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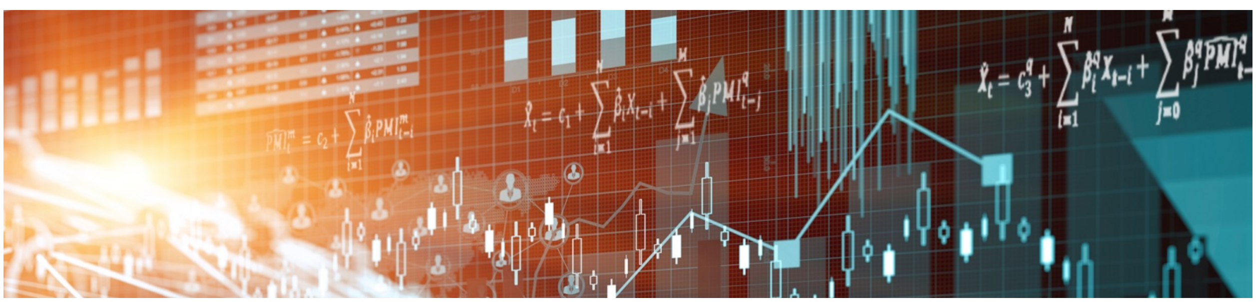
Interpreting Volatility Shocks as Preference Shocks



by Shaofeng Xu

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Interpreting Volatility Shocks as Preference Shocks

by

Shaofeng Xu

Canadian Economic Analysis Department
Bank of Canada
Ottawa, Ontario, Canada K1A 0G9
sxu@bankofcanada.ca

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Abstract

This paper examines the relationship between volatility shocks and preference shocks in an analytically tractable endogenous growth model with recursive preferences and stochastic volatility. I show that there exists an explicit mapping between volatility shocks and preference shocks, and a rise in volatility generates the same impulse responses of macroeconomic aggregates as a negative preference shock.

Bank topics: Business fluctuations and cycles; Economic models
JEL codes: E2; E3

Résumé

Cet article examine la relation entre les chocs de volatilité et les chocs de préférence dans un modèle à croissance endogène résoluble analytiquement qui intègre des préférences récursives et une volatilité stochastique. L'auteur montre qu'il existe une correspondance explicite entre les chocs de volatilité et les chocs de préférence, et qu'une hausse de la volatilité génère les mêmes profils de réaction d'agrégats macroéconomiques qu'un choc de préférence négatif.

Sujets : Cycles et fluctuations économiques; Modèles économiques
Codes JEL : E2, E3

Non-Technical Summary

Since the late 2000s, both empirical and theoretical studies have shown that time-varying volatility has significant effects on economic activity. The current research on volatility shocks has been primarily based on quantitative dynamic stochastic general equilibrium models. However, the complex nature of these models makes it difficult to develop intuition regarding the implications of volatility risk. This paper develops an analytically tractable model useful for both economic researchers and policy-makers to better understand the nature of volatility risk prevailing in the economy.

The main findings of the paper are the following. In the developed model, a linear function exists that maps volatility shocks into preference shocks, where the latter refer to shocks affecting the utility of households and thus their willingness to consume. The form of the function implies that an increase in uncertainty generates the same impulse responses of macroeconomic aggregates as a negative preference shock, resulting in slower growth of both consumption and output. The existence of the mapping thus offers a theoretical justification for the demand-shock nature of volatility shocks as documented in the literature.

1 Introduction

Since the late 2000s, time-varying volatility has attracted the attention of macroeconomists, with both empirical and theoretical studies showing significant negative effects of a rise in uncertainty on economic activity.[1] The current research on volatility shocks has been primarily based on quantitative dynamic stochastic general equilibrium (DSGE) models. However, they are difficult to use to obtain intuition about the properties of volatility risk. In this paper, I attempt to make some progress along this line by providing an intuitive interpretation of volatility shocks through an analytic model.

The paper constructs a tractable continuous-time endogenous growth model to analyze the relationship between volatility shocks and preference shocks. The model economy consists of infinitely lived households and firms. The representative household has recursive preferences over a numeraire consumption good, and makes consumption-saving decisions over time. The representative firm produces the consumption good using a production technology that is linear in capital, and faces uncertainty about the capital depreciation rate. Capital depreciation shocks exhibit stochastic volatility, which follows a mean-reverting Cox-Ingersoll-Ross (CIR) process. The Pareto optimal allocation of the economy is solved in closed form up to a second-order ordinary differential equation. The model highlights that an increase in volatility can slow the growth of output and consumption in the absence of any level shocks.

The main findings of the paper are twofold. First, I prove that there exists a linear function mapping volatility shocks into preference shocks. To show this, I construct a hypothetical economy identical to the benchmark economy, except that the representative household faces some properly specified preference shocks and the variance of capital depreciation shocks faced by the representative firm is constant. The form of the linear function suggests that an increase in volatility in the benchmark economy resembles a negative preference shock in the constructed economy, where the size of the implied preference shock is proportional to the risk aversion of the household and the deviation of volatility from its long-run average. The second major finding is that a temporary increase in uncertainty generates exactly the same impulse responses of macroeconomic aggregates as a properly defined negative preference shock. At the heart of this equivalence result is that the household would adjust its consumption, and thus saving, in the same manner under the two types of shocks, which results in an equivalent subsequent growth path of investment, capital and output. Therefore, one can interpret volatility shocks as a class of preference shocks characterized by the linear function described above.

This paper is closely related to the strand of literature that examines the economic impact of time-varying uncertainty. Examples include Bloom (2009), Fernández-Villaverde et al. (2011), Arellano et al. (2012), Basu and Bundick (2012), Bloom et al. (2012), Christiano et al. (2014),

[1] This paper uses the terms "volatility" and "uncertainty" interchangeably, both meaning "variance."

Fernández-Villaverde et al. (2015), and Leduc and Liu (2015). These studies show that a rise in volatility alone can have sizable negative effects on economic activity in the absence of any level shocks. This paper differentiates itself from the existing studies along two notable dimensions. First, it proves that there exists an explicit mapping between volatility shocks and preference shocks. This equivalence result provides a theoretical justification for the claim made by Leduc and Liu (2015) that uncertainty shocks are aggregate demand shocks.[2] It also helps us understand why volatility shocks have been successful in explaining empirical business cycle regularities, since earlier studies have shown that preference shocks are typically effective in producing the classic business cycle patterns. Second, in contrast to existing papers mainly using quantitative DSGE models, this paper characterizes the impulse responses of volatility shocks in an analytic fashion, and derives the conditions under which the negative relationship between uncertainty and economic activity holds.

The remainder of this paper is organized as follows. Section 2 describes the benchmark model. Section 3 constructs an economy with preference shocks, and shows an explicit relationship between volatility shocks and a certain class of preference shocks. Section 4 concludes.

2 The Model

This section lays out a continuous-time endogenous growth model with recursive preferences and stochastic volatility in production.

2.1 Preferences

The representative infinitely lived household values a stochastic process $\{C_t\}$ of a numeraire consumption good according to a recursive utility function introduced by Duffie and Epstein (1992), which is a continuous-time analogue of the discrete-time recursive utility proposed by Epstein and Zin (1989):

$$V_t = E_t\left[\int_t^{\infty} f\left(C_s, V_s\right) ds\right], \tag{1}$$

where the function f is a normalized aggregator of the form

$$f\left(C, V\right) = \rho \frac{1-\gamma}{1-\psi^{-1}} V \left(\frac{C^{1-\psi^{-1}}}{\left(\left(1-\gamma\right)V\right)^{\left(1-\psi^{-1}\right)/(1-\gamma)}} - 1 \right). \tag{2}$$

Here ρ is the rate of time preference, ψ is the intertemporal elasticity of substitution (IES), and γ is the coefficient of relative risk aversion. The recursive utility function (1) generalizes

[2] The authors find both in the data and in a quantitative DSGE model that an increase in uncertainty acts like a negative aggregate demand shock by raising unemployment and lowering inflation.

the standard time-additive constant-relative-risk-aversion (CRRA) expected utility function by separating the risk aversion from the IES, and nests the CRRA utility as a special case for $\gamma = \psi^{-1}$. As will be clear later, disentangling the risk aversion and the IES is important for the model to generate plausible impulse responses to volatility shocks. In the economy, the household forms its expectation based on information available at a given time t, i.e., $E_t[\cdot] = E[\cdot\,|\mathcal{F}_t]$, where the information sets over time are represented by a filtration $\{\mathcal{F}_t\}$ associated with a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, with $\mathcal{F}$ and $\mathcal{P}$ denoting, respectively, the σ-algebra on Ω and the probability measure on $\mathcal{F}$.

2.2 Production

There is a representative firm endowed with an AK production technology of the form

$$Y_t = AK_t, \tag{3}$$

where A is a positive productivity parameter, and capital K_t is the sole factor for producing output Y_t that can be used for both consumption C_t and investment I_t. The capital stock of the firm is assumed to evolve as

$$dK_t = I_t dt + K_t \sqrt{v_t} dB_{1t}, \tag{4}$$

where $\{B_{1t}\}$ is a standard Brownian motion in $\mathbb{R}$ driving capital depreciation shocks.[3] Except for time-varying volatility, technology specifications in (3) and (4) are common in the literature, such as Epaularda and Pommeret (2003).

The variable v_t in (4), which denotes the variance of capital depreciation shocks, follows a mean-reverting CIR process proposed by Cox, Ingersoll and Ross (1985):[4]

$$dv_t = \alpha (m - v_t) dt + \beta \sqrt{v_t} dB_{2t}, \tag{5}$$

where $\{B_{2t}\}$ is another standard Brownian motion in $\mathbb{R}$, and parameters α, m and β denote, respectively, the mean-reverting rate, the mean volatility level, and the volatility of volatility.

The two random variables B_{1t} and B_{2t} characterize, respectively, the level risk and the

[3] The specification of capital depreciation shocks helps keep the model analytically tractable. This type of shock has been used as a reduced-form way to model business cycles in the literature, e.g., Epaularda and Pommeret (2003), Barro (2009), and Gertler and Karadi (2011). Capital depreciation shocks are realistic for wars or natural disasters, but they can also be interpreted as shocks to the quality of existing capital by affecting the effective units of capital brought from the previous period.

[4] Apart from its analytical simplicity, the CIR process in (5) has been broadly adopted in finance to capture the dynamics of stock price volatility, one important measure of macroeconomic uncertainty proposed by the existing literature of uncertainty shocks, and thus is a reasonable volatility specification.

volatility risk in capital accumulation, since shocks $\{dB_{1t}\}$ in (4) directly determine the level of future capital, whereas shocks $\{dB_{2t}\}$ in (5) indirectly affect capital accumulation by changing the dispersion of level shocks. For expositional simplicity, I assume that level shocks and volatility shocks are independent, so the paper focuses on connections between volatility shocks and economic activity that arise through the model's internal economic structure rather than through purely statistical channels between the two types of shocks. Denote

$$\xi = \frac{2\alpha m}{\beta^2} \text{ and } \theta = \frac{\beta^2}{2\alpha}. \tag{6}$$

As shown in Cox, Ingersoll and Ross (1985), when $\alpha > 0$ and $m > 0$, the CIR process $\{v_t\}$ in (5) stays non-negative after starting with a positive value, and has a stationary Gamma distribution with the shape parameter ξ and the scale parameter θ. The associated stationary density function is

$$\eta(v) = \frac{e^{-\frac{v}{\theta}} v^{\xi-1}}{\Gamma(\xi)\, \theta^{\xi}}, \tag{7}$$

where $\Gamma(x) = \int_0^\infty e^{-z} z^{x-1} dz$ represents the Gamma function. The unconditional mean and variance of $\{v_t\}$ in the stationary distribution (7) equal m and $\frac{m\beta^2}{2\alpha}$, respectively.

2.3 The Social Planner's Problem

The Pareto optimal allocation of the economy is that given initial capital stock K and volatility level v, the social planner chooses an optimal consumption stream to maximize the representative household's utility given by (1) and (2), subject to feasibility constraints (3) to (5). Let $J : \mathbb{R}_2 \to \mathbb{R}$, $(K, v) \mapsto J(K, v)$ and $C : \mathbb{R}_2 \to \mathbb{R}$, $(K, v) \mapsto C(K, v)$ be the value function and the consumption function of the social planner's problem, which is characterized by the following Hamilton-Jacobi-Bellman (HJB) equation:[5]

$$0 = \max_C \left\{ f(C, J) + (AK - C) J_K + \alpha (m - v) J_v + \frac{1}{2} v K^2 J_{KK} + \frac{1}{2} \beta^2 v J_{vv} \right\}. \tag{8}$$

Applying the conjecture-verify method, one can show that

$$J(K, v) = g(v) \frac{K^{1-\gamma}}{1-\gamma},\; C(K, v) = \rho^{\psi} g(v)^{\frac{1-\psi}{1-\gamma}} K, \tag{9}$$

[5] For a given function $u(x, y)$, I use u_x, u_y, u_{xx} and u_{yy} to denote derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$, respectively.

where $g(\cdot)$ is a function satisfying the following ordinary differential equation (ODE):

$$\frac{1}{2}\beta^2 v g''(v) + \alpha(m-v)g'(v) + \left(\begin{array}{c} \rho\omega^{-1}\left(\rho^{\psi-1}g(v)^{-\omega\psi} - 1\right) + \\ (1-\gamma)\left(A - \rho^{\psi}g(v)^{-\omega\psi} - \frac{1}{2}\gamma v\right) \end{array}\right) g(v) = 0, \tag{10}$$

with the parameter ω defined as

$$\omega = \frac{1-\psi^{-1}}{1-\gamma}. \tag{11}$$

I now use the benchmark model to analyze the consequences of a temporary increase in uncertainty. Such an examination is useful, since it can evaluate whether the model is capable of generating empirically plausible predictions. The results are summarized in Proposition 1.

Proposition 1 *Suppose at time t_0, the economy is hit by a one-time adverse volatility shock, i.e., $dB_{2t_0} > 0$, $dB_{2t} = 0$ for $t > t_0$ (uncertainty increases suddenly at t_0), and there are no level shocks, i.e., $dB_{1t} = 0$ for $t \geq t_0$. It follows that*

1. *if $\psi < 1$, then the growth rates of output and consumption increase immediately at t_0 and decline afterwards.*
2. *if $\psi > 1$, then the growth rate of output first decreases at t_0 and then gradually picks up as time passes; if, in addition, the volatility shock at t_0 is large enough, i.e., $v_{t_0} > v^*$ for some number v^*, then consumption growth also falls at t_0 and rises gradually afterwards.*

Proof. I first derive an approximate analytical solution to ODE (10) using an approximation technique similar to that adopted in Chacko and Viceira (2005). Define $d(v) = \log\left(\frac{C(K,v)}{K}\right)$, $d_0 = E_v[d(v)]$, and $q_0 = e^{d_0}$, where $C(K,v) = \rho^{\psi}g(v)^{-\omega\psi}K$ is the optimal consumption function given in (9). Note that q_0 is positive by construction and its value is endogenously determined within the model. Linearizing the function $\frac{C(K,v)}{K} = e^{d(v)}$ around d_0 yields $\rho^{\psi}g(v)^{-\omega\psi} = \frac{C(K,v)}{K} \approx e^{d_0}(1-d_0) + e^{d_0}d(v)$, by which (10) can be rewritten as

$$\frac{1}{2}\beta^2 v\hat{g}''(v) + \alpha(m-v)\hat{g}'(v) + \left(\begin{array}{c} e(q_0) - q_0\log\hat{g}(v) + \\ (1-\gamma)\left(\mu - \frac{1}{2}\gamma v\right) \end{array}\right)\hat{g}(v) = 0. \tag{12}$$

Here $\mu = \rho\log\rho + A - \rho$, $e(\cdot)$ is a function defined as

$$e(q) = \omega^{-1}\psi^{-1}\left(q\left(1+\psi\log\rho - \log q\right) - \rho\left(1+\psi\log\rho - \log\rho\right)\right), \tag{13}$$

and q_0 is a constant given by $q_0 = \exp\left\{E_v\left[\log\left(\rho^{\psi}g(v)^{-\omega\psi}\right)\right]\right\}$, where $E_v[\cdot]$ denotes the expectation with respect to the invariant distribution (7), and $g(\cdot)$ is the exact solution to

(10). It is straightforward to show that the approximate ODE in (12) is solved by $\hat{g}(v) = \exp\{\hat{a}_0 + \hat{a}_1 v\}$, where parameters $\hat{a}_0$ and $\hat{a}_1$ are given by

$$\hat{a}_0 = \frac{\alpha m \hat{a}_1 + \mu(1-\gamma) + e(q_0)}{q_0}, \ \hat{a}_1 = \frac{\alpha + q_0 - \sqrt{(\alpha+q_0)^2 + \beta^2\gamma(1-\gamma)}}{\beta^2}. \tag{14}$$

I next use the approximate solution to examine the impact of the given volatility shock at t_0 on the growth rates of output and consumption. Note by (3) and (4), one has

$$\frac{dY_t}{Y_t} = f_Y(v_t)\,dt + \sqrt{v_t}dB_{1t}, \quad f_Y(v_t) = A - \rho^{\psi}\exp\left\{\frac{1-\psi}{1-\gamma}(\hat{a}_0 + \hat{a}_1 v_t)\right\}. \tag{15}$$

Given $dB_{1t} = 0$ for $t > t_0$, it follows from (15) that the growth rate of output equals $f_Y(v_t)$ after t_0. By (15), $f_Y(v_t)$ decreases (increases) with v_t if $\psi > (<)\,1$. This implies that if $\psi > 1$, the growth rate of output decreases at t_0 and then picks up as volatility reverts down to its pre-shock level, whereas the opposite holds if $\psi < 1$. Meanwhile, by Ito's lemma, one obtains

$$\frac{dC_t}{C_t} = f_C(v_t)\,dt + \sqrt{v_t}dB_{1t} + \frac{1-\psi}{1-\gamma}\hat{a}_1\beta\sqrt{v_t}dB_{2t}, \tag{16}$$

$$f_C(v_t) = f_Y(v_t) + \frac{1-\psi}{1-\gamma}\left(\alpha(m - v_t)\hat{a}_1 + \frac{1-\psi}{2(1-\gamma)}\hat{a}_1^2\beta^2 v_t\right). \tag{17}$$

Given $dB_{1t} = dB_{2t} = 0$ for $t > t_0$, the growth rate of consumption equals $f_C(v_t)$ by (16) after t_0. One can show that $f_C'(v_t) > 0$ if $\psi < 1$, and $f_C'(v_t) < 0$ if $\psi > 1$ and

$$v_t > v^* = \left(\frac{1-\psi}{1-\gamma}\hat{a}_1\right)^{-1}\log\left(\rho^{-\psi}e^{-\frac{1-\psi}{1-\gamma}\hat{a}_0}\left(\frac{1-\psi}{1-\gamma}\frac{\hat{a}_1\beta^2}{2} - \alpha\right)\right). \tag{18}$$

Thus, if $\psi < 1$, the growth rate of consumption increases after the volatility shock, while if $\psi > 1$, the opposite is true as long as the shock is large enough. ■

Proposition 1 shows that when the IES is greater than one, which is an empirically plausible assumption,[6] a large increase in volatility can decelerate the growth of output and consumption in the absence of any level shocks, a result consistent with related findings in the existing literature of uncertainty shocks. In the face of elevated uncertainty about future capital returns, the risk-averse representative household would prefer to spend a greater fraction of its current wealth on consumption,[7] with the resulting decreased investment in the firm slowing the pace of capital accumulation and output growth after the volatility shock. Subsequently, future consumption falls relative to current consumption, implying a slower growth in consumption.

[6] See, for example, Bansal and Yaron (2004) and Barro (2009).

[7] If the IES is greater (smaller) than one, the household's marginal propensity to consume increases (decreases) with volatility, due to the interactions of substitution effect and income effect arising from changes in volatility.

The proposition also suggests that for the model to generate plausible impulse responses to volatility shocks, the IES has to be greater than one. This justifies the use of recursive utility rather than CRRA utility in the analysis, since in the latter case the IES is the reciprocal of the risk aversion, and a risk aversion greater than one, a widely accepted assumption, would necessarily imply an IES below unity. The recursive utility function, however, does not impose the same restriction, because the two parameters are determined separately.

3 Volatility Shocks as Preference Shocks

This section presents the main results of the paper by establishing an equivalence result between volatility shocks and a certain class of preference shocks.

Consider an economy differing from its benchmark counterpart in Section 2 in two dimensions. First, instead of varying over time as in (5), the variance of capital depreciation shocks is fixed at m, which equals the steady-state volatility level of the CIR process in (5). Accordingly, the capital stock accumulates over time as

$$d\tilde{K}_t = \tilde{I}_t dt + \tilde{K}_t \sqrt{m} d\tilde{B}_{1t}, \tag{19}$$

where $\left\{\tilde{B}_{1t}\right\}$ is a standard Brownian motion on $\mathbb{R}$ representing capital depreciation shocks. For easy differentiation, a tilde is used to denote a variable in the newly constructed economy throughout the paper. Mathematically, equation (19) is a special case of (4) by setting either the mean-reverting rate $\alpha = \infty$ or the volatility of volatility $\beta = 0$ in (5), so volatility shocks are turned off.

The second difference is that in this new economy the representative household is subject to periodic preference shocks, and its utility process $\left\{\tilde{V}_t\right\}$ has a recursive representation as

$$\tilde{V}_t = E_t\left[\int_t^\infty \tilde{f}\left(\tilde{C}_s, \varphi_s, \tilde{V}_s\right) ds\right], \tag{20}$$

where the function $\tilde{f}$ takes the form

$$\tilde{f}\left(\tilde{C}, \varphi, \tilde{V}\right) = \rho \frac{1-\gamma}{1-\psi^{-1}} \tilde{V} \left(\frac{\tilde{C}^{1-\psi^{-1}}}{\left(\left(1-\gamma\right)\tilde{V}\right)^{\left(1-\psi^{-1}\right)/(1-\gamma)}} + \left(1-\psi^{-1}\right)\varphi - 1\right), \tag{21}$$

and the process $\{\varphi_t\}$ evolves according to

$$d\varphi_t = -\alpha\varphi_t dt - \frac{\gamma}{2\rho}\beta\sqrt{m - \frac{2\rho}{\gamma}\varphi_t} d\tilde{B}_{2t}, \tag{22}$$

with $\left\{\tilde{B}_{2t}\right\}$ being a standard Brownian motion on $\mathbb{R}$ independent of $\left\{\tilde{B}_{1t}\right\}$.

By comparing (21) to (2), it is evident that the term φ in (21) represents a preference shock to the household, where all else equal a larger φ raises the utility of the household as $\tilde{f}_\varphi\left(\tilde{C},\varphi,\tilde{V}\right)>0$. As will be clear later, the preference-shock term φ influences the household's consumption-saving decisions by affecting its marginal utility of consumption, a mechanism similar in spirit to studies based on recursive preferences, such as Albuquerque et al. (2016). For expositional convenience, this newly constructed economy is named the economy with preference shocks, defined together by equations (3), and (19) to (22).

Proposition 2 *Let the Pareto optimal allocation in the economy with preference shocks be represented by the value function* $\tilde{J}:\mathbb{R}_2\to\mathbb{R}$, $\left(\tilde{K},\varphi\right)\mapsto\tilde{J}\left(\tilde{K},\varphi\right)$ *and the consumption function* $\tilde{C}:\mathbb{R}_2\to\mathbb{R}$, $\left(\tilde{K},\varphi\right)\mapsto\tilde{C}\left(\tilde{K},\varphi\right)$. *Then, it holds that*

$$J\left(K,v\right)=\tilde{J}\left(K,\varphi\left(v\right)\right)\ \text{ and } C\left(K,v\right)=\tilde{C}\left(K,\varphi\left(v\right)\right),\text{ for all } \left(K,v\right)\in\mathbb{R}_2 \tag{23}$$

where the function φ *is defined as*

$$\varphi\left(v\right)=\frac{\gamma}{2\rho}\left(m-v\right), \tag{24}$$

and functions $J\left(\cdot\right)$ *and* $C\left(\cdot\right)$ *are given in (9).*

Proof. By definition, the value function $\tilde{J}$ and the consumption function $\tilde{C}$ must satisfy the following HJB equation:

$$0=\max_{\tilde{C}}\left\{\tilde{f}\left(\tilde{C},\varphi,\tilde{J}\right)+\left(A\tilde{K}-\tilde{C}\right)\tilde{J}_{\tilde{K}}-\alpha\varphi\tilde{J}_\varphi+\frac{1}{2}m\tilde{K}^2\tilde{J}_{\tilde{K}\tilde{K}}+\frac{\gamma^2}{8\rho^2}\beta^2\left(m-\frac{2\rho}{\gamma}\varphi\right)\tilde{J}_{\varphi\varphi}\right\}. \tag{25}$$

Conjecture $\tilde{J}$ is of the form

$$\tilde{J}\left(\tilde{K},\varphi\right)=h\left(\varphi\right)\frac{\tilde{K}^{1-\gamma}}{1-\gamma} \tag{26}$$

for some unknown function h. The first-order condition of equation (25) with respect to $\tilde{C}$ is $\tilde{f}_{\tilde{C}}\left(\tilde{C},\varphi,\tilde{J}\right)=\tilde{J}_{\tilde{K}}\left(\tilde{K},\varphi\right)$, which by the functional forms of $\tilde{f}$ and $\tilde{J}$ imply the optimal consumption function

$$\tilde{C}\left(\tilde{K},\varphi\right)=\rho^\psi h\left(\varphi\right)^{\frac{1-\psi}{1-\gamma}}\tilde{K}. \tag{27}$$

Substituting (26) and (27) into (25) leads to that $h\left(\cdot\right)$ satisfies the following ODE:

$$\frac{\gamma^2}{8\rho^2}\beta^2\left(m-\frac{2\rho}{\gamma}\varphi\right)h''\left(\varphi\right)-\alpha\varphi h'\left(\varphi\right)+\left(\begin{array}{c}\rho\omega^{-1}\left(\rho^{\psi-1}h\left(\varphi\right)^{-\omega\psi}-1+\varphi\left(1-\psi^{-1}\right)\right)+\\ \left(1-\gamma\right)\left(A-\rho^\psi h\left(\varphi\right)^{-\omega\psi}-\frac{1}{2}\gamma m\right)\end{array}\right)h\left(\varphi\right)=0. \tag{28}$$

One can directly verify that the function

$$h(\varphi) = g\left(m - \frac{2\rho}{\gamma}\varphi\right) \tag{29}$$

solves ODE (28), where $g(\cdot)$ is defined by (10). Therefore, by (24) and (29), one has $h(\varphi(v)) = g(v)$, and thus $J(K,v) = \tilde{J}(K,\varphi(v))$ and $C(K,v) = \tilde{C}(K,\varphi(v))$ from (9), (26) and (27). ■

Proposition 2 says that all else equal, the household in the benchmark economy with volatility equal to v would choose the same amount of consumption and achieve the same level of lifetime utility as the household in the economy with preference shocks when its associated preference term equals $\varphi(v)$. This result stems from the equivalence in the household's marginal utility of consumption in the two economies, since $f_C(C, J(K,v)) = \tilde{f}_{\tilde{C}}\left(\tilde{C}, \varphi(v), \tilde{J}(K,\varphi(v))\right)$ for all $(K,v) \in \mathbb{R}_2$. As a consequence, volatility shocks can be mapped into a special class of preference shocks with the underlying link represented by the linear function $\varphi(\cdot)$ in (24). According to the form of the function, an increase in volatility $(dv > 0)$ resembles a decrease in the preference term $(d\varphi = -\frac{\gamma}{2\rho}dv < 0)$, whose size increases with the risk aversion γ and decreases with the rate of time preference ρ. Intuitively, the more risk-averse (larger γ) or the more patient (smaller ρ) the representative household, the greater the impact of changes in uncertainty about future capital returns. Note by (24) the unconditional mean of preference shocks $\{\varphi(v_t)\}$ is zero since the long-run average of the process $\{v_t\}$ in (5) equals m.

The next proposition describes the key result of this paper of equivalence in the way macroeconomic aggregates respond to volatility shocks and preference shocks.

Proposition 3 *Suppose at the beginning of time t_0, the benchmark economy has capital stock K_{t_0} and volatility level m, while the economy with preference shocks has capital stock K_{t_0} and preference term $\varphi = 0$. At time t_0, the two economies are hit, respectively, by a one-time adverse volatility shock, i.e., $dB_{2t_0} > 0$, $dB_{2t} = 0$ for $t > t_0$ (uncertainty increases suddenly at t_0), and a one-time negative preference shock, i.e., $d\tilde{B}_{2t_0} = dB_{2t_0} > 0$, $d\tilde{B}_{2t} = 0$ for $t > t_0$ (current-period utility decreases immediately at t_0). There are no level shocks in either economy, i.e., $dB_{1t} = d\tilde{B}_{1t} = 0$ for $t \geq t_0$. Then, it holds that*

$$\varphi_t = \varphi(v_t)\,, \; K_t = \tilde{K}_t,\; Y_t = \tilde{Y}_t,\; C_t = \tilde{C}_t,\; I_t = \tilde{I}_t,\; t \geq t_0, \tag{30}$$

where φ_t represents the preference term evolving according to (22).

Proof. By (5) and the condition $dB_{2t} = 0$ for $t > t_0$, the volatility process right after the volatility shock at t_0 can be written as $dv_t = \alpha(m - v_t)\,dt$ with $v_{t_0} = m + \beta\sqrt{m}dB_{2t_0}$, which implies that $v_t = e^{\alpha(t_0 - t)}v_{t_0} + \left(1 - e^{\alpha(t_0-t)}\right)m$, $t > t_0$. Similarly, the dynamics of the preference term φ_t after t_0 can be expressed as $d\varphi_t = -\alpha\varphi_t dt$ with $\varphi_{t_0} = -\frac{\gamma}{2\rho}\beta\sqrt{m}d\tilde{B}_{2t_0}$, which implies that $\varphi_t = \varphi_{t_0} e^{\alpha(t_0 - t)}$, $t > t_0$. Given $d\tilde{B}_{2t_0} = dB_{2t_0}$, one has

$$\varphi_t = \frac{\gamma}{2\rho}(m - v_t) = \varphi(v_t), \; t \geq t_0. \tag{31}$$

By (3), (4) and (9), the growth rate of capital in the benchmark economy is

$$\frac{dK_t}{K_t} = \left(A - \rho^{\psi} g(v_t)^{\frac{1-\psi}{1-\gamma}}\right) dt + \sqrt{v_t} dB_{1t} = \left(A - \rho^{\psi} g(v_t)^{\frac{1-\psi}{1-\gamma}}\right) dt, \; t > t_0, \tag{32}$$

where the last equality is due to the absence of level shocks, i.e., $dB_{1t} = 0$ for $t \geq t_0$. Similarly, the growth rate of capital in the economy with preference shocks equals

$$\frac{d\tilde{K}_t}{\tilde{K}_t} = \left(A - \rho^{\psi} h(\varphi_t)^{\frac{1-\psi}{1-\gamma}}\right) dt, \; t > t_0. \tag{33}$$

By (24), (29) and (31), it follows that

$$h(\varphi_t) = h(\varphi(v_t)) = g\left(m - \frac{2\rho}{\gamma}\varphi(v_t)\right) = g(v_t), \tag{34}$$

implying that $dK_t/K_t = d\tilde{K}_t/\tilde{K}_t$ for $t > t_0$. Because capital equals K_{t_0} at time t_0 in both economies, one has for $t \geq t_0$, $K_t = \tilde{K}_t$ and thus $Y_t = AK_t = A\tilde{K}_t = \tilde{Y}_t$. Consequently, by (9), (27) and (34), it holds that

$$C_t = C(K_t, v_t) = \rho^{\psi} g(v_t)^{\frac{1-\psi}{1-\gamma}} K_t = \rho^{\psi} h(\varphi_t)^{\frac{1-\psi}{1-\gamma}} \tilde{K}_t = \tilde{C}\left(\tilde{K}_t, \varphi_t\right) = \tilde{C}_t, \; t \geq t_0$$

showing that the paths of consumption are identical in the two economies. Since investment is the difference between output and consumption, $I_t = \tilde{I}_t$ for $t \geq t_0$. ■

Proposition 3 states that the impulse responses of macroeconomic aggregates to an adverse volatility shock in the benchmark economy are exactly the same as those to a negative preference shock in the economy with preference shocks. More precisely, as seen in the proof of the proposition, the growth dynamics of capital, output, consumption and investment driven by a sudden increase in volatility $dv_{t_0} = \beta\sqrt{m} dB_{2t_0} > 0$ (due to a one-time innovation dB_{2t_0} in the volatility process (5)) are equivalent to that driven by a sudden decrease in the current-period utility $d\varphi_{t_0} = -\frac{\gamma}{2\rho}\beta\sqrt{m} d\tilde{B}_{2t_0} < 0$ (due to a one-time innovation $d\tilde{B}_{2t_0} = dB_{2t_0}$ in the preference process (22)). Lying at the heart of this equivalence result is that as shown in (23), given the same initial capital stock, the household adjusts its consumption to the subsequent series of volatility $\{v_t\}_{t \geq t_0}$ in the same manner as it does to the subsequent series of preference term $\{\varphi(v_t)\}_{t \geq t_0}$, which leads to identical paths of investment, capital and output.

The above revealed equivalence between volatility shocks and preference shocks is not only interesting in theory, but is also useful for gaining insight into the economic impact of volatility shocks. Propositions 2 and 3 together uncover an explicit mapping from volatility shocks to preference shocks, such that an increase in uncertainty acts like a negative preference shock with regard to the impulse responses of macroeconomic aggregates. The existence of such a mapping allows one to interpret volatility shocks as preference shocks, which makes it straightforward to understand why a rise in volatility has negative effects on economic activity as shown in the literature. The finding of the mapping also closely relates the current paper to Leduc and Liu (2015), who argue that volatility shocks operate as aggregate demand shocks. The authors document from the data that an increase in uncertainty leads to higher unemployment and lower inflation, a result robust to alternative uncertainty measures, identification strategies, and model specifications. They show that the demand-shock-like macroeconomic effects of volatility shocks can arise in a DSGE model with labor search frictions and nominal rigidities. This paper complements the work of Leduc and Liu (2015) by deriving an exact mapping between volatility shocks and preference shocks, and thus providing a theoretical justification for the demand-shock nature of volatility shocks.[8]

4 Conclusion

This paper presents an intuitive interpretation of volatility shocks using a tractable endogenous growth model with recursive preferences and stochastic volatility. I show that there exists an explicit mapping between volatility shocks and preference shocks, such that a rise in volatility generates the same impulse responses of macroeconomic aggregates as a properly specified negative preference shock. To preserve tractability and economic intuition, the model is deliberately kept simple, so the paper can explicitly analyze the impact of volatility shocks. There are limitations in the study. For example, the model assumes linear production technology and abstracts from labor markets. It would be very interesting to explore the existence or even characterize the properties of some mapping between volatility shocks and preference shocks in a model that incorporates features such as nonlinear technology or labor supply. In light of the findings in Leduc and Liu (2015) and this paper, it is reasonable to believe that the close relationship between the two type of shocks would still hold in a more general setting.

[8] The similarity between volatility shocks and preference shocks also sheds light on why volatility shocks have success in explaining empirical regularities observed in the data. For example, Basu and Bundick (2012) argue that volatility shocks can generate comovements among key macroeconomic variables consistent with business cycles. However, this would not be a surprise if one thinks of volatility shocks as preference shocks, since earlier studies, such as Gali (1999) and Ireland (2004), have shown that preference shocks, or more broadly demand shocks, are typically effective in producing the classic business cycle patterns.

References

[1] Albuquerque, R., M. Eichenbaum, V. Luo, and S. Rebelo (2016), "Valuation Risk and Asset Pricing," *Journal of Finance*, forthcoming.

[2] Arellano, C., Y. Bai, and P. Kehoe (2012), "Financial Markets and Fluctuations in Uncertainty," Federal Reserve Bank of Minneapolis Research Department Staff Report.

[3] Bansal, R. and A. Yaron (2004), "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles," *Journal of Finance*, 59(4), 1481–1509.

[4] Barro, R.J. (2009), "Rare Disasters, Asset Prices, and Welfare Costs," *American Economic Review*, 99(1), 243–264.

[5] Basu, S. and B. Bundick (2012), "Uncertainty Shocks in a Model of Effective Demand," NBER Working Paper, No. 18420.

[6] Bloom, N. (2009), "The Impact of Uncertainty Shocks," *Econometrica*, 77(3), 623–685.

[7] Bloom, N., M. Floetotto, N. Jaimovich, I. Saporta-Eksten, and S.J. Terry (2012), "Really Uncertain Business Cycles," NBER Working Paper, No. 18245.

[8] Chacko, G. and L.M. Viceira (2005), "Dynamic Consumption and Portfolio Choice with Stochastic Volatility in Incomplete Markets," *Review of Financial Studies*, 18(4), 1369–1402.

[9] Christiano, L.J., R. Motto, and M. Rostagno (2014), "Risk Shocks," *American Economic Review*, 104(1), 27–65.

[10] Cox, J.C., J.E. Ingersoll, and S.A. Ross (1985), "A Theory of the Term Structure of Interest Rates," *Econometrica*, 53(2), 385–407.

[11] Duffie, D. and L.G. Epstein (1992), "Stochastic Differential Utility," *Econometrica*, 60(2), 353–394.

[12] Epaularda, A. and A. Pommeret (2003), "Recursive utility, Endogenous Growth, and the Welfare Cost of Volatility," *Review of Economic Dynamics*, 6(3), 672–684.

[13] Epstein, L.G. and S.E. Zin (1989), "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, 57(4), 937–969.

[14] Fernández-Villaverde, J., P. Guerrón-Quintana, J. Rubio-Ramírez, and M. Uribe (2011), "Risk Matters: The Real Effects of Volatility Shocks," *American Economic Review*, 101(6), 2530–2561.

[15] Fernández-Villaverde, J., P. Guerrón-Quintana, K. Kuester, and J. Rubio-Ramírez (2015), "Fiscal Volatility Shocks and Economic Activity," *American Economic Review*, 105(11), 3352–3384.

[16] Gali, J. (1999), "Technology, Employment, and the Business Cycle: Do Technology Shocks Explain Aggregate Fluctuations?" *American Economic Review*, 89(1), 249–271.

[17] Gertler, M. and P. Karadi (2011), "A Model of Unconventional Monetary Policy," *Journal of Monetary Economics*, 58(1), 17–34.

[18] Ireland, P.N. (2004), "Technology Shocks in the New Keynesian Model," *Review of Economics and Statistics*, 86(4), 923–936.

[19] Leduc, S. and Z. Liu (2015), "Uncertainty Shocks are Aggregate Demand Shocks," Federal Reserve Bank of San Francisco Working Paper, No. 2012-10.