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Abstract

Plain vanilla options have a single underlying asset and a single condition on the payoff at the expiration date. For this class of options, a well-known result of Duffie, Pan and Singleton (2000) shows how to invert the characteristic function to obtain a closed-form formula for their prices. However, multiple-asset and multiple-condition derivatives such as rainbow options cannot be priced within this framework. Utilizing inversion of the Fourier transform – and resorting to neither the Black-Scholes framework nor the affine models settings – the authors provide an analytical solution for options whose payoffs depend on two or more conditions. Numerical experiments based on the multiple-asset and multiple-condition derivatives are provided to illustrate the usefulness of the proposed approach.

JEL classification: G12
Bank classification: Asset pricing

Résumé

Les options classiques n'ont qu'un sous-jacent, et leur valeur à l'échéance est déterminée par une seule condition. Un des apports bien connus de l'étude de Duffie, Pan et Singleton (2000) est d'avoir montré comment utiliser la transformée inverse de la fonction caractéristique pour obtenir une formule analytique des prix dans cette classe d'options. Or, les prix des dérivés à plusieurs sous-jacents et conditions, tels que les options arc-en-ciel, ne peuvent être évalués à l'intérieur de ce cadre. En s'appuyant sur la formule d'inversion de la transformée de Fourier, mais sans recourir au modèle de Black et Scholes ni aux modèles affines, les auteurs proposent une solution analytique pour calculer le prix d'options dont la valeur à l'échéance est déterminée par deux conditions ou plus. Ils procèdent à des expériences numériques afin d'illustrer l'utilité de l'approche proposée dans le cas des dérivés à plusieurs sous-jacents et conditions.

Classification JEL : G12
Classification de la Banque : Évaluation des actifs

Non-Technical Summary

Derivatives (or options) pricing is an important topic for both academics and practitioners. Two approaches are generally considered when pricing a derivative, namely numerical methods and closed-form expressions. There are numerous advantages of having closed-form derivatives prices, among them: (1) they enable a quick and efficient computation of the price, (2) they can be used to evaluate derivatives' price sensitivity to a given parameter, and (3) they are useful in qualitative analysis. Unfortunately, combining closed-form and realistic dynamics (on assets underlying the derivative) has proven very challenging.

Other works in the literature evaluate derivatives written on a single stock. However, this paper proposes a closed-form approach to evaluate derivatives written on several stock prices. Besides providing users with an analytical solution to multivariate derivative pricing, we decompose the resulting prices and interpret each component.

In a setting where a derivative written on two assets has been evaluated, we can show that the resulting price has three intuitive components. The first, termed the "constant'' component, changes mostly with the price of the underlying assets. The "constant" component can be assimilated to the premium that derivatives buyers are willing to pay to hedge against variations in the level of the stock. The second is referred to as the "volatility'' component, and is found to be mostly correlated to the average volatility across underlying assets. It can be interpreted as the premium paid by options buyers to hedge against variation in individual volatilities. The final component is designated as the "correlation'' component, since it co-moves with correlation among underlying assets. Hence we interpret this last piece as the premium paid by options buyers to hedge against variations in the correlation of underlying assets.

The paper highlights several interesting methodological aspects, but it leaves out some real-world issues such as risk-neutral parameters calibration and risk-neutral distribution fitting. Our approach would be useful in these and other areas as more options data become available.

1 Introduction

In a contingent claims valuation, the lack of closed-form solutions for derivatives may undermine its practical use and pose significant implementation hurdles. The analytical formula embedded in the Black-Scholes settings[1] substantially lessens the computational burden of options pricing. Unfortunately, the closed-form price derived under the Black-Scholes assumptions cannot be carried over, even for one-dimensional derivatives, when accounting for well-known stylized facts about returns.[2] It is much more challenging to price options with several dimensions and multiple payoff conditions, since the high dimensionality often raises the computational cost.

Furthermore, adding flexibility to benchmark models such as Black-Scholes often comes with a cost: the loss of its applicability. Heston (1993) shows how to get a closed-form price to European options when the underlying asset features both stochastic volatility and the leverage effect. Duffie et al. (2000) generalize Heston's framework to any univariate time-series model with a closed-form characteristic function. Yet, this paper goes beyond the univariate setting, by proposing an analytical solution to option pricing within a multivariate framework where, in addition to time variations of individual volatilities, we also have time-varying correlations. When departing from the Black-Scholes valuation framework, professionals often resort to numerical schemes and simulations to value exotic options (e.g., Glasserman, 2003; Duffy, 2009). However, numerical techniques and simulations share common flaws: dimensionality[3] and the high computational burden.

This research underscores the usefulness of the Fourier-Stieltjes transform[4] in deriving a semi-analytical solution where the computational task of options pricing comes down to the evaluation of the Laplace transform. The relevance of our semi-analytical solution is based on two main arguments. First, its empirical applicability is enhanced by virtue of an analytical formula, which can be used in calibration problems. Thus, some parameters of underlying asset distribution (implied parameters) can be recovered directly from available options prices. Moreover, contrary to the historical data on underlying assets, options data are forward looking, since they can be used to infer the underlying asset distribution. Thus we

[1] The main assumptions of the Black-Scholes model are: a continuous stochastic process, constant volatility, the efficient market hypothesis and log-normal distributions.

[2] Empirical facts that do not support Black-Scholes assumptions include: stochastic volatility, jumps, fat tails, skewness and leverage effects.

[3] It is worth noting that the Monte Carlo simulations do not exhibit the curse of dimensionality. However, as we will show in our empirical section, the high computation time in the Monte Carlo simulations is a concern.

[4] In the remainder of the paper, we will be using Laplace transform, characteristic function and Fourier-Stieltjes transform interchangeably.

can use our closed-form options prices to fit observed prices, and provide new insights which are missing from historical underlying assets data.[5] Second, one of the greatest virtues of the Laplace transform is its versatility. Indeed, unlike numerical methods, the Fourier transform can be implemented in a numerically stable way (Davies, 1973). In fact, whenever the characteristic function of the state variable exists in closed form, how to infer a closed-form formula for options prices is well known (Duffie et al., 2000; Dufresne et al., 2009). However, to the best of our knowledge, inferring multiple-condition contingent claims with the Fourier transform has not yet been addressed in the literature. This omission is particularly glaring, since: (a) basket options are by far the most liquid contingent claims for investors,[6] and (b) the correlation and volatility parameters are the crucial drivers for pricing basket options (see Buraschi et al., 2014; Qu, 2010), particularly under adverse market conditions. In order to fill this void, this paper provides the framework to value options on multiple-asset and multiple-payoff conditions.

Our work is closely related to that of Duffie et al. (2000), where the closed-form price for a single condition is derived. Those authors exploit the well-established exponential affine form of the characteristic function embedded in the affine process in order to compute option prices. Although the affine processes have attracted much attention from both professional and academic audiences, for the theory developed in this paper, the only requirement is the existence of the Laplace transform in closed form.[7] For the empirical investigation, we use structured products written on the NYSE and NASDAQ indices. Since both indices are market-value weighted, they can be used in risk management and portfolio optimizations.

To illustrate concretely how our pricing formulas work, we develop a simple, theoretically grounded and broadly applicable multivariate model (affine realized variance, or ARV) that captures individual and joint dynamics in several stock

[5] For example, Navatte and Villa (2000) demonstrate that the recovered implied moments of the CAC 40 significantly improved out-of-sample pricing performance; Buss and Vilkov (2012) find that using options data to construct implied factor betas significantly outperforms historical betas in the capital asset pricing model; Kempf et al. (2014) report substantial portfolio gains when relying solely on the plain vanilla options to calibrate a covariance matrix; Buraschi et al. (2014) demonstrate the usefulness of implied correlations in the factor returns construction; and van Binsbergen et al. (2012) use well-known put-call parity (Stoll, 1969) to construct a term structure analog of dividend yields for the market portfolio by calibrating implied dividend yields from the option contracts.

[6] In Canada, for example, Structured-Retails-Products (2013) report that the notional amount of options written on basket was 2.865 billion CAD, whereas that of single shares was only 16 million CAD in 2013.

[7] Many papers compute the closed-form expression of the characteristic function: notably, the affine models literature, which includes Bates (1996), Bakshi et al. (1997), Duffie et al. (2003), and Christoffersen et al. (2006). Other papers, such as Feunou and Meddahi (2009), provide a general framework that characterizes the infinite-order affine model. Some prominent examples of non-affine models are Ahn and Gao (1999); Feunou and Fontaine (2010).

market returns. Because the ARV model is affine, its conditional characteristic function at any horizon is available in closed form. We find conclusive evidence that options prices in the ARV model are similar to prices in the standard benchmark, non-affine dynamic conditional correlation (DCC) model of Engle (2000). Within the ARV model, Monte Carlo simulations are used as an alternative to our closed-form approach. While both approaches provide similar options prices, the computation time for the Monte Carlo approach is about twelve times that of ours. Furthermore, beyond elegant mathematical expression, the advantage of transform analysis is its ability to break down multivariate options prices into intuitive components. Thus, this paper represents a significant step toward better understanding multivariate options pricing drivers and their impacts on effective risk assessment.

The remainder of this paper is organized as follows. Section 2 is devoted to the set-up. Section 3 states the main results and section 4 introduces the ARV model. Section 5 describes how to use our theoretical framework to price rainbow options within the ARV model. Section 6 provides concluding remarks. To preserve the main thread of the paper, we have placed many of the technical proofs and details in appendices.

2 The set-up

2.1 Notation and setting

We consider a financial market with derivatives depending on multivariate state variables:

- $X_t = (X_1, X_2, \ldots, X_n)_t$ is a column vector of n state variables at time t.
- $(X_i)_{t \geq 0}$ denotes the process of asset $i, i = 1, 2, \ldots, n$ defined in $D \subseteq \mathbb{R}^n$.
- $u = (u_1, u_2, \ldots, u_n) \in \mathbb{C}^n$ is a row vector of complex variables with the same dimension as the state vector X_t at each time t.
- T is the option's maturity.

Given available information up to time t, $\mathscr{F}_t$, for $t \leq T$, we assume that the conditional discounted moment-generating function of X_T can be expressed as

$$\psi(u, X_t, t, T) = \mathbb{E}_t \left[\exp\left(-\int_t^T R(X_s) ds \right) e^{u.X_T} \right], \tag{1}$$

where $\mathbb{E}_t$ denotes the conditional expectation given the information set ($\mathscr{F}_t$) up to t. In other words, the conditional expectation is the function of u, X_t, t, T. We would like to emphasize that the class of affine jump diffusions from Duffie et al. (2000)

is a special case of our framework where $\psi(\cdot)$ can be computed using an affine models framework.[8]

2.2 Common multiple-condition contingent claims

Below are some examples of rainbow options that are of sufficient interest in the context of basket options.[9] The pricing formulas for two assets are provided in section 4.3.

❶ "Best/worst of assets or cash," paying the maximum/minimum of two or several securities and cash at maturity: $\max/\min(X_1, X_2, \ldots, X_n, K)$ (for details, see Johnson, 1987; Martzoukous, 2001).

❷ "Call on max/min," which entitles the owner to buy the maximum/minimum asset at a given strike at expiry: $\max(\max/\min(X_1, X_2, \ldots, X_n) - K)$ (see Johnson, 1987).

❸ "Put on max/min," giving the holder the right to sell the maximum/minimum asset at a given strike at expiry: $\max(K - \max/\min(X_1, X_2, \ldots, X_n))$ (see Johnson, 1987).

❹ "Exchange one asset for another and earn the spread between the two," which enables the long-position investor to sell/buy an asset strike price given by the price of another: $\max/\min\ (X_1 - X_2, 0)$ (see Margrabe, 1978; Gay and Manaster, 1984).

3 Main results

In this section, we review analytical solutions of increasing difficulty. Firstly, in section 3.1 we address the benchmark single condition on the terminal payoff. Secondly, within the multiple-condition framework, we introduce in section 3.2 the benchmark bivariate case. Finally, the extension to more than two conditions is provided in section 3.3.

3.1 One-condition derivatives pricing

In this section, we review one condition in the option payoff, as in Duffie et al. (2000). Given $(x, T, a, b) \in D \times [0, +\infty] \times \mathbb{R}^n \times \mathbb{R}^n$, let $G_{a,b}(., x, T) : \mathbb{R} \to \mathbb{R}_+$ be

[8] We implicitly assume that the expectation is well defined. This assumption is fully covered on page 1351 of Duffie et al. (2000).

[9] Although this section focuses on popular rainbow options, it is important to grasp that other examples can be amended to cater for multiple conditions on the payoff.

defined as follows:

$$G_{a,b}(y,X_t,T) = \mathbb{E}_t\left[\exp\left(-\int_t^T R(X_s)ds\right)e^{a.X_T}1_{bX_T\leq y}\right]. \tag{2}$$

We recall the following Proposition 2.2 from Duffie et al. (2000), which involves a single condition on the payoff.[10]

Proposition 1

$$G_{a,b}(y,x,T) = \frac{\psi(a,x,t,T)}{2} + \frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{e^{ivy}\psi(a-ivb,x,t,T)-e^{-ivy}\psi(a+ivb,x,t,T)}{iv}dv. \tag{3}$$

Proof. See Appendix A of Duffie et al. (2000).

Obviously, the single-condition payoff examined previously cannot be used to price multiple-condition options. Before we rush to tackle the multiple-condition payoff, let us get more comprehensive insights from two conditions, because they emerge as a special case of the multiple-condition payoff.

3.2 Semi-closed-form for two conditions

In this section, we provide a formula to price derivatives with two conditions on the options payoff. For expositional purposes, we provide details for the two conditions in Appendix A.

Given $(x,T,a,b_1,b_2)\in D\times[0,+\infty]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$, let us define $G_{a,b_1,b_2}(.,x,T):$ $\mathbb{R}^2\to\mathbb{R}_+$ as follows:

$$G_{a,b_1,b_2}(y_1,y_2,x,T) = \mathbb{E}_t\left[\exp\left(-\int_t^T R(X_s)ds\right)e^{a.X_T}1_{b_1X_T\leq y_1}1_{b_2X_T\leq y_2}\right]. \tag{4}$$

The goal is to compute $G_{a,b_1,b_2}(y_1,y_2,x,T)$ given by expression (4). We recover a single-dimension case of Duffie et al. (2000) (Proposition 1) by letting one of y_1 or y_2 become ∞.

At this stage, we recognize that Monte Carlo simulations can be used from the outset of the option payoff. Notwithstanding the issue of complex options with double conditions, Monte Carlo simulations and other numerical schemes are

[10]We can name this contingent claim payoff (2) the "*One condition option,*" since it involves only one indicator function.

more computationally intensive than our Fourier-Stieltjes transform. The Fourier-Stieltjes transform is well defined by

$$\begin{aligned}\mathscr{G}_{a,b_1,b_2}(v_1,v_2,X_t,T) &= \int_{\mathbb{R}^2} e^{iv_1y_1+iv_2y_2} G_{a,b_1,b_2}(dy_1,dy_2,X_t,T) \\ &= \mathbb{E}_t\left[\exp\left(-\int_t^T R(X_s)ds\right)\exp^{(a+iv_1b_1+iv_2b_2)X_T}\right] \\ &= \psi(a+iv_1b_1+iv_2b_2,X_t,t,T).\end{aligned} \tag{5}$$

For $y_1,y_2 \in \mathbb{R}$, let us define the expression $\mathbb{I}_{\{1,2\}}$ by the following relation:

$$\begin{aligned}\mathbb{I}_{\{1,2\}} \equiv &\int_{\mathbb{R}^2} \frac{e^{iv_1y_1+iv_2y_2}\psi(a-iv_1b_1-iv_2b_2)-e^{-iv_1y_1+iv_2y_2}\psi(a+iv_1b_1-iv_2b_2)}{(iv_1)(iv_2)}dv_1dv_2 \\ &+\int_{\mathbb{R}^2} \frac{-e^{iv_1y_1-iv_2y_2}\psi(a-iv_1b_1+iv_2b_2)+e^{-iv_1y_1-iv_2y_2}\psi(a+iv_1b_1+iv_2b_2)}{(iv_1)(iv_2)}dv_1dv_2.\end{aligned} \tag{6}$$

The option price formula we have been looking at can then be reduced to the following proposition.

Proposition 2

$$\begin{aligned}G_{a,b_1,b_2}(y_1,y_2,x,T) =&\frac{\psi(a,x,t,T)}{4}+\frac{1}{16\pi^2}\mathbb{I}_{\{1,2\}} \\ &+\frac{1}{8\pi}\int_{-\infty}^{+\infty}\frac{e^{ivy_2}\psi(a-ivb_2,x,t,T)-e^{-ivy_2}\psi(a+ivb_2,x,t,T)}{iv}dv \\ &+\frac{1}{8\pi}\int_{-\infty}^{+\infty}\frac{e^{ivy_1}\psi(a-ivb_1,x,t,T)-e^{-ivy_1}\psi(a+ivb_1,x,t,T)}{iv}dv.\end{aligned} \tag{7}$$

Let us denote $Re(\cdot)$ and $Im(\cdot)$, respectively, as the real and the imaginary part of a complex number. Using *sin* and *cos* properties along with the parity of the integrand, we can derive the following, more compact, formulas for Proposition 2:

$$
\begin{aligned}
& G_{a,b_1,b_2}(y_1,y_2,x,T) \\
= \quad & \frac{\psi(a,x,t,T)}{4} \\
& -\frac{1}{2\pi}\int_0^{+\infty}\frac{Im\left[e^{-ivy_1}\psi\left(a+ivb_1\right)+e^{-ivy_2}\psi\left(a+ivb_2\right)\right]}{v}dv \\
& -\frac{1}{2\pi^2}\int_0^{+\infty}\int_0^{+\infty}\frac{Re\left[e^{-iv_1y_1-iv_2y_2}\psi\left(a+iv_1b_1+iv_2b_2\right)\right]}{v_1v_2}dv_1dv_2 \\
& -\frac{1}{2\pi^2}\int_0^{+\infty}\int_0^{+\infty}\frac{Re\left[e^{-iv_1y_1+iv_2y_2}\psi\left(a+iv_1b_1-iv_2b_2\right)\right]}{v_1v_2}dv_1dv_2.
\end{aligned}
\tag{8}
$$

Proof. The proof of Proposition 2 relies on two-dimensional differential calculus and trigonometric equations (see Appendix A).

In this section, we concentrate on elucidating the case of the two-conditions payoff, which greatly strengthens our intuition of multiple conditions and proves to be a useful connection with some derivatives (e.g., secured debt valuation in Chang et al., 2006, and cost-of-living contracts in Stulz, 1982). Let us abstract from the two-conditions framework above to examine the valuation of multiple-condition contingent claims in a more general setting.

3.3 Pricing formulas with more than two conditions

In this section, we investigate the valuation of multiple-condition contingent claims in the general setting. To characterize a solution effectively, we give the problem a little more structure:

- Denote $E = \{1, 2, \ldots, m\}$ as a set of m members, where m denotes the exercise domain; i.e., the number of condition indicator functions determining the payoff. In the sequel, $|.|$ stands for the cardinality of the members of any subset of E, and E_k denotes a subset of E with k members ($|E_k| = k$).
- Define $\mathbb{E}$ as the power set of E. We have $|\mathbb{E}| = 2^m$. For each $A \in \mathbb{E}$, A^c is the complementary of A; in other words, it is the subset of members of E that are not in A.

Our goal is to compute the multiple-condition valuation for the following expectation on the terminal payoff:

$$G_{a,b}(y, X_t, T) = \mathbb{E}_t\left[\exp\left(-\int_t^T R(X_s)ds\right)e^{a.X_T}\prod_{j=1}^{m} 1_{b_j X_T \leq y_j}\right], \tag{9}$$

where $b_j \in \mathbb{R}^n, \forall j = 1, 2, \ldots, m$. Recall that n is the number of state variables in our pricing formula, while m is the number of conditions.

Set $\mathbb{I}_A, A \in \mathbb{E}$ as follows:

$$\mathbb{I}_A = \int_{\mathbb{R}^{|A|}} \frac{\sum_{B\subseteq A}(-1)^{|B|}e^{iv_{(B)}y_{(B)}-iv_{(A\setminus B)}y_{(A\setminus B)}}\psi\left(a - iv_{(B)}b_{(B)} + iv_{(A\setminus B)}b_{(A\setminus B)}\right)}{\prod_{j\in A}(iv_j)}\prod_{j\in A} dv_j, \tag{10}$$

where $A \setminus B$ is a subset of members in A that are not in B and $|B|$ is the cardinality of B and $v_{(B)}y_{(B)} = \sum_{j\in B} v_j y_j$. The summation $\sum_{B\subseteq A}$ in (10) is the sum over the subset of A with $|B|$ members.[11]

Lemma 1 *Let Q_A be defined by the following expression:*

$$Q_A \equiv \frac{\sum_{B\subseteq A}(-2)^{|B|}e^{iv_{(B)}\left(y_{(B)}-z_{(B)}\right)-iv_{(A\setminus B)}\left(y_{(A\setminus B)}-z_{(A\setminus B)}\right)}}{\prod_{j\in A}(iv_j)}.$$

We then have the following relation:

$$Q_A = (-2)^{|A|}\prod_{j=1}^{|A|}\frac{\sin\left(v_j\left(y_j - z_j\right)\right)}{v_j}. \tag{11}$$

The proof of Lemma 1 rests on the combinatory identity (refer to Appendix B for more details). With the help of Lemma 1, we can state the following lemma.

Lemma 2 *Denote as $G_{a,b}(y_{(A)})$ the option value where only the y_i with $i \in A$* ***are not*** *$+\infty$; then we have*

$$\mathbb{I}_A = (-2\pi)^{|A|}\sum_{B\subseteq A}(-2)^{|B|}G_{a,b}(y_{(B)}). \tag{12}$$

---

[11] It is worth noting that for a null set, $G_{a,b}(y_{(\varnothing)}) = G_{a,b}(\overbrace{+\infty, +\infty, \ldots, +\infty}^{\text{m times}}) = \psi(a, x, t, T)$. Accordingly, we set $\mathbb{I}_A = \psi(a, x, t, T)$ whenever A is a null set.

Proof. The proof of Lemma 2 resorts to Lemma 1 (see Appendix C).

We are now in a position to price any given number of multiple conditions on options payoff. We have the following proposition.

Proposition 3 *For* $y = (y_1, y_2, \ldots, y_m) \in \mathbb{R}^m$,

$$2^m G_{a,b}(y_1, y_2, \ldots, y_m, x, T) = \sum_{A \subseteq E} \frac{1}{(2\pi)^{|A|}} \mathbb{I}_A, \tag{13}$$

where $\mathbb{I}_A, A \in \mathbb{E}$ *is given by (10)*

The proof of Proposition 3 is given in Appendix D.

Proposition 3 is the main theoretical contribution of our paper. It derives the price of derivatives up to numerical integrations where the only requirement is the existence of the Fourier transform in closed form.

4 A multivariate affine model for stock prices

In this section, we introduce the return and realized covariance data, as well as a multivariate model that describes the joint dynamic of several asset returns. The model is the multivariate analog of the realized volatility (ARV) introduced in Christoffersen et al. (2014). The model is estimated in section 5 and used to illustrate how to concretely apply our theory to price rainbow options.[12] To avoid excessive departures from the main purpose of the current section, we provide only the main ingredients of the pricing and save the technical part of the model for Appendix E.

4.1 The affine realized variance (ARV) model

The affine model has gained widespread acceptance due to its analytical tractability, as well as its flexibility in coping with some stylized facts about returns (time-varying returns, stochastic volatility, volatility risk premium, etc). Thus, in order to skirt some restrictions implicit in the Black-Scholes framework, the affine model can provide a theoretical benchmark for the joint dynamic of returns and volatility.

R_t is a vector of log-returns ($R_t = \ln(S_t/S_{t-1})$) of dimension n, and RV_t a realized variance-covariance $n \times n$ matrix, both observed at the end of day t. It is well known (see Andersen and Andreasen (2000), Andersen et al. (2001) and Forsberg

[12] But it is worth noting that the framework introduced in this paper is not restricted to the particular choice of data, dynamics or options to be priced.

and Bollerslev (2002)) that the distribution of returns standardized by the realized variance is Gaussian. Hence we can write

$$R_t = \mu_{t-1} + (RV_t - \Sigma_{t-1})\,\delta + RV_t^{1/2} z_{1,t},$$

where the $z_{1,t}$ are *iid* and normally distributed ($\mathbb{N}(0, I_n)$), Σ_{t-1} is the conditional expectation of the realized variance

$$\Sigma_{t-1} = E_{t-1}[RV_t],$$

and μ_{t-1} is the conditional expectation of log returns

$$\mu_{t-1} = E_{t-1}[R_t].$$

We will assume that the shock in the realized variance follows a standard Wishart distribution (widely used to model the variance-covariance matrix; see Gourieroux, 2006; Gourieroux and Sufana, 2010; Buraschi et al., 2010):

$$RV_t = \Sigma_{t-1} + \sigma\,[W_t - E_{t-1}(W_t)]\,\sigma',$$

where σ is an $n \times n$ matrix, and the W_t are *iid* and Wishart distributed with a degree of freedom $p > n-1$ and variance V. Denote $W_t \sim W_n(V, p)$, where V is a symmetric positive definite matrix.

4.2 Conditional expectations

4.2.1 Conditional expectations of the realized variance-covariance matrix

The goal here is to specify the dynamic of $\Sigma_{t-1} = E_{t-1}[RV_t]$, though it is important to mention that Σ_{t-1} is not only the expectation of the realized variance-covariance matrix, but also the conditional variance-covariance matrix of the returns

$$\Sigma_{t-1} = E_{t-1}[RV_t] = Var_{t-1}[R_t].$$

Consistent with recent literature (e.g., Shephard and Sheppard, 2010; Christoffersen et al., 2014), we assume that Σ_t is updated through RV_t; hence,

$$\Sigma_t = \omega + \beta\Sigma_{t-1}\beta' + \alpha RV_t\alpha', \tag{14}$$

with ω a symmetric, semi-definite positive matrix. This specification is very similar to the GARCH specification, the only difference being that we use the realized variance to update the volatility instead of the noisy daily squared-returns. We can further express (14) more explicitly (see Appendix E.2).

4.2.2 Conditional expectations of log returns

Denote the risk-free rate as r_f, the price of risk as λ, and an $n-$column vector that has 1 at row i and 0 elsewhere as e_i. We have the following proposition.

Proposition 4 *If the conditional mean at $t-1$, μ_{t-1}, is chosen such that*

$$E_{t-1}[\exp\left(e_i' R_t\right)] = \exp\left(r_f + \lambda_i e_i' \Sigma_{t-1} e_i\right),$$

then

$$\begin{aligned} e_i'\mu_{t-1} &= r_f + \left(\lambda_i - \frac{1}{2}\right) e_i'\Sigma_{t-1}e_i \\ &\quad + p e_i'\sigma V \sigma'\delta + \frac{p}{2} e_i'\sigma V \sigma' e_i \\ &\quad + \frac{p}{2}\ln\left[\det\left(I_n - 2\left(\sigma'\delta e_i'\sigma + \frac{1}{2}\sigma' e_i e_i'\sigma\right)V\right)\right]. \end{aligned}$$

Proof. The proof of Proposition 4 is given in Appendix E.3.

4.2.3 Moment-generating function

We express the one-step-ahead, conditional-moment-generating function by the following proposition.

Proposition 5 *The one-step-ahead, conditional-moment-generating function is given by*

$$\begin{aligned} & E_{t-1}\left[\exp\left(u'R_t + Tr\left(\theta\Sigma_t\right)\right)\right] \\ = \; & \exp\left(A\left(u,\theta\right) + Tr\left(B\left(u,\theta\right)\Sigma_{t-1}\right)\right), \end{aligned}$$

with

$$\begin{aligned} A\left(u,\theta\right) &= \begin{array}{c} \sum_{i=1}^{n} u_i \left(\begin{array}{c} r_f + p e_i'\sigma V\sigma'\delta + \frac{p}{2} e_i'\sigma V \sigma' e_i \\ + \frac{p}{2}\ln\left[\det\left(I_n - 2\left(\sigma'\delta e_i'\sigma + \frac{1}{2}\sigma' e_i e_i'\sigma\right)V\right)\right] \end{array} \right) \\ -\frac{p}{2}\ln\left[\det\left(I_n - 2\left(\frac{1}{2}\sigma' u u'\sigma + \sigma'\delta u'\sigma + \sigma'\alpha'\theta\alpha\sigma\right)V\right)\right] \\ -p u'\sigma V\sigma'\delta - \frac{p}{2} u'\sigma V \sigma' u + Tr\left(\theta\left(\omega - p\alpha\sigma V\sigma'\alpha'\right)\right) \end{array} \\ B\left(u,\theta\right) &= \beta'\theta\beta + \alpha'\theta\alpha + \frac{1}{2}uu' + \sum_{i=1}^{n} e_i u_i \left(\lambda_i - \frac{1}{2}\right) e_i'. \end{aligned}$$

Proof. The proof of Proposition 5 is given in Appendix E.4.1.

We also provide the multi-steps-ahead moment-generating function in the following proposition.

Proposition 6

$$E_t\left[\exp\left(u'\sum_{i=1}^{\tau}R_{t+i}\right)\right]=\exp\left(C\left(u;\tau\right)+Tr\left(D\left(u;\tau\right)\Sigma_t\right)\right),$$

with

$$\begin{aligned}C\left(u;1\right)&=A\left(u,0\right)\\ D\left(u;1\right)&=B\left(u,0\right),\end{aligned}$$

and

$$\begin{aligned}C\left(u;\tau+1\right)&=C\left(u;\tau\right)+A\left(u,D\left(u;\tau\right)\right)\\ D\left(u;\tau+1\right)&=B\left(u,D\left(u;\tau\right)\right).\end{aligned}$$

Proof. The proof of Proposition 6 is given in Appendix E.4.2. We next use the ARV model to illustrate concretely how to evaluate rainbow contracts in the bivariate case ($n=2$).

4.3 Bivariate option pricing ($n=2$)

In this section, we give the pricing formula for the bivariate case and show how it can be used to price the examples listed in section 2.2.

4.3.1 Options formulas for the bivariate case

Given that the price of an option is the discounted expectation of future payoff under the risk-neutral measure Q, option pricing thus does require a change of probability. That change is routinely done in the literature through the Esscher transform (see Gerber and Shiu, 1994 for more details). We follow a similar pattern, and find that the risk-neutral dynamic is almost the same as the historical up to one parameter, λ. Under Q, we should make sure to fix $\lambda = 0$, which implies that

$$E_{t-1}^{Q}[\exp(R_t)] = \exp(r_f).$$

For a given time t ($0 \leq t \leq T$), S_{1t} and $S_{2,t}$ are the price at t of two stocks. Define

$$X_t = (\ln(S_{1t}),\ \ln(S_{2t}))'. \tag{15}$$

Our expression (1) implies that

$$\begin{aligned}
\psi(u;X_t,\Sigma_t) &= E_t^Q\left[\exp\left(-(T-t)r_f + u'X_T\right)\right] \\
&= E_t^Q\left[\exp\left(-(T-t)r_f + u'X_t + u'(X_T - X_t)\right)\right] \\
&= \exp\left(-(T-t)r_f + u'X_t\right) E_t^Q\left[\exp\left(u'(X_T - X_t)\right)\right] \\
&= \exp\left(-(T-t)r_f + u'X_t + C(u;T-t) + Tr(D(u;T-t)\Sigma_t)\right),
\end{aligned}$$

where $C(u;T-t)$ and $D(u;T-t)$ are given by Proposition 6, and Σ_t is given by (14).

Following Proposition 2, we have the following expression:

$$\begin{aligned}
& G_{a,b_1,b_2}(y_1,y_2;X_t,\Sigma_t) \\
= \ & \frac{\psi(a;X_t,\Sigma_t)}{4} && (16) \\
& -\frac{1}{2\pi}\int_0^{+\infty} \frac{Im\left[e^{-ivy_1}\psi(a+ivb_1) + e^{-ivy_2}\psi(a+ivb_2)\right]}{v} dv && (17) \\
& -\frac{1}{2\pi^2}\int_0^{+\infty}\int_0^{+\infty} \frac{Re\left[e^{-iv_1y_1 - iv_2y_2}\psi(a+iv_1b_1+iv_2b_2)\right]}{v_1v_2} dv_1dv_2 \\
& -\frac{1}{2\pi^2}\int_0^{+\infty}\int_0^{+\infty} \frac{Re\left[e^{-iv_1y_1 + iv_2y_2}\psi(a+iv_1b_1-iv_2b_2)\right]}{v_1v_2} dv_1dv_2. && (18)
\end{aligned}$$

As shown in the above formula, the closed-form price can be decomposed into three components. The first component (16) is the conditional-moment-generating function. We designate it the "Constant-Part" and we expect very little time-series variation. The second component (17) is an integral, which we refer to as the "Volatility-Part." Intuitively, we expect its time-series variation to be mostly driven by variation in the individual variances. The third and final component (18) is a double integral designated "Correlation-Part." We conjecture its time-series variation to be mostly driven by variations in the correlation. In the following section, we use the formula to price well-known options.

4.3.2 Pricing common bivariate contingent claims

This section provides the pricing formulas in the case of two assets for the options listed in section 2.2. Throughout this section, e_1 is a vector $(1,0)$ and e_2 is a vector $(0,1)$. To ease the notations, we drop the arguments (X_t,Σ_t) in the function $G_{a,b_1,b_2}(y_1,y_2;X_t,\Sigma_t)$.

❶ "Best of assets or cash," paying the maximum of two securities and cash at maturity. The payoff is given by

$$\begin{aligned}
& \max(S_{1T},S_{2T},K) \\
= \; & S_{1T}1_{[S_{1T}>S_{2T}]}1_{[S_{1T}>K]}+S_{2T}1_{[S_{2T}>S_{1T}]}1_{[S_{2T}>K]}+K1_{[S_{1T}<K]}1_{[S_{2T}<K]} \\
= \; & e^{e_1'X_T}1_{[\ln S_{2T}-\ln S_{1T}<0]}1_{[-\ln S_{1T}<-\ln K]}+e^{e_2'X_T}1_{[\ln S_{1T}-\ln S_{2T}<0]}1_{[-\ln S_{2T}<-\ln K]} \\
+ \; & K1_{[\ln S_{1T}<\ln K]}1_{[\ln S_{2T}<\ln K]}.
\end{aligned}$$

Hence, the price of this option at time t is the sum of three quantities:

$$\begin{aligned}
& E_t\left[\exp\left(-r_f(T-t)\right)\max(S_{1T},S_{2T},,K)\right] \\
= \; & G_{e_1,e_2-e_1,-e_1}(0,-\ln K)+G_{e_2,e_1-e_2,-e_2}(0,-\ln K)+KG_{0,e_1,e_2}(\ln K,\ln K).
\end{aligned}$$

❷ "Call on max," which entitles the owner to buy the maximum of two assets at a given strike at expiry. The payoff here is

$$\begin{aligned}
& \max(\max(S_{1T},S_{2T})-K,0) \\
= \; & (\max(S_{1T},S_{2T})-K)\,1_{[K<\max(S_{1T},S_{2T})]} \\
= \; & \max(S_{1T},S_{2T})1_{[K<\max(S_{1T},S_{2T})]}-K1_{[K<\max(S_{1T},S_{2T})]} \\
= \; & (S_{1T}-K)\,1_{[K<S_{1T}]}1_{[S_{1T}>S_{2T}]}+(S_{2T}-K)\,1_{[K<S_{2T}]}1_{[S_{1T}<S_{2T}]}.
\end{aligned}$$

Then, the price of this option at time t is the sum of four quantities:

$$\begin{aligned}
& E_t\left[\exp\left(-r_f(T-t)\right)\max(\max(S_{1T},S_{2T})-K,0)\right] \\
= \; & G_{e_1,e_2-e_1,-e_1}(0,-\ln K)-KG_{0,e_2-e_1,-e_1}(0,-\ln K) \\
& +G_{e_2,e_1-e_2,-e_2}(0,-\ln K)-KG_{0,e_1-e_2,-e_2}(0,-\ln K).
\end{aligned}$$

❸ "Call on min," which entitles the owner to buy the minimum of two assets at a given strike at expiry. The payoff here is

$$\begin{aligned}
& \max(\min(S_{1T},S_{2T})-K,0) \\
= \; & (\min(S_{1T},S_{2T})-K)\,1_{[K<\min(S_{1T},S_{2T})]} \\
= \; & \min(S_{1T},S_{2T})1_{[K<\min(S_{1T},S_{2T})]}-K1_{[K<\min(S_{1T},S_{2T})]} \\
= \; & (S_{2T}-K)\,1_{[K<S_{2T}]}1_{[S_{1T}>S_{2T}]}+(S_{1T}-K)\,1_{[K<S_{1T}]}1_{[S_{1T}<S_{2T}]}.
\end{aligned}$$

The price of this option at time t is the sum of four quantities:

$$\begin{aligned}
& E_t\left[\exp\left(-r_f(T-t)\right)\max(\max(S_{1T},S_{2T})-K,0)\right] \\
= \;& G_{e_1,e_1-e_2,-e_1}(0,-\ln K)-KG_{0,e_1-e_2,-e_1}(0,-\ln K) \\
& +G_{e_2,e_2-e_1,-e_2}(0,-\ln K)-KG_{0,e_2-e_1,-e_2}(0,-\ln K).
\end{aligned}$$

❹ "Put on max," giving the holder the right to sell the maximum of two assets at a given strike at expiry. The payoff here is

$$\begin{aligned}
& \max(K-\max(S_{1T},S_{2T}),0) \\
= \;& (K-\max(S_{1T},S_{2T}))\,1_{[K>\max(S_{1T},S_{2T})]} \\
= \;& (K-S_{1T})\,1_{[K>S_{1T}]}1_{[S_{1T}>S_{2T}]}+(K-S_{2T})\,1_{[K>S_{2T}]}1_{[S_{1T}<S_{2T}]}.
\end{aligned}$$

The price of this option at time t is the sum of four quantities:

$$\begin{aligned}
& E_t\left[\exp\left(-r_f(T-t)\right)\max(\max(S_{1T},S_{2T})-K,0)\right] \\
= \;& -G_{e_1,e_2-e_1,e_1}(0,\ln K)+KG_{0,e_2-e_1,e_1}(0,\ln K) \\
& -G_{e_2,e_1-e_2,e_2}(0,\ln K)+KG_{0,e_1-e_2,e_2}(0,\ln K).
\end{aligned}$$

❺ "Put on min," giving the holder the right to sell the minimum of two assets at a given strike at expiry. The payoff here is

$$\begin{aligned}
& \max(K-\min(S_{1T},S_{2T}),0) \\
= \;& (K-\min(S_{1T},S_{2T}))\,1_{[K>\min(S_{1T},S_{2T})]} \\
= \;& (K-S_{1T})\,1_{[K>S_{1T}]}1_{[S_{1T}<S_{2T}]}+(K-S_{2T})\,1_{[K>S_{2T}]}1_{[S_{1T}>S_{2T}]}.
\end{aligned}$$

The price of this option at time t is obtained by adding four quantities:

$$\begin{aligned}
& E_t\left[\exp\left(-r_f(T-t)\right)\max(\max(S_{1T},S_{2T})-K,0)\right] \\
= \;& -G_{e_1,e_1-e_2,e_1}(0,\ln K)+KG_{0,e_1-e_2,e_1}(0,\ln K) \\
& -G_{e_2,e_2-e_1,e_2}(0,\ln K)+KG_{0,e_2-e_1,e_2}(0,\ln K).
\end{aligned}$$

❻ "Exchange one asset for another and earn the spread between the two." The payoff here is

$$\begin{aligned}
& \max(S_{1T}-S_{2T},0) \\
= \;& (S_{1T}-S_{2T})\,1_{[S_{1T}>S_{2T}]}.
\end{aligned}$$

The price of this option at time t is the difference of two quantities:

$$\begin{aligned}
& E_t\left[\exp\left(-r_f(T-t)\right)\max(S_{1T}-S_{2T},0)\right] \\
= \;& G_{e_1,e_2-e_1,0}(0,0)-G_{e_2,e_2-e_1,0}(0,0).
\end{aligned}$$

5 Data, estimation and options pricing

In this section, we describe the data used in our empirical exercise. We estimate model parameters by maximizing the likelihood function and use the estimates as inputs in pricing rainbow contracts.

5.1 Data

For the empirical illustration, we use return data from 8 March 1971 to 14 August 2013 on the NASDAQ and NYSE composite indices, among the most important financial indices. It is worth noting that the two indices reflect the performance of two quite distinct markets. In particular, NASDAQ loads on technology stocks, such as Microsoft and Intel, whereas NYSE contains a large proportion of mostly well-established large industrial companies, such as General Electric and Ford.

Figure 1 shows the time series of realized volatility for the two indices. The period under consideration covers the 1973 oil crisis, the stock market crash of October 1987, the dot-com bubble burst of the early 2000s and the most recent financial crisis. From a purely descriptive point of view, the realized volatility and the realized correlation between the two indices clearly show that they share some well-documented, stylized facts about volatility. In particular, the time series of the volatility of these indices is found to be consistent with time-varying volatility as well as the volatility clustering effect. In addition, both series share similar patterns during the most recent financial crisis, which has unsurprisingly led to a sustained period of higher volatilities and correlations. Nevertheless, the figure also reveals that the NYSE and NASDAQ indices can also react rather differently to some kinds of extreme events. In fact, the NASDAQ market was hit more strongly by the bursting of the high-tech bubble in the early 2000s, whereas the crash of October 1987 mostly affected the NYSE stocks.

5.2 Estimation and historical model analysis

Parameters are estimated through the maximum likelihood procedure and reported in Table 1. All the parameters are significant (see the standard errors in parentheses); to avoid identification issues, we impose matrices α, β and σ to be lower triangular. The Wishart distribution degrees-of-freedom parameter n is set to 2 and the variance matrix V to $1/p$ times the identity matrix ($V = (1/p)I_n$).

As expected, the variance-covariance matrix is persistent as eigenvalues of $\beta\beta' + \alpha\alpha'$ are close to 1. The prices of risk (λ) are all positive and have the expected sign, and thus market participants expect positive returns for risks taken.

The NASDAQ's λ is higher than that of the NYSE, which means that market participants expect more reward for risks taken in the technology stocks compared to industrial companies. The δ measures the correlation between a shock in returns and a shock in realized variances, and gauges the so-called leverage effect. A negative δ indicates a negative instantaneous correlation between returns and realized variances. The δ of the NYSE is negative, as expected, while that of NASDAQ is surprisingly positive, which implies that periods of high volatility in the technology sector coincide with periods of high returns.

Table 1: Estimation of the MARV model on daily NASDAQ and NYSE returns and RV (1971-2013)

λ	1.361	1.077	δ	5.025	-0.907
	$(5.19E-01)$	$(6.17E-02)$		$(1.87E+00)$	$(1.49E+00)$
ω	2.97E-05	2.05E-05	α	5.83E-07	0.00E+00
	$(3.84E-07)$	$(3.34E-07)$		$(2.85E-08)$	
	2.05E-05	2.68E-05		5.30E-07	-1.62E-06
	$(3.34E-07)$	$(2.79E-07)$		$(3.48E-08)$	$(2.63E-07)$
β	0.971	0.000	σ	3.91E-03	0.00E+00
	$(2.27E-04)$	$(0.00E+00)$		$(1.51E-06)$	$(0.00E+00)$
	0.000	0.966		2.70E-03	2.56E-03
	$(2.94E-04)$	$(3.28E-04)$		$(1.63E-06)$	$(7.74E-07)$

The key ingredient in multivariate options pricing is the conditional variance-covariance matrix. How well our model is able to price options accurately depends on how accurately our model forecasts the future realized covariance matrix. Given that the conditional variance-covariance matrix is also the conditional expectation of future realized covariance, the model fit can be evaluated by comparing the ex-ante measure with the ex-post. Figure 2 compares the time series of both the conditional volatilities and correlations extracted from our affine model with the observed realized volatilities and correlations. Our model's conditional moments forecast accurately the ex-post moments.

We exploit the well-established tractability and flexibility properties of the affine processes, raising the natural question of whether the cost to get the closed-form price (and, thus, to have an affine model) is too high. To answer this question, we consider the dynamic of the non-affine conditional correlation model (DCC) of Engle (2000), which is the benchmark for conditional covariance matrix modelling. Options pricing in the DCC can only be done through simulations. Figure 3 compares the time series of both the conditional volatilities and correlations extracted from our affine model with those of the DCC ones. Despite its affine nature, our model's conditional moments are very close to those of the DCC and we expect the two models to generate similar options prices.

In the following section, we provide empirical illustrations that demonstrate the real potential of our approach in options pricing.

5.3 Options pricing

In this subsection, we compute options prices using two methodologies, namely Monte Carlo simulation and our closed-form formula. Note, though, that the goal here is to show how our pricing formula really works in practice. We consider rainbow options with three different maturities: 1, 2 and 3 months. For each of these maturities we consider five different strike prices, from 80 to 120, in increments of 10. Moreover, to put the two indices on an equal footing, it is assumed that the starting values are 100 for both indices. This has the effect of essentially considering options on the worst-performing of the indices. We compute the prices on each of the last 100 days in our sample.

Table 2: Average (across time) options prices

	Moneyness (S/X)				
Maturity (in months)	1.25	1.11	1.00	0.91	0.83
			Bests of assets		
1M	103.33	103.60	105.70	111.53	120.26
2M	105.50	106.34	109.01	114.23	121.69
3M	107.44	108.69	111.60	116.54	123.41
			Call on max		
1M	23.33	13.60	5.70	1.53	0.26
2M	25.50	16.34	9.01	4.23	1.69
3M	27.44	18.69	11.60	6.54	3.41
			Call on min		
1M	17.59	8.50	2.59	0.46	0.05
2M	16.92	9.12	4.01	1.44	0.43
3M	16.94	9.88	5.10	2.35	0.99
			Put on max		
1M	0.01	0.28	2.38	8.21	16.95
2M	0.14	0.99	3.65	8.86	16.33
3M	0.39	1.64	4.56	9.52	16.36
			Put on min		
1M	0.06	0.97	5.07	12.93	22.52
2M	0.63	2.84	7.73	15.15	24.16
3M	1.45	4.39	9.60	16.85	25.49

Table 2 reports the average (across the time-series dimension) prices. As expected, those prices generally increase with maturity. In the money call ($S/X > 1$), contracts are much more expensive than out of the money, while out of the money puts ($S/X > 1$) are cheaper than in the money. "Put on min" contracts are generally less expensive than those for "Put on max," and the same result applies to call contracts. This is as expected, since the premium in the "Put on min" is lower than that of the "Put on max." Beyond simply pricing options, we make clear throughout the following subsection that our results provide valuable loading insights by which options market participants bet on specific factors of the returns dynamic.

5.4 Disentangling our closed-form option formula

To gain additional insights into our pricing formula, we next decompose the overall price into three components: the "Constant-Part" that does not require any integral (see equation (16)), the "Volatility-Part" that requires computing a univariate integral (see equation (17)) and finally the "Correlation-Part" that requires computing a double integral (see equation (18)). We study the main drivers in the time-series motion of all three components. To our surprise, there is almost no time-series variation in equation (16), which implies that changes in the variance-covariance matrix have no effect on that equation. For that reason we designate it the "Constant-Part," since it primarily drives only the level of the contract price.

Table 3: Correlation (across time) between the “Volatility-Part” of options prices and the average conditional volatility

	Moneyness (S/X)				
Maturity (in months)	1.25	1.11	1.00	0.91	0.83
Bests of assets					
1M	0.95	0.99	0.99	0.99	0.99
2M	0.95	0.99	0.99	0.99	0.99
3M	0.96	0.99	0.99	0.99	0.99
Call on max					
1M	0.95	0.99	0.99	0.99	0.99
2M	0.95	0.99	0.99	0.99	0.99
3M	0.96	0.99	0.99	0.99	0.99
Call on min					
1M	0.98	1.00	1.00	1.00	1.00
2M	0.98	1.00	1.00	1.00	1.00
3M	0.98	1.00	1.00	1.00	1.00
Put on max					
1M	0.98	1.00	1.00	1.00	1.00
2M	0.98	1.00	1.00	1.00	1.00
3M	0.98	1.00	1.00	1.00	1.00
Put on min					
1M	0.95	0.99	0.99	0.99	0.99
2M	0.95	0.99	0.99	0.99	0.99
3M	0.96	0.99	0.99	0.99	0.99

Table 3 reports the correlation (across time) between the "Volatility-Part" of options prices and the average volatility level. Looking across maturities, moneyness and contract types, we see that the correlation between the average (across assets) of volatilities and the volatility component of options prices is almost 1. That part of the contract price is the cost paid by options buyers to hedge against time variations in individual stock volatilities.

Table 4: Correlation (across time) between the “Correlation-Part” of options prices and the conditional correlation

	Moneyness (S/X)				
Maturity (in months)	1.25	1.11	1.00	0.91	0.83
		Bests of assets			
1M	-0.97	0.80	0.87	0.88	0.85
2M	-0.98	0.75	0.86	0.87	0.87
3M	-0.98	0.68	0.85	0.87	0.88
		Call on max			
1M	-0.87	-0.89	-0.83	0.90	0.89
2M	-0.95	-0.97	-0.96	0.97	0.97
3M	-0.97	-0.98	-0.98	0.98	0.98
		Call on min			
1M	0.87	0.89	0.83	-0.90	-0.89
2M	0.95	0.97	0.96	-0.97	-0.97
3M	0.97	0.98	0.98	-0.98	-0.98
		Put on max			
1M	-0.87	-0.89	-0.83	0.90	0.89
2M	-0.95	-0.97	-0.96	0.97	0.97
3M	-0.97	-0.98	-0.98	0.98	0.98
		Put on min			
1M	0.87	0.89	0.83	-0.90	-0.89
2M	0.95	0.97	0.96	-0.97	-0.97
3M	0.97	0.98	0.98	-0.98	-0.98

Table 4 reports the correlation (across time) between the "Correlation-Part" of options prices and the conditional correlation. Looking across maturities, moneyness and contract types, we see that the co-movement between the conditional correlation and the "Correlation-Part" of options prices is very high. That part of the contract price is the cost paid by options buyers to hedge against time variations in correlations among stocks.

Overall, results suggest that our decomposition can shed light on some fundamental drivers in options valuation.

5.5 Closed-form vs. simulated options prices

In this section, we carry out numerical experiments to illustrate the accuracy of our multivariate closed-form options price by comparing it to the simulated options price.[13] Given computational budget constraints often encountered in business environments, the model's compliance with reasonable pricing time becomes an important issue.

It should be stressed that as we increase the number of simulations, we clearly increase the computation time. To illustrate this point, we report in Table 5 the computation times in function of the number of simulations, and sum the time required to compute all the contract prices, across all the dimensions. For 10,000 paths, it takes approximately one hour. The running time for our pricing formula is about five minutes.

Table 5: Options prices computation time (in min) by number of simulated paths (in thousands)

NSP	Time
1	5.95
2	13.24
3	23.64
4	30.18
5	37.81
6	43.64
7	53.08
8	58.38
9	60.38
10	64.50
Closed form	5.18

[13]The software used is Matlab 2013 on a 3.5 GHz computer with 16 GB of memory

Table 6: Average (across time) of the differences between simulated and closed-form options prices

	Moneyness (S/X)				
Maturity (in months)	1.25	1.11	1.00	0.91	0.83
			Bests of assets		
1M	0.05	0.05	0.03	-0.01	-0.01
2M	-0.01	0.01	-0.01	-0.02	0.00
3M	0.10	0.05	0.02	0.02	0.01
			Call on max		
1M	0.05	0.05	0.03	-0.01	-0.01
2M	-0.01	0.01	-0.01	-0.02	0.00
3M	0.10	0.05	0.02	0.02	0.01
			Call on min		
1M	-0.01	0.01	0.01	-0.01	-0.00
2M	-0.06	-0.04	-0.01	-0.02	-0.01
3M	0.04	0.02	0.02	0.03	0.00
			Put on max		
1M	0.00	-0.00	-0.01	-0.02	-0.06
2M	0.00	-0.01	-0.01	-0.01	0.01
3M	-0.01	-0.01	0.00	-0.01	0.01
			Put on min		
1M	0.00	-0.00	-0.00	-0.01	0.01
2M	-0.00	0.01	-0.01	0.00	-0.01
3M	0.00	-0.01	-0.01	-0.01	0.02

Table 6 reports the difference between the simulated and our proposed options prices, both computed from the ARV model. The main point from this table is that both options prices are similar, which reinforces the validity of our theory.

6 Conclusion

This paper extends the single-condition derivatives pricing framework of Duffie et al. (2000), based on the Fourier transform. Options pricing formulas are given up to numerical integrations. Our approach allows for quite general aggregate contingent claims pricing with several sources of randomness (including stochastic volatility and jump). Our theoretical methodology provides a valuable tool in the options pricing literature. It is noteworthy that a significant improvement over Monte Carlo in computational efficiency is attained without sacrificing pricing accuracy. Moreover, this paper disentangles options prices into intuitive components that enable traders to adequately assess their exposures to each options price driver.

The paper highlights some interesting methodological aspects of the Fourier inversion formula in a highly stylized options pricing setting, but it leaves out several real-world issues such as risk-neutral parameters calibration and risk-neutral distribution fitting. Our approach would be useful in these and other areas as more options data become available.

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A Proof of Proposition 2

For $0 < \tau_1, \tau_2 < +\infty$ and $y_1, y_2 \in \mathbb{R}$, let us define the expression $\mathbb{I}$ by the following relation:

$$\mathbb{I} \equiv \int_{-\tau_1}^{\tau_1}\int_{-\tau_2}^{\tau_2} \quad \frac{e^{iv_2y_2}}{iv_2}\left[\frac{e^{iv_1y_1}\psi(a-iv_1b_1-iv_2b_2,x,t,T)-e^{-iv_1y_1}\psi(a+iv_1b_1-iv_2b_2,x,t,T)}{iv_1}\right] \\ -\frac{e^{-iv_2y_2}}{iv_2}\left[\frac{e^{iv_1y_1}\psi(a-iv_1b_1+iv_2b_2,x,t,T)-e^{-iv_1y_1}\psi(a+iv_1b_1+iv_2b_2,x,t,T)}{iv_1}\right]dv_1dv_2. \tag{19}$$

Because $\psi(a+iv_1b_1+iv_2b_2,x,t,T) = \int_{\mathbb{R}^2} e^{iv_1z_1+iv_2z_2}G_{a,b_1,b_2}(dz_1,dz_2;x,T)$, the relation (19) can be expressed as

$$\mathbb{I} = \int_{-\tau_1}^{\tau_1}\int_{-\tau_2}^{\tau_2}\int_{\mathbb{R}^2} \frac{e^{iv_1y_1+iv_2y_2-iv_1z_1-iv_2z_2} - e^{-iv_1y_1+iv_2y_2+iv_1z_1-iv_2z_2}}{(iv_1)(iv_2)} \\ \frac{-e^{iv_1y_1-iv_2y_2-iv_1z_1+iv_2z_2} + e^{-iv_1y_1-iv_2y_2+iv_1z_1+iv_2z_2}}{(iv_1)(iv_2)}G_{a,b_1,b_2}(dz_1,dz_2;x,T)dv_1dv_2.$$

Since we disregard the exact order in which the integral has been composed, integrand permutations conserve the value of the integral, and $\mathbb{I}$ can be expressed as follows:

$$\mathbb{I} = \int_{\mathbb{R}^2}\int_{-\tau_1}^{\tau_1}\int_{-\tau_2}^{\tau_2} \frac{e^{iv_1y_1+iv_2y_2-iv_1z_1-iv_2z_2} - e^{-iv_1y_1+iv_2y_2+iv_1z_1-iv_2z_2}}{(iv_1)(iv_2)} \\ \frac{-e^{iv_1y_1-iv_2y_2-iv_1z_1+iv_2z_2} + e^{-iv_1y_1-iv_2y_2+iv_1z_1+iv_2z_2}}{(iv_1)(iv_2)}dv_1dv_2G_{a,b_1,b_2}(dz_1,dz_2;x,T). \tag{20}$$

To proceed with the determination of $\mathbb{I}$, we make use of the following relation. For $\tau_1 > 0, \tau_2 > 0$,

$$\int_{-\tau_1}^{\tau_1}\int_{-\tau_2}^{\tau_2} \frac{e^{iv_1y_1+iv_2y_2-iv_1z_1-iv_2z_2} - e^{-iv_1y_1+iv_2y_2+iv_1z_1-iv_2z_2} - e^{iv_1y_1-iv_2y_2-iv_1z_1+iv_2z_2} + e^{-iv_1y_1-iv_2y_2+iv_1z_1+iv_2z_2}}{(iv_1)(iv_2)}dv_1dv_2 \\ = \int_{-\tau_1}^{\tau_1}\int_{-\tau_2}^{\tau_2} \frac{e^{iv_1y_1+iv_2y_2-iv_1z_1-iv_2z_2} + e^{-iv_1y_1-iv_2y_2+iv_1z_1+iv_2z_2}}{iv_1iv_2} - \frac{e^{-iv_1y_1+iv_2y_2+iv_1z_1-iv_2z_2} + e^{iv_1y_1-iv_2y_2-iv_1z_1+iv_2z_2}}{iv_1iv_2}dv_1dv_2. \tag{21}$$

Using the usual trigonometric identity, (21) becomes (22), as follows:

$$
\begin{aligned}
&\int_{-\tau_1}^{\tau_1}\int_{-\tau_2}^{\tau_2}\frac{e^{iv_1y_1+iv_2y_2-iv_1z_1-iv_2z_2}-e^{-iv_1y_1+iv_2y_2+iv_1z_1-iv_2z_2}-e^{iv_1y_1-iv_2y_2-iv_1z_1+iv_2z_2}+e^{-iv_1y_1-iv_2y_2+iv_1z_1+iv_2z_2}}{(iv_1)(iv_2)}dv_1dv_2\\
&=\int_{-\tau_1}^{\tau_1}\int_{-\tau_2}^{\tau_2}-\frac{2\cos\left(v_1(y_1-z_1)+v_2(y_2-z_2)\right)-2\cos\left(v_1(y_1-z_1)-v_2(y_2-z_2)\right)}{v_1v_2}dv_1dv_2\\
&=\int_{-\tau_1}^{\tau_1}\int_{-\tau_2}^{\tau_2}2\left[\frac{\cos\left(v_1(y_1-z_1)-v_2(y_2-z_2)\right)-\cos\left(v_1(y_1-z_1)+v_2(y_2-z_2)\right)}{v_1v_2}\right]dv_1dv_2\\
&=\int_{-\tau_1}^{\tau_1}\int_{-\tau_2}^{\tau_2}2\left[\frac{\cos\left(v_1(y_1-z_1)\right)\cos\left(v_2(y_2-z_2)\right)+\sin\left(v_1(y_1-z_1)\right)\sin\left(v_2(y_2-z_2)\right)}{v_1v_2}\right]\\
&-2\left[\frac{\cos\left(v_1(y_1-z_1)\right)\cos\left(v_2(y_2-z_2)\right)-\sin\left(v_1(y_1-z_1)\right)\sin\left(v_2(y_2-z_2)\right)}{v_1v_2}\right]dv_1dv_2\\
&=\int_{-\tau_1}^{\tau_1}\int_{-\tau_2}^{\tau_2}4\frac{\sin\left(v_1(y_1-z_1)\right)\sin\left(v_2(y_2-z_2)\right)}{v_1v_2}dv_1dv_2\\
&=4\int_{-\tau_1}^{\tau_1}\frac{\sin\left(v_1(y_1-z_1)\right)}{v_1}dv_1\int_{-\tau_2}^{\tau_2}\frac{\sin\left(v_2(y_2-z_2)\right)}{v_2}dv_2.
\end{aligned}
\tag{22}
$$

Furthermore, the following result holds for every $\tau>0$:

$$
\int_{-\tau}^{\tau}\frac{e^{-iv(z-y)}-e^{iv(z-y)}}{iv}dv=\int_{-\tau}^{\tau}\frac{-2\sin\left(v(z-y)\right)}{v}dv=-2sgn(z-y)\int_{-\tau}^{\tau}\frac{-\sin\left(v|z-y|\right)}{v}dv.
\tag{23}
$$

And $\lim_{\tau\to\infty}\int_{\tau}^{\tau}\frac{\sin\left(v|z-y|\right)}{v}dv=\pi.$

Pooling together the relation (23) with the bounded convergence theorem and using the fact that $\lim_{(y_1,y_2)\to(\infty,\infty)}G_{a,b_1,b_2}(y_1,y_2;x,T)=\psi(a,x,0,T)$, $\mathbb{I}$ as defined in (19), when letting $\tau_1,\tau_2,\to\infty$, this brings about

$$
\begin{aligned}
\frac{\lim_{(\tau_1,\tau_2)\to(\infty,\infty))}\mathbb{I}}{4\pi^2} &= \int_{\mathbb{R}^2}sgn(z_1-y_1)sgn(z_2-y_2)G_{a,b_1,b_2}(dz_1,dz_2;x,T)\\
&=\underbrace{\int_{y_1}^{+\infty}\int_{y_2}^{+\infty}G_{a,b_1,b_2}(dz_1;dz_2)}_{I}-\underbrace{\int_{y_1}^{+\infty}\int_{-\infty}^{y_2}G_{a,b_1,b_2}(dz_1;dz_2)}_{II}\\
&-\underbrace{\int_{y_2}^{+\infty}\int_{-\infty}^{y_1}G_{a,b_1,b_2}(dz_1;dz_2)}_{III}+\underbrace{\int_{-\infty}^{y_1}\int_{-\infty}^{y_2}G_{a,b_1,b_2}(dz_1;dz_2)}_{IV}.
\end{aligned}
\tag{24}
$$

We determine each of the expressions above in order to compute $\mathbb{I}$. Firstly, we proceed by computing the expression I

$$
\begin{aligned}
I=&\int_{y_1}^{+\infty}\int_{y_2}^{+\infty}G_{a,b_1,b_2}(dz_1;dz_2)=\int_{y_1}^{\infty}\left[G_{a,b_1,b_2}(dz_1;+\infty)-G_{a,b_1,b_2}(dz_1;y_2)\right]\\
=&\int_{y_1}^{\infty}G_{a,b_1,b_2}(dz_1;+\infty)-\int_{y_1}^{\infty}G_{a,b_1,b_2}(dz_1;y_2)\\
=&\,G_{a,b_1,b_2}(+\infty;+\infty)-G_{a,b_1,b_2}(y_1;+\infty)-G_{a,b_1,b_2}(+\infty;y_2)+G_{a,b_1,b_2}(y_1;y_2).
\end{aligned}
$$

We know that, whenever one of y_1 or y_2 is $+\infty$, we recover Duffie et al. (2000) one-condition framework; then, from their Proposition 2, we have

$$
\begin{aligned}
G_{a,b_1,b_2}(y_1;+\infty) &= \frac{\psi(a,x,t,T)}{2}+\frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{e^{ivy_1}\psi^{\chi}(a-ivb_1,x,t,T)-e^{-ivy_1}\psi(a+ivb_1,x,t,T)}{iv}dv\\
G_{a,b_1,b_2}(+\infty;y_2) &= \frac{\psi(a,x,t,T)}{2}+\frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{e^{ivy_2}\psi^{\chi}(a-ivb_2,x,t,T)-e^{-ivy_2}\psi(a+ivb_2,x,t,T)}{iv}dv.
\end{aligned}
$$

Plugging these relations into I yields

$$I = G_{a,b_1,b_2}(y_1;y_2) \quad -\frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{e^{ivy_1}\psi(a-ivb_1,x,t,T)-e^{ivy_1}\psi(a+ivb_1,x,t,T)}{iv}dv \\ -\frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{e^{ivy_2}\psi(a-ivb_2,x,t,T)-e^{ivy_2}\psi(a+ivb_2,x,t,T)}{iv}dv. \tag{25}$$

The second expression, II, is derived as follows:

$$\begin{aligned} II= &\quad \int_{y_1}^{+\infty}\int_{-\infty}^{y_2}\partial^2 G_{a,b_1,b_2}(z_1;z_2) = \int_{y_1}^{+\infty} dG_{a,b_1,b_2}(z_1;y_2-) \\ = &\quad G_{a,b_1,b_2}(+\infty;y_2-) - G_{a,b_1,b_2}(y_1;y_2-), \end{aligned}$$

where $G_{a,b_1,b_2}(-y_1,y_2,x,T) = \lim_{z_1\to y_1,z_1<y_1;z_2\to y_2,z_2>y_2} G_{a,b_1,b_2}(z_1,z_2,x,T)$ and $G_{a,b_1,b_2}(-y_1,-y_2,x,T) = \lim_{z_1\to y_1,z_1<y_1;z_2\to y_2,z_2<y_2} G_{a,b_1,b_2}(z_1,z_2,x,T)$.

From Proposition 2 of Duffie et al. (2000) we also have

$$G_{a,b_1,b_2}(+\infty;y_2-) = \frac{\psi(a,x,t,T)}{2} + \frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{e^{ivy_2}\psi(a-ivb_2,x,t,T)-e^{-ivy_2}\psi(a+ivb_2,x,t,T)}{iv}dv$$

and

$G_{a,b_1,b_2}(y_1;y_2-) = G_{a,b_1,b_2}(y_1;y_2)$. The expression II then becomes

$$II = \frac{\psi(a,x,t,T)}{2} + \frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{e^{ivy_2}\psi(a-ivb_2,x,t,T)-e^{-ivy_2}\psi(a+ivb_2,x,t,T)}{iv}dv - G_{a,b_1,b_2}(y_1;y_2). \tag{26}$$

Likewise, III is computed as

$$\begin{aligned} III= &\quad \int_{y_2}^{+\infty}\int_{-\infty}^{y_1} G_{a,b_1,b_2}(dz_1;dz_2) = \int_{y_2}^{+\infty} dG_{a,b_1,b_2}(y_1-;z_2) \\ = &\quad G_{a,b_1,b_2}(y_1-;+\infty) - G_{a,b_1,b_2}(y_1-;y_2) \\ = &\quad \frac{\psi(a,x,t,T)}{2} + \frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{e^{ivy_1}\psi(a-ivb_1,x,t,T)-e^{-ivy_1}\psi(a+ivb_1,x,t,T)}{iv}dv - G_{a,b_1,b_2}(y_1;y_2). \end{aligned} \tag{27}$$

Finally, the last term, IV, is expressed as

$$IV = \int_{-\infty}^{y_1}\int_{-\infty}^{y_2} G_{a,b_1,b_2}(dz_1;dz_2) = \int_{-\infty}^{y_1} dG_{a,b_1,b_2}(z_1;y_2-) = G_{a,b_1,b_2}(y_1-;y_2-) = G_{a,b_1,b_2}(y_1;y_2). \tag{28}$$

In conclusion, we can express the following limit:

$$\frac{1}{4\pi^2}\lim_{(\tau_1,\tau_2)\to(\infty,\infty))}\mathbb{I} = \quad 4G_{a,b_1,b_2}(y_1;y_2) - \psi(a,x,t,T) - \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{ivy_2}\psi(a-ivb_2,x,t,T)-e^{ivy_2}\psi(a+ivb_2,x,t,T)}{iv}dv \\ -\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{ivy_1}\psi(a-ivb_1,x,t,T)-e^{-ivy_1}\psi(a+ivb_1,x,t,T)}{iv}dv. \tag{29}$$

B Proof of Lemma 1

The proof of Lemma 1 relies on the following combinatory identity:

$$\prod_{j\in A}(r_j+s_j)=\sum_{B\subseteq A}\left(\prod_{j\in B}r_j\right)\left(\prod_{j\in A\setminus B}s_j\right). \tag{30}$$

Substituting $r_j \to -\frac{e^{iv_j(y_j-z_j)}}{2i}, s_j \to \frac{e^{-iv_j(y_j-z_j)}}{2i}$ and using the fact that $\sin\left(v_j(y_j-z_j)\right)=\frac{e^{iv_j(y_j-z_j)}-e^{-iv_j(y_j-z_j)}}{2i}$, we have $-2\frac{\sin\left(v_j(y_j-z_j)\right)}{v_j}=\frac{e^{-iv_j(y_j-z_j)}-e^{iv_j(y_j-z_j)}}{iv_j}$ and (30) becomes

$$(-2)^{|A|}\prod_{j\in A}\frac{\sin\left(v_j(y_j-z_j)\right)}{v_j}=\frac{\sum_{B\subseteq A}(-1)^{|B|}e^{iv_{(B)}\left(y_{(B)}-z_{(B)}\right)-iv_{(A\setminus B)}\left(y_{(A\setminus B)}-z_{(A\setminus B)}\right)}}{\prod_{j\in A}(iv_j)}. \tag{31}$$

We therefore show equation (11) of Lemma 1 by combinatory identity.

Substituting Q_A by its expression (11) in $\mathbb{I}_A$ leads to

$$\begin{aligned}
\mathbb{I}_A &= \int_{\mathbb{R}^{|A|}}\int_{\mathbb{R}^{|A|}}(-2)^{|A|}\prod_{j=1}^{|A|}\frac{\sin\left(v_j\left(y_j-z_j\right)\right)}{v_j}\prod_{j\in A}dv_jG_{a,b}(dz_{(A)},x,T)\\
&= (-2)^{|A|}\int_{\mathbb{R}^{|A|}}\prod_{j\in A}\left[\int\frac{\sin\left(v_j\left(y_j-z_j\right)\right)}{v_j}dv_j\right]G_{a,b}(dz_{(A)},x,T)\\
&= (-2\pi)^{|A|}\int_{\mathbb{R}^{|A|}}\prod_{j\in A}sgn\left(z_j-y_j\right)G_{a,b}(dz_{(A)},x,T).
\end{aligned}$$

C Proof of Lemma 2

Henceforth, to simplify our notations, we drop the multiple occurrences of X_t, T, χ from the expression of $G_{a,b}(z,X_t,T)$. By definition, we have

$$\mathbb{I}_A=\int_{\mathbb{R}^{|A|}}\frac{\sum_{k=0}^{|A|}(-1)^k\sum_{A_k\subseteq A}e^{iv_{(A_k)}y_{(A_k)}-iv_{(A\setminus A_k)}y_{(A\setminus A_k)}}\psi\left(a-iv_{(A_k)}b_{(A_k)}+iv_{(A\setminus A_k)}b_{(A\setminus A_k)}\right)}{\prod_{j\in A}(iv_j)}\prod_{j\in A}dv_j.$$

Using the fact that

$$\psi\left(a-iv_{(A_k)}b_{(A_k)}+iv_{(A\setminus A_k)}b_{(A\setminus A_k)}\right)=\int_{\mathbb{R}^{|A|}}e^{-iv_{(A_k)}z_{(A_k)}+iv_{(A\setminus A_k)}z_{(A\setminus A_k)}}G_{a,b}(dz_{(A)},X_t,T),$$

we have

$$
\begin{aligned}
& \mathbb{I}_A \\
= \quad & \int_{\mathbb{R}^{|A|}} \frac{\sum_{k=0}^{|A|}(-1)^k \sum_{A_k \subseteq A} e^{iv_{(A_k)}y_{(A_k)} - iv_{(A\setminus A_k)}y_{(A\setminus A_k)}} \int_{\mathbb{R}^{|A|}} e^{-iv_{(A_k)}z_{(A_k)} + iv_{(A\setminus A_k)}z_{(A\setminus A_k)}} G_{a,b}(dz_{(A)})}{\prod_{j\in A}(iv_j)} \prod_{j\in A} dv_j \\
= \quad & \int_{\mathbb{R}^{|A|}} \int_{\mathbb{R}^{|A|}} \frac{\sum_{k=0}^{|A|}(-1)^k \sum_{A_k \subseteq A} e^{iv_{(A_k)}\left(y_{(A_k)} - z_{(A_k)}\right) - iv_{(A\setminus A_k)}\left(y_{(A\setminus A_k)} - z_{(A\setminus A_k)}\right)}}{\prod_{j\in A}(iv_j)} \prod_{j\in A} dv_j G_{a,b}(dz_{(A)}) \\
= \quad & \int_{\mathbb{R}^{|A|}} \int_{\mathbb{R}^{|A|}} Q_A \prod_{j\in A} dv_j G_{a,b}(dz_{(A)}, X_t, T),
\end{aligned}
$$

where

$$
Q_A \equiv \frac{\sum_{k=0}^{|A|}(-1)^k \sum_{A_k \subseteq A} e^{iv_{(A_k)}\left(y_{(A_k)} - z_{(A_k)}\right) - iv_{(A\setminus A_k)}\left(y_{(A\setminus A_k)} - z_{(A\setminus A_k)}\right)}}{\prod_{j\in A}(iv_j)}.
$$

Replacing Q_A by its expression in $\mathbb{I}_A$ leads to

$$
\begin{aligned}
\mathbb{I}_A \quad = \quad & \int_{\mathbb{R}^{|A|}} \int_{\mathbb{R}^{|A|}} (-2)^{|A|} \prod_{j=1}^{|A|} \frac{\sin\left(v_j\left(y_j - z_j\right)\right)}{v_j} \prod_{j\in A} dv_j G_{a,b}(dz_{(A)}, X_t, T) \\
= \quad & (-2)^{|A|} \int_{\mathbb{R}^{|A|}} \prod_{j\in A} \left[\int \frac{\sin\left(v_j\left(y_j - z_j\right)\right)}{v_j} dv_j\right] G_{a,b}(dz_{(A)}, X_t, T) \\
= \quad & (-2\pi)^{|A|} \int_{\mathbb{R}^{|A|}} \prod_{j\in A} sgn\left(z_j - y_j\right) G_{a,b}(dz_{(A)}, X_t, T).
\end{aligned}
$$

We are now ready to demonstrate the lemma. The goal is to show that

$$
\mathbb{I}_A = (-2\pi)^{|A|} \sum_{k=0}^{|A|} (-2)^k \sum_{A_k \subseteq A} G_{a,b}(y_{(A_k)}).
$$

In other words, we endeavor to prove that

$$
\mathbb{J}_A \equiv \int_{\mathbb{R}^{|A|}} \prod_{j\in A} sgn\left(z_j - y_j\right) G_{a,b}(dz_{(A)}) = \sum_{k=0}^{|A|} (-2)^k \sum_{A_k \subseteq A} G_{a,b}(y_{(A_k)}). \qquad (32)
$$

To do so, we proceed by induction. Suppose that the relation holds for each subset of E whose member's cardinality is less than or equal to $|A|$. Let us show that the relation is also fulfilled for the subset with $|A|+1$ elements. Denoting $A+1 \equiv A \cup \{y_{|A|+1}\}$, we have

$$\begin{aligned}\mathbb{J}_{A+1} &= \int_{\mathbb{R}^{|A|+1}} \prod_{j\in A+1} sgn\left(z_j - y_j\right) G_{a,b}(dz_{(A+1)}) \\ &= \int_{\mathbb{R}} sgn(z_{|A|+1} - y_{|A|+1}) d\left[\int_{\mathbb{R}^{|A|}} \prod_{j\in A} sgn\left(z_j - y_j\right) G_{a,b}(dz_{(A)}; z_{|A|+1})\right].\end{aligned}$$

By induction, we can apply (32) to $\int_{\mathbb{R}^{|A|}} \prod_{j\in A} sgn\left(z_j - y_j\right) G_{a,b}(dz_{(A)}; z_{|A|+1})$. This implies that

$$\int_{\mathbb{R}^{|A|}} \prod_{j\in A} sgn\left(z_j - y_j\right) G_{a,b}(dz_{(A)}; z_{|A|+1}) = \sum_{k=0}^{|A|} (-2)^k \sum_{A_k \subseteq A} G_{a,b}(y_{(A_k)}; z_{|A|+1}), \quad (33)$$

thus

$$\mathbb{J}_{A+1} = \int_{\mathbb{R}} sgn(z_{|A|+1} - y_{|A|+1}) \left[\sum_{k=0}^{|A|} (-2)^k \sum_{A_k \subseteq A} G_{a,b}(y_{(A_k)}; dz_{|A|+1})\right],$$

hence

$$\begin{aligned}&\mathbb{J}_{A+1} \\ = \; & \sum_{k=0}^{|A|} (-2)^k \sum_{A_k \subseteq A} \int_{\mathbb{R}} sgn(z_{|A|+1} - y_{|A|+1}) G_{a,b}\left(y_{(A_k)}; dz_{|A|+1}\right) \\ = \; & \sum_{k=0}^{|A|} (-2)^k \sum_{A_k \subseteq A} \left[-\int_{-\infty}^{y_{|A|+1}} G_{a,b}\left(y_{(A_k)}; dz_{|A|+1}\right) + \int_{y_{|A|+1}}^{\infty} G_{a,b}\left(y_{(A_k)}; dz_{|A|+1}\right)\right] \\ = \; & \sum_{k=0}^{|A|} (-2)^k \sum_{A_k \subseteq A} \left[-2G_{a,b}\left(y_{(A_k)}; y_{|A|+1}\right) + G_{a,b}\left(y_{(A_k)}; -\infty\right) + G_{a,b}\left(y_{(A_k)}; \infty\right)\right].\end{aligned}$$

By the definition of $G_{a,b}(\cdot)$, we have

$$G_{a,b}\left(y_{(A_k)}; -\infty\right) = 0, \quad G_{a,b}\left(y_{(A_k)}; \infty\right) = G_{a,b}\left(y_{(A_k)}\right),$$

and therefore

$$\begin{aligned}\mathbb{J}_{A+1} &= \sum_{k=0}^{|A|} (-2)^k \sum_{A_k \subseteq A} \left[-2G_{a,b}\left(y_{(A_k)}; y_{|A|+1}\right) + G_{a,b}\left(y_{(A_k)}\right)\right] \\ &= \sum_{k=0}^{|A|} (-2)^{k+1} \sum_{A_k \subseteq A} G_{a,b}\left(y_{(A_k)}; y_{|A|+1}\right) + \mathbb{J}_A \\ &= (-2)^{|A|+1} G_{a,b}(y_{(A+1)}) + \sum_{k=0}^{|A|-1} (-2)^{k+1} \sum_{A_k \subseteq A} G_{a,b}\left(y_{(A_k)}; y_{|A|+1}\right) + \mathbb{J}_A.\end{aligned}$$

Replacing $\sum_{A_k \subseteq A} G_{a,b}\left(y_{(A_k)}; y_{|A|+1}\right)$ by

$$\sum_{A_k \subseteq A} G_{a,b}\left(y_{(A_k)}; y_{|A|+1}\right) = \left[\begin{array}{c} \sum_{A_k \subseteq A} G_{a,b}\left(y_{(A_k)}; y_{|A|+1}\right) \\ + \sum_{A_{k+1} \subseteq A} G_{a,b}\left(y_{(A_{k+1})}\right) - \sum_{A_{k+1} \subseteq A} G_{a,b}\left(y_{(A_{k+1})}\right) \end{array} \right]$$

implies that

$$\mathbb{J}_{A+1} = (-2)^{|A|+1} G_{a,b}(y_{(A+1)}) + \sum_{k=0}^{|A|-1} (-2)^{k+1} \left[\begin{array}{c} \sum_{A_k \subseteq A} G_{a,b}\left(y_{(A_k)}; y_{|A|+1}\right) \\ + \sum_{A_{k+1} \subseteq A} G_{a,b}\left(y_{(A_{k+1})}\right) - \sum_{A_{k+1} \subseteq A} G_{a,b}\left(y_{(A_{k+1})}\right) \end{array} \right] + \mathbb{J}_A.$$

Hence

$$\begin{aligned} \mathbb{J}_{A+1} \quad = \quad & (-2)^{|A|+1} G_{a,b}(y_{(A+1)}) + \sum_{k=0}^{|A|-1} (-2)^{k+1} \left[\sum_{A_k \subseteq A} G_{a,b}\left(y_{(A_k)}; y_{|A|+1}\right) + \sum_{A_k \subseteq A+1} G_{a,b}\left(y_{(A_k)}\right) \right] \\ & - \sum_{k=0}^{|A|-1} (-2)^{k+1} \left[\sum_{A_{k+1} \subseteq A} G_{a,b}\left(y_{(A_{k+1})}\right) \right] + \mathbb{J}_A. \end{aligned}$$

By noting that

$$\sum_{A_k \subseteq A} G_{a,b}\left(y_{(A_k)}; y_{|A|+1}\right) + \sum_{A_k \subseteq A+1} G_{a,b}\left(y_{(A_k)}\right) = \sum_{A_j \subseteq A+1} G_{a,b}\left(y_{(A_j)}\right),$$

we can rewrite $\mathbb{J}_{A+1}$ as the following:

$$\begin{aligned} \mathbb{J}_{A+1} \quad = \quad & (-2)^{|A|+1} G_{a,b}(y_{(A+1)}) + \sum_{k=0}^{|A|-1} (-2)^{k+1} \sum_{A_{k+1} \subseteq A+1} G_{a,b}\left(y_{(A_{k+1})}\right) \\ & - \sum_{k=0}^{|A|-1} (-2)^{k+1} \left[\sum_{A_{k+1} \subseteq A} G_{a,b}\left(y_{(A_{k+1})}\right) \right] + \mathbb{J}_A. \end{aligned}$$

Rearranging the summation leads to

$$\begin{aligned} \mathbb{J}_{A+1} \quad = \quad & (-2)^{|A|+1} G_{a,b}(y_{(A+1)}) + \sum_{j=1}^{|A|} (-2)^{j} \sum_{A_j \subseteq A+1} G_{a,b}\left(y_{(A_j)}\right) \\ & - \sum_{k=0}^{|A|-1} (-2)^{k+1} \left[\sum_{A_{k+1} \subseteq A} G_{a,b}\left(y_{(A_{k+1})}\right) \right] + \mathbb{J}_A, \end{aligned}$$

which implies

$$\mathbb{J}_{A+1} = \sum_{j=1}^{|A|+1} (-2)^{j} \sum_{A_j \subseteq A+1} G_{a,b}\left(y_{(A_j)}\right) + \mathbb{J}_A - \sum_{j=1}^{|A|} (-2)^{j} \left[\sum_{A_j \subseteq A} G_{a,b}\left(y_{(A_j)}\right) \right].$$

Using the induction assumption up to $\mathbb{J}_A$, we have

$$\mathbb{J}_A - \sum_{j=1}^{|A|}(-2)^j \left[\sum_{A_j \subseteq A} G_{a,b}\left(y_{(A_j)}\right)\right] = G_{a,b}\left(y_{(\emptyset)}\right) = \psi(a,x,t,T).$$

Thus

$$\mathbb{J}_{A+1} = \sum_{j=1}^{|A|+1}(-2)^j \sum_{A_j \subseteq A+1} G_{a,b}\left(y_{(A_j)}\right) + \psi(a,x,t,T).$$

In other words,

$$\mathbb{J}_{A+1} = \sum_{j=0}^{|A|+1}(-2)^j \sum_{A_j \subseteq A+1} G_{a,b}\left(y_{(A_j)}\right).$$

This ends the proof of Lemma 2.

D Proof of Proposition 3

The proof of Proposition 3 relies on Lemma 2 and an application of the *Möbius inversion* for the Boolean algebra of subsets of a finite set (see Hazewinkel, 2002; Rota, 1964).

Define $A \subseteq E$,

$$\mathbb{G}(A) = 2^{|A|} G_{a,b}(y_{(A)}), \qquad \mathbb{F}(A) = (-2\pi)^{-|A|}\mathbb{I}_A, \tag{34}$$

where $\mathbb{I}_A, A \subseteq E$ is given by (10).

Lemma 2 implies that

$$\mathbb{F}(A) = \sum_{B \subseteq A}(-1)^{|A|-|B|}\mathbb{G}(B). \tag{35}$$

The *Möbius inversion* says that (35) is equivalent to

$$\mathbb{G}(A) = \sum_{B \subseteq A}\mathbb{F}(B) \qquad \forall B \subseteq A, \tag{36}$$

and the proof of Proposition 3 is completed with $A = E$ in (36).

E Some ARV model features

Below we provide some ARV model properties.

E.1 The density

The joint conditional density of returns (R_t) and realized variance-covariance matrix (RV_t) is

$$f_{t-1}(R_t, RV_t) = f_{t-1}(R_t|RV_t)\, f_{t-1}(RV_t)$$

$$f_{t-1}(R_t|RV_t) = (2\pi)^{-\frac{n}{2}} |RV_t|^{-\frac{1}{2}} \exp\left(-\frac{1}{2} z_{1,t}' z_{1,t}\right),$$

where

$$z_{1,t} = RV_t^{-1/2}\left[R_t - \mu_{t-1} - (RV_t - \Sigma_{t-1})\,\delta\right],$$

and

$$f_{t-1}(RV_t) = 2^{-pn/2} \left|\sigma V \sigma'\right|^{-p/2} \Gamma_n\left(\frac{p}{2}\right)^{-1} |W_t|^{\frac{p-n-1}{2}} \exp\left(-\frac{Tr\left((\sigma V \sigma')^{-1} W_t\right)}{2}\right),$$

where

$$W_t = \sigma^{-1}(RV_t - \Sigma_{t-1})\left(\sigma^{-1}\right)' + pV.$$

E.2 Model characteristics

It follows from (14) that the conditional variance can be expressed as

$$\begin{aligned}
\Sigma_t &= \omega + \beta\Sigma_{t-1}\beta' + \alpha RV_t \alpha' \\
&= \omega + \beta\Sigma_{t-1}\beta' + \alpha\left\{\Sigma_{t-1} + \sigma\left[W_t - pV\right]\sigma'\right\}\alpha' \\
&= \omega + \beta\Sigma_{t-1}\beta' + \alpha\Sigma_{t-1}\alpha' + \alpha\sigma W_t \sigma'\alpha' - p\alpha\sigma V\sigma'\alpha' \\
&= \omega - p\alpha\sigma V\sigma'\alpha' + \beta\Sigma_{t-1}\beta' + \alpha\Sigma_{t-1}\alpha' + \alpha\sigma W_t\sigma'\alpha'.
\end{aligned}$$

Given that

$$\Sigma_t = \omega - p\alpha\sigma V\sigma'\alpha' + \beta\Sigma_{t-1}\beta' + \alpha\Sigma_{t-1}\alpha' + \alpha\sigma W_t\sigma'\alpha',$$

the model is well defined (in the sense that the support of distribution of Σ_t is the symmetric positive definite real matrix) whenever the following condition is fulfilled:

$$\omega - p\alpha\sigma V\sigma'\alpha' \geq 0,$$

and we can express ω as

$$\omega = p\alpha\sigma V\sigma'\alpha' + \gamma\gamma'.$$

Further, given that

$$E_{t-1}[\Sigma_t] = \omega + \beta\Sigma_{t-1}\beta' + \alpha\Sigma_{t-1}\alpha',$$

the variance matrix is covariance-stationary if all the eigenvalues of $\beta\beta' + \alpha\alpha'$ are less than 1.

E.3 Proof of Proposition 4, the conditional expectation of log returns

$$E_{t-1}[\exp(e_i'R_t)] = \exp(r_f + \lambda_i e_i'\Sigma_{t-1}e_i)$$

$$\begin{aligned}
& E_{t-1}[\exp(e_i'R_t)] \\
= \;& E_{t-1}\left[\exp\left(e_i'\mu_{t-1} + e_i'(RV_t - \Sigma_{t-1})\delta + e_i'RV_t^{1/2}z_{1,t}\right)\right] \\
= \;& E_{t-1}\left[\exp\left(e_i'\mu_{t-1} + e_i'\sigma[W_t - pV]\sigma'\delta + e_i'RV_t^{1/2}z_{1,t}\right)\right] \\
= \;& E_{t-1}\left[\exp\left(e_i'\mu_{t-1} + e_i'\sigma[W_t - pV]\sigma'\delta + \frac{1}{2}e_i'RV_te_i\right)\right] \\
= \;& E_{t-1}\left[\exp\left(\begin{array}{c} e_i'\mu_{t-1} + e_i'\sigma[W_t - pV]\sigma'\delta \\ +\frac{1}{2}e_i'(\Sigma_{t-1} + \sigma[W_t - pV]\sigma')e_i \end{array}\right)\right] \\
= \;& E_{t-1}\left[\exp\left(\begin{array}{c} e_i'\mu_{t-1} + e_i'\sigma[W_t - pV]\sigma'\delta \\ +\frac{1}{2}(e_i'\Sigma_{t-1}e_i + e_i'\sigma[W_t - pV]\sigma'e_i) \end{array}\right)\right]
\end{aligned}$$

$$\begin{aligned}
= \;& E_{t-1}\left[\exp\left(\begin{array}{c} e_i'\mu_{t-1} + e_i'\sigma W_t\sigma'\delta - pe_i'\sigma V\sigma'\delta \\ +\frac{1}{2}e_i'\Sigma_{t-1}e_i + \frac{1}{2}e_i'\sigma W_t\sigma'e_i - \frac{p}{2}e_i'\sigma V\sigma'e_i \end{array}\right)\right] \\
= \;& E_{t-1}\left[\exp\left(\begin{array}{c} e_i'\mu_{t-1} + \frac{1}{2}e_i'\Sigma_{t-1}e_i - pe_i'\sigma V\sigma'\delta - \frac{p}{2}e_i'\sigma V\sigma'e_i \\ +e_i'\sigma W_t\sigma'\delta + \frac{1}{2}e_i'\sigma W_t\sigma'e_i \end{array}\right)\right] \\
= \;& E_{t-1}\left[\exp\left(\begin{array}{c} e_i'\mu_{t-1} + \frac{1}{2}e_i'\Sigma_{t-1}e_i - pe_i'\sigma V\sigma'\delta - \frac{p}{2}e_i'\sigma V\sigma'e_i \\ +Tr\left(\left(\sigma'\delta e_i'\sigma + \frac{1}{2}\sigma'e_ie_i'\sigma\right)W_t\right) \end{array}\right)\right] \\
= \;& E_{t-1}\left[\exp\left(\begin{array}{c} e_i'\mu_{t-1} + \frac{1}{2}e_i'\Sigma_{t-1}e_i - pe_i'\sigma V\sigma'\delta - \frac{p}{2}e_i'\sigma V\sigma'e_i \\ -\frac{p}{2}\ln\left[\det\left(I_n - 2\left(\sigma'\delta e_i'\sigma + \frac{1}{2}\sigma'e_ie_i'\sigma\right)V\right)\right] \end{array}\right)\right].
\end{aligned}$$

Hence

$$\begin{aligned}
e_i'\mu_{t-1} \;=\;& r_f + \left(\lambda_i - \frac{1}{2}\right)e_i'\Sigma_{t-1}e_i \\
& +pe_i'\sigma V\sigma'\delta + \frac{p}{2}e_i'\sigma V\sigma'e_i \\
& +\frac{p}{2}\ln\left[\det\left(I_n - 2\left(\sigma'\delta e_i'\sigma + \frac{1}{2}\sigma'e_ie_i'\sigma\right)V\right)\right].
\end{aligned}$$

E.4 Proof of the moment-generating functions

E.4.1 Proof of Proposition 5

$$\begin{aligned}
& E_{t-1}\left[\exp\left(u'R_t + Tr(\theta\Sigma_t)\right)\right] \\
= \;& \exp\left(A(u,\theta) + Tr(B(u,\theta)\Sigma_{t-1})\right)
\end{aligned}$$

with

$$
\begin{aligned}
A(u,\theta) &= \begin{array}{c} \sum_{i=1}^{n} u_i \left(\begin{array}{c} r_f + pe_i'\sigma V\sigma'\delta + \frac{p}{2}e_i'\sigma V\sigma' e_i \\ +\frac{p}{2}\ln\left[\det\left(I_n - 2\left(\sigma'\delta e_i'\sigma + \frac{1}{2}\sigma' e_i e_i'\sigma\right)V\right)\right] \end{array} \right) \\ -\frac{p}{2}\ln\left[\det\left(I_n - 2\left(\frac{1}{2}\sigma' uu'\sigma + \sigma'\delta u'\sigma + \sigma'\alpha'\theta\alpha\sigma\right)V\right)\right] \\ -pu'\sigma V\sigma'\delta - \frac{p}{2}u'\sigma V\sigma' u + Tr\left(\theta\left(\omega - p\alpha\sigma V\sigma'\alpha'\right)\right) \end{array} \\
B(u,\theta) &= \beta'\theta\beta + \alpha'\theta\alpha + \frac{1}{2}uu' + \sum_{i=1}^{n} e_i u_i \left(\lambda_i - \frac{1}{2}\right) e_i'.
\end{aligned}
$$

In the following, we provide more details on the proof of that result:

$$
\begin{aligned}
& E_{t-1}\left[\exp\left(u'R_t + Tr\left(\theta\Sigma_t\right)\right)\right] \\
= \; & E_{t-1}\left[\exp\left(\begin{array}{c} u'\left(\mu_{t-1} + \left(RV_t - \Sigma_{t-1}\right)\delta + RV_t^{1/2} z_{1,t}\right) \\ +Tr\left[\theta\left(\omega - p\alpha\sigma V\sigma'\alpha' + \beta\Sigma_{t-1}\beta' + \alpha\Sigma_{t-1}\alpha' + \alpha\sigma W_t\sigma'\alpha'\right)\right] \end{array} \right)\right] \\
= \; & E_{t-1}\left[\exp\left(\begin{array}{c} u'\left(\mu_{t-1} + \sigma W_t\sigma'\delta - p\sigma V\sigma'\delta\right) + \frac{1}{2}u' RV_t u \\ +Tr\left[\theta\left(\omega - p\alpha\sigma V\sigma'\alpha' + \beta\Sigma_{t-1}\beta' + \alpha\Sigma_{t-1}\alpha' + \alpha\sigma W_t\sigma'\alpha'\right)\right] \end{array} \right)\right] \\
= \; & E_{t-1}\left[\exp\left(\begin{array}{c} u'\left(\mu_{t-1} + \sigma W_t\sigma'\delta - p\sigma V\sigma'\delta\right) + \frac{1}{2}u'\Sigma_{t-1}u + \frac{1}{2}u'\sigma W_t\sigma' u - \frac{p}{2}u'\sigma V\sigma' u \\ +Tr\left[\theta\left(\omega - p\alpha\sigma V\sigma'\alpha' + \beta\Sigma_{t-1}\beta' + \alpha\Sigma_{t-1}\alpha' + \alpha\sigma W_t\sigma'\alpha'\right)\right] \end{array} \right)\right] \\
= \; & E_{t-1}\left[\exp\left(\begin{array}{c} u'\mu_{t-1} - pu'\sigma V\sigma'\delta - \frac{p}{2}u'\sigma V\sigma' u + Tr\left(\theta\left(\omega - p\alpha\sigma V\sigma'\alpha'\right)\right) \\ +Tr\left(\left(\beta'\theta\beta + \alpha'\theta\alpha + \frac{1}{2}uu'\right)\Sigma_{t-1}\right) \\ +Tr\left[\left(\frac{1}{2}\sigma' uu'\sigma + \sigma'\delta u'\sigma + \sigma'\alpha'\theta\alpha\sigma\right)W_t\right] \end{array} \right)\right]
\end{aligned}
$$

$$
\begin{aligned}
= \; & \exp\left(\begin{array}{c} u'\mu_{t-1} - pu'\sigma V\sigma'\delta - \frac{p}{2}u'\sigma V\sigma' u + Tr\left(\theta\left(\omega - p\alpha\sigma V\sigma'\alpha'\right)\right) \\ +Tr\left(\left(\beta'\theta\beta + \alpha'\theta\alpha + \frac{1}{2}uu'\right)\Sigma_{t-1}\right) \\ -\frac{p}{2}\ln\left[\det\left(I_n - 2\left(\frac{1}{2}\sigma' uu'\sigma + \sigma'\delta u'\sigma + \sigma'\alpha'\theta\alpha\sigma\right)V\right)\right] \end{array} \right) \\
= \; & \exp\left(\begin{array}{c} \sum_{i=1}^{n} u_i e_i'\mu_{t-1} - pu'\sigma V\sigma'\delta - \frac{p}{2}u'\sigma V\sigma' u + Tr\left(\theta\left(\omega - p\alpha\sigma V\sigma'\alpha'\right)\right) \\ +Tr\left(\left(\beta'\theta\beta + \alpha'\theta\alpha + \frac{1}{2}uu'\right)\Sigma_{t-1}\right) \\ -\frac{p}{2}\ln\left[\det\left(I_n - 2\left(\frac{1}{2}\sigma' uu'\sigma + \sigma'\delta u'\sigma + \sigma'\alpha'\theta\alpha\sigma\right)V\right)\right] \end{array} \right)
\end{aligned}
$$

$$
\begin{aligned}
= \; & \exp\left(\begin{array}{c} \sum_{i=1}^{n} u_i \left(\begin{array}{c} r_f + \left(\lambda_i - \frac{1}{2}\right)e_i'\Sigma_{t-1}e_i + pe_i'\sigma V\sigma'\delta + \frac{p}{2}e_i'\sigma V\sigma' e_i \\ +\frac{p}{2}\ln\left[\det\left(I_n - 2\left(\sigma'\delta e_i'\sigma + \frac{1}{2}\sigma' e_i e_i'\sigma\right)V\right)\right] \end{array} \right) \\ -pu'\sigma V\sigma'\delta - \frac{p}{2}u'\sigma V\sigma' u + Tr\left(\theta\left(\omega - p\alpha\sigma V\sigma'\alpha'\right)\right) \\ +Tr\left(\left(\beta'\theta\beta + \alpha'\theta\alpha + \frac{1}{2}uu'\right)\Sigma_{t-1}\right) \\ -\frac{p}{2}\ln\left[\det\left(I_n - 2\left(\frac{1}{2}\sigma' uu'\sigma + \sigma'\delta u'\sigma + \sigma'\alpha'\theta\alpha\sigma\right)V\right)\right] \end{array} \right) \\
= \; & \exp\left(\begin{array}{c} \sum_{i=1}^{n} u_i \left(\begin{array}{c} r_f + pe_i'\sigma V\sigma'\delta + \frac{p}{2}e_i'\sigma V\sigma' e_i \\ +\frac{p}{2}\ln\left[\det\left(I_n - 2\left(\sigma'\delta e_i'\sigma + \frac{1}{2}\sigma' e_i e_i'\sigma\right)V\right)\right] \end{array} \right) \\ -\frac{p}{2}\ln\left[\det\left(I_n - 2\left(\frac{1}{2}\sigma' uu'\sigma + \sigma'\delta u'\sigma + \sigma'\alpha'\theta\alpha\sigma\right)V\right)\right] \\ -pu'\sigma V\sigma'\delta - \frac{p}{2}u'\sigma V\sigma' u + Tr\left(\theta\left(\omega - p\alpha\sigma V\sigma'\alpha'\right)\right) \\ +Tr\left[\left(\beta'\theta\beta + \alpha'\theta\alpha + \frac{1}{2}uu' + \sum_{i=1}^{n} e_i u_i \left(\lambda_i - \frac{1}{2}\right)e_i'\right)\Sigma_{t-1}\right] \end{array} \right).
\end{aligned}
$$

E.4.2 Proof of Proposition 6

$$
\begin{aligned}
E_t\left[\exp\left(u'\sum_{i=1}^{\tau+1}R_{t+i}\right)\right] &= E_t\left[\exp\left(u'R_{t+1}\right)E_{t+1}\left[\exp\left(u'\sum_{i=2}^{\tau+1}R_{t+i}\right)\right]\right] \\
&= E_t\left[\exp\left(u'R_{t+1}\right)E_{t+1}\left[\exp\left(u'\sum_{j=1}^{\tau}R_{t+1+i-1}\right)\right]\right] \\
&= E_t\left[\exp\left(u'R_{t+1}+C(u;\tau)+Tr\left(D(u;\tau)\Sigma_{t+1}\right)\right)\right] \\
&= \exp\left(C(u;\tau)\right)E_t\left[\exp\left(u'R_{t+1}+Tr\left(D(u;\tau)\Sigma_{t+1}\right)\right)\right] \\
&= \exp\left(C(u;\tau)+A\left(u,D(u;\tau)\right)+Tr\left(B\left(u,D(u;\tau)\right)\Sigma_t\right)\right).
\end{aligned}
$$

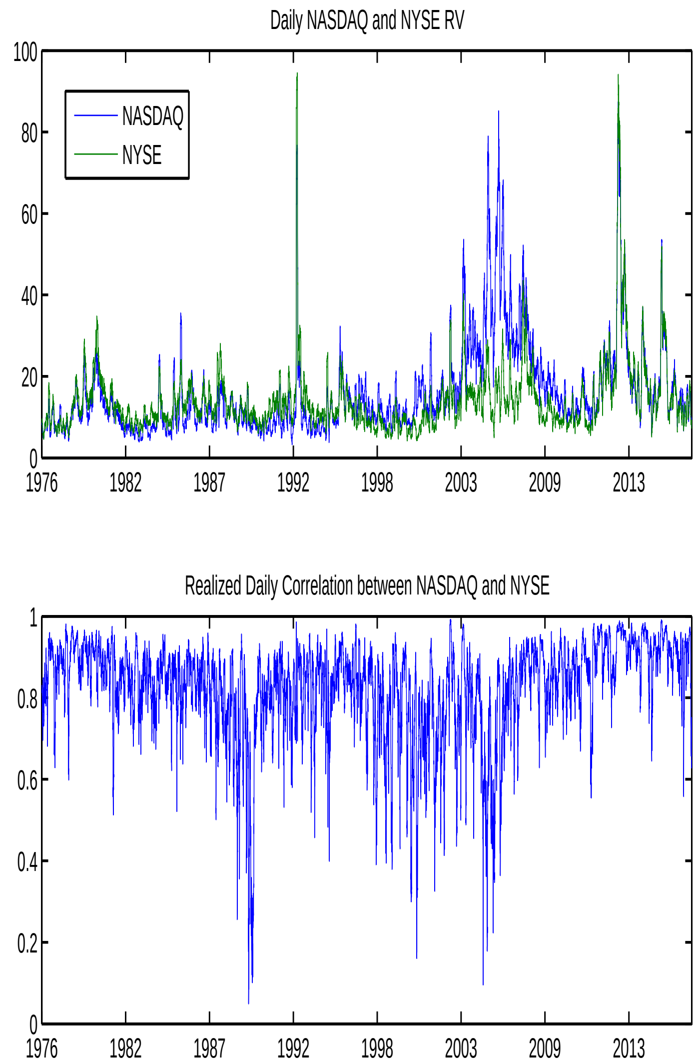

Figure 1: **Realized covariance**. Daily measures of realized volatilities and correlation, from 8 March 1971 to 14 August 2013.

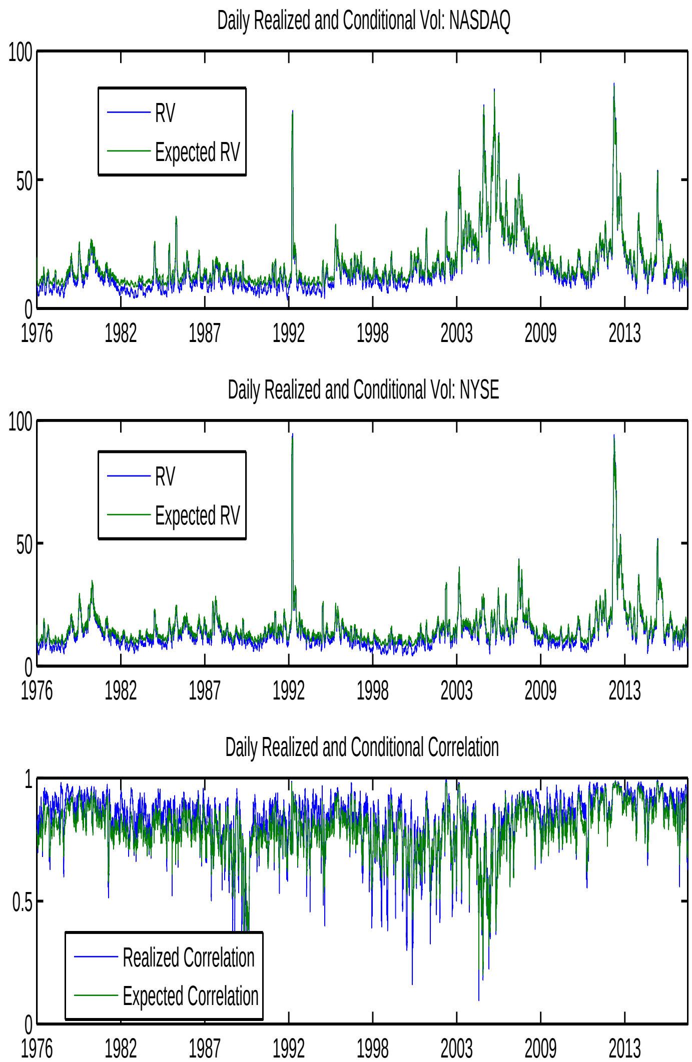

Figure 2: **Realized covariance vs. conditional covariance matrix**. Daily measures of realized and conditional volatilities, as well as correlation, from 8 March 1971 to 14 August 2013.

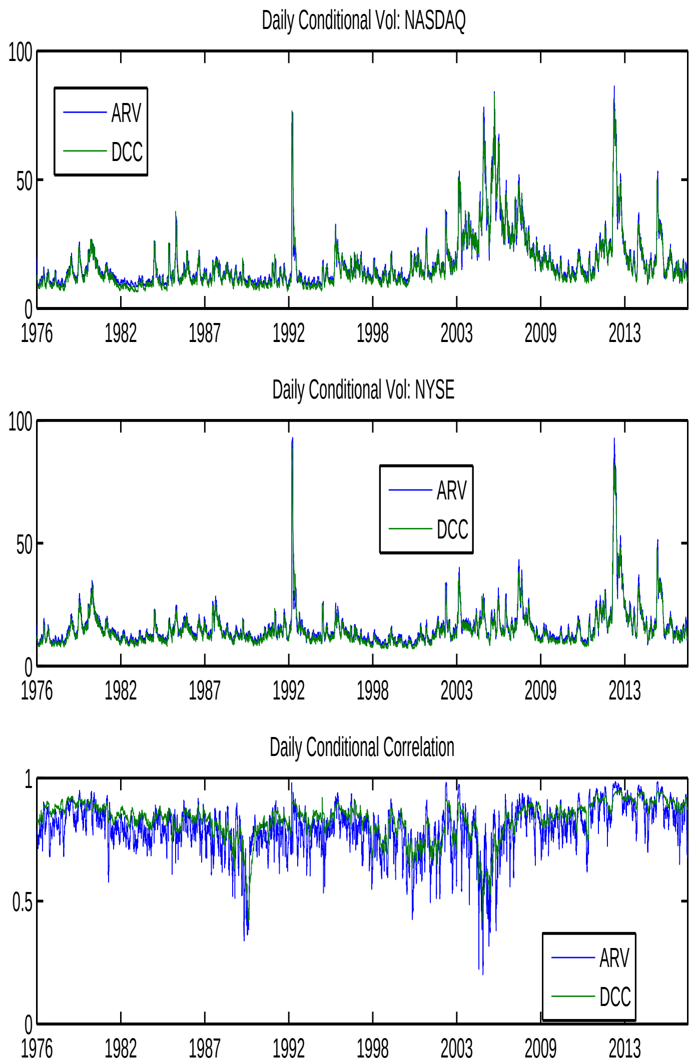

Figure 3: **Conditional covariance matrix: ARV vs. DCC**. Daily measures of conditional volatilities and correlation extracted from the ARV and DCC, from 8 March 1971 to 14 August 2013.