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Volatility Risk and Economic Welfare

by Shaofeng Xu

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Shaofeng Xu

Canadian Economic Analysis Department
Bank of Canada
Ottawa, Ontario, Canada K1A 0G9
sxu@bankofcanada.ca

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Abstract

This paper examines the effects of time-varying volatility on welfare. I construct a tractable endogenous growth model with recursive preferences, stochastic volatility, and capital adjustment costs. The model shows that a rise in volatility can decelerate growth in the absence of any level shocks. In contrast to level risk, which is always welfare reducing for a risk-averse household, volatility risk can increase or decrease welfare, depending on model parameters. When calibrated to U.S. data, the model finds that the welfare cost of volatility risk is largely negligible under plausible model parameterizations.

Bank topics: Business fluctuations and cycles; Economic models

JEL codes: E2; E3

Résumé

La présente étude traite des effets sur le bien-être de la volatilité qui varie dans le temps. L'auteur construit un modèle à croissance endogène résoluble qui intègre des préférences récursives, une volatilité stochastique et des coûts d'ajustement du capital. Ce modèle montre qu'une hausse de la volatilité peut ralentir la croissance en l'absence de chocs de niveau. Contrairement au risque de niveau, qui conduit toujours à une perte de bien-être pour les ménages réfractaires au risque, le risque de volatilité peut faire augmenter ou diminuer le bien-être, selon les paramètres du modèle. Lorsque le modèle est calibré en fonction des données se rapportant aux États-Unis et nourri de paramètres plausibles, les résultats indiquent que la perte de bien-être liée au risque de volatilité est somme toute négligeable.

Sujets : Cycles et fluctuations économiques ; Modèles économiques

Codes JEL : E2 ; E3

Non-Technical Summary

Since the late 2000s, there has been a growing literature that highlights time-varying volatility as a driver of business cycles. For example, studies have shown that the massive increase in uncertainty following the recent financial crisis was an important factor driving the depth of the Great Recession. The literature has so far mainly focused on the short-run real effects of volatility shocks. Little is known about the long-run welfare effects of volatility risk. Although it is widely accepted that larger uncertainty decreases growth and welfare, studies of the welfare impact of volatility risk, which implies the possibility of experiencing both high and low volatility, have been largely absent.

This paper aims to fill part of the gap by analyzing the relationship between volatility risk and economic welfare in an analytically tractable growth model. The main findings of the paper are twofold. First, I show that in contrast to level risk, which is always welfare reducing for a risk-averse household, volatility risk can increase or decrease welfare depending on model parameters, such as the magnitude of risk aversion. Second, as far as the U.S. economy is considered, the welfare impact of volatility risk is largely negligible. The calibrated model estimates that the welfare cost of volatility risk is equivalent to a 0.0062 percent decrease in annual consumption.

1 Introduction

Since the late 2000s, there has been a growing literature that highlights time-varying volatility as an important driver of business cycles.[1] For example, studies have shown that the massive spike in uncertainty following the recent financial crisis played a notable role in accounting for the depth of the Great Recession. The literature has so far mainly focused on the short-run real effects of volatility shocks. Little is known about the long-run welfare effects of volatility risk. Although it is widely accepted that larger uncertainty decreases growth and welfare, studies of the welfare impact of volatility risk, which implies the possibility of experiencing both high and low volatility, have been largely absent.

This paper aims to fill part of the gap by analyzing the relationship between volatility risk and economic welfare in a tractable continuous-time stochastic endogenous growth model. The model economy consists of infinitely lived households and firms. The representative household has recursive preferences over a consumption good, and makes dynamic consumption-saving decisions. The recursive utility disentangles the effects of risk aversion and intertemporal elasticity of substitution (IES). The representative firm produces the consumption good using a production technology that is linear in capital, and is subject to quadratic capital adjustment costs. The firm faces uncertainty about the capital depreciation rate. Capital depreciation shocks exhibit stochastic volatility, which follows a mean-reverting Cox-Ingersoll-Ross (CIR) process. In the paper, risks associated with shocks to the capital depreciation rate and shocks to the volatility are denoted as level risk and volatility risk, respectively.

The Pareto optimal allocation of the economy is solved in closed form up to a second-order ordinary differential equation (ODE). The model shows explicitly that a temporary rise in volatility can decelerate growth in the absence of any level shocks. In the face of elevated uncertainty about future capital returns, a risk-averse household becomes less willing to invest in the firm, which slows the pace of capital accumulation and output growth. The paper measures the welfare cost of volatility risk as the maximum fraction of annual consumption that the representative household accepts to give up in return for eliminating volatility shocks. I also derive in a similar manner the welfare costs of level risk and total uncertainty in production.

The main findings from the welfare analysis are twofold. First, the model shows that in contrast to level risk, which is always welfare reducing for a risk-averse household, volatility risk can increase or decrease welfare, depending on model parameters. When the IES equals unity, I prove that volatility risk is welfare neutral if the coefficient of relative risk aversion is equal to one, i.e., the case of log utility, but it would increase (decrease) welfare if the coefficient is smaller (greater) than one. The intuition behind the results is as follows. The existence of level risk generates fluctuations in capital depreciation rates and thus the growth

[1] This paper uses the terms "volatility" and "uncertainty" interchangeably, both meaning "variance".

rates of capital and consumption. Therefore, level risk decreases the utility of a risk-averse household. By contrast, when the IES equals one, the conditional expected growth rate of capital is a constant, and volatility risk affects capital accumulation by changing the dispersion of capital depreciation shocks. This introduces time-variation in the investment opportunity set for capital. Since a household with logarithmic preferences is indifferent to fluctuations in the investment opportunity set, volatility risk has no welfare impact. If the household is less risk averse than logarithmic households, it becomes willing to trade off periods of low welfare associated with high volatility for periods of high welfare associated with low volatility, and the household views the presence of volatility shocks welfare beneficial. By the same token, if the household is more risk-averse than logarithmic households, volatility risk becomes welfare reducing. When the IES differs from unity, numerical results further confirm that the welfare impact of volatility risk can be in any direction. However, its impact is dominated by that of level risk, resulting in an overall reduction in welfare due to uncertainty in production.

The second finding is that the welfare cost of volatility risk is largely negligible for the U.S. economy. I calibrate the model to match important moments in the U.S. data between 1956 and 2014. The parameterized model estimates that the representative household would willingly give up 2.2 percent of its annual consumption to eliminate the level risk, but only 0.0062 percent of consumption to remove the volatility risk. I further conduct a sensitivity analysis of welfare calculations, and find that as far as the U.S. economy is considered, the welfare impact of volatility risk is very limited under plausible model parameterizations.

This paper is mostly related to two strands of the literature. The first strand examines channels through which changes in uncertainty affect economic activity. Examples include Bloom (2009), Fernández-Villaverde et al. (2011), Arellano et al. (2012), Basu and Bundick (2012), Bloom et al. (2012), Christiano et al. (2014), Fernández-Villaverde et al. (2015), Leduc and Liu (2016), and Xu (2016). These studies in general find a negative impact of uncertainty on economic activity, emphasizing, for example, its real-options effect, risk-premium effect, or precautionary-savings effect on the choices of economic agents. My paper contributes to this literature along two important dimensions. First, the literature focuses on the short-run real effects of changes in uncertainty, and to my best knowledge, this paper is the first attempt to examine the long-run welfare impact of time-variation in uncertainty. Second, in contrast to existing studies mainly using quantitative dynamic stochastic general equilibrium (DSGE) models, the paper provides an analytically tractable model that allows for an intuitive characterization of the impact of volatility risk.

This paper is also related to the literature investigating the welfare cost of business cycles. Lucas (1987) argues that the welfare gain from eliminating consumption fluctuations is negligible. Since then, a large body of subsequent studies has examined whether relaxing the assumptions in Lucas's computation might result in business cycles being more costly. Examples include Imrohoroğlu (1989), Obstfeld (1994), Campbell and Cochrane (1999), Krusell and Smith (1999), Tallarini (2000), Otrok (2001), Epaularda and Pommeret (2003), Barro (2009), Krusell et al. (2009), and Pindyck and Wang (2013). These papers typically find a larger welfare cost of business cycles than Lucas (1987) with various degrees of magnitude. By contrast, recent studies by Lester et al. (2014) and Cho et al. (2015) show that welfare can be higher in a more volatile economy, since agents might make purposeful use of uncertainty in their favor. The current paper differs in nature from the cost-of-business-cycle literature in that it studies the welfare effects of higher-frequency two-sided movements in volatility, whereas the literature focuses on that of more secular one-sided changes in volatility. More precisely, the literature examines the welfare gain from permanently decreasing volatility from a certain level to zero, and thus the welfare impact studied therein essentially corresponds to the welfare cost of level risk considered in my paper.

The remainder of this paper is organized as follows. Section 2 describes the benchmark model. Section 3 solves the model analytically, and characterizes the effects of a temporary rise in volatility. Section 4 analyzes the welfare implications of volatility risk. Section 5 calibrates the model to U.S. data, and estimates the welfare cost of volatility risk. Section 6 concludes. Technical details are relegated to the appendix.

2 The Model

To provide an economic structure for the welfare analysis, this section lays out a continuous-time endogenous growth model with recursive preferences and stochastic volatility in production.

2.1 Preferences

The economy is populated by a continuum of identical infinitely lived households of total mass equal to one. The representative household values a consumption process $\{C_t\}$ according to a recursive utility function introduced by Duffie and Epstein (1992), which is a continuous-time analog of the discrete-time recursive utility proposed by Epstein and Zin (1989):

$$V_t = E_t\left[\int_t^{\infty} f\left(C_s, V_s\right) ds\right], \tag{1}$$

where the function f is a normalized aggregator of the form

$$f\left(C, V\right) = \begin{cases} \rho\left(\log C - V\right) & \text{if } \psi = 1 \text{ and } \gamma = 1 \\ \rho\left(1-\gamma\right)V\left(\log C - \frac{1}{1-\gamma}\log\left(\left(1-\gamma\right)V\right)\right) & \text{if } \psi = 1 \text{ and } \gamma \neq 1 \\ \rho\frac{1-\gamma}{1-\psi^{-1}}V\left(\frac{C^{1-\psi^{-1}}}{\left(\left(1-\gamma\right)V\right)^{\left(1-\psi^{-1}\right)/\left(1-\gamma\right)}} - 1\right) & \text{if } \psi \neq 1 \end{cases}. \tag{2}$$

Here ρ is the rate of time preference, ψ is the IES, and γ is the coefficient of relative risk aversion. The recursive utility function (1) generalizes the standard time-additive constant-relative-risk-aversion (CRRA) expected utility function by separating the risk aversion from the IES, and nests the CRRA utility as a special case for $\gamma = \psi^{-1}$, in which case equation (1) takes the familiar form $V_t = \rho E_t \left[\int_t^\infty e^{-\rho(s-t)} \frac{C_s^{1-\gamma}}{1-\gamma} ds \right]$. As will be clear later, the separation of the risk aversion from the IES is important since it allows one to disentangle the individual effects of the two parameters. It also helps the model generate plausible impulse responses to volatility shocks. In the economy, the household forms its expectation based on information available at a given time t, i.e., $E_t[\cdot] = E[\cdot \,|\mathcal{F}_t]$, where the information sets over time are represented by a filtration $\{\mathcal{F}_t\}$ associated with a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, with $\mathcal{F}$ and $\mathcal{P}$ denoting, respectively, the σ-algebra on Ω and the probability measure on $\mathcal{F}$.

2.2 Production

There is a representative firm endowed with an AK production technology of the form

$$Y_t = AK_t, \tag{3}$$

where A is a positive productivity parameter, and capital K_t is the sole factor for producing output Y_t that can be used for both consumption C_t and investment I_t. The capital stock of the firm is assumed to evolve as

$$dK_t = \Phi\left(I_t, K_t\right) dt + K_t \sqrt{v_t} dB_{1t}, \tag{4}$$

where B_{1t} is a standard Brownian motion in $\mathbb{R}$ driving capital depreciation shocks.[2] Except for time-varying volatility, technology specifications in (3) and (4) are common in the literature, such as Epaularda and Pommeret (2003), Barro (2009), and Pindyck and Wang (2013).

The term $\Phi\left(I_t, K_t\right)$ in (4) represents the flow of net investment defined as

$$\Phi\left(I_t, K_t\right) = I_t - Q\left(I_t, K_t\right), \tag{5}$$

where $Q\left(I_t, K_t\right)$ denotes capital adjustment costs. I assume that $Q\left(I, K\right)$ takes the form

$$Q\left(I, K\right) = q\left(i\right) K,\; q\left(i\right) = \frac{1}{2}\kappa i^2, \tag{6}$$

[2]The specification of capital depreciation shocks helps keep the model analytically tractable. This type of shock has been used as a reduced-form way to model business cycles in the literature, e.g., Epaularda and Pommeret (2003), Barro (2009), Gertler and Karadi (2011), and Pindyck and Wang (2013). Capital depreciation shocks are realistic for wars or natural disasters, but they can also be interpreted as shocks to the quality of existing capital by affecting the effective units of capital brought from the previous period.

where the parameter κ is non-negative, and the variable $i = \frac{I}{K}$ measures the investment-capital ratio. For expositional simplicity, one can rewrite the function Φ in (5) as

$$\Phi(I, K) = \phi(i) K, \ \phi(i) = i - \frac{1}{2}\kappa i^2. \tag{7}$$

The variable v_t in (4), which denotes the variance of capital depreciation shocks, follows a mean-reverting CIR process proposed by Cox et al. (1985):[3]

$$dv_t = \alpha(m - v_t) dt + \beta\sqrt{v_t} dB_{2t}, \tag{8}$$

where B_{2t} is another standard Brownian motion in $\mathbb{R}$, and parameters α, m, and β denote, respectively, the mean-reverting rate, the mean volatility level, and the volatility of volatility.

The two random variables B_{1t} and B_{2t} characterize, respectively, the level risk and the volatility risk in capital accumulation, since shocks $\{dB_{1t}\}$ in (4) directly determine the level of future capital, whereas shocks $\{dB_{2t}\}$ in (8) indirectly affect capital accumulation by changing the dispersion of level shocks. In the current analysis, I assume that level shocks and volatility shocks are independent, so the paper focuses on connections between volatility shocks and economic activity that arise through the model's internal economic structure rather than through purely statistical channels between the two types of shocks. Denote

$$\xi = \frac{2\alpha m}{\beta^2} \text{ and } \theta = \frac{\beta^2}{2\alpha}. \tag{9}$$

As shown in Cox et al. (1985), when $\alpha > 0$ and $m > 0$, the CIR process in (8) stays non-negative after starting with a positive value, and has a stationary Gamma distribution with the shape parameter ξ and the scale parameter θ. The associated stationary density function is

$$\eta(v) = \frac{e^{-\frac{v}{\theta}} v^{\xi-1}}{\Gamma(\xi)\theta^{\xi}}, \tag{10}$$

where $\Gamma(x) = \int_0^{\infty} e^{-z} z^{x-1} dz$ represents the Gamma function. The unconditional mean and variance of $\{v_t\}$ in the stationary distribution (10) equal m and $\frac{m\beta^2}{2\alpha}$, respectively.

2.3 The Social Planner's Problem

Since the focus of this paper is on the welfare impact of volatility risk, it is convenient to work directly with the social planner's problem. The Pareto optimal allocation of the economy is that

[3] Apart from its analytical tractability, the CIR process in (8) has been broadly adopted in finance to capture the dynamics of stock price volatility, one important measure of macroeconomic uncertainty proposed by the existing literature of uncertainty shocks, and thus is a reasonable volatility specification.

given initial capital stock K and volatility v, the social planner chooses an optimal consumption process to maximize the representative household's utility given by (1) and (2), subject to feasibility constraints (3), (4) and (8). Let $J : \mathbb{R}_2 \to \mathbb{R}$, $(K, v) \mapsto J(K, v)$ and $C : \mathbb{R}_2 \to \mathbb{R}$, $(K, v) \mapsto C(K, v)$ be the value function and the consumption function of the social planner's problem, which is characterized by the following Hamilton-Jacobi-Bellman (HJB) equation:[4]

$$0 = \max_{C} \left\{ f(C, J) + \Phi(I, K) J_K + \alpha(m - v) J_v + \frac{1}{2} v K^2 J_{KK} + \frac{1}{2}\beta^2 v J_{vv} \right\}. \tag{11}$$

The right-hand side of (11) is the maximum sum of the household's utility flow $f(C, J)$ and the instantaneous changes in its value function. The optimality of consumption requires the sum to be zero. Among those instantaneous changes in the value function, terms $\Phi(I, K) J_K$ and $\frac{vK^2}{2} J_{KK}$ together measure the marginal increase of the household's value function due to a unit increase in the capital stock, whereas the sum of terms $\alpha(m - v) J_v$ and $\frac{\beta^2 v}{2} J_{vv}$ represents the effects of a unit increase in the volatility level on the household's value function.

3 Solving the Model

This section solves the social planner's problem (11), and characterizes how the economy would respond to a temporary increase in uncertainty.

3.1 The Analytical Solution

I first present a lemma that will be used to solve the model.

Lemma 1 *For a given $x \in (0, \infty)$, there exists a unique number $y_\psi(x) \in \left(-\infty, \frac{1}{\kappa}\right)$ satisfying the following equation for y:*

$$A - y = (1 - \kappa y)^{-\psi} x. \tag{12}$$

The associated solution function $y_\psi(x)$ decreases with x.[5]

The following proposition derives an analytical solution to problem (11).

Proposition 1 *The HJB equation in (11) is solved by the value function $J : \mathbb{R}_2 \to \mathbb{R}$ and the consumption function $C : \mathbb{R}_2 \to \mathbb{R}$ such that*

1. *If $\psi = 1$ and $\gamma = 1$, then*

$$J(K, v) = \log K - \frac{1}{2(\rho + \alpha)} v + \frac{1}{\rho}\left(\mu - \frac{\alpha m}{2(\rho + \alpha)}\right), \; C(K, v) = (A - i_1^*) K, \tag{13}$$

[4]For a given function $u(x, y)$, I use u_x, u_y, u_{xx} and u_{yy} to denote derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$, respectively.
[5]The dependence of $y_\psi(x)$ on A and κ is omitted for notational simplicity.

where parameters

$$i_1^* = y_1(\rho), \ \mu = \rho \log(A - i_1^*) + \phi(i_1^*), \tag{14}$$

with functions y_1 and ϕ defined in (12) and (7), respectively.

2. *If $\psi = 1$ and $\gamma \neq 1$, then*

$$J(K, v) = \exp\{a_0 + a_1 v\} \frac{K^{1-\gamma}}{1-\gamma}, \ C(K, v) = (A - i_1^*) K, \tag{15}$$

where parameters a_0 and a_1 are given by

$$a_0 = \frac{\alpha m a_1 + \mu(1-\gamma)}{\rho}, \ a_1 = \frac{\alpha + \rho - \sqrt{(\alpha+\rho)^2 + \beta^2 \gamma (1-\gamma)}}{\beta^2}. \tag{16}$$

3. *If $\psi \neq 1$, then*

$$J(K, v) = g(v) \frac{K^{1-\gamma}}{1-\gamma}, \ C(K, v) = \left(A - i_\psi^*(v)\right) K, \tag{17}$$

where

$$i_\psi^*(v) = y_\psi \left(\rho^\psi g(v)^{\frac{1-\psi}{1-\gamma}} \right), \tag{18}$$

and g is a function satisfying the following ODE:

$$\frac{1}{2}\beta^2 v g''(v) + \alpha(m - v) g'(v) + \left(\begin{array}{c} \rho\omega^{-1} \left(\left(A - i_\psi^*(v)\right)^{1-\psi^{-1}} g(v)^{-\omega} - 1 \right) + \\ (1-\gamma)\left(\phi\left(i_\psi^*(v)\right) - \frac{1}{2}\gamma v\right) \end{array} \right) g(v) = 0, \tag{19}$$

with the parameter ω defined as

$$\omega = \frac{1 - \psi^{-1}}{1-\gamma}. \tag{20}$$

Proposition 1 characterizes the Pareto optimal allocation of the economy. When the IES is not equal to one, the allocation is determined by the solution function g of ODE (19), which can be solved numerically by a projection method described later. Figure 1 illustrates two examples of the function g, where except for γ the parameter values are based on Table 2.

It is worth mentioning that the model has an interesting implication for the potential long-run effects of time-varying volatility. To see this more precisely, consider the general case with the IES different than unity. By (3), (4), (17) and Ito's lemma, one can show that the optimal

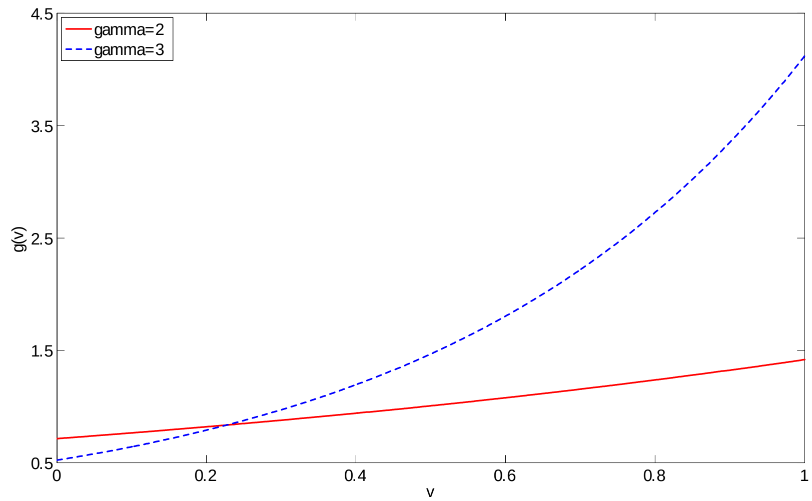

Figure 1: Solution function $g(v)$

growth rates of capital and output are

$$\frac{dY_t}{Y_t} = \frac{dK_t}{K_t} = f_Y(v_t)\,dt + \sqrt{v_t}dB_{1t}, \tag{21}$$

$$f_Y(v_t) = \phi\left(i^*_\psi(v_t)\right) = \phi\left(y_\psi\left(\rho^\psi g(v_t)^{\frac{1-\psi}{1-\gamma}}\right)\right), \tag{22}$$

and that of consumption is

$$\frac{dC_t}{C_t} = f_C(v_t)\,dt + \sqrt{v_t}dB_{1t} + c^*_\psi(v_t)^{-1}\,c^{*\prime}_\psi(v_t)\,\beta\sqrt{v_t}dB_{2t}, \tag{23}$$

$$f_C(v_t) = \phi\left(i^*_\psi(v_t)\right) + c^*_\psi(v_t)^{-1}\left(c^{*\prime}_\psi(v_t)\,\alpha(m-v_t) + \frac{1}{2}c^{*\prime\prime}_\psi(v_t)\,\beta^2 v_t\right),\ c^*_\psi(v_t) = A - i^*_\psi(v_t). \tag{24}$$

Equations (21) and (23) indicate that the conditional expected growth rates of capital, output and consumption vary over time, driven by fluctuations in volatility. This result arises because as shown in (17), the representative household would adjust its consumption and thus saving in response to changes in volatility, which consequently induces persistent time-variation in the conditional expected growth rates of the above aggregates.[6]

[6]This finding relates the model to the long-run risk literature, since it provides a theoretical justification for the existence of long-run consumption risk initially proposed by Bansal and Yaron (2004). As opposed to most studies in the literature, where the long-run risk is imposed exogenously, (23) shows that such a risk can arise endogenously in the presence of volatility risk. In contrast to Kaltenbrunner and Lochstoer (2010) and Croce (2014), who examine the endogeneity of long-run consumption risk in a quantitative model with productivity shocks, this paper uncovers analytically the link between volatility risk and long-run consumption risk.

3.2 The Effects of a Rise in Volatility

This subsection characterizes the consequences of a temporary increase in uncertainty. Such an examination is important since it can evaluate whether the model is capable of generating plausible predictions. To establish intuition for the mechanisms that determine how the representative household's saving choices and the economy as a whole respond to volatility shocks, it is instructive to focus the analysis on a simple special case without capital adjustment costs, i.e., $\kappa = 0$. Under this assumption, one can derive an approximate analytical solution to ODE (19). The approximation is useful since it allows for an analytical description of the effects of changes in volatility on the household's consumption-saving behavior and the macroeconomy.

When $\kappa = 0$, equation (12) is solved by $y_\psi(x) = A - x$ for any $\psi > 0$. Subsequently, the value function and the consumption function in (17) become

$$J(K, v) = g(v) \frac{K^{1-\gamma}}{1-\gamma}, \; C(K, v) = \rho^\psi g(v)^{\frac{1-\psi}{1-\gamma}} K, \tag{25}$$

where the function g satisfies the following ODE:

$$\frac{1}{2}\beta^2 v g''(v) + \alpha(m - v) g'(v) + \begin{pmatrix} \rho\omega^{-1}\left(\rho^{\psi-1} g(v)^{-\omega\psi} - 1\right) + \\ (1-\gamma)\left(A - \rho^\psi g(v)^{-\omega\psi} - \frac{1}{2}\gamma v\right) \end{pmatrix} g(v) = 0. \tag{26}$$

The next proposition derives an approximate analytical solution to the above ODE, following the approximation technique in Chacko and Viceira (2005).[7]

Proposition 2 *The ODE in (26) has an approximate solution*

$$\hat{g}(v) = \exp\{\hat{a}_0 + \hat{a}_1 v\}, \tag{27}$$

where parameters $\hat{a}_0$ and $\hat{a}_1$ are given by

$$\hat{a}_0 = \frac{\alpha m \hat{a}_1 + \mu(1-\gamma) + e(q_0)}{q_0}, \; \hat{a}_1 = \frac{\alpha + q_0 - \sqrt{(\alpha + q_0)^2 + \beta^2\gamma(1-\gamma)}}{\beta^2}. \tag{28}$$

Here $\mu = \rho \log\rho + A - \rho$, e is a function defined as

$$e(q) = \omega^{-1}\psi^{-1}\left(q(1 + \psi\log\rho - \log q) - \rho(1 + \psi\log\rho - \log\rho)\right), \tag{29}$$

[7] The household's problem presented here differs from that in Chacko and Viceira (2005). For example, in the model of Chacko and Viceira (2005), markets are incomplete and the return on the risk-free asset is exogenous and constant over time. By contrast, using the two welfare theorems, one can show that the household's problem in the current model with $\kappa = 0$ is equivalent to a setting, where the household has access to a bond and a stock, and the interest rate on the bond varies with volatility.

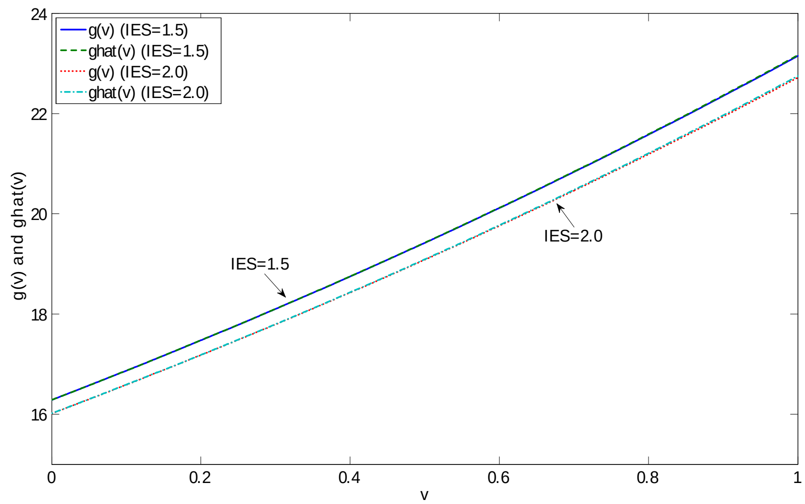

Figure 2: Solution function $g(v)$ and its approximation $\hat{g}(v)$

and q_0 is a constant given by

$$q_0 = \exp\left\{E_v\left[\log\left(\rho^\psi g(v)^{\frac{1-\psi}{1-\gamma}}\right)\right]\right\},$$

where $E_v[\cdot]$ denotes the expectation with respect to the distribution in (10).

The accuracy of the above approximation depends on the degree of the variation of the optimal log consumption-capital ratio around its unconditional mean. As shown in the appendix, the variance of the ratio is

$$Var_v\left[\log\frac{C(K,v)}{K}\right] \approx \left(\frac{1-\psi}{1-\gamma}\right)^2 \hat{a}_1^2 \frac{m\beta^2}{2\alpha}, \tag{30}$$

where $\hat{a}_1$ is defined in (28). This implies that the approximation is accurate if ψ is close to 1 and exact if $\psi = 1$. Figure 2 illustrates the approximate analytical solution $\hat{g}(v)$ and its exact counterpart $g(v)$ under different levels of the IES.[8] It shows that $\hat{g}(v)$ captures the magnitude and shape of $g(v)$ reasonably well, and thus is a reliable tool for characterizing how the household's consumption-saving behavior and the growth dynamics of the economy respond to exogenous changes in volatility.

First, I analyze the properties of the household's marginal propensity to consume (MPC). According to (25), at each point in time, the household would consume a fraction $\rho^\psi g(v)^{\frac{1-\psi}{1-\gamma}}$ of its capital, which coincides with the household's wealth in the model without capital adjustment costs. Replacing $g(v)$ in $\rho^\psi g(v)^{\frac{1-\psi}{1-\gamma}}$ with its approximation $\hat{g}(v)$ in (27) implies that

[8] Except for ψ, the parameter values used in Figure 2 are $\rho = 0.08$, $\gamma = 2$, $A = 0.084$, $\alpha = 2.8$, $m = 0.027$, and $\beta = 0.55$. The approximation error $\max_{v\in[0,1]}\left|\frac{\hat{g}(v)-g(v)}{g(v)}\right|$ is 6.2×10^{-4} for $\psi = 1.5$ and 1.4×10^{-3} for $\psi = 2.0$.

the household's MPC equals $\rho^{\psi} \exp\left\{\frac{1-\psi}{1-\gamma}\left(\hat{a}_0 + \hat{a}_1 v\right)\right\}$. By the definition of $\hat{a}_1$ in (28), it is straightforward to verify that $\frac{\hat{a}_1}{1-\gamma} < 0$ for $\gamma > 0$, and thus the MPC increases (decreases) with volatility if $\psi > (<)\, 1$. This result follows from the interactions of the substitution effect and the income effect associated with changes in volatility. On the one hand, a rise in volatility makes it more attractive for the household to consume today rather than tomorrow to avoid future uncertainty. The substitution effect implies that the household would consume a larger proportion of its current wealth. On the other hand, a larger variability of level shocks implies a higher probability of experiencing low consumption tomorrow, against which the household would protect itself by saving more today, the more averse it is to intertemporal fluctuations in consumption. The income effect would lower the household's current consumption. Which of these two conflicting effects dominates depends on the magnitude of the IES ψ. When ψ is greater than one, the substitution effect dominates and the MPC increases with volatility. When ψ is smaller than one, the income effect dominates and the relationship between MPC and uncertainty is negative. In the borderline case with ψ equal to unity, the two effects offset each other, leading to a time-invariant MPC.

I now analyze the macroeconomic consequences of a temporary increase in uncertainty. The results are summarized in Proposition 3.

Proposition 3 *Assume $\kappa = 0$. Suppose at time t_0, the economy is hit by a one-time adverse volatility shock, i.e., $dB_{2t_0} > 0$, $dB_{2t} = 0$ for $t > t_0$ (uncertainty increases suddenly at t_0), and there are no level shocks, i.e., $dB_{1t} = 0$ for $t \geq t_0$. It follows that*

1. *If $\psi < 1$, then the growth rates of output and consumption increase immediately at t_0 and decline afterwards.*

2. *If $\psi = 1$, then the volatility shock has no impact on the growth dynamics.*

3. *If $\psi > 1$, then the growth rate of output first decreases at t_0 and then gradually picks up as time passes; if, in addition, the volatility shock is large enough, i.e., $v_{t_0} > v^*$ for some number v^*, then consumption growth also falls at t_0 and rises gradually afterwards.*

Proposition 3 shows that volatility shocks alone can drive fluctuations in economic activity in the absence of any level shocks. Figure 3 illustrates the percentage point changes of consumption growth following a temporary rise in volatility in the benchmark model.[9] It is shown that when the IES is greater than one, which is an empirically plausible assumption discussed later, a mere large increase in uncertainty can decelerate the growth of output and consumption. The mechanism of how this adverse volatility shock affects the economy operates as follows. In the face of elevated uncertainty about future capital returns, the risk-averse representative

[9] Except for the IES, the parameter values used in Figure 3 are based on Table 2.

household would prefer to spend a greater fraction of its current wealth on consumption, with the resulting decreased investment in the firm slowing the pace of capital accumulation and output growth. Subsequently, future consumption falls relative to current consumption, implying a slower growth in consumption. As time passes, volatility gradually reverts down to its pre-shock level, which drives up the growth rates of output and consumption.

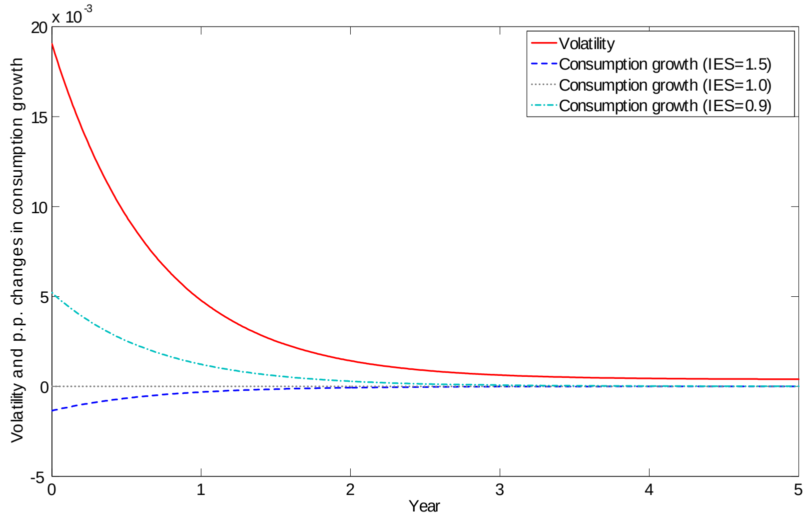

Figure 3: Impulse responses of consumption growth to a volatility shock

Before concluding this section, it is worth mentioning an interesting implication of Proposition 3 for the magnitude of the IES, whose value is still under debate in the literature. Hall (1988) finds that the estimated IES is quite small and close to zero using aggregate consumption data. Conversely, Attanasio and Vissing-Jorgensen (2003) suggest that the IES for stockholders is typically above one. Bansal and Yaron (2004) argue that estimates of the IES based on aggregate data would be seriously downward biased in the presence of time-varying consumption volatility, and in addition, an IES below unity would result in a counterfactual prediction that higher uncertainty increases prices of risky assets. As shown in Proposition 3 and Figure 3, in order for the model to generate plausible impulse responses to volatility shocks, the IES has to be greater than one. Therefore, the model suggests a new line of evidence for an IES above unity. The requirement for a high IES also justifies the use of recursive utility rather than CRRA utility in the analysis, since in the latter case the IES is the reciprocal of the risk aversion, and a risk aversion greater than one, a widely accepted assumption, would necessarily imply the IES below unity. The recursive utility function, however, does not impose the same restriction, because the two parameters are determined separately.

4 Volatility Risk and Welfare

This section examines the welfare implications of volatility risk. In the welfare analysis, I compare the benchmark economy to an otherwise identical economy except that volatility shocks in production are turned off. More precisely, in this counterfactual economy, the capital stock accumulates over time as

$$dK_t = \Phi(I_t, K_t)\,dt + K_t\sqrt{m}d\bar{B}_{1t}, \tag{31}$$

where $\bar{B}_{1t}$ is a standard Brownian motion on $\mathbb{R}$. In contrast to (4), the variance of level shocks is fixed at m in (31), which equals the steady-state volatility level of process (8). Mathematically, process (31) is a special case of (4) by setting either the mean-reverting rate $\alpha = \infty$ or the volatility of volatility $\beta = 0$ in (8), and thus there is only level risk in the economy.

Let $J(\cdot; m) : \mathbb{R} \to \mathbb{R}$, $K \mapsto J(K; m)$ and $C(\cdot; m) : \mathbb{R} \to \mathbb{R}$, $K \mapsto C(K; m)$ denote, respectively, the value function and the consumption function characterizing the Pareto optimal allocation in the counterfactual economy without volatility risk. Then, functions $J(\cdot; m)$ and $C(\cdot; m)$ satisfy the following HJB equation:

$$0 = \max_C \left\{ f(C, J(K; m)) + \Phi(I, K)\, J_K(K; m) + \frac{1}{2} m K^2 J_{KK}(K; m) \right\}. \tag{32}$$

The solution to the above equation is presented in Proposition 4. Since the proof uses the same method as that for Proposition 1, it is omitted for space consideration.

Proposition 4 *The HJB equation in (32) is solved by the value function $J(\cdot; m) : \mathbb{R} \to \mathbb{R}$ and the consumption function $C(\cdot; m) : \mathbb{R} \to \mathbb{R}$ such that*

1. *If $\psi = 1$ and $\gamma = 1$, then*

$$J(K; m) = \log K + \frac{1}{\rho}\left(\mu - \frac{m}{2}\right),\; C(K; m) = (A - i_1^*)\, K, \tag{33}$$

where μ and i_1^* are given by (14).

2. *If $\psi = 1$ and $\gamma \neq 1$, then*

$$J(K; m) = \exp\left\{ \frac{(1-\gamma)\left(\mu - \frac{1}{2}\gamma m\right)}{\rho} \right\} \frac{K^{1-\gamma}}{1-\gamma},\; C(K; m) = (A - i_1^*)\, K. \tag{34}$$

3. *If $\psi \neq 1$, then*

$$J(K; m) = \exp\left\{ h^{(m)} \right\} \frac{K^{1-\gamma}}{1-\gamma},\; C(K; m) = \left(A - i_\psi^*\right) K, \tag{35}$$

where

$$i_\psi^* = y_\psi\left(\rho^\psi e^{\frac{1-\psi}{1-\gamma}h^{(m)}}\right), \tag{36}$$

and $h^{(m)}$ is a constant solving for x in the following equation:

$$\frac{\rho}{1-\psi^{-1}}\left(\left(A - y_\psi\left(\rho^\psi e^{\frac{1-\psi}{1-\gamma}x}\right)\right)^{1-\psi^{-1}} e^{-\omega x} - 1\right) + \phi\left(y_\psi\left(\rho^\psi e^{\frac{1-\psi}{1-\gamma}x}\right)\right) - \frac{1}{2}\gamma m = 0. \tag{37}$$

For expositional convenience, in the counterfactual economy without volatility risk, let $U^\delta(K_t; m)$ denote the time-t utility of a household whose Pareto optimal consumption $C(K_s; m)$ is decreased by a fraction δ for all $s \geq t$. By definition, one has

$$U^\delta(K_t; m) = E_t\left[\int_t^\infty f\left((1-\delta)C(K_s; m), U^\delta(K_s; m)\right) ds\right] = J((1-\delta)K_t; m). \tag{38}$$

The last equality in (38) follows from the proportionality of the optimal consumption and investment with respect to capital as shown in Proposition 4, as well as the specification of the capital accumulation process in (31). The equality implies that a decrease in all future consumption by a fraction δ is equivalent to giving up a fraction δ of its current capital stock.

I now define the total welfare cost of uncertainty in production and its individual components related to level risk and volatility risk, respectively.

Definition 1 *The welfare costs of uncertainty in production are defined as follows:*

1. *The welfare cost of volatility risk is the fraction λ_v of consumption the representative household is willing to give up at all future periods to be as well off in a world facing only level risk with constant volatility m in production as it is in a world with both level risk and volatility risk, i.e.,*

$$U^{\lambda_v}(K; m) = E_v[J(K, v)], \tag{39}$$

where $E_v[\cdot]$ is the expectation with respect to the distribution in (10).

2. *The welfare cost of level risk is the fraction λ_l of consumption the representative household is willing to give up at all future periods to be as well off in a deterministic world as it is in a world facing only level risk with constant volatility m in production, i.e.,*

$$U^{\lambda_l}(K; 0) = J(K; m). \tag{40}$$

3. *The total welfare cost of uncertainty in production is the fraction λ of consumption the representative household is willing to give up at all future periods to be as well off in a*

deterministic world as it is in a world facing both level risk and volatility risk, i.e.,

$$U^{\lambda}(K;0) = E_v[J(K,v)]. \tag{41}$$

Two remarks about the above definition are in order before proceeding. First, the welfare cost of volatility risk measures the long-run welfare effects of the risk of experiencing higher-frequency two-sided movements in volatility. The value of λ_v represents the welfare gain from eliminating the time-variation in volatility. The analysis thus differs fundamentally from the cost-of-business-cycle exercises in Lucas (1987) and subsequent papers. This literature examines the long-run welfare effects of more secular one-sided changes in volatility, which is typically a model parameter. Most often, the literature considers the welfare gain from permanently decreasing volatility from a certain level to zero, in which case the welfare impact studied therein essentially corresponds to the welfare cost of level risk as defined in (40).

Second, the right-hand side of equation (39) represents the household's expected lifetime utility with respect to the distribution in (10). This certainty equivalent of the stochastic value function $J(K,v)$ for a given K is consistent with recursive preferences (1), because as shown in Duffie and Epstein (1992), the certainty equivalent functional associated with the normalized aggregator (2) coincides with the expectation functional.

The next proposition derives the expressions of the three welfare costs defined above.

Proposition 5 *The following hold that*

$$1-\lambda = (1-\lambda_l)(1-\lambda_v). \tag{42}$$

1. *If $\psi = 1$ and $\gamma = 1$, then*

$$\lambda = \lambda_l = 1 - \exp\left\{-\frac{m}{2\rho}\right\},\ \lambda_v = 0. \tag{43}$$

2. *If $\psi = 1$ and $\gamma \neq 1$, then*

$$\lambda = 1 - \exp\left\{\frac{\alpha m a_1}{\rho(1-\gamma)}\right\}(1-a_1\theta)^{-\frac{\xi}{1-\gamma}}, \tag{44}$$

$$\lambda_l = 1 - \exp\left\{-\frac{\gamma m}{2\rho}\right\}, \tag{45}$$

$$\lambda_v = 1 - \exp\left\{\frac{\alpha m a_1 + \frac{1}{2}(1-\gamma)\gamma m}{\rho(1-\gamma)}\right\}(1-a_1\theta)^{-\frac{\xi}{1-\gamma}}, \tag{46}$$

where ξ and θ are given by (9), and a_1 is defined in (16).

3. If $\psi \neq 1$, then

$$\lambda = 1 - \exp\left\{\frac{\log E_v\left[g\left(v\right)\right] - h^{(0)}}{1-\gamma}\right\}, \tag{47}$$

$$\lambda_l = 1 - \exp\left\{\frac{h^{(m)} - h^{(0)}}{1-\gamma}\right\}, \tag{48}$$

$$\lambda_v = 1 - \exp\left\{\frac{\log E_v\left[g\left(v\right)\right] - h^{(m)}}{1-\gamma}\right\}, \tag{49}$$

where the function g solves (19), and parameters $h^{(m)}$ and $h^{(0)}$ are defined by (37).

Equality (42) suggests that if the product of λ_l and λ_v is small, $\lambda \approx \lambda_l + \lambda_v$. That is, the total welfare cost of uncertainty in production is the sum of the welfare cost associated with level risk and that with volatility risk. Somewhat surprising is when the representative household has logarithmic preferences, $\lambda_v = 0$, i.e., volatility risk has no welfare impact at all. As a matter of fact, the next proposition shows that the presence of volatility risk can increase or decrease welfare, depending on model parameters.

Proposition 6 *If $\psi = 1$, then*

$$\lambda > 0,\ \lambda_l > 0,\ \lambda_v \begin{cases} < 0 & \text{if } \gamma < 1 \\ = 0 & \text{if } \gamma = 1 \\ > 0 & \text{if } \gamma > 1 \end{cases}. \tag{50}$$

Proposition 6 says that when the IES is equal to one, level risk is always welfare reducing for a risk-averse household, but the welfare impact of volatility risk can be in any direction.[10] If the household has logarithmic preferences, i.e., $\gamma = 1$, volatility risk has no welfare effects. If the household is less risk-averse than logarithmic households, the existence of volatility risk improves the household's welfare. If the household is more risk-averse than logarithmic households, fluctuations in uncertainty reduce the household's welfare. Meanwhile, the total welfare cost of uncertainty in production is positive, suggesting that the welfare impact of volatility risk is dominated by that of level risk.

I now provide some economic intuition for the signs of the three welfare costs in Proposition 6. First of all, when the IES is one, the growth rates of capital and consumption equal $\phi\left(i_1^*\right)dt + \sqrt{v_t}dB_{1t}$, where i_1^* is defined by (14). All else equal, the level risk, as represented by the random

[10] There is a large literature, as discussed in Bloom (2014), that finds a positive impact of uncertainty on investment. For example, uncertainty can promote growth since firms might make purposeful use of uncertainty in their favor by expanding in response to positive shocks and contracting in response to negative shocks. The positive impact found in this literature reflects the consequences of a permanent increase in uncertainty. By contrast, the beneficial welfare effects of volatility risk in Proposition 6 say that mean-reverting two-sided movements in volatility could in net lead to higher welfare, though uncertainty itself is welfare reducing ($\lambda_l > 0$).

variable dB_{1t}, generates fluctuations in the speed of capital depreciation and thus the growth rates of capital and consumption. Hence, the presence of the risk necessarily reduces the welfare of a risk-averse household. Second, the volatility risk, as represented by the random variable dB_{2t} in (8), affects capital accumulation by changing the dispersion of capital depreciation shocks, and thus essentially introduces time-variation in the investment opportunity set for capital. According to previous studies on dynamic optimal consumption-portfolio allocation problems, such as Campbell and Viceira (2001), Chacko and Viceira (2005), and Bhamra and Uppal (2006), a household with a unity coefficient of relative risk aversion, i.e., the case of log utility, is indifferent to changes in the investment opportunity set, and thus volatility risk is welfare neutral. When the household's risk aversion is smaller than one, it is willing to trade off periods of low welfare associated with high volatility for periods of high welfare associated with low volatility, and the household would view the existence of volatility shocks welfare beneficial. By the same token, when the household's risk aversion exceeds one, volatility risk becomes welfare reducing. Lastly, despite the possible welfare gain associated with the presence of volatility risk, the total welfare cost of uncertainty arising from both level risk and volatility risk is positive. This is intuitive as removing both the level risk and the volatility risk after all eliminates fluctuations in capital accumulation and thus consumption growth, which is naturally welcome by a risk-averse household.

Table 1: Welfare costs $(\lambda_v, \lambda_l, \lambda)$ (%)

ψ	γ			
	0.5	0.75	2	3
1.0	$(-0.00011, 0.33, 0.33)$	$(-0.00013, 0.49, 0.49)$	$(0.0037, 1.3, 1.3)$	$(0.017, 2.0, 2.0)$
1.5	$(-0.00020, 0.55, 0.55)$	$(-0.00022, 0.82, 0.82)$	$(0.0062, 2.2, 2.2)$	$(0.028, 3.2, 3.3)$
2.0	$(-0.00030, 0.83, 0.83)$	$(-0.00034, 1.2, 1.2)$	$(0.0094, 3.3, 3.3)$	$(0.042, 4.8, 4.9)$
3.0	$(-0.00062, 1.7, 1.7)$	$(-0.00069, 2.6, 2.6)$	$(0.019, 6.6, 6.6)$	$(0.085, 9.6, 9.7)$

When ψ is different than unity, predictions about the signs of the three welfare costs are difficult to establish, since there are no closed-form solutions to equations (19) and (37). The model has to be solved numerically in this case. Note that with $\psi \neq 1$, volatility risk induces not only time-varying dispersions of capital depreciation shocks, but also fluctuations in the conditional expected growth rates of capital and consumption by (21) and (23). Table 1 reports the welfare numbers for various levels of risk aversion and IES, while the other parameter values are displayed in Table 2. The numerical exercise confirms the findings in Proposition 6: volatility risk can increase or decrease welfare; its impact, however, is dominated by that of level risk, resulting in an overall reduction in welfare due to uncertainty in production.

5 Numerical Results

This section calibrates the benchmark model to U.S. data, and uses it to quantify the welfare impact of volatility risk.

5.1 Calibration

There are in total eight parameters in the benchmark model, namely, the rate of time preference ρ, the IES ψ, the coefficient of relative risk aversion γ, the productivity level A, the capital adjustment cost κ, the mean-reverting rate α, the mean volatility level m, and the volatility of volatility β. The first three parameters characterize the representative household's preferences, whereas the last five specify the representative firm's production technology. In the calibration, I split the set of model parameters into two subsets. The first subset consists of those parameters whose values are standard in the literature or can be estimated directly without solving the model. The second subset includes parameters that are calibrated endogenously to match selected moments in U.S. data. The calibration details are described as follows, where parameters and moments are all annualized.

I set the rate of time preference ρ at 0.03. This is equivalent to a discount rate of 0.97 in discrete-time models. The coefficient of relative risk aversion γ is fixed at 2 in the benchmark model, a number well within the consensus range of the parameter. I choose the IES ψ to be 1.5. This value is consistent with those used in the literature such as Bansal and Yaron (2004). The productivity parameter A is calibrated to match the long-run output-capital ratio of 0.37, a value in line with U.S. experience such as that shown in D'Adda and Scorcu (2003).

Table 2: Benchmark parameters

Parameter	Description	Value
ρ	Rate of time preference	0.03
ψ	Intertemporal elasticity of substitution	1.5
γ	Coefficient of relative risk aversion	2.0
A	Productivity level	0.37
κ	Capital adjustment cost	13.9
α	Mean-reverting rate	1.4
m	Mean volatility level	0.00039
β	Volatility of volatility	0.11

The remaining four parameters $(\kappa, m, \alpha, \beta)$ are calibrated jointly to match four selected moments in the 1956-2014 U.S. data, namely, the standard deviation of real output growth rates, the mean of real consumption growth rates, the standard deviation of real consumption growth rates, and the mean of consumption-investment ratios. These moments are chosen because they are closely related to the dynamic behavior of consumption, which is important for the subsequent welfare analysis. Since the IES is different than one in the benchmark model, calibrating the above four parameters requires computing the model numerically. In this paper, I use a projection method to solve ODE (19). In particular, I approximate its solution function $g(v)$ by an n-th degree polynomial $\hat{g}(v)$, whose coefficients are determined such that $\hat{g}(v)$ satisfies (19) on $n+1$ different points in the v-space. The relevant unconditional moments are calculated by taking expectation with respect to the density function in (10).

Table 2 summarizes the benchmark model parameterization. Note that the calibrated value of the capital adjustment cost lies within the range of the estimates of the parameter in the literature.[11] Table 3 compares the unconditional moments generated by the parameterized model with those observed in the data. As seen in Panel A, the targeted moments are well matched by the model. Panel B reports statistics that the model is not calibrated to fit. The model-implied standard deviation of investment growth is lower than its data counterpart, whereas consumption growth and investment growth both display stronger comovements with output growth than documented in the data. The disparity stems from the fact that in the model, variations in the growth rates of output, consumption and investment are largely driven by the common level shocks $\sqrt{v_t}dB_{1t}$. Thus, the growth rates of these aggregates have comparable volatilities and they also comove strongly with each other. This is an inevitable limitation of the current parsimonious model. In sum, the calibrated model captures the key features of the data reasonably well, and thus provides a suitable laboratory for the following welfare analysis.

Table 3: Unconditional moments

Panel A: Targeted moments	Data	Model
Standard deviation of output growth	0.022	0.020
Mean of consumption growth	0.033	0.035
Standard deviation of consumption growth	0.018	0.020
Mean of consumption-investment ratio	4.4	4.4
Panel B: Untargeted moments		
Standard deviation of investment growth	0.089	0.020
Correlation of consumption growth and output growth	0.91	1.0
Correlation of investment growth and output growth	0.87	1.0

5.2 Welfare Costs

This subsection uses the calibrated model to estimate the welfare cost of volatility risk in the U.S. and compares it to that of level risk. The calculation of welfare costs is based on the

[11] Depending on methods and data sets, the existing estimates of κ vary significantly, ranging from below one in Cooper and Haltiwanger (2006) to over twenty in Hayashi (1982).

corresponding formulas derived in Proposition 5.

As displayed in the first row of Table 4, the benchmark model finds that the welfare costs of volatility risk, level risk, and total uncertainty in production are $\lambda_v = 6.2 \times 10^{-5}$, $\lambda_l = 0.022$, and $\lambda = 0.022$. In other words, the representative household would willingly give up, respectively, 0.0062 and 2.2 percent of its annual consumption to eliminate the volatility risk and the level risk, while in total the household would accept to forego 2.2 percent of its annual consumption to remove both risks. The calculation thus suggests that volatility risk and level risk are both welfare reducing for the U.S. economy. But the welfare gain from eliminating the former is very small. Note that the welfare cost of level risk deduced herein is higher than the corresponding estimates (well below 1 percent) as in Lucas (1987) and Krusell and Smith (1999), but is largely comparable to findings in related studies based on recursive utility. For example, the estimated welfare cost of level risk mostly ranges from 1 to 20 percent of consumption, as in Obstfeld (1994), Tallarini (2000), Epaularda and Pommeret (2003), and Barro (2009).

In what follows, I conduct a sensitivity analysis of how the computed welfare effects depend on model parameters. In the analysis, except for parameters explicitly under investigation, the rest are all set at their benchmark values.

Table 4: Sensitivity analysis (%)

	Volatility risk (λ_v)	Level risk (λ_l)	Total uncertainty (λ)
(1) benchmark	0.0062	2.2	2.2
(2) $\gamma = 3$	0.028	3.2	3.3
(3) $\psi = 2$	0.0094	3.3	3.3
(4) $\alpha = 5$	0.00052	2.2	2.2
(5) $m = 0.002$	0.031	11	11
(6) $\beta = 0.2$	0.021	2.2	2.2

Note: In the benchmark, $\gamma = 2$, $\psi = 1.5$, $\alpha = 1.4$, $m = 0.00039$, $\beta = 0.11$.

I first examine the effects of risk aversion and IES on the welfare cost estimation. First, the coefficient of relative risk aversion is found to have a measurable impact on the welfare evaluation. As seen in the second row of Table 4, a rise in risk aversion increases the welfare gains from eliminating either the volatility risk or the level risk. For example, the welfare benefit from removing the volatility risk rises to 0.028 percent of annual consumption at $\gamma = 3$, up from 0.0062 percent in the benchmark with $\gamma = 2$, while the corresponding benefit from eliminating the level risk increases to 3.2 percent of annual consumption, up from 2.2 percent. Second, a larger IES increases the welfare costs of both volatility risk and level risk. The third row of the table shows that the welfare gain from eliminating the volatility risk increases from 0.0062 percent of annual consumption in the benchmark with $\psi = 1.5$ to 0.0094 percent at $\psi = 2$, whereas that for removing the level risk rises from 2.2 percent of consumption to 3.3

percent. Both λ_v and λ_l are increasing in ψ because a higher IES implies the household's optimal consumption becomes more sensitive to shocks affecting capital accumulation.

The last three rows of Table 4 display how the characteristics of the volatility process affect the welfare cost estimation. First, the fourth row shows that an increase in the mean-reverting rate α results in a smaller welfare cost of volatility risk, but has no impact on the welfare evaluation of level risk.[12] An increase in α means that shocks to volatility are less persistent. All else equal, this makes the volatility process in (8) less volatile, and accordingly it becomes less welfare beneficial to eliminate the volatility risk. However, since level shocks and volatility shocks are independent, the parameter α only determines how volatility fluctuates around its long-run average level m, and changes in α would not affect the welfare cost of level risk. Second, the fifth row reestimates the welfare costs for a higher average volatility level m. A larger m increases not only the average size of level shocks, but also the stationary variance $m\frac{\beta^2}{2\alpha}$ of the volatility process. Thus, the household would find it more beneficial to remove either the volatility risk or the level risk. Lastly, as seen in the sixth row, a rise in the volatility of volatility β implies a larger welfare cost of volatility risk. Intuitively, a larger β means volatility varies more widely around its long-run average, and thus eliminating the volatility risk becomes more beneficial. However, the parameter β does not affect a household's welfare evaluation of level risk, following a similar argument for the mean-reverting rate α.

In summary, as far as the U.S. economy is considered, time-variation in volatility has a very limited welfare impact under plausible model parameterizations. Before concluding, two remarks are in order with regard to the above welfare results. First, the welfare cost of volatility risk measures the consequences of long-term two-sided fluctuations in volatility, but is silent on the effects of short-term one-sided changes in volatility, such as a temporary spike in uncertainty. It is possible that variations in uncertainty might have more significant welfare implications in the short run according to studies such as Bloom (2014). Second, the finding of a negligible welfare cost of volatility risk does not come as a complete surprise, since the magnitude of business cycles in the U.S. during the sample years (1956-2014), measured by the standard deviation of the annual growth rate of either output or consumption, is relatively small. However, this may not be the case for economies where volatility is notably higher. Fernández-Villaverde et al. (2011) recently presented clear evidence that volatility shocks may be an important mechanism behind business cycles in some emerging economies, and the welfare cost of volatility risk could thus be much larger for these countries.

[12] Mathematically, one can see from (48) that besides the risk aversion parameter γ, the welfare cost of level risk depends on the values of $h^{(m)}$ and $h^{(0)}$, which by (37) are both independent of α.

6 Conclusion

This paper studies the welfare effects of time-variation in volatility. I propose an analytically tractable endogenous growth model with recursive preferences, stochastic volatility, and capital adjustment costs. The model shows explicitly that changes in volatility can affect growth in the absence of any level shocks. It also finds that volatility risk can increase or decrease welfare, depending on model parameters. When calibrated to U.S. data, the model shows that the welfare cost of volatility risk is largely negligible under plausible model parameterizations.

To preserve tractability and economic intuition, the model is deliberately kept simple so I can characterize the properties of the model in an analytical fashion. There are clearly limitations in the study. For example, the paper abstracts from labor markets, and their interaction with volatility shocks is potentially an important channel through which volatility risk would affect welfare. Endogenous labor supply would give households an additional instrument to insure against changes in uncertainty. By contrast, when firms face labor adjustment costs, the wait-and-see effect would amplify the impact of volatility shocks as shown in Bloom et al. (2012), and thus likely make volatility risk more welfare costly. Meanwhile, the current welfare analysis is conducted within a representative agent framework. Introducing heterogeneity of households or firms adds a dimension of interest, since the welfare consequences of fluctuations in uncertainty vary across individual agents. As suggested by previous welfare exercises such as Krusell et al. (2009), the presence of the asymmetric effects of volatility shocks might imply a larger welfare cost of volatility risk than what would be predicted by a representative agent model. To address these problems, one could probably consider a full-fledged DSGE model, and use the intuition obtained in this paper to go further in the exploration of the welfare effects of volatility risk. I leave improvement along these lines for future research.

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Appendix

Proof of Lemma 1

For a given $x > 0$, denote $z(y, x) = A - y - (1 - \kappa y)^{-\psi} x$. Then, one has

$$\lim_{y \downarrow -\infty} z(y, x) = \infty, \ \lim_{y \uparrow \frac{1}{\kappa}} z(y, x) = -\infty, \ z_y(y, x) = -1 - \psi\kappa (1 - \kappa y)^{-\psi - 1} x < 0.$$

Thus, there exists a unique number $y_\psi(x) < \frac{1}{\kappa}$ such that $z(y_\psi(x), x) = 0$. Note that in the absence of capital adjustment costs, i.e., $\kappa = 0$, $y_\psi(x) = A - x$ by (12). Applying the Implicit Function Theorem to equation $z(y, x) = 0$ yields

$$y'_\psi(x) = -\frac{z_x(y, x)}{z_y(y, x)} = -\frac{(1 - \kappa y)^{-\psi}}{1 + \psi\kappa (1 - \kappa y)^{-\psi - 1} x} < 0,$$

implying that the function $y_\psi(x)$ decreases with x.

Proof of Proposition 1

First consider the case with $\psi = 1$. When $\gamma \neq 1$, the value function J is conjectured to take the form in (15) for some unknown constants a_0 and a_1. The first-order condition (FOC) of the

HJB equation in (11) is

$$f_C(C,J) = \Phi_I(I,K)\, J_K(K,v)\,. \tag{51}$$

Substituting the functional forms of f, Φ and J into (51) yields that the optimal investment-capital ratio i_1^* satisfies

$$A - i_1^* = (1-\kappa i_1^*)^{-1}\rho,$$

which by Lemma 1 implies that $i_1^* = y_1(\rho)$, and accordingly, the optimal consumption-capital ratio $c_1^* = A - i_1^*$. As a consequence, the optimal consumption function is $C(K,v) = c_1^* K$, which gives the consumption function as stated in (15). Evaluating (11) with $C(K,v) = c_1^* K$ and the conjectured functional form of J leads to

$$\frac{1}{2}\beta^2 v a_1^2 + \alpha(m-v)a_1 + (1-\gamma)\left(\mu - \frac{1}{2}\gamma v\right) - \rho(a_0 + a_1 v) = 0, \tag{52}$$

where the parameter μ is defined in (14). Collecting constants and terms in v results in the following system of equations for (a_0, a_1):

$$\rho a_0 - \alpha m a_1 - \mu(1-\gamma) = 0,$$
$$\frac{1}{2}\beta^2 a_1^2 - (\alpha+\rho)a_1 - \frac{1}{2}\gamma(1-\gamma) = 0,$$

which implies that

$$a_1 = \frac{(\alpha+\rho) \pm \sqrt{(\alpha+\rho)^2 + \beta^2\gamma(1-\gamma)}}{\beta^2},\ \ a_0 = \frac{\alpha m a_1 + \mu(1-\gamma)}{\rho}.$$

For the value function J to exist, the term $(\alpha+\rho)^2 + \beta^2\gamma(1-\gamma)$ must be non-negative: if $\gamma < 1$, this condition holds; if $\gamma > 1$, the condition holds provided that $(\alpha+\rho)^2 + \beta^2\gamma(1-\gamma) \geq 0$. While the presence of two roots for a_1 suggests multiple solutions, only the one with the negative square root has a well-defined limit as $\beta \to 0$. When $\gamma = 1$, i.e., in the case of log utility, one can similarly derive the associated value function and consumption function as stated in (13).

In the case with $\psi \neq 1$, conjecture the value function J is of the form (17) for some unknown function g. The FOC in (51) implies that the optimal investment-capital ratio $i_\psi^*(v)$ satisfies

$$A - i_\psi^*(v) = \left(1 - \kappa i_\psi^*(v)\right)^{-\psi} \rho^\psi g(v)^{\frac{1-\psi}{1-\gamma}},$$

which by Lemma 1 implies that $i_\psi^*(v) = y_\psi\left(\rho^\psi g(v)^{\frac{1-\psi}{1-\gamma}}\right)$, and the optimal consumption-capital ratio $c_\psi^*(v) = A - i_\psi^*(v)$. Thus, the optimal consumption function is $C(K,v) = c_\psi^*(v)K$, which gives the consumption function as stated in (17). Evaluating (11) with $C(K,v) = c_\psi^*(v)K$ and the conjectured functional form of J leads to that g must satisfy (19).

Proof of Proposition 2

Define $d(v) = \log\left(\frac{C(K,v)}{K}\right)$, $d_0 = E_v[d(v)]$, and $q_0 = e^{d_0}$. Note that the parameter q_0 is positive by construction and is endogenously determined within the model. Linearizing the function $\frac{C(K,v)}{K} = e^{d(v)}$ around d_0 yields

$$\rho^{\psi} g(v)^{-\omega\psi} = \frac{C(K,v)}{K} \approx e^{d_0} + e^{d_0}(d(v) - d_0) = e^{d_0}(1 - d_0) + e^{d_0} d(v).$$

Substituting the above approximation for $\rho^{\psi} g(v)^{-\omega\psi}$ into (26) results in an approximate ODE

$$\frac{1}{2}\beta^2 v \hat{g}''(v) + \alpha(m - v)\hat{g}'(v) + \begin{pmatrix} e(q_0) - q_0 \log \hat{g}(v) + \\ (1-\gamma)\left(\mu - \frac{1}{2}\gamma v\right) \end{pmatrix} \hat{g}(v) = 0, \tag{53}$$

where the function e is defined in (29). Similar to the derivation for (15), one can show that ODE (53) is solved by $\hat{g}(v) = \exp\{\hat{a}_0 + \hat{a}_1 v\}$, where $\hat{a}_0$ and $\hat{a}_1$ are defined in (28).

Note that the above approximation is based on the expansion of the optimal log consumption-capital ratio $d(v)$ around its unconditional mean d_0. Thus, the accuracy of the approximation is determined by the variance of $d(v)$. By definition, one has

$$Var_v[d(v)] = Var_v\left[\log\left(\rho^{\psi} g(v)^{-\omega\psi}\right)\right] \approx Var_v[-\omega\psi(\hat{a}_0 + \hat{a}_1 v)] = \left(\frac{1-\psi}{1-\gamma}\right)^2 \hat{a}_1^2 \frac{m\beta^2}{2\alpha},$$

suggesting that the approximation is accurate if ψ is close to 1 and exact if $\psi = 1$. One can also show that as $\psi \to 1$, $q_0 = e^{d_0} \to \rho$, $e(q_0) \to 0$, $\tilde{a}_0 \to a_0$, $\tilde{a}_1 \to a_1$, and thus $\hat{g}(v) \to g(v)$.

Proof of Proposition 3

First consider the case with $\psi \neq 1$. When $\kappa = 0$, the optimal consumption-investment ratio $c_{\psi}^{*}(v) = \rho^{\psi}\hat{g}(v_t)^{\frac{1-\psi}{1-\gamma}}$, where $\hat{g}(v)$ is the approximate analytical solution in (27). Substituting $c_{\psi}^{*}(v)$ into (21) and (23) yields

$$\frac{dY_t}{Y_t} = f_Y(v_t)\,dt + \sqrt{v_t}dB_{1t}, \tag{54}$$

$$\frac{dC_t}{C_t} = f_C(v_t)\,dt + \sqrt{v_t}dB_{1t} + \frac{1-\psi}{1-\gamma}\hat{a}_1\beta\sqrt{v_t}dB_{2t}, \tag{55}$$

where the conditional expected growth rates of output and consumption are given by

$$f_Y(v_t) = A - \rho^{\psi} \exp\left\{\frac{1-\psi}{1-\gamma}(\hat{a}_0 + \hat{a}_1 v_t)\right\}, \tag{56}$$

$$f_C(v_t) = f_Y(v_t) + \frac{1-\psi}{1-\gamma}\left(\alpha(m - v_t)\hat{a}_1 + \frac{1-\psi}{2(1-\gamma)}\hat{a}_1^2\beta^2 v_t\right). \tag{57}$$

Given $dB_{1t} = dB_{2t} = 0$ for $t > t_0$, after the volatility shock at t_0, the growth rate of output equals $f_Y(v_t)$ by (54), whereas that of consumption equals $f_C(v_t)$ by (55). I now examine how these growth rates evolve over time. First, by the definition of $\hat{a}_1$ in (28), one has that $\frac{\hat{a}_1}{1-\gamma} < 0$ for all $\gamma > 0$, and thus $f_Y(v_t)$ in (56) decreases (increases) with v_t if $\psi > (<) 1$. Meanwhile, taking derivative of $f_C(v_t)$ in (57) with respective to v_t leads to

$$f_C'(v_t) = \frac{1}{2}\left(\frac{1-\psi}{1-\gamma}\right)^2 \hat{a}_1^2\beta^2 - \frac{1-\psi}{1-\gamma}\hat{a}_1\left(\rho^{\psi}\exp\left\{\frac{1-\psi}{1-\gamma}(\hat{a}_0 + \hat{a}_1 v_t)\right\} + \alpha\right).$$

If $\psi < 1$, $f_C'(v_t) > 0$ since $\frac{\hat{a}_1}{1-\gamma} < 0$ for all $\gamma > 0$. If $\psi > 1$, $f_C'(v_t) < 0$ as long as

$$\frac{1}{2}\left(\frac{1-\psi}{1-\gamma}\right)^2 \hat{a}_1^2\beta^2 < \frac{1-\psi}{1-\gamma}\hat{a}_1\left(\rho^{\psi}\exp\left\{\frac{1-\psi}{1-\gamma}(\hat{a}_0 + \hat{a}_1 v_t)\right\} + \alpha\right),$$

which is equivalent to

$$v_t > v^* = \left(\frac{1-\psi}{1-\gamma}\hat{a}_1\right)^{-1} \log\left(\rho^{-\psi} e^{-\frac{1-\psi}{1-\gamma}\hat{a}_0}\left(\frac{1-\psi}{1-\gamma}\frac{\hat{a}_1\beta^2}{2} - \alpha\right)\right). \tag{58}$$

In the case with $\psi = 1$, by Proposition 1, one can easily verify that $\frac{dC_t}{C_t} = \frac{dY_t}{Y_t} = \phi(i_1^*)\,dt + \sqrt{v_t}dB_{1t}$, where i_1^* is defined by (14). Thus, consumption and output grow at the same rate. Given $dB_{1t} = 0$ for $t \geq t_0$, the common growth rate equals $\phi(i_1^*)$ at time t_0 and afterwards, implying that the volatility shock does not affect the growth dynamics. Note that the proof of the results for $\psi = 1$ holds for the general case with $\kappa \neq 0$.

Proof of Proposition 5

Equality (42) directly follows from (38) and the functional forms of value functions $J(K, v)$ and $J(K; m)$. In the case with $\psi = 1$ and $\gamma = 1$, evaluating (40) with the functional forms of $J(\cdot; m)$ and $J(\cdot; 0)$ in (33) yields the expression of λ_l. By (13) and (39), one has

$$\log((1-\lambda_v)K) + \frac{1}{\rho}\left(\mu - \frac{1}{2}m\right) = E_v\left[\frac{1}{\rho}\log K - \frac{1}{2(\rho+\alpha)}v + \frac{1}{\rho}\left(\mu - \frac{\alpha m}{2(\rho+\alpha)}\right)\right],$$

which by $E_v[v] = m$ leads to $\lambda_v = 0$. According to (42), it holds that $\lambda = \lambda_l$.

When $\psi = 1$ and $\gamma \neq 1$, the expression of λ_l follows directly by using the functional forms of $J(\cdot; m)$ and $J(\cdot; 0)$ in (34). Meanwhile, (39) requires that $\exp\left\{\frac{(1-\gamma)\left(\mu-\frac{1}{2}\gamma m\right)}{\rho}\right\}\frac{((1-\lambda_v)K)^{1-\gamma}}{1-\gamma} = E_v\left[\exp\{a_0 + a_1 v\}\frac{K^{1-\gamma}}{1-\gamma}\right]$. By the definitions of a_0 and a_1 in (16), one has

$$\begin{aligned}
(1-\lambda_v)^{1-\gamma} &= \frac{\exp\left\{\frac{\alpha m a_1 + \frac{1}{2}\gamma(1-\gamma)m}{\rho}\right\}}{\Gamma(\xi)\,\theta^{\xi}}\int_0^{\infty} e^{-qv} v^{\xi-1} dv, \text{ by (10), } q = \frac{1}{\theta} - a_1 \\
&= \exp\left\{\frac{\alpha m a_1 + \frac{1}{2}\gamma(1-\gamma)m}{\rho}\right\}(\theta q)^{-\xi}, \text{ by } \Gamma(\xi) = \int_0^{\infty} e^{-z} z^{\xi-1} dz, \qquad (59)
\end{aligned}$$

which leads to the expression of λ_v in (46) and then that of λ in (44) by (42). Here the number q is assumed to be positive to guarantee that the expectation $E_v[\exp\{a_1 v\}]$ is well defined. It is straightforward to see that this condition is equivalent to $\alpha + \sqrt{(\alpha+\rho)^2 + \beta^2\gamma(1-\gamma)} > \rho$, which is always true for $\gamma \leq 1$ and holds for $\gamma > 1$ if $\alpha > \rho$.

In the case with $\psi \neq 1$, one can directly derive the desired expressions of λ_v, λ_l and λ by applying the functional forms of $J(\cdot)$, $J(\cdot; m)$ and $J(\cdot; 0)$ in Definition 1.

Proof of Proposition 6

When $\gamma = 1$, corresponding results in (50) follow directly from (43). When $\gamma \neq 1$, one first has $\lambda_l > 0$ by its expression in (45). To prove $\lambda > 0$, it is sufficient to show $(1 - a_1\theta)^{-\frac{\xi}{1-\gamma}} < 1$ since $\frac{a_1}{1-\gamma} < 0$ and then $\exp\left\{\frac{\alpha m a_1}{\rho(1-\gamma)}\right\} < 1$. Given $\frac{a_1}{1-\gamma} < 0$, it holds that $1 - a_1\theta > (<)\, 1$ if $\gamma < (>)\, 1$ and thus $(1 - a_1\theta)^{-\frac{\xi}{1-\gamma}} < 1$ for $\gamma \neq 1$ as desired. To deduce the sign of λ_v, from the proof of Proposition 5, one has

$$\begin{aligned}
(1-\gamma)\log(1-\lambda_v) &= \frac{\alpha m a_1 + \frac{1}{2}(1-\gamma)\gamma m}{\rho} - \xi\log(1 - a_1\theta) \\
&> \frac{\alpha m a_1 + \frac{1}{2}(1-\gamma)\gamma m}{\rho} + \xi a_1\theta, \text{ by } \log(1 - a_1\theta) < -a_1\theta \\
&= \frac{(\alpha+\rho)\, a_1 m + \frac{1}{2}(1-\gamma)\gamma m}{\rho} \\
&= \frac{m\gamma(1-\gamma)}{\rho}\left(\frac{1}{2} - \frac{\alpha+\rho}{\alpha+\rho+\sqrt{(\alpha+\rho)^2 + \beta^2\gamma(1-\gamma)}}\right) > 0,
\end{aligned}$$

where the last inequality follows from $\sqrt{(\alpha+\rho)^2 + \beta^2\gamma(1-\gamma)} > (<)\, \alpha + \rho$ if $\gamma < (>)\, 1$. Therefore, it holds that $\lambda_v < (>)\, 0$ if $\gamma < (>)\, 1$.