma)s_q(w,q)\big]dq - K(1+\lambda(\gamma)).$$

As shown above, $\bar{q}(w,\gamma_c) \leq \tilde{q}(w,\gamma_c)$ for all w for which $e(w,\gamma_c)=1$. One needs to show that the derivative of the above expression with respect to w is positive. Therefore, it suffices to show

$$\int_{\bar{q}(w,\gamma)}^{\tilde{q}(w,\gamma)} \big[\underbrace{f_{wq}(w,q)}_{\geq 0} + \lambda(\gamma_c)\underbrace{s_{wq}(w,q)}_{\geq 0}\big]dq$$

$$+\frac{\partial\tilde{q}(w,\gamma_c)}{\partial w}\int_{q_0}^{\tilde{q}(w,\gamma_c)} \big[f_q(w,q) + \lambda(\gamma_c)s_q(w,q)\big]dq - \frac{\partial\bar{q}(w,\gamma_c)}{\partial w}\int_{q_0}^{\bar{q}(w,\gamma_c)} \big[f_q(w,q) + \lambda(\gamma_c)s_q(w,q)\big]dq \geq 0.$$

Define $\tilde{\alpha}$ and $\bar{\alpha}$ as follows:

$$\tilde{q}(w,\gamma) = Q(\tilde{\alpha}w) \text{ where } \tilde{\alpha} \equiv \frac{-(1-\beta)(1-\theta) + \beta\sigma(\theta + 1/\lambda(\gamma))}{(1-\beta)\theta + \beta\sigma(\theta + 1/\lambda(\gamma))},$$

$$\bar{q}(w,\gamma) = Q(\bar{\alpha}w) \text{ where } \bar{\alpha} \equiv \frac{-(\gamma-\beta)(1-\theta)+\beta\sigma\theta}{(\gamma-\beta)\theta+\beta\sigma\theta}.$$

But $\tilde{\alpha} \geq \bar{\alpha}$ as $\bar{q}(w,\gamma) \leq \tilde{q}(w,\gamma)$. It is shown in Lemma 4 below that $\alpha Q'(\alpha w)$ is increasing in α, so $\tilde{\alpha}Q'(\tilde{\alpha}w) \geq \bar{\alpha}Q'(\bar{\alpha}w)$. Therefore,

$$\Rightarrow \frac{\partial \tilde{q}(w,\gamma)}{\partial w} = \frac{\partial Q(\tilde{\alpha}w)}{\partial w} = \tilde{\alpha}Q'(\tilde{\alpha}w) \geq \bar{\alpha}Q'(\bar{\alpha}w) = \frac{\partial Q(\bar{\alpha}w)}{\partial w} = \frac{\partial \bar{q}(w,\gamma)}{\partial w}.$$

Finally, $\tilde{q}(w,\gamma) \geq \bar{q}(w,\gamma)$, $Q'(.)$ is positive, and $f_q(w,q)+\lambda(\gamma)s_q(w,q) > 0$ for $q < \tilde{q}(w,\gamma)$, so the inequality is established. □

Lemma 4. *If* $\frac{rQ''(r)}{Q'(r)} \geq -1$, *then* $\alpha Q'(\alpha w)$ *is increasing in* α.

Proof.

$$\frac{\partial}{\partial \alpha}\Big(\alpha Q'(\alpha w)\Big) \geq 0 \Leftrightarrow Q'(\alpha w) + \alpha w Q''(\alpha w) \geq 0 \Leftrightarrow -\frac{\alpha w Q''(\alpha w)}{Q'(\alpha w)} \leq 1.$$

□

Derivation of the constraint of the planner's problem in the two-type example

The first constraint is simplified to:

$$t_{e2} = (\gamma_e - 1)z_{e2} + (1-\pi_2)/\pi_2(\gamma_c - 1)D_{w_1}(q_1) - ((1-\pi_2)\gamma_c + \pi)t_c/\pi,$$

where z_{c1} is substituted out from the third constraint. Using the other constraints, one yields:

$$t_c + z_{e2} + (\gamma_e - 1)z_{e2} + (1-\pi_2)/\pi_2(\gamma_c - 1)D_{w_1}(q_1) - ((1-\pi_2)\gamma_c + \pi)t_c/\pi = D_{w_2}(q_2)$$

$$\Rightarrow \gamma_e z_{e2} = D_{w_2}(q_2) - (1-\pi_2)/\pi_2(\gamma_c - 1)D_{w_1}(q_1) + (1-\pi_2)\gamma_c t_c/\pi_2$$

$$\Rightarrow \gamma_e t_{e2} = (\gamma_e - 1)D_{w_2}(q_2) + (1-\pi_2)/\pi_2(\gamma_c - 1)D_{w_1}(q_1) - ((1-\pi_2)\gamma_c + \pi_2)t_c/\pi_2.$$

Proof of Proposition 10. Assume that $\gamma_c \leq 1$. Then,

$$\max_q\{-(\gamma_c - \beta)D_{w_2}(q) + \beta\sigma\theta(w_2 u(q) - c(q))\}$$

$$\geq \max_q\{-(1-\beta)D_{w_2}(q) + \beta\sigma\theta(w_2 u(q) - c(q))\}$$

$$> -(1-\beta)D_{w_2}(q_2) + \beta\sigma\theta(w_2 u(q_2) - c(q_2)) - K + (1-\pi_2)/\pi_2(\gamma_c - 1)D_{w_1}(q_1) - \gamma_c t_c/\pi_2,$$

where the first inequality comes from $\gamma_c \leq 1$, and the second inequality comes from $K > 0$ and $t_c \geq 0$. This is a contradiction with the optimality of co-existence, because type w_2 must use CBDC (by Proposition 9), but it it is shown here that type w_2 is better off using cash. □

Proof of Proposition 11. $\bar{B}(\pi)$ is a continuous function of π around $\pi = 1$, using the theorem of the maximum. $E(\pi)$ is also continuous. By Assumption 2, $\bar{B}(1) < E(1)$. Therefore, there exists a neighborhood around $\pi = 1$ such that $\bar{B}(\pi) < E(\pi)$ for $\pi > \tilde{\pi}$ where $\tilde{\pi} \in (0, 1)$. This completes the proof. □

Proof of Preposition 12. If condition (25) is satisfied, then the first best can be achieved using the following policy: $\gamma_c = \beta$, γ_e is set sufficiently high ($\gamma_e > 2 - \beta$), and

$$t_e = \begin{cases} (1 - \frac{(2-\beta)}{\gamma_e})D(q^*) & z_e > (2-\beta)/\gamma_e D(q^*) \\ 0 & \text{otherwise} \end{cases}.$$

□

Proof of Proposition 13. **Proposition 13 (a):**

Suppose by way of contradiction that cash is not valued. I propose the following allocation and show that it achieves higher welfare: $q_e = q^*$, and $z_c = \epsilon > 0$ where ϵ is chosen sufficiently close to 0 that (24) and (23) are satisfied. Since $\alpha_b = 0$, (23) is reduced to

$$-(\gamma_e - \beta)z_e + \beta t_e + \beta\sigma\theta\alpha_e f(q_e) \geq 0.$$

Since in the proposed allocation, $q_e = q^*$, so we must have $z_e + t_e = D(q^*)$. Together with (22), this implies $\gamma_e z_e + (\gamma_c - 1)z_c = D(q^*)$; therefore,

$$-(1-\beta)D(q^*) + \beta\sigma\theta\alpha_e f(q^*) \geq (1 - \gamma_c)z_c. \tag{31}$$

By assumption, the LHS is strictly positive, so if z_c is sufficiently small, this constraint is satisfied. Finally, we need to check that $z_c > 0$ is a solution to $\max\{-(\gamma_c - \beta)z_c + \beta\sigma\theta\alpha_c f(q)\}$. This condition is satisfied if $-(\gamma_c - \beta)(1-\theta) + \beta\sigma\theta\alpha_c > 0$, but γ_c can be sufficiently close to β for this inequality to hold.

In this allocation, the first best is achieved in CBDC meetings, and the production level in cash meetings is strictly positive. Therefore, the welfare level in this allocation is above the welfare level in any allocation in which the production level in cash meetings is zero, and this production level is zero only if cash is not valued.

Now assume that CBDC is not valued in the optimal allocation. Consider the same allocation but with $q'_e = \epsilon$. Equation (31) must hold with ϵ replacing q^*. If $\gamma_c > 1$, then this equation holds because its LHS is positive, following the facts that (25) holds and also $\epsilon < q^*$. If $\gamma_c \leq 1$, since $\gamma_c \geq \beta$ and $z_e < q^*$, then if ϵ is set sufficiently small, this inequality holds following (25).

Finally, it is not possible that neither cash nor CBDC is valued, because any of the allocations valued above yield strictly higher welfare than zero. This completes the proof.

Proposition 13 (b):

In this proof, I assume for simplicity that $\alpha_e = 0$. The proof can be easily extended for $\alpha_e > 0$. I write the LHS of (23) in the following format:

$$\max_{z_c} \big\{-(\gamma_c-\beta)(z_c+z_e+t_e)+\beta\sigma\theta\big(\alpha_c f(q_c)+\alpha_b f(q_b)\big)\big\}+(\gamma_c-\beta)(z_e+t_e)-(\gamma_e-\beta)z_e+\beta t_e(z_e)+\beta\sigma\theta\alpha_e f(q_e).$$

Assuming that $q_b \leq q^*$, we will have $z_c + z_e + t_e = D(q_b)$. Next, using (22), the LHS of (23) can be written as: $\max_{z_c} \big\{-(\gamma_c-\beta)(z_c+z_e+t_e)+\beta\sigma\theta\big(\alpha_c f(q_c)+\alpha_b f(q_b)\big)\big\}+(\gamma_c-1)D(q_b)$.

Now, I write the RHS of (23) as follows:

$$\begin{aligned}
&\max_{q} \big\{-(\gamma_c-\beta)D(q)+\beta\sigma\theta(\alpha_c+\alpha_b)f(q)\big\}\\
&\geq \max_{z_c} \big\{-(\gamma_c-\beta)(z_c+z_e+t_e)+\beta\sigma\theta(\alpha_c+\alpha_b)f(D^{-1}(z_c+z_e+t_e))\big\}\\
&> \max_{z_c}\{-(\gamma_c-\beta)(z_c+z_e+t_e)+\beta\sigma\theta(\alpha_c f(D^{-1}(z_c))+\alpha_b f(D^{-1}(z_c+z_e+t_e)))\}\\
&= \max_{z_c}\{-(\gamma_c-\beta)(z_c+z_e+t_e)+\beta\sigma\theta\big(\alpha_c f(q_c)+\alpha_b f(q_b)\big)\big\},
\end{aligned}$$

where the first inequality is derived by optimality of q and the second one is obtained by the fact that $z_e + t_e > 0$. Altogether, the constraint implies that $(\gamma_c - 1)D(q_b) > 0$, and consequently $\gamma_c > 1$. □

Proof of Proposition 14. The constraint of the planner's problem can be written as:

$$\max_{z_c} \big\{-(\gamma_c-\beta)z_c+\beta\sigma\theta\big(\alpha_c f(q_c)+\alpha_b f(q_b)\big)\big\}+(\gamma_c-1)z_c-(1-\beta)(z_e+t_e)$$

$$\geq \max_{q} \big\{-(\gamma_c-\beta)D(q)+\beta\sigma\theta(\alpha_c+\alpha_b)f(q)\big\}.$$

By way of contradiction, assume that for any $\alpha_b > 0$, CBDC is valued in the optimal allocation. Now, I propose another allocation in which cash inflation and the real CBDC balances in the DM (post-transfers) are slightly lower, but the value of the objective function is higher:

$$\gamma_c' = \gamma_c - \Delta_1,$$

$$z_e' + t_e' = z_e + t_e - \Delta_2.$$

Since $\alpha_e = 0$, the first best cannot be achieved following Proposition 12. If $\gamma_c = \beta$, then buyers can buy the first level of DM production by carrying only cash; therefore, we must have $\gamma_c > \beta$. Also, CBDC is valued, so $z_e + t_e > 0$. Hence, the proposed allocation and policy with γ_c' (and sufficiently high γ_e) are feasible.

I argue here that if α_b is sufficiently small (or, equivalently, α_c is sufficiently large), then sufficiently small values for Δ_1 and Δ_2 can be found that the welfare level is higher in the proposed allocation relative to the original allocation and the constraint continues to be satisfied. If Δ_1 and Δ_2 are small, then one can calculate the changes in the LHS and RHS of the constraint by simply taking a derivative. Hence, we need to show the following:

$$-\Big(-z_c+z_c+(\gamma_c-1)D'(q_1)\frac{\partial q_1}{\partial \gamma_c}\Big)\Delta_1-\Big(\beta\sigma\theta\alpha_b f'(q_b)\frac{\partial q_b}{\partial(z_e+t_e)}+(\gamma_c-1)\frac{\partial z_c}{\partial(z_e+t_e)}-(1-\beta)\Big)\Delta_2 \geq D(\tilde{q})\Delta_1,$$

where

$$\tilde{q} \equiv \arg\max_q \big\{-(\gamma_c-\beta)D(q)+\beta\sigma\theta(\alpha_c+\alpha_b)f(q)\big\}.$$

For writing this inequality, I differentiated both sides with respect to γ_c and (z_e+t_e), and I also used the envelope theorem. After simple algebra, we need to show:

$$\Big((1-\beta)-\beta\sigma\theta\underbrace{\alpha_b}_{\text{small}}f'(q_b)\frac{\partial q_b}{\partial(z_e+t_e)}-(\gamma_c-1)\underbrace{\frac{\partial z_c}{\partial(z_e+t_e)}}_{\text{small}}\Big)\Delta_2 \geq \Big(D(\tilde{q})+(\gamma_c-1)D'(q_1)\frac{\partial q_1}{\partial\gamma_c}\Big)\Delta_1.$$

If α_b is set sufficiently small, then the coefficient in the LHS will become positive, because z_c is not affected by much when α_b is small. Therefore, we can find a sufficiently small value for Δ_1/Δ_2 that this inequality holds. The objective function has now increased, since the dominant term is $\alpha_c f(q_c)$ (because α_b is small), and q_c has increased (because cash inflation is lower). This is a contradiction because we could find a feasible solution with a higher value for the objective function. This completes the proof.

Proof of part (b) is similar to (and easier than) part (a), so I skip it. Assumption $u'(0) < \infty$ is needed to ensure that when agents' CBDC balances are reduced, the reduction in the utility level is bounded. □

On the Calibration Exercise

I elaborate on some details of the calibration exercise in Section 6. As a reminder regarding the notation, we have:

$$O(w,\gamma) \equiv \max_q\{-(\gamma-\beta)D_w(q)+\beta\sigma\theta(wu(q)-c(q))\}, \tag{32}$$

$$\bar{q}(w,\gamma) \equiv \arg\max_q\{-(\gamma-\beta)D_w(q)+\beta\sigma\theta(wu(q)-c(q))\}. \tag{33}$$

Denote

$$i = \frac{\gamma-\beta}{\beta}.$$

One can write the first-order condition for a given inflation rate as:

$$(-i(1-\theta)+\sigma\theta)wu'(q)-(i+\sigma)\theta c'(q)=0.$$

Assume the following functional forms:

$$u(q)=q^{1-\eta}, c(q)=c_1 q.$$

Hence, $\bar{q}(w,\gamma)$ is given by:

$$\bar{q}(w,\gamma)=\left(\frac{(-i(1-\theta)+\sigma\theta)w(1-\eta)}{(i+\sigma)\theta c_1}\right)^{1/\eta}.$$

Also,

$$z(w,i)\equiv D_w(\bar{q}(w,\gamma))=(1-\theta)wu(\bar{q})+\theta c(\bar{q})=\bar{q}((1-\theta)w\bar{q}^{-\eta}+\theta c_1)$$
$$\Rightarrow z(w,i)=\bar{q}(w,\gamma)\theta c_1\left(\frac{(i+\sigma)(1-\theta)}{(-i(1-\theta)+\sigma\theta)(1-\eta)}+1\right).$$

Price over marginal cost in the DM is thus given by

$$\mu(w,i)=\frac{z(w,i)}{\bar{q}(w,\gamma)c_1}=\frac{(i+\sigma)(1-\theta)}{(-i(1-\theta)+\sigma\theta)(1-\eta)}+1.$$

The production function in the CM was assumed to be linear in the benchmark model. However, with the linear function, the level of production in the CM is indeterminate. To eliminate this indeterminacy, and following the literature, I assume for this empirical exercise that the production function in the CM is $U(X)=Aln(X)$. This implies that the level of production in the CM is $X^*=A$. The parameters (η,A,θ) are estimated using the standard method developed by Lucas (2001). The idea is to use the relationship between the nominal interest rate and money demand; i.e., $L(i)=M/(PY)$. Assume the population is composed of two types, w_1 and w_2. In the benchmark model,

$$\hat{L}(i)=\frac{M/P}{Y}=\frac{\pi_1 z_1(w_1,i)+\pi_2 z_2(w_1,i)}{A+\sigma(\pi_1 z_1(w_1,i)+\pi_2 z_2(w_2,i))}=\left(\frac{A}{\pi_1 z_1(w_1,i)+\pi_2 z_2(w_2,i)}+\sigma\right)^{-1}. \quad (34)$$

The parameters are estimated by minimizing the distance between the data- and model-generated money demand. Hence, the following problem is solved:

$$\min_{(\eta,A,\theta)} \quad \textstyle\sum_{t=1}^{T}(L(i_t)-\hat{L}(i_t))^2 \quad (35)$$
$$\text{s.t.} \quad \pi_1\mu(w_1,i_0)+\pi_2\mu(w_2,i_0)=\mu_0, \quad (36)$$

where $L(i_t)$ denotes the $M/(PY)$ from the data at time t. $\hat{L}(i_t)$ is calculated from (34) using the nominal interest rate at time t. The markup at 2% inflation rate (or $i_0\approx 5\%$) is set to $\mu_0=1.10$.

I use M1 to represent M, nominal GDP to represent PY, and the rate on the three-month T-bill to represent i. For the US, the data span 1900 to 2000 and are taken from Craig and Rocheteau (2006). They use the same data set that Lucas used; however, Lucas had included data until only 1994, so Craig and Rocheteau (2006) extended the data set to include data up to 2000. For Canada, the data are from CANSIM. I use series v41552795 for M1 (which includes currency outside banks, chartered bank chequable deposits, less inter-bank chequable deposits). I use series v646937 for nominal GDP. The rate on the three-month T-bill is taken from Table 176-0043. The data span 1967 to 2008. I did not include earlier dates for Canada, because the M1 data series was discontinued and there were some inconsistencies between earlier versions and the current one.

It is worth mentioning that the length of data used here for Canada is short, so a longer data horizon would provide more reliable estimates. The problem with inclusion of data from the distant past is that it is likely that the underlying parameters of the environment become different from the parameters of the environment now, so the estimates may not be useful for any counter-factual analysis; e.g., the estimated cost of 10% inflation. This is not only a general point that may be true in many empirical settings, but it is specifically true in this economy as $M/(PY)$ has decreased from the pre-1980 period to the post-1980 period, because of extensive usage of electronic means of payment (like credit cards) and less demand to hold currency or its equivalents for transaction purposes.