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Abstract

This paper studies the use of information for incentives and risk sharing in agency problems. When the principal is risk neutral or the outcome is contractible, risk sharing is unnecessary or completely taken care of by a contract on the outcome. In this case, information systems are ranked according to their informativeness of the agent's action. When the outcome is noncontractible, however, the principal has to rely on imperfect information for both incentives and risk sharing. Under the first-order approach, we characterize a problem-independent ranking of information systems, which is relaxed from Gjesdal's (1982) criterion. We also find sufficient conditions justifying the first-order approach.

JEL classification: D8
Bank classification: Economic models

Résumé

L'auteur se penche sur l'utilisation de l'information à des fins d'incitation et de partage des risques pour résoudre les problèmes d'agence. Lorsque le mandant est neutre à l'égard du risque ou que le résultat de l'action du mandataire est « contractualisable », le partage des risques est superflu ou ses modalités sont entièrement établies par un contrat ayant pour objet ce résultat. Dans ce dernier cas, les systèmes d'information sont classés selon l'information qu'ils apportent sur l'action du mandataire. Par contre, lorsque le résultat n'est pas contractualisable, le mandant doit se fier à une information imparfaite à des fins tant d'incitation que de partage des risques. L'auteur retient l'approche du premier ordre et propose un classement des systèmes d'information indépendant des problèmes d'agence, ce qui représente un assouplissement par rapport au critère formulé par Gjesdal (1982). Il expose en outre les conditions suffisantes sous lesquelles cette approche est valable.

Classification JEL : D8
Classification de la Banque : Modèles économiques

Non-technical summary

The "informativeness principle" in contract theory predicts that an agent should only be paid according to performance measures that reveal information about the agent's effort level.

There are, however, real-life cases where the agent is paid not only according to measures of his effort, but also to measures of the outcome. For example, in many medical malpractice cases, the fee that the client (the principal) pays the lawyer (the agent) is usually based on both the time the lawyer has spent on the case—a measure of the lawyer's effort—and the outcome of the case. Project managers are usually paid according to both the time they spent on projects, as well as the outcome of the projects.

In this paper, we argue that the "informativeness principle" is suitable when the principal is risk neutral. In this case, the only purpose of the contract is to motivate the agent to work hard. Paying the agent based on anything that is not informative of the agent's effort will expose the agent to unnecessary compensation risks but provide no incentive.

A new theorem is therefore developed for the case where the principal is risk averse. A risk-averse principal uses the compensation contract for two purposes: (i) to motivate the agent to work hard, and (ii) to share risks in the outcome with the agent. Therefore, the optimal compensation contract should include measures of the agent's effort, as well as the outcome of the agent's effort.

1 Introduction

The principal-agent problem consists of two stages: the first stage is the principal's choice of an information system, which generates contractible (i.e., commonly observable and verifiable) signals or performance measures. The second stage is the optimal design of a contract based on the information system chosen in the first stage. Much of the literature focuses on the second stage, assuming that the information system is given. In this paper, we focus on the first stage and study the choice of information systems by the principal.

In general, information systems should be evaluated according to their efficiency in serving two potential purposes: (i) providing incentives to the agent's action and (ii) allocating the risk in the outcome of the agent's action. There are, however, two cases where risk sharing is unnecessary or straightforward, so information systems are in effect evaluated by their efficiency in providing incentives only. The first case involves a risk-neutral principal, who uses information systems for incentives only. In the second case where the principal is strictly risk-averse and the outcome is contractible, risk sharing can be completely taken care of by a contract contingent solely on the outcome. Additional observables may be used for incentives if they provide more information about the agent's action beyond that conveyed by the outcome. In this case, information systems are ranked by how informative the additional observables are about the agent's action.

In many cases, however, the outcome is noncontractible. Gjesdal (1982) gives three main reasons for this.[1] First, the outcome may be unobservable at the time when the agent is paid. For instance, the manager of a firm may be paid irreversibly before the outcome of his action is observed. Second, contracting on the outcome may be too costly. For instance, perfect auditing of income tax returns is expensive. Lastly, the outcome is often an imperfect estimate of the "real" outcome. For instance, the quality measure of a project is an imperfect estimate of the "real" quality.

[1] See also Mirrlees (1976) and Maskin (2002). Baker (2002) also justifies noncontractible outcomes for organizations where the total value of the organization is the noncontractible outcome.

When the outcome is noncontractible, a strictly risk-averse principal has to reply on imperfect information for both incentives and risk sharing. Following this line of reasoning, we develop a criterion that ranks information systems by how informative they are of the agent's action and the outcome.

Our criterion is developed under the first-order approach, by which the principal can predict the agent's action using the agent's first-order conditions alone. The first-order approach can be justified by conditions ensuring that the agent's utility is concave in his action.[2] Finding these conditions for the case with a risk-averse principal and a noncontractible outcome presents technical difficulties: the interaction between the two roles of information systems may cause the agent's utility to be nonconcave in his action. Nevertheless, we show that, by imposing restrictions on the information structure and both parties' risk aversions, we can justify the first-order approach.

Comparing information systems in agency problems was first raised by Holmström (1979), and further studied by Kim (1995), Jewitt (1997, 2007), Dewatripont, Jewitt and Tirole (1999), Demougin and Fluet (2001), Fagart and Sinclair-Desgagne (2007), and Xie (2011). These studies assume either a risk-neutral principal or a contractible outcome. Therefore, these studies are about information systems for incentive as opposed to risk-sharing purposes. The main prediction of these studies is the "informativeness principle," which says that an information system is valuable if and only if it provides information about the agent's action.[3] This paper studies an alternative scenario where the principal is risk averse and the outcome is noncontractible. We find that the "informativeness principle" does not hold in the current context, and a new criterion is proposed.

Gjesdal (1982) compares information systems in the same context as the current paper, noting that when both the action and the outcome are noncontractible, a risk-averse principal

[2] See Rogerson (1985), Jewitt (1988), Sinclair-Desgagne (1994), and more recently, Conlon (2009) for justification of the first-order approach in cases where the principal is risk neutral and/or the outcome is contractible.

[3] See also Shavell (1979), Laffont and Martimort (2002), Tirole (2006), and Bolton and Dewatripont (2005).

has to rely on imperfect information for risk-sharing as well as incentive purposes. Gjesdal develops a sufficient statistic criterion that is more restrictive than our criterion, but does not rely on the first-order approach. In other words, we relax Gjasdal's sufficient statistic criterion under the first-order approach.

The rest of the paper is organized as follows. We set up the model in Section 2. In Section 3, we study the two cases where risk sharing is not a challenge. In Section 4, we study the case where information systems are ranked for both the incentive and risk-sharing purposes. We justify the first-order approach in Section 5, and conclude in Section 6. All proofs are in the Appendix.

2 The Model

A principal faces a feasible set Γ of information systems, each of which can be represented by a random vector $\tilde{\mathbf{x}}$. After choosing an $\tilde{\mathbf{x}} \in \Gamma$, the principal makes a take-it-or-leave-it contract, $s_{\tilde{\mathbf{x}}}(\cdot) \in [\underline{s}, \bar{s}]$, with an agent, who has an outside reservation utility of 0. The parameters $\underline{s}$ and $\bar{s}$ are the lower and upper bounds of the agent's payments, respectively.[4] If the contract is accepted, the agent chooses an unobservable real-valued action $a \in R^+$, and incurs private cost $c(a)$ with $c' > 0$ and $c'' \geq 0$. The action a stochastically generates a real-valued outcome $\tilde{b}$ on a fixed support $b \in [\underline{b}, \bar{b}]$,[5] and is imperfectly correlated with the information system $\tilde{\mathbf{x}}$.

The principal's utility $v(b - s_{\tilde{\mathbf{x}}})$ is defined on her residual, which is the outcome minus the compensation to the agent. The agent derives utility from the received payment minus the private cost of action, $u(s_{\tilde{\mathbf{x}}}) - c(a)$. We assume that $v' > 0$, $v'' \leq 0$, $u' > 0$, $u'' < 0$, $c' > 0$, and $c'' > 0$. In particular, the agent is strictly risk-averse, while the principal could

[4] The upper and the lower bounds are introduced to avoid nonexistence problems (see Page, 1987). In addition, contracts in real life are always bounded due to legal and other constraints on what payments the parties can make.

[5] $\tilde{b}$ is measured in monetary units and can be interpreted as the principal's willingness to pay for the outcome, if the outcome itself, e.g., quality, is not monetary.

be risk-neutral or strictly risk-averse.

We assume that all information systems in Γ are contractible – commonly observable and verifiable by both the principal and the agent. We thereby exclude the cases considered by Maskin and Tirole (1999) and Maskin (2002) where information systems are either observable by one party but not the other, or are commonly observable but nonverifiable. As shown by Maskin and Tirole (1999) and Maskin (2002), the principal's problems in these cases are mechanism design problems mixed with moral hazard, which are not the focus of this paper.

All distribution and density functions are denoted by F and f, respectively, with the subscript indicating which random variables are intended. For instance, $F_{\tilde{b}}(b|a)$ is the marginal distribution function of $\tilde{b}$, given the agent's action a, and $f_{(\tilde{b},\tilde{\mathbf{x}})}(b,\mathbf{x}|a)$ is the joint density of $\tilde{b}$ and $\tilde{\mathbf{x}}$, given a. All density functions are positive and continuous on their corresponding supports, and are differentiable in each of their arguments to the order needed. All distributions are common knowledge.

Following the literature, we focus on ranking information systems for inducing a given action a. Given a, the principal solves for $s_{\tilde{\mathbf{x}}}$ in the following program:

$$V(\tilde{\mathbf{x}}) = \max_{s_{\tilde{\mathbf{x}}} \in [\underline{s},\bar{s}]} \iint v(b - s_{\tilde{\mathbf{x}}}(\mathbf{x})) f_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, b|a)\,db\,d\mathbf{x}$$

$$\text{(1)} \qquad s.t. \quad \int u(s_{\tilde{\mathbf{x}}}(\mathbf{x})) f_{\tilde{\mathbf{x}}}(\mathbf{x}|a)\,d\mathbf{x} - c(a) \geq 0, \quad \text{and}$$

$$\text{(2)} \qquad a \in \max_{\hat{a}} \int u(s_{\tilde{\mathbf{x}}}(\mathbf{x})) f_{\tilde{\mathbf{x}}}(\mathbf{x}|\hat{a})\,d\mathbf{x} - c(\hat{a}).$$

Constraint (1) is the participation constraint, which states that the agent's expected utility must be no less than his outside reservation utility. Constraint (2) is the incentive compatibility constraint, which states that, given the payment schedule $s_{\tilde{\mathbf{x}}}$, a maximizes the agent's expected utility.

A technical challenge is the infinite number of constraints imposed by (2). A common solution is to use the first-order approach, which replaces (2) with the first-order necessary condition that

$$\int u(s_{\tilde{\mathbf{x}}}(\mathbf{x}))f_{\tilde{\mathbf{x}}_a}(\mathbf{x}|a)d\mathbf{x} - c'(a) = 0, \tag{3}$$

where the subscript a denotes partial derivative in a. Equation (3) is called the relaxed incentive compatibility constraint. We do not restrict our analysis to the first-order approach. We will instead evaluate how the validity of the first-order approach, or the lack thereof, affects the results that can be obtained about ranking of information systems, and then find conditions under which the first-order approach is valid.

Before proceeding with the analysis, we need to define precisely the concept of a "more valuable" information system.

Definition 1. Given two information systems, $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}} \in \Gamma$, $\tilde{\mathbf{x}}$ is *weakly more valuable at a* than $\tilde{\mathbf{y}}$ if $V(\tilde{\mathbf{x}}, a) \geq V(\tilde{\mathbf{y}}, a)$. $\tilde{\mathbf{x}}$ is *strictly more valuable at a* than $\tilde{\mathbf{y}}$ if $V(\tilde{\mathbf{x}}, a) > V(\tilde{\mathbf{y}}, a)$.

The principal uses information systems for two potential purposes: (i) providing incentives for the agent's action and (ii) allocating the risk in the outcome between the two parties. To serve the two purposes, an information system needs to be informative of the agent's action and the outcome, respectively, though the statistical meaning of "being informative" may be different for the two purposes and in different scenarios. There are, however, cases where risk sharing is unnecessary or straightforward, so information systems will be evaluated by how informative they are of the agent's action only. We will study these cases in the next section, before jumping into the more complicated case where information systems are ranked by their relative informativeness of both the agent's action and the outcome.

3 Ranking of Information Systems When Risk-Sharing Is Not a Challenge

There are two cases where risk sharing is not a challenge: (i) when the principal is risk neutral, and (ii) when the principal is risk averse and the outcome is contractible. In the first case, risk sharing is unnecessary so information systems will simply be evaluated by how informative they are of the agent's action. In the second case, risk sharing can be completely taken care of by the outcome, and ranking of information systems is determined by how informative the other observables (besides the outcome) in the information system are about the agent's action.

Definition 2. Information system $\tilde{\mathbf{x}}$ is *sufficient for* $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ *when estimating* a if the conditional distribution of $\tilde{\mathbf{y}}$ given $\mathbf{x}$ is invariant with respect to a, i.e.,

$$f_{\tilde{\mathbf{y}}}(\mathbf{y}|\mathbf{x}, a) = f_{\tilde{\mathbf{y}}}(\mathbf{y}|\mathbf{x}), \quad \textit{for almost all} \quad \mathbf{x}, \mathbf{y}, \;\; \textit{and} \;\; a.$$

Definition 3. Let $L_{\tilde{\mathbf{x}}}(\tilde{\mathbf{x}}|a)$ denote *the likelihood ratio of* $\tilde{\mathbf{x}}$ *given* a, i.e.,

$$L_{\tilde{\mathbf{x}}}(\tilde{\mathbf{x}}|a) \equiv \frac{f_{\tilde{\mathbf{x}}a}(\tilde{\mathbf{x}}|a)}{f_{\tilde{\mathbf{x}}}(\tilde{\mathbf{x}}|a)}.$$

Similarly, let $L_{\tilde{\mathbf{x}}}(\tilde{\mathbf{x}}|a, b)$ denote *the likelihood ratio of* $\tilde{\mathbf{x}}$ *given* a *and* b*,*

$$L_{\tilde{\mathbf{x}}}(\tilde{\mathbf{x}}|a, b) \equiv \frac{f_{\tilde{\mathbf{x}}a}(\tilde{\mathbf{x}}|a, b)}{f_{\tilde{\mathbf{x}}}(\tilde{\mathbf{x}}|a, b)}.$$

Note that both $L_{\tilde{\mathbf{x}}}(\tilde{\mathbf{x}}|a, b)$ and $L_{\tilde{\mathbf{x}}}(\tilde{\mathbf{x}}|a)$ are random variables, as they are functions of the random vector $\tilde{\mathbf{x}}$.

Definition 4. Given two random variables $\tilde{\mathrm{x}}$ and $\tilde{\mathrm{y}}$, the distribution of $\tilde{\mathrm{x}}$ is a *mean-preserving* *spread* of the distribution of $\tilde{y}$, if $\tilde{x}$ and $\tilde{y}$ have the same mean and

$$\int n(x)dF_{\tilde{x}}(x) \leq \int n(y)dF_{\tilde{y}}(y),$$

for all increasing and concave functions $n(\cdot)$.

Rothschild and Stiglitz (1970) show that the above mean-preserving spread condition is equivalent to the existence of a random variable $\tilde{\epsilon}$ such that

$$\tilde{x} \overset{d}{=} \tilde{y} + \tilde{\epsilon}, \text{ and } E[\tilde{\epsilon}|y] = 0, \ \forall y, \tag{4}$$

where $\overset{d}{=}$ means "has the same distribution as." If we define $\tilde{z} \equiv \tilde{y} + \tilde{\epsilon}$, then (4) is further equivalent to the following martingale condition:

$$E[\tilde{z}|y] = E[\tilde{y} + \tilde{\epsilon}|y] = y, \ \forall y. \tag{5}$$

In sum, the mean-preserving spread condition is equivalent to the existence of an artificial random variable $\tilde{z}$, which has the same marginal distribution as $\tilde{x}$, and for which the martingale condition (5) holds.

3.1 The Case Where the Principal Is Risk-Neutral

This case has been well studied in the literature (see Gjesdal, 1982; Grossman and Hart, 1983; Kim, 1995; and Jewitt, 2007). A risk-neutral principal in effect solves the following cost-minimization problem:

$$\min_{s_{\tilde{\mathbf{x}}}, a} \int s_{\tilde{\mathbf{x}}}(\mathbf{x}) f_{\tilde{\mathbf{x}}}(\mathbf{x}|a) d\mathbf{x}, \tag{6}$$

subject to (1) and (2). Since the object of the principal is to minimize the expected payment, risk sharing is unnecessary. Therefore, information systems are ranked by how informative they are about the agent's action only. Let $\tilde{\mathbf{x}} \stackrel{d}{=}_a \tilde{\mathbf{y}}$ stand for "$\tilde{\mathbf{x}}$ has the same conditional distribution as $\tilde{\mathbf{y}}$, given a." We have the following proposition:

Proposition 1. *Information system $\tilde{\mathbf{x}}$ is weakly more valuable at a than $\tilde{\mathbf{y}}$ for all risk-neutral principals, if there exists an artificial density function $\phi(\mathbf{y}|\mathbf{x})$, such that*

$$f_{\tilde{\mathbf{y}}}(\mathbf{y}|a) = \int \phi(\mathbf{y}|\mathbf{x}) f_{\tilde{\mathbf{x}}}(\mathbf{x}|a) d\mathbf{x}, \quad \textit{for almost all } \mathbf{y}, \tag{7}$$

or equivalently, if there exists an artificial random vector $\tilde{\mathbf{z}}$, such that

$$\tilde{\mathbf{z}} \stackrel{d}{=}_a \tilde{\mathbf{y}}, \textit{ and } \tilde{\mathbf{x}} \textit{ is sufficient for } (\tilde{\mathbf{x}}, \tilde{\mathbf{z}}) \textit{ when estimating } a. \tag{8}$$

The sufficiency of (7) was first proven by Gjesdal (1982) and Grossman and Hart (1983) using the following strategy: if (7) holds, then for an arbitrary contract $s_{\tilde{\mathbf{y}}}$, we can construct a contract $s_{\tilde{\mathbf{x}}}$, which is equivalent to $s_{\tilde{\mathbf{y}}}$ from the agent's perspective, but weakly improves the principal's welfare.

Condition (8) is an artificial-random-vector representation of Condition (7). Note that while (7) is more often presented in principal-agent textbooks, (8) is in effect more intuitive: the first part of (8) implies that $\tilde{\mathbf{z}}$ and $\tilde{\mathbf{y}}$ are identical from the principal's perspective, due to their identical marginal distribution given a. The sufficient-statistic condition in the second part of (8) suggests that $\tilde{\mathbf{x}}$ is preferred to $\tilde{\mathbf{z}}$. Then by the transitivity of preferences, $\tilde{\mathbf{x}}$ is preferred to $\tilde{\mathbf{y}}$.

Condition (8) also suggests that if $\tilde{\mathbf{x}}$ is sufficient for $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ when estimating a, then $\tilde{\mathbf{x}}$ is more valuable than $\tilde{\mathbf{y}}$ for all risk-neutral principals, but not vice versa. In particular, $\tilde{\mathbf{x}}$ could be more valuable than $\tilde{\mathbf{y}}$, even though $\tilde{\mathbf{y}}$ provides additional information about a beyond that contained in $\tilde{\mathbf{x}}$. Note that if we replace the first part of (8) with a stronger condition that $\tilde{\mathbf{z}} = \tilde{\mathbf{y}}$, then (8) becomes the condition that $\tilde{\mathbf{x}}$ is sufficient for $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ when estimating a. Apparently, $\tilde{\mathbf{z}} = \tilde{\mathbf{y}}$ implies $\tilde{\mathbf{z}} \stackrel{d}{=}_a \tilde{\mathbf{y}}$, but not vice versa.

When the first-order approach is valid, we can relax the ranking conditions as follows:

Proposition 2. *Assume that the first-order approach is valid. Information system* $\tilde{\mathbf{x}}$ *is weakly more valuable at a than* $\tilde{\mathbf{y}}$ *for all risk-neutral principals, if and only if*

(9) *the distribution of* $L_{\tilde{\mathbf{x}}}(\tilde{\mathbf{x}}|a)$ *is a mean-preserving spread of that of* $L_{\tilde{\mathbf{y}}}(\tilde{\mathbf{y}}|a)$,

or equivalently, there exists an artificial random vector $\tilde{\mathbf{z}}$, *such that*

$$\tilde{\mathbf{z}} \overset{d}{=}_a \tilde{\mathbf{y}}, \text{ and } L_{\tilde{\mathbf{z}}}(\mathbf{z}|a) = E\left[L_{\tilde{\mathbf{x}}}(\tilde{\mathbf{x}}|a)|L_{\tilde{\mathbf{z}}}(\mathbf{z}|a)\right]. \tag{10}$$

The proof follows from Jewitt (2007): by conjugate duality, the principal's expected cost can be expressed as a concave function of the likelihood ratio of the information system. Since all likelihood ratios have zero means, a mean-preserving spread condition on likelihood ratio functions is both necessary and sufficient for ranking information systems. Following a different approach, Kim (1995) proves the sufficiency of Condition (9), but not the necessity.

Condition (10) is an artificial-random-vector representation of Condition (9): there exists an artificial random vector $\tilde{\mathbf{z}}$ such that the martingale condition in (10) holds.

Both Propositions 1 and 2 provide incentive conditions implying that $\tilde{\mathbf{x}}$ is more informative about a than $\tilde{\mathbf{y}}$ is. The statistical meaning of being "more informative," however, is different in the two propositions. Specifically, (7) implies (9), but not vice versa (see for instance Propositions 4 and 5 in Kim, 1995). That is, the first-order approach relaxes the incentive condition from (7) to (9). The economic intuition of this relaxation is that, under the first-order approach, instead of comparing the implemented action to every other action, the principal only needs to compare the implemented action "locally" to an arbitrarily close action. For this "local" incentive purpose, the principal needs an information system that allows her to easily identify any small deviation from the target action (large $|f_a|$) at low cost (small f) or, equivalently, a large variation of the likelihood ratio f_a/f for a given effort level a. This is also the reason why Condition (9) is imposed on likelihood ratios, instead of on information systems themselves.

3.2 The Case Where the Principal Is Strictly Risk-Averse and the Outcome Is Contractible

Apparently, the principal always wants to contract on $\tilde{b}$. That is, information system $(\tilde{\mathbf{x}}, \tilde{b})$ is always weakly more valuable than $\tilde{\mathbf{x}}$, because the former provides weakly more information. The following proposition shows that $(\tilde{\mathbf{x}}, \tilde{b})$ is in effect strictly more valuable than $\tilde{\mathbf{x}}$, as long as $\tilde{\mathbf{x}}$ is not perfectly informative of $\tilde{b}$:

Proposition 3. *Information system $(\tilde{\mathbf{x}}, \tilde{b})$ is strictly more valuable at a than $\tilde{\mathbf{x}}$ for all strictly risk-averse principals, if there does not exist a function $g(\cdot, \cdot)$, such that $f_{\tilde{b}}(b|\mathbf{x}, a)$ is a point mass at $g(\mathbf{x}, a)$ for almost all $\mathbf{x}$.*

We prove Proposition 3 by showing that, given the optimal contract contingent solely on $\tilde{\mathbf{x}}$, a modified contract based on both $\tilde{\mathbf{x}}$ and $\tilde{b}$ is Pareto improving.

Proposition 3 suggests that $\tilde{b}$ strictly improves risk sharing in the information system $(\tilde{\mathbf{x}}, \tilde{b})$, as long as $\tilde{\mathbf{x}}$ is not perfectly informative of $\tilde{b}$. Note that Proposition 3 holds whether $\tilde{b}$ is informative about a or not. That is, $\tilde{b}$ may or may not improve incentives, but it strictly improves risk sharing if $f_{\tilde{b}}(b|\mathbf{x}, a)$ is not a point mass, or equivalently, if $\tilde{\mathbf{x}}$ is not perfectly informative of $\tilde{b}$.

In what follows, we assume that none of the contractible information systems, other than $\tilde{b}$, is perfectly informative of $\tilde{b}$. According to Proposition 3, $\tilde{b}$ should always be written into the contract. We now wonder how the principal ranks information systems $(\tilde{\mathbf{x}}, \tilde{b})$ and $(\tilde{\mathbf{y}}, \tilde{b})$. Since risk sharing has been completely taken care of by $\tilde{b}$, any additional observables, i.e., $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$, are used for incentives only. Therefore, $(\tilde{\mathbf{x}}, \tilde{b})$ is weakly more valuable than $(\tilde{\mathbf{y}}, \tilde{b})$ if, given $\tilde{b}$, $\tilde{\mathbf{x}}$ provides more information about $\tilde{b}$ than $\tilde{\mathbf{y}}$ does. We have the following proposition:

Proposition 4. *Information system $(\tilde{\mathbf{x}}, \tilde{b})$ is weakly more valuable at a than $(\tilde{\mathbf{y}}, \tilde{b})$ for all strictly risk-averse principals, regardless of the distribution of $\tilde{b}$, if there exists an artificial* *density function* $\phi(\mathbf{y}|\mathbf{x}, b)$*, such that*

$$f_{\tilde{\mathbf{y}}}(\mathbf{y}|a,b) = \int \phi(\mathbf{y}|\mathbf{x},b) f_{\tilde{\mathbf{x}}}(\mathbf{x}|a,b) d\mathbf{x}, \quad \textit{for almost all } b \textit{ and } \mathbf{y}, \tag{11}$$

or equivalently, if there exists an artificial random vector $\tilde{\mathbf{z}}$*, such that*

$$\tilde{\mathbf{z}} \stackrel{d}{=}_a \tilde{\mathbf{y}}, \textit{ and } \tilde{\mathbf{x}} \textit{ is sufficient for } (\tilde{\mathbf{x}}, \tilde{\mathbf{z}}) \textit{ when estimating } a, \textit{ given } b. \tag{12}$$

Proof. The proof is similar to that of Proposition 1, so is omitted. □

Proposition 4 is similar to Proposition 1 in several aspects. First, The proofs are similar. Second, both propositions involve cases where risk sharing is not a challenge, so the only concern is to rank information systems for the incentive purpose. The difference is that in Proposition 4, $\tilde{b}$ is contractible, so (11) and (12) involve b, while in Proposition 1, $\tilde{b}$ is noncontractible, but the principal is risk neutral, so the conditions do not involve b at all.

Proposition 5. *Assume that the first-order approach is valid. Information system* $(\tilde{\mathbf{x}}, \tilde{b})$ *is weakly more valuable at* a *than* $(\tilde{\mathbf{y}}, \tilde{b})$ *regardless of the distribution of* $\tilde{b}$*, for all risk-averse principals, if and only if*

$$\textit{the distribution of } L_{\tilde{\mathbf{x}}}(\mathbf{x}|a,b) \textit{ is a mean-preserving spread of that of } L_{\tilde{\mathbf{y}}}(\mathbf{y}|a,b), \tag{13}$$

or equivalently, there exists an artificial random vector $\tilde{\mathbf{z}}$*, such that*

$$\tilde{\mathbf{z}} \stackrel{d}{=}_a \tilde{\mathbf{y}}, \textit{ and } L_{\tilde{\mathbf{z}}}(\mathbf{z}|a,b) = E\left[L_{\tilde{\mathbf{x}}}(\tilde{\mathbf{x}}|a,b) | L_{\tilde{\mathbf{z}}}(\mathbf{z}|a,b)\right].$$

The proof of Proposition 5 is similar to that of Proposition 2. In particular, by conjugate duality, the principal's expected utility can be transferred to an increasing and convex function of the likelihood ratio function, $L_{\tilde{\mathbf{x}}}(\mathbf{x}|a,b)$, which has a zero mean. Therefore, the mean-preserving spread condition (13) is both necessary and sufficient for $\tilde{\mathbf{x}}$ to be more valuable than $\tilde{\mathbf{y}}$. Again, the first-order approach relaxes the incentive condition from an integral condition (11) to a mean-preserving spread condition (13).

Different from Propositions 1 and 2, Propositions 4 and 5 show that when the outcome is contractible, what matters for the comparison of information systems is not how informative the other variables ($\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$) are, but how much additional information about a they can provide beyond that contained in $\tilde{b}$.

4 Ranking of Information Systems for Both the Incentive and Risk-Sharing Purposes

When the principal is strictly risk averse and there is no perfect information about either the agent's action or the outcome, the principal has to rely on imperfect information for both incentives and risk sharing. This suggests that information systems should be evaluated according to their informativeness about the agent's action and the outcome. Following this line of reasoning, Gjesdal (1982) proves that $\tilde{\mathbf{x}}$ is weakly more valuable than $\tilde{\mathbf{y}}$ for all strictly risk-averse principals, if there exists an artificial density function $\phi(\mathbf{y}|\mathbf{x})$ such that

$$f_{\tilde{\mathbf{y}}}(\mathbf{y}|a,b) = \int \phi(\mathbf{y}|\mathbf{x}) f_{\tilde{\mathbf{x}}}(\mathbf{x}|a,b) d\mathbf{x}, \quad \forall \mathbf{y} \text{ and } b.$$

Gjesdal's (1982) condition has an equivalent artificial-random-vector representation as follows: Let $\tilde{\mathbf{x}} \stackrel{d}{=}_{a,b} \tilde{\mathbf{y}}$ stand for "$\tilde{\mathbf{x}}$ has the same marginal distribution as $\tilde{\mathbf{y}}$, given a and b." There exists an artificial random vector $\tilde{\mathbf{z}} \stackrel{d}{=}_{a,b} \tilde{\mathbf{y}}$, such that $\tilde{\mathbf{x}}$ is sufficient for $(\tilde{\mathbf{x}}, \tilde{\mathbf{z}})$ when estimating both a and $\tilde{b}$, i.e.,

$$(14) \qquad f_{\tilde{\mathbf{z}}}(\mathbf{z}|\mathbf{x},a,b) = f_{\tilde{\mathbf{z}}}(\mathbf{z}|\mathbf{x}), \quad \forall \mathbf{z}, \mathbf{x}, \text{ and } b.$$

This sufficient statistic condition can be broken down to an incentive condition and a risk-sharing condition separately, as shown in the following proposition:

Proposition 6. *Information system* $\tilde{\mathbf{x}}$ *is weakly more valuable at* a *than* $\tilde{\mathbf{y}}$ *for all strictly risk-averse principals, if there exists an artificial random vector* $\tilde{\mathbf{z}}$*, such that* $\tilde{\mathbf{z}} \overset{d}{=}_{a,b} \tilde{\mathbf{y}}$*, and*

$$f_{\tilde{\mathbf{z}}}(\mathbf{z}|\mathbf{x}, a) = f_{\tilde{\mathbf{z}}}(\mathbf{z}|\mathbf{x}), \quad \forall \mathbf{z} \text{ and } \mathbf{x}, \quad \text{and} \tag{15}$$

$$f_{\tilde{\mathbf{z}}}(\mathbf{z}|\mathbf{x}, a, b) = f_{\tilde{\mathbf{z}}}(\mathbf{z}|\mathbf{x}, a), \quad \forall \mathbf{z}, \mathbf{x}, \text{ and } b. \tag{16}$$

Condition (15) is an incentive condition, saying that $\tilde{\mathbf{x}}$ is sufficient for $(\tilde{\mathbf{x}}, \tilde{\mathbf{z}})$ when estimating a. Condition (16) is a risk-sharing condition, saying that given a, $\tilde{\mathbf{x}}$ is sufficient for $(\tilde{\mathbf{x}}, \tilde{\mathbf{z}})$ when estimating $\tilde{b}$.

Condition (15) is identical to (8)—the incentive condition in the risk-neutral principal case. Recall that in Proposition 2, (8) is relaxed to (10) under the first-order approach. Therefore, it is natural to wonder if the same relaxation extends to the current case where the principal is strictly risk averse. The following proposition shows that, indeed, under the first-order approach, (15) can be relaxed to a condition that is similar to but more restrict than (10).

Proposition 7. *Assume that the first-order approach is valid. Information system* $\tilde{\mathbf{x}}$ *is weakly more valuable at* a *than* $\tilde{\mathbf{y}}$ *for all strictly risk-averse principals, if there exists an artificial conditional density function* $\phi(\mathbf{y}|\mathbf{x}, a)$ *such that*

$$\int \phi_a(\mathbf{y}|\mathbf{x}, a) f(\mathbf{x}|a) d\mathbf{x} = 0, \quad \text{for almost all } \mathbf{y}, \quad \text{and} \tag{17}$$

$$\int \phi(\mathbf{y}|\mathbf{x}, a) f(\mathbf{x}|a, b) d\mathbf{x} = f(\mathbf{y}|a, b), \quad \text{for almost all } \mathbf{y} \text{ and } b. \tag{18}$$

Or equivalently, if there exists an artificial random vector $\tilde{\mathbf{z}}$ *with* $\tilde{\mathbf{z}} \overset{d}{=}_{a,\tilde{b}} \tilde{\mathbf{y}}$*, and*

$$\int f_{\tilde{\mathbf{z}}a}(\mathbf{z}|\mathbf{x}, a) f_{\tilde{\mathbf{x}}}(\mathbf{x}|a) d\mathbf{x} = 0, \quad \text{and} \tag{19}$$

$$f_{\tilde{\mathbf{z}}}(\mathbf{z}|\mathbf{x}, a, b) = f_{\tilde{\mathbf{z}}}(\mathbf{z}|\mathbf{x}, a). \tag{20}$$

We prove the sufficiency of the set of conditions (17) and (18) by showing that if (17) holds, then for any contract $s_{\tilde{\mathbf{y}}}$, we can construct a contract $s_{\tilde{\mathbf{x}}}$ that induces the same action and the same welfare for the agent. If, in addition, (18) holds, then $s_{\tilde{\mathbf{x}}}$ generates higher welfare for the principal than $s_{\tilde{\mathbf{y}}}$ does. Conditions (19) and (20) are an artificial-random-vector representation of Conditions (17) and (18).

Condition (19) is an incentive condition, as it is relaxed from (15), which is the incentive condition in Gjesdal's (1982) theorem. Condition (15) requires that $f_{\tilde{\mathbf{z}}a}(\mathbf{z}|\mathbf{x}, a) = 0$ at all values of $\mathbf{z}$ and $\mathbf{x}$, while (19) requires only that the integral of $f_{\tilde{\mathbf{z}}a}(\mathbf{z}|\mathbf{x}, a)$ with respect to $F_{\tilde{\mathbf{x}}}(\mathbf{x}|a)$ be zero. On the other hand, Propositions 6 and 7 have the same risk-sharing conditions, i.e., (20) and (16) are identical.

Also note that (19) is more restrictive than (10), which is the relaxed incentive condition for the case with a risk-neutral principal. To see this result, we need to transform (19) to a format similar to (10), as in the following lemma:

Lemma 1. *Condition* (19) *is equivalent to*

$$L_{\tilde{\mathbf{z}}}(\mathbf{z}|a) = E\left[L_{\tilde{\mathbf{x}}}(\tilde{\mathbf{x}}|a)\,|\mathbf{z}\right]. \tag{21}$$

Note that (21) is exactly Formula (3.2) in Dewatripont, Jewitt and Tirole (1999). Condition (21) is close to but more restrictive than (10). Both conditions say that given the value of a certain function $g(\mathbf{z})$, the conditional expectation of $L_{\tilde{\mathbf{x}}}(\tilde{\mathbf{x}}|a)$ is equal to $L_{\tilde{\mathbf{z}}}(\mathbf{z}|a)$. The function g, however, is different in the two conditions: in (21), $g(\mathbf{z}) = \mathbf{z}$, while in (10), $g(\mathbf{z}) = L_{\tilde{\mathbf{z}}}(\mathbf{z}|a)$. Since $L_{\tilde{\mathbf{z}}}(\mathbf{z}|a)$ is a function of $\mathbf{z}$, (21) implies (10), but not vice versa. If, however, $L_{\tilde{\mathbf{x}}}(\mathbf{x}|a)$ is a one-to-one function of $\mathbf{x}$ and similarly for $L_{\tilde{\mathbf{z}}}(\mathbf{z}|a)$—e.g., if $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{z}}$ are random variables, and a monotone likelihood ratio property holds so that $L_{\tilde{\mathbf{x}}}(\mathbf{x}|a)$ and $L_{\tilde{\mathbf{z}}}(\mathbf{z}|a)$ are increasing in $\mathbf{x}$ and $\mathbf{z}$, respectively—then (21) coincides with (10).

In sum, if the first-order approach is valid, the incentive condition (15) in Gjesdal's (1982) criterion can be relaxed to (19), which is nevertheless more restrictive than (10), the relaxed incentive condition for the case with risk-neutral principals.

4.1 Being More Informative of the Outcome Is Necessary for Being More Valuable

Both Gjesdal's criterion (Proposition 6) and the relaxed criterion (Proposition 7) are sufficient conditions requiring that the more valuable information system be more informative about the outcome. However, it is possible that both criteria are simply too strong, in that being more informative about the outcome is not necessary for an information system to be more valuable. We can, however, rule out this possibility. In particular, we prove by three means that in the current context a problem-independent ranking criterion necessarily requires the more valuable information system to be more informative about the outcome. First, from the principal's first-order condition in $s_{\tilde{\mathbf{x}}}(\cdot)$, we will see why an incentive condition alone is not sufficient and why a problem-independent ranking necessarily involves certain statistical conditions on the outcome. Second, we prove that being informative of both the agent's action and the outcome in terms of Holmström's (1979) informativeness condition is both necessary and sufficient for comparing inclusive information systems, where one information system is a subset of another. Third, we provide counterexamples.

4.1.1 The Principal's First-Order Condition

In this subsection, we will study the principal's first-order condition with respect to $s_{\tilde{\mathbf{x}}}(\cdot)$ in the three cases we have considered: (i) when the principal is risk neutral; (ii) when the principal is strictly risk averse and the outcome is contractible; and (iii) when the principal is strictly risk averse and the outcome is noncontractible. We will see why an incentive condition alone is sufficient for ranking information systems in the first two cases, but not in the last case, and why some additional condition on $\tilde{b}$ is necessarily involved in the last case.

First, when the principal is risk neutral, the first-order condition for $s_{\tilde{\mathbf{x}}}(\mathbf{x})$ is[6]

$$\frac{1}{u'\left(s_{\tilde{\mathbf{x}}}(\mathbf{x})\right)} = \lambda + \mu L_{\tilde{\mathbf{x}}}(\mathbf{x}|a).$$

Clearly, $s_{\tilde{\mathbf{x}}}(\mathbf{x})$ in the above equality depends exclusively on $L_{\tilde{\mathbf{x}}}(\mathbf{x}|a)$, implying that $L_{\tilde{\mathbf{x}}}(\tilde{\mathbf{x}}|a)$ is equivalent to $\tilde{\mathbf{x}}$ in a sense that $\tilde{\mathbf{x}}$ delivers no additional useful information beyond that conveyed by $L_{\tilde{\mathbf{x}}}(\tilde{\mathbf{x}}|a)$. This is the reason why Kim's mean-preserving spread condition is imposed on $L_{\tilde{\mathbf{x}}}(\tilde{\mathbf{x}}|a)$ instead of on $\tilde{\mathbf{x}}$.

Second, when the principal is risk averse and the outcome is contractible, the first-order condition for $s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, b)$ is

$$\frac{v'\left(b - s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, b)\right)}{u'\left(s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, b)\right)} = \lambda + \mu L_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, b|a).$$

Therefore, $s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, b)$ depends exclusively on $L_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, b|a)$ and $\tilde{b}$. That is, given $L_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, b|a)$ and b, there is nothing to be gained by observing $\mathbf{x}$. Because $L_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, b|a) \equiv \frac{f_{(\tilde{\mathbf{x}},\tilde{b})_a}}{f_{(\tilde{\mathbf{x}},\tilde{b})}}(\mathbf{x}, b|a) = \frac{f_{\tilde{b}a}}{f_{\tilde{b}}}(b|a) + \frac{f_{\tilde{\mathbf{x}}a}}{f_{\tilde{\mathbf{x}}}}(\mathbf{x}|b, a) \equiv L_{\tilde{b}}(b|a) + L_{\tilde{\mathbf{x}}}(\mathbf{x}|b, a)$, and because $\tilde{b}$ is the common observable in the two information systems for comparison, the ranking criterion is imposed on $L_{\tilde{\mathbf{x}}}(\mathbf{x}|b, a)$.

Finally, when the principal is strictly risk averse and the outcome is noncontractible, the first-order condition with respect to $s_{\tilde{\mathbf{x}}}(\mathbf{x})$ is

$$\frac{\int v'\left(b - s_{\tilde{\mathbf{x}}}(\mathbf{x})\right) f_{\tilde{b}}(b|\mathbf{x}, a)db}{u'\left(s_{\tilde{\mathbf{x}}}(\mathbf{x})\right)} = \lambda + \mu L_{\tilde{\mathbf{x}}}(\mathbf{x}|a).$$

This condition implies that $s_{\tilde{\mathbf{x}}}(\mathbf{x})$ not only depends on $L_{\tilde{\mathbf{x}}}(\mathbf{x}|a)$, but is also affected by $f_{\tilde{b}}(b|\mathbf{x}, a)$, which is indexed by $\mathbf{x}$. Since $s_{\tilde{\mathbf{x}}}(\mathbf{x})$ cannot be expressed as a function of $L_{\tilde{\mathbf{x}}}(\mathbf{x}|a)$ alone, the mean-preserving spread condition (which is imposed on $L_{\tilde{\mathbf{x}}}(\mathbf{x}|a)$) is not sufficient for ranking information systems. Certain conditions on $f_{\tilde{b}}(b|\mathbf{x}, a)$ are needed as well. Indeed,

[6]This is Formula (1) in Kim (1995). See also Holmström (1979); Shavell (1979); and Bolton and Dewatripont (2005).

Proposition 7 and Lemma 1 show that the combination of the sufficient statistic condition (20) and the martingale condition (21) is sufficient.

4.1.2 Ranking Inclusive Information Systems

We show that being more informative of the noncontractible outcome is a necessary condition for ranking inclusive information systems where one information system generates more random variables than the other, so that the latter is a proper subset of the former. For example, $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ and $\tilde{\mathbf{x}}$ are inclusive information systems. Apparently, $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ is always weakly more valuable than $\tilde{\mathbf{x}}$ because the former provides more information. An interesting question is therefore, under what conditions is $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ strictly more valuable than $\tilde{\mathbf{x}}$ for certain strictly risk-averse principals? This question is answered by the following proposition, extended from Holmström's (1979) informativeness criterion.

Proposition 8. *Under the first-order approach, there exists a strictly risk-averse principal, for whom* $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ *is strictly more valuable at* a *than* $\tilde{\mathbf{x}}$*, if and only if* $\tilde{\mathbf{x}}$ *is not sufficient for* $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ *when estimating both* a *and* $\tilde{b}$*, or equivalently,*

$$(22) \qquad f_{\tilde{\mathbf{y}}}(\mathbf{y}|\mathbf{x}, a, b) \neq f_{\tilde{\mathbf{y}}}(\mathbf{y}|\mathbf{x}), \quad \textit{for certain } (\mathbf{x}, \mathbf{y}, b) \textit{ with a positive measure.}$$

We prove Proposition 8 in two steps. We first prove the sufficiency of (22) by showing that if (22) holds, then there exists a strictly risk-averse principal, for whom a small deviation $\Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ to the optimal contract $s_{\tilde{\tilde{\mathbf{x}}}}(\tilde{\mathbf{x}})$ is Pareto improving. We then prove the necessity of (22) by showing that if (22) does not hold, then for an arbitrary contract $s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}$, there exists a contract $s_{\tilde{\mathbf{x}}}$ that weakly Pareto dominates $s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}$.

Proposition 8 states that $\tilde{\mathbf{y}}$ is valuable in $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ if and only if $\tilde{\mathbf{y}}$ provides additional information about either the agent's action or the outcome beyond that conveyed by $\tilde{\mathbf{x}}$. Given $\tilde{\mathbf{x}}$, being noninformative about the agent's action is not sufficient for $\tilde{\mathbf{y}}$ to be valueless to the principal, because it is still unclear how well $\tilde{\mathbf{y}}$ performs for the risk-sharing purpose.

If $\tilde{\mathbf{y}}$ is also noninformative about the outcome, then it is safe to say that $\tilde{\mathbf{y}}$ has no value in $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$.

4.1.3 Examples

The purpose of this subsection is to show, by means of two numerical examples, that being more informative of the outcome is a necessary condition for ranking information systems when the principal is risk averse and the outcome is noncontractible. In addition, we show how the comparison results depend on the principal's risk aversion relative to the agent's.

In both examples, we assume two one-dimensional information systems, $\tilde{x}$ and $\tilde{y}$. In Example 1, $\tilde{x}$ is more informative of the agent's action, while $\tilde{y}$ is more informative of the outcome. We show that $\tilde{x}$ is preferred to $\tilde{y}$ by risk-neutral principals, but they are not uniformly comparable for risk-averse principals: a barely risk-averse principal prefers $\tilde{x}$, but starts to prefer $\tilde{y}$ as she becomes more risk-averse.

In Example 2, $\tilde{y}$ is more informative of the outcome than $\tilde{x}$, but is as informative of the agent's action as $\tilde{x}$. We show that $\tilde{x}$ and $\tilde{y}$ generate the same welfare for a risk-neutral principal. A risk-averse principal, however, always prefers $\tilde{y}$ to $\tilde{x}$.

Example 1

Suppose the agent is risk averse, with utility given by $u(s) = 1 - e^{-s}$, and can take action $a \in \{0, 1\}$, where action 0 is costless and action 1 costs the agent $c = 0.05$. Action a stochastically affects a noncontractible outcome $\tilde{b}$, which has two possible values, s and f, worth 2 and 1, respectively, to the principal. Action a also stochastically affects two information systems $\tilde{x}$ and $\tilde{y}$, both of which have two possible values: n and y. The probability mass functions $f(b|a)$, $f(x|b, a)$ and $f(y|b, a)$ are shown in Table 1.

It is clear from Table 1 that, given a, $\tilde{x}$ is independent of $\tilde{b}$, while $\tilde{y}$ is not, suggesting that given a, $\tilde{y}$ is more informative of $\tilde{b}$ than $\tilde{x}$ is. On the other hand, $\tilde{x}$ is more informative of a than $\tilde{y}$ is. This fact is easier to see from Table 2, which shows the probability mass functions

Table 1: The Probability Mass Functions in Example 1

a	b	$f(b\|a)$	x	$f(x\|b,a)$	y	$f(y\|b,a)$
0	f	4/5	n	4/5	n	3/5
			y	1/5	y	2/5
	s	1/5	n	4/5	n	3/5
			y	1/5	y	2/5
1	f	1/2	n	1/5	n	4/5
			y	4/5	y	1/5
	s	1/2	n	1/5	n	1/5
			y	4/5	y	4/5

of $\tilde{x}$ and $\tilde{y}$, given a: apparently, an increase in a has an larger impact on the probability mass of $\tilde{x}$ than on $\tilde{y}$.

Table 2: The Probability Mass Functions in Example 1

a	x	$f(x\|a)$	y	$f(y\|a)$
0	n	4/5	n	3/5
	y	1/5	y	2/5
1	n	1/5	n	1/2
	y	4/5	y	1/2

In sum, $\tilde{x}$ is more informative of the action a, while $\tilde{y}$ is more informative of the outcome $\tilde{b}$. Given an information system $\tilde{z} = \tilde{x}$ or $\tilde{y}$, in order to induce positive effort (i.e., a=1), the principal solves the following problem:

$$
\begin{aligned}
V(\tilde{z}) = \max_{s(\cdot)} & \sum_{b=\{s,f\}} \sum_{z=\{\text{n, y}\}} v(b - s(z)) f(z|b, a = 1) f(b|a = 1) \\
\text{s.t.} & \sum_{z=\{\text{n, y}\}} \left(1 - e^{-s(z)}\right) f(z|a = 1) - c \geq 0, \quad \text{and} \\
& \sum_{z=\{\text{n, y}\}} \left(1 - e^{-s(z)}\right) f(z|a = 1) - c \geq \sum_{z=\{\text{n, y}\}} \left(1 - e^{-s(z)}\right) f(z|a = 0).
\end{aligned}
$$

We consider two cases: (i) the principal is risk neutral, with $v(b-s) = b-s$, and (ii) the principal is risk averse, with $v(b - s) = 1 - e^{-r(b-s)}$, where r is the principal's absolute risk aversion. The numerical solutions of the principal's expected utilities in different cases are listed in Table 3.

Table 3: The Principal's Expected Utilities Under $\tilde{x}$ and $\tilde{y}$, Respectively, in Example 1

		(i) Risk-Neutral Principal	(ii) Risk-Averse Principal					
			r = 0.01	r = 0.1	r = 0.5	r = 1	r = 2	r = 3
$V(\tilde{z})$	$\tilde{z} = \tilde{x}$	1.4481	0.0144	0.1337	0.4999	0.7348	0.9146	0.9693
	$\tilde{z} = \tilde{y}$	1.4128	0.0140	0.1311	0.4966	0.7362	0.9186	0.9709

Table 3 shows that $\tilde{x}$ and $\tilde{y}$ are not uniformly comparable for risk-averse principals. More specifically, a barely risk-averse principal prefers $\tilde{x}$ to $\tilde{y}$, but starts to prefer $\tilde{y}$ over $\tilde{x}$ as her risk aversion r increases above 1. This numerical finding is intuitive: as the principal becomes more risk averse, risk sharing becomes relatively more important and eventually dominates incentives. Therefore, the information system that is more informative of the outcome $\tilde{y}$ is preferred.

On the other hand, a risk-neutral principal gets higher welfare from $\tilde{x}$, which is more informative of the agent's action.

Example 2

Example 2 is the same as Example 1, except that we assume a different distribution for $\tilde{y}$, as shown in Table 4.

Table 4: The Probability Mass Functions in Example 2

a	b	$f(b\|a)$	x	$f(x\|b,a)$	y	$f(y\|b,a)$
0	f	4/5	n	4/5	n	4/5
			y	1/5	y	1/5
	s	1/5	n	4/5	n	4/5
			y	1/5	y	1/5
1	f	1/2	n	1/5	n	7/20
			y	4/5	y	13/20
	s	1/2	n	1/5	n	1/20
			y	4/5	y	19/20

Table 4 shows that, given a, $\tilde{x}$ is independent of $\tilde{b}$, while $\tilde{y}$ is not. This condition implies that given a, $\tilde{y}$ is more informative of $\tilde{b}$ than $\tilde{x}$ is. On the other hand, the distribution of $\tilde{y}$ given a is the same as that of $\tilde{x}$. This fact is easier to see from Table 5, which shows the probability mass functions of $\tilde{x}$ and $\tilde{y}$, given a.

Table 5: The Probability Mass Functions in Example 2

a	x	$f(x\|a)$	y	$f(y\|a)$
0	n	4/5	n	4/5
	y	1/5	y	1/5
1	n	1/5	n	1/5
	y	4/5	y	4/5

Again, we consider two cases: (i) the principal is risk neutral with $v(b-s) = b - s$; and (ii) the principal is risk averse with $v(b-s) = 1 - e^{-r(b-s)}$, where r is the principal's absolute risk aversion. The numerical solutions of the principal's expected utilities in different cases are listed in Table 6.

Table 6: The Principal's Expected Utilities Under $\tilde{x}$ and $\tilde{y}$, Respectively, in Example 2

		(i) Risk-Neutral Principal	(ii) Risk-Averse Principal					
			r = 0.01	r = 0.1	r = 0.5	r = 1	r = 2	r = 3
$V(\tilde{z})$	$\tilde{z} = \tilde{x}$	1.4481	0.0144	0.1337	0.4999	0.7348	0.9146	0.9693
	$\tilde{z} = \tilde{y}$	1.4481	0.0145	0.1338	0.5008	0.7369	0.9168	0.9706

As we have expected, $\tilde{x}$ and $\tilde{y}$ are indifferent for a risk-neutral principal, as they are informative of the agent's action to the same extent. Risk-averse principals, on the other hand, always prefer $\tilde{y}$ to $\tilde{x}$, as $\tilde{y}$ is more informative of the outcome.

5 Justifying the First-Order Approach

5.1 Background

In the previous sections, we learned that ranking criteria can be relaxed under the first-order approach. Therefore, justification of the first-order approach is important, but it is also nontrivial. The technical difficulty has been well studied by Rogerson (1985), Jewitt (1988), Sinclair-Desgagne (1994), and Conlon (2009) for cases where the principal is risk neutral and/or the outcome is contractible. In this section, we justify the first-order approach for the other case where the principal is risk averse and the outcome is noncontractible.

In this section, we focus on the information system $\tilde{\mathbf{x}}$. Therefore, without creating confusion, we can omit the subscript of the contracts and density and distribution functions. For instance, $s(\mathbf{x}) \equiv s_{\tilde{\mathbf{x}}}(\mathbf{x})$ and $f(\mathbf{x}|a) \equiv f_{\tilde{\mathbf{x}}}(\mathbf{x}|a)$. Following the literature, we justify the first-order approach for the optimal action only. That is, instead of solving for s for a given action a, the principal solves for both s and a in the following program

$$\max_{s\in[\underline{s},\bar{s}],a} \iint v(b - s(\mathbf{x}))f(\mathbf{x}, b|a)db\, d\mathbf{x} \tag{23}$$

subject to Constraints (1) and (2).

The first-order approach replaces (2) with its first-order necessary condition (3). Note that (3) is a necessary condition of (2) only if the optimal action a is finite. To avoid the extreme case of $a = \infty$, we make the following two assumptions: (i) the distribution function $F_{\tilde{b}}(b|a)$ is convex in a for all b, and (ii) $\lim_{a\to\infty} c'(a) = \infty$. The first assumption is equivalent to the CISP condition in Conlon (2009) for a random variable (instead of a random vector). According to Lemma 1 of Conlon (2009), this assumption implies that the principal's gross benefit $\int v(b)f_{\tilde{b}}(b|a)db$ is concave in a, i.e., the marginal benefit of a decreases with a. The second assumption means that the marginal cost of a increases with a to infinity. Therefore, the optimal action a will be finite, even in the counterfactual case where the principal puts in her own effort and reaps all the benefit of the outcome.

We will provide two sets of conditions, which are extended from Conlon's generalization of Rogerson's and Jewitt's conditions. In particular, we follow the same approach as adopted by the previous studies: to ensure that (3) implies (2), it suffices to prove that the agent's expected utility $\int u(s^*(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x}$ is a concave function of his action a. To that end, they first find conditions which imply that the function $u(s^*(\mathbf{x}))$ is in some restricted class, and then find conditions on the density $f(\mathbf{x}|a)$ such that the mapping, $\int u(s^*(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x}$, maps this restricted class into concave functions. The difference is that Rogerson chooses $u(s^*(\mathbf{x}))$

to be nondecreasing, while Jewitt chooses $u(s^*(\mathbf{x}))$ to be nondecreasing and concave.

Following Rogerson (1985), I refer to the program consisting of (23), (1), and (2) as *the unrelaxed program*, and the program (23), (1), (3) as *the relaxed program.* Let $s^*(\cdot)$, a^*, λ^*, and μ^* be the solution to the relaxed program, where λ^* and μ^* are the Lagrange multipliers for the participation and relaxed incentive comparability constraints, (1) and (3), respectively. As in Conlon (2009), our extensions rely on the result that $\mu^* \geq 0$.

5.2 Ensuring that $\mu^* \geq 0$

The result of $\mu^* \geq 0$ has been proven by both Rogerson (1985) and Jewitt (1988) for the standard case where the principal is risk neutral or the outcome is contractible. Jewitt's (1988) proof cannot be easily extended to the current context. In particular, Jewitt shows that the relaxed incentive compatibility constraint (3) is equivalent to $COV\left(u(s^*(\mathbf{x})), \frac{1}{u'(s^*(\mathbf{x}))}\right) = \mu^* c_a(a)$, where COV stands for covariance (see Lemma 1 in Jewitt, 1988). Since u and $1/u'$ are both monotone in the same direction, they have a nonnegative covariance, and thereby $\mu^* \geq 0$. In the current context, the same calculation as in Jewitt (1988) shows that (3) is equivalent to $COV\left(u(s^*(\mathbf{x})), \frac{\int v'(b-s^*(\mathbf{x}))f(b|\mathbf{x},a)db}{u'(s^*(\mathbf{x}))}\right) = \mu^* c'(a)$. Unfortunately the covariance on the left-hand side is not necessarily positive.

We instead follow Rogerson's (1985) strategy of proof. In particular, we replace (3) with the following *doubly relaxed constraint*:

$$\int u(s(\mathbf{x}))f_a(\mathbf{x}|a)d\mathbf{x} - c'(a) \geq 0. \tag{24}$$

We refer to the program (23), (1), and (24) as the *doubly relaxed program.* Let $s^{**}(\cdot)$, a^{**}, λ^{**}, and μ^{**} be the solution to the doubly relaxed program, where λ^{**} and μ^{**} are the Lagrange multipliers for the participation and doubly relaxed incentive comparability constraints, (1) and (24), respectively. Since $\mu^{**} \geq 0$ by default, $\mu^* \geq 0$ as well if one can prove that any solution to the doubly relaxed program is also a solution to the relaxed program. Some definitions used in the proof need to be introduced.

Definition 5. *The first-order stochastic dominance (FOSD) condition of* $F(b|\mathbf{x}, a)$ in $\mathbf{x}$ is satisfied, if $\int U(b)dF(b|\hat{\mathbf{x}}, a) \geq \int U(b)dF(b|\mathbf{x}, a)$ for all nondecreasing function U whenever $\hat{\mathbf{x}} \geq \mathbf{x}$ (i.e., $\hat{\mathrm{x}}_i \geq \mathrm{x}_i$, $i = 1, 2, ..., n$).

Definition 5 involves an FOSD condition of $\tilde{b}$ in $\mathbf{x}$, taking a as given. It states that, for a given a, a larger value of $\mathbf{x}$ implies a larger value of b. Therefore, a risk-sharing contract $s^{**}_{\tilde{\mathbf{x}}}(\mathbf{x})$ should be nondecreasing in $\mathbf{x}$. This condition is based on a fixed a because when the principal chooses information systems and designs contracts for the risk-sharing purpose, she takes the agent's action as given. This condition suggests the existence of some "common shocks" that impact both $\tilde{b}$ and $\tilde{\mathbf{x}}$ in the same direction. For instance, in the principal-agent relationship between a firm's board and its CEO, the firm's value is the noncontractible outcome the board cares about, and the stock price is an information system on which the CEO's pay is based. Both the firm's value and the stock price are affected by market conditions and macroeconomic factors.

Definition 6. *The second-order stochastic dominance (SOSD) condition of* $F(b|\mathbf{x}, a)$ in a is satisfied, if $\int U(b)dF(b|\mathbf{x}, \hat{a}) \geq \int U(b)dF(b|\mathbf{x}, a)$ for all nondecreasing and concave functions U whenever $\hat{a} > a$.

Definition 6 involves an SOSD condition of $\tilde{b}$ in a, taking $\mathbf{x}$ as fixed. It states that, for a given value of $\mathbf{x}$, a higher action is associated with a "better" distribution of $\tilde{b}$, in the sense that any individual with a nondecreasing and concave utility in b prefers a higher action value.

We say that the set $\mathbf{E} \subseteq \mathbf{R}^n$ is an *increasing set* (Milgrom and Weber, 1982; Conlon, 2009) if $\mathbf{x} \in \mathbf{E}$ and $\mathbf{y} \geq \mathbf{x}$ imply $\mathbf{y} \in \mathbf{E}$.

Definition 7. $f(\mathbf{x}|a)$ satisfies the *nondecreasing increasing set probability (NISP)* condition for $\mathbf{x}$ in a, if for every increasing set $\mathbf{E}$, the probability $\text{Prob}(\tilde{\mathbf{x}} \in \mathbf{E}|a)$ is nondecreasing in a.

Note that Definition 5 defines FOSD in $\tilde{b}$ relative to $\mathbf{x}$ taking a as given, while Definition 6 defines SOSD in $\tilde{b}$ relative to a taking $\mathbf{x}$ as given, and Definition 7 defines NISP in $\mathbf{x}$ relative to a.

The following lemma related to the NISP condition is introduced by Conlon (2009).

Lemma 2. *The density $f(\mathbf{x}|a)$ satisfies NISP if and only if, for the transformation*

$$\varphi^T(a) = \int \varphi(\mathbf{x}) f(\mathbf{x}|a) d\mathbf{x} = E[\varphi(\tilde{\mathbf{x}})|a],$$

$\varphi^T(a)$ is nondecreasing in a for any nondecreasing function $\varphi(\mathbf{x})$.

The lemma is proven by showing that any nondecreasing function $\varphi(\mathbf{x})$ can be approximated uniformly by the sum $\sum_i I_{\mathbf{E}_i}(\mathbf{x})$, where the $\mathbf{E}_i$ are increasing sets and $I_{\mathbf{E}_i}(\mathbf{x})$ is an indicator function, which equals 1 if $\mathbf{x} \in \mathbf{E}_i$ and zero otherwise. This suggests that $E[\varphi(\tilde{\mathbf{x}})|a]$ can be approximated by $\sum_i E[I_{\mathbf{E}_i}(\mathbf{x})] = \sum_i \text{Prob}(\tilde{\mathbf{x}} \in \mathbf{E}|a)$, which is nondecreasing in a by the definition of NISP.

Denote the principal's absolute risk aversion by $r_P(s) \equiv -\frac{v''(s)}{v'(s)}$, and similarly for the agent $r_A(s) \equiv -\frac{u''(s)}{u'(s)}$. In addition, define $L \equiv \min(\underline{s}, \underline{b} - \bar{s})$ and $U \equiv \max(\bar{s}, \bar{b} - \underline{s})$. Apparently, $[L, U] = [\underline{s}, \bar{s}] \cup [\underline{b} - \bar{s}, \bar{b} - \underline{s}]$. Recall that $[\underline{s}, \bar{s}]$ is the range of the payment to the agent, and $[\underline{b} - \bar{s}, \bar{b} - \underline{s}]$ is the range of the remaining outcome left for the principal after the payment. Therefore, both the principal's and the agent's income fall in the range of $[L, U]$. With these conditions and notations in hand, we are ready to give conditions for $\mu^* \geq 0$.

Proposition 9. *Suppose that the FOSD, SOSD, and NISP conditions hold. If there exist positive constants β and K, such that for each $s \in [L, U]$,*

$$r_A(s) \geq \frac{K}{\beta}, \text{ and} \tag{25}$$

$$\left(1 - \frac{1}{\beta}\right) K \leq r_P(s) \leq K, \tag{26}$$

then any solution to the doubly relaxed program is also a solution to the relaxed program, and *therefore* $\mu^* \geq 0$. *In particular,* $\mu^* \geq 0$ *if the principal has constant absolute risk aversion, or if the range of the absolute risk aversion of the principal is not higher than that of the agent, i.e.,* $r_P(s) \leq K \leq r_A(s), \forall s \in [L, U]$.

The proposition is proven by showing that in the doubly relaxed program, the principal's welfare is nondecreasing in a at $a = a^{**}$. Then the agent's welfare cannot be increasing in a, because otherwise an increase of a would be Pareto improving and a^{**} cannot be a solution. Therefore, the doubly relaxed incentive compatibility constraint (24) is binding, and any solution to the doubly relaxed program is also a solution to the relaxed program.

The key, therefore, is to prove that in the doubly relaxed program, the principal's welfare is nondecreasing in the agent's action. This is also the place where the current proof (of $\mu^* \geq 0$) departs from the standard proof, so it is worthwhile to investigate how the standard proof and the current proof differ in this part.

In the standard moral hazard model where $\tilde{\mathbf{x}} = \tilde{b}$, the principal's first-order condition, for the case $\mu^{**} = 0$, is

$$\frac{v'(b - s^{**}(b))}{u'(s^{**}(b))} = \lambda^{**}.$$

The above equality is Formula (7) in Holmström (1979). Implicit differentiation with respect to b then gives

$$s_b^{**}(b) = \frac{v''(b - s^{**}(b))}{v''(b - s^{**}(b)) - r_A(s^{**}(b))v'(b - s^{**}(b))}. \tag{27}$$

Because $v''(\cdot)$ is in both the numerator and the denominator, $s_b^{**}(b)$ is bounded from above by one, implying that $b - s^{**}(b)$ is increasing in b, and so is $v(b - s^{**}(b))$. Then, by the NISP condition of $\tilde{b}$ in a, the principal's payoff is increasing in a.

In the current model, (27) is replaced by

$$s_{x_i}^{**}(\mathbf{x}) = \frac{\int v'(b - s^{**}(\mathbf{x}))dF_{x_i}(b|\mathbf{x}, a^{**})}{\int v''(b - s^{**}(\mathbf{x}))dF(b|\mathbf{x}, a^{**}) - r_A(s^{**}(\mathbf{x})) \int v'(b - s^{**}(\mathbf{x}))dF(b|\mathbf{x}, a^{**})}. \tag{28}$$

In addition, we need to prove that $s^{**}_{x_i}(\mathbf{x})$ is bounded from above as

$$(29) \qquad s^{**}_{x_i}(\mathbf{x}) \leq \frac{\int v(b - s^{**}(\mathbf{x}))dF_{x_i}(b|\mathbf{x}, a^{**})}{\int v'(b - s^{**}(\mathbf{x}))dF(b|\mathbf{x}, a^{**})}.$$

To see why we have to impose an upper bound on $s^{**}_{x_i}(\mathbf{x})$, we need to understand the two opposing effects on the principal's welfare of an increase in the agent's action, a. On the one hand, a increases the outcome b, and this is a marginal gain to the principal. On the other hand, a increases $\mathbf{x}$ and thereby increases the expectation of the payment, $s^{**}(\mathbf{x})$, to the agent, and this is a marginal loss to the principal. If $s^{**}_{x_i}(\mathbf{x})$ is very large so that the marginal loss from the increased payment exceeds the marginal gain from the increased outcome, the principal's welfare will decrease with the agent's effort at the optimum. Therefore, to make sure the principal's payoff is nondecreasing in the agent's action, we have to impose an upper bound on $s^{**}_{x_i}(\mathbf{x})$.

However, (28) is hard to handle because it has $\int v'(b - s^{**}(\mathbf{x}))dF_{x_i}(b|\mathbf{x}, a^{**})$ in the numerator, but a different term $\int v''(b - s^{**}(\mathbf{x}))dF(b|\mathbf{x}, a^{**})$ in the denominator. Therefore, we deviate from the standard approach here. The intuition of the new approach is easier to see by looking at the two particular cases assumed at the end of Proposition 9: (i) the principal has constant risk aversion, and (ii) the range of the absolute risk aversion of the principal is not higher than that of the agent. First, if we ignore the second term in the denominator of (28), (28) implies

$$(30) \qquad s^{**}_{x_i}(\mathbf{x}) \leq \frac{\int v'(b - s^{**}(\mathbf{x}))dF_{x_i}(b|\mathbf{x}, a^{**})}{\int v''(b - s^{**}(\mathbf{x}))dF(b|\mathbf{x}, a^{**})}.$$

Then, if the principal has a constant risk aversion, i.e., $v'' = -r_P v' = r_P^2 v$, (30) is equivalent to (29) as we desired.

Note that the numerator and the denominator in (30) are closely related, and reduce to the same term $v''(b - s^{**}(b))$ in the standard case where $\tilde{b}$ is contractible.[7]

[7] In the standard case where $\tilde{b}$ is contractible, the formula corresponding to (30) is (27). Note that

On the other hand, if we ignore the first term in the denominator of (28), it yields

$$s^{**}_{x_i}(\mathbf{x}) \leq \frac{\int v'(b - s^{**}(\mathbf{x}))dF_{x_i}(b|\mathbf{x}, a^{**})}{-r_A(s^{**}(\mathbf{x})) \int v'(b - s^{**}(\mathbf{x}))dF(b|\mathbf{x}, a^{**})}. \tag{31}$$

We need to replace $r_A(s^{**}(\mathbf{x}))$ in the denominator of (31). According to the assumptions that $r_P(s) \leq K \leq r_A(s), \forall s \in [L, U]$, and the fact that $s^{**}(\mathbf{x}) \in [L, U]$ and $b - s^{**}(\mathbf{x}) \in [L, U]$, we have $r_A\big(s^{**}(\mathbf{x})\big) \geq r_P\big(b - s^{**}(\mathbf{x})\big) \equiv -\frac{v''(b-s^{**}(\mathbf{x}))}{v'(b-s^{**}(\mathbf{x}))}$. By rearranging the above inequality, we have $r_A\big(s^{**}(\mathbf{x})\big)v'(b - s^{**}(\mathbf{x})) + v''(b - s^{**}(\mathbf{x})) \geq 0$, or equivalently $\frac{\partial r_A(s^{**}(\mathbf{x}))v(b-s^{**}(\mathbf{x}))+v'(b-s^{**}(\mathbf{x}))}{\partial b} \geq 0$. This inequality, together with the FOSD condition of $F_{\tilde{b}}(b|\mathbf{x}, a^{**})$ in $\mathbf{x}$, implies that

$$\int \big[r_A\big(s^{**}(\mathbf{x})\big)v(b - s^{**}(\mathbf{x})) + v'(b - s^{**}(\mathbf{x}))\big]dF_{x_i}(b|\mathbf{x}, a^{**}) \geq 0.$$

By rearranging the above inequality, we have

$$r_A(s^{**}(\mathbf{x})) \geq -\frac{\int v'(b - s^{**}(\mathbf{x}))dF_{x_i}(b|\mathbf{x}, a^{**})}{\int v(b - s^{**}(\mathbf{x}))dF_{x_i}(b|\mathbf{x}, a^{**})}.$$

Finally by substituting the above inequality in (31), we get (29), as desired.

The two particular cases correspond to two extreme values of β in Conditions (25) and (26). When $\beta = \infty$, Conditions (25) and (26) are equivalent to the case that the principal has a constant risk aversion. When $\beta = 1$, Conditions (25) and (26) are equivalent to the case that the range of the absolute risk aversion of the principal is not higher than that of the agent. The combination of (25) and (26) with $\beta \in (1, \infty)$ covers all the intermediate cases between the two particular cases.

Lastly, note that both (27) and (28) are decreasing with r_A. That is, the optimal contract becomes less responsive to the signal as the agent becomes more risk averse relative to the principal. This result is intuitive. A less-responsive contract provides less incentive but is

if we ignore the second term in the denominator of (27), the numerator and the denominator are both $v''(b - s^{**}(b))$.

also less risky to the agent. As the agent becomes more risk averse relative to the principal, a less-responsive—therefore less-risky—contract is Pareto improving.

5.3 An Extension of the Mirrlees-Rogerson Conditions

Definition 8. $f(\mathbf{x}|a)$ satisfies the *monotone likelihood ratio (MLR)* condition if $\frac{f_a}{f}(\mathbf{x}|a)$ is nondecreasing in $\mathbf{x}$.

Definition 9. $f(\mathbf{x}|a)$ satisfies the *concave increasing-set probability (CISP)* condition for $\mathbf{x}$ in a if, for every increasing set $\mathbf{E}$, the probability $\text{Prob}(\tilde{x} \in \mathbf{E}|a)$ is concave in a.

Lemma 3. *The density $f(\cdot|a)$ satisfies CISP if and only if the transformation*

$$\varphi^T(a) = \int \varphi(\mathbf{x}) f(\mathbf{x}|a) d\mathbf{x} = E[\varphi(\tilde{\mathbf{x}})|a]$$

is concave in a for any nondecreasing function $\varphi(\mathbf{x})$.

Proof. This is Lemma 1 in Conlon (2009). □

Proposition 10. *Suppose that the FOSD, SOSD, NISP, MLR and CISP conditions hold, as well as the two conditions in Proposition 9. Then any solution to the relaxed program also solves the unrelaxed program, and the first-order approach is valid.*

5.4 An Extension of Jewitt's Conditions

Definition 10. $f(\mathbf{x}|a)$ satisfies the *monotone concave likelihood ratio (MCLR)* condition, if $\frac{f_a}{f}(\mathbf{x}|a)$ is nondecreasing and concave in $\mathbf{x}$.

Definition 11. $f(\mathbf{x}|a)$ satisfies the *concave increasing convex set probability (CICSP)* condition for $\mathbf{x}$ in a if, for every increasing and *convex* set $\mathbf{E}$, the probability $\text{Prob}(\tilde{\mathbf{x}} \in \mathbf{E}|a)$ is concave in a.

Lemma 4. *If $f(\mathbf{x}|a)$ satisfies the CICSP condition, then the transformation*

$$\varphi^T(a) = \int \varphi(\mathbf{x}) f(\mathbf{x}|a) d\mathbf{x} = E[\varphi(\tilde{\mathbf{x}})|a]$$

is concave in a for any nondecreasing and quasi-concave function $\varphi(\mathbf{x})$.

Proof. The proof is similar to the "only if" part of the proof of Lemma 2. □

Let $K(s; \mathbf{x}, a) \equiv \frac{\int v'(b-s) f(b|\mathbf{x}, a) db}{u'(s)}$, and define $\omega(\cdot\ ; \mathbf{x}, a) \equiv K^{-1}(\cdot\ ; \mathbf{x}, a)$ as the inverse function of $K(s; \mathbf{x}, a)$ in s. Because

$$K_s(s; \mathbf{x}, a) = -\frac{\int v''(b-s) f(b|\mathbf{x}, a) db \cdot u'(s) + \int v'(b-s) f(b|\mathbf{x}, a) db \cdot u''(s)}{u'(s)^2} > 0,$$

ω is well defined and increasing in z.

Proposition 11. *Suppose that the FOSD, SOSD, NISP, MCLR and CICSP conditions hold, that the two conditions in Proposition 9 hold, and that $u(\omega(z; \mathbf{x}, a))$ is concave in z and $\mathbf{x}$, $\forall a$, then any solution to the relaxed program also solves the original program, i.e., the first-order approach is valid.*

It is clear from the proof that the condition on the ω-function in the current paper is used in the same way as the condition on ω in Jewitt (1988) and Conlon (2009): to extend the nondecreasing and concave property from the likelihood ratio to the optimal contract.

6 Conclusion

This paper has studied the use of information for incentives and risk sharing in agency problems. We distinguish among three cases: (i) when the principal is risk neutral; (ii) when the principal is strictly risk averse and the outcome is contractible; and (iii) when the principal is strictly risk averse and the outcome is noncontractible. We show that in the first two cases, risk sharing does not impose a challenge: while risk sharing is unnecessary for a risk-neutral principal, for a strictly risk-averse principal, risk sharing can be completely taken care of by a contract solely on the outcome. Information systems are then ranked by how additionally informative the other observables are of the agent's action. In the third case, since the outcome is noncontractible, the strictly risk-averse principal has to rely on imperfect information for both incentives and risk sharing. Accordingly, a problem-independent ranking of information systems necessarily involves two conditions: (i) an incentive condition comparing information systems by how informative they are about the agent's action, and (ii) a risk-sharing condition comparing information systems by how informative they are about the outcome. Under the first-order approach, we develop a criterion, in which the incentive condition is relaxed from that in Gjesdal's (1982) sufficient statistic criterion. Both criteria feature the same risk-sharing condition. We then provide sufficient conditions justifying the first-order approach.

There are a number of caveats regarding our findings and areas for future research. First, we justify the first-order approach by finding conditions under which the agent's utility is globally concave in his action. However, as shown in Jewitt (2007) and Conlon (2009), the conditions ensuring the global concavity are very restrictive, especially when the information system is multi-dimensional. Therefore, it is interesting to seek new conditions to validate the first-order approach without ensuring the global concavity.

Second, it also remains an open question how one can relax Gjesdal's (1982) sufficient statistic criterion without relying on the first-order approach. More generally, what is the necessary and sufficient condition for ranking information systems in agency problems when the first-order approach is invalid? Gjesdal's (1982) criterion is an extension of Blackwell's (1953) sufficient statistic condition, which is known to be both necessary and sufficient for ranking information systems in decision problems. So one may conjecture that Gjesdal's (1982) criterion, or equivalently, Blackwell's (1953) sufficient statistic condition, is a necessary condition for ranking information systems in agency problems as well. Gjesdal (1982), however, has provided evidence rejecting this conjecture, because the agency problem is different from the decision problem: the agency problem is a game, and actions are chosen by the agent, instead of given by nature. In addition, the agency problem contains restrictions on utility functions not encountered in the decision problem.

Appendix

Proof of Proposition 1

Proof. The proof has two parts. In the first part, we prove the sufficiency of (7), and in the second part, we prove the equivalence between (7) and (8).

Part I: We show that if (7) holds, then for any given contract $s_{\tilde{\mathbf{y}}}$, we can construct a contract $s_{\tilde{\mathbf{x}}}$ which generates the same welfare for the agent as $s_{\tilde{\mathbf{y}}}$ does, but weakly improves the principal's welfare.

Let

$$u(s_{\tilde{\mathbf{x}}}(\mathbf{x})) = \int u(s_{\tilde{\mathbf{y}}}(\mathbf{y}))\phi(\mathbf{y}|\mathbf{x})d\mathbf{y}. \tag{32}$$

Then we have

$$\begin{aligned}
E[u(s_{\tilde{\mathbf{x}}}(\mathbf{x}))|a] &= \int u(s_{\tilde{\mathbf{x}}}(\mathbf{x}))f_{\tilde{\mathbf{x}}}(\mathbf{x}|a)d\mathbf{x} \\
&= \iint u(s_{\tilde{\mathbf{y}}}(\mathbf{y}))\phi(\mathbf{y}|\mathbf{x})d\mathbf{y} f_{\tilde{\mathbf{x}}}(\mathbf{x}|a)\, d\mathbf{x} \\
&= \int u(s_{\tilde{\mathbf{y}}}(\mathbf{y})) \int \phi(\mathbf{y}|\mathbf{x})f_{\tilde{\mathbf{x}}}(\mathbf{x}|a)d\mathbf{x}\, d\mathbf{y} \\
&= \int u(s_{\tilde{\mathbf{y}}}(\mathbf{y}))f_{\tilde{\mathbf{y}}}(\mathbf{y}|a)d\mathbf{y} \\
&= E[u(s_{\tilde{\mathbf{y}}}(\mathbf{y}))|a], \quad \forall a,
\end{aligned}$$

where the second equality follows from (32), and the second-last equality follows from (7). Since $s_{\tilde{\mathbf{x}}}$ and $s_{\tilde{\mathbf{y}}}$ result in the same welfare for the agent at all values of a, they are identical from the agent's perspective.

On the other hand, applying Jensen's inequality to (32) gets $s_{\tilde{\mathbf{x}}}(\mathbf{x}) \leq \int s_{\tilde{\mathbf{y}}}(\mathbf{y})\phi(\mathbf{y}|\mathbf{x})d\mathbf{y}$, or equivalently,

$$\begin{aligned}
E[s_{\tilde{\mathbf{x}}}(\mathbf{x})|a] = \int s_{\tilde{\mathbf{x}}}(\mathbf{x}) f_{\tilde{\mathbf{x}}}(\mathbf{x}|a) d\mathbf{x} &\leq \iint s_{\tilde{\mathbf{y}}}(\mathbf{y}) \phi(\mathbf{y}|\mathbf{x}) d\mathbf{y} f_{\tilde{\mathbf{x}}}(\mathbf{x}|a) d\mathbf{x} \\
&= \int s_{\tilde{\mathbf{y}}}(\mathbf{y}) \int \phi(\mathbf{y}|\mathbf{x}) f_{\tilde{\mathbf{x}}}(\mathbf{x}|a) d\mathbf{x} d\mathbf{y} \\
&= \int s_{\tilde{\mathbf{y}}}(\mathbf{y}) f_{\tilde{\mathbf{y}}}(\mathbf{y}|a) d\mathbf{y} \\
&= E[s_{\tilde{\mathbf{y}}}(\mathbf{y})|a],
\end{aligned}$$

where the second-last equality follows from (7). That is, the contract $s_{\tilde{\mathbf{x}}}(\mathbf{x})$ is less costly to the principal than $s_{\tilde{\mathbf{y}}}(\mathbf{y})$ is. Therefore, $s_{\tilde{\mathbf{x}}}(\mathbf{x})$ weakly Pareto dominates $s_{\tilde{\mathbf{y}}}(\mathbf{y})$.

Part II. The equivalence between (7) and (8) can be proven as follows. If (7) holds, we define $\tilde{\mathbf{z}}$ in such a way that $f_{\tilde{\mathbf{z}}}(\mathbf{z}|\mathbf{x}, a) = \phi(\mathbf{z}|\mathbf{x})$. According to Definition 2, $\tilde{\mathbf{x}}$ is sufficient for $(\tilde{\mathbf{x}}, \tilde{\mathbf{z}})$ when estimating a. In addition, we have

$$f_{\tilde{\mathbf{z}}}(\mathbf{z}|a) = \int f_{\tilde{\mathbf{z}}}(\mathbf{z}|\mathbf{x}, a) f_{\tilde{\mathbf{x}}}(\mathbf{x}|a) d\mathbf{x} = \int \phi(\mathbf{z}|\mathbf{x}) f_{\tilde{\mathbf{x}}}(\mathbf{x}|a) d\mathbf{x} = f_{\tilde{\mathbf{y}}}(\mathbf{z}|a), \quad \forall \mathbf{z}, \tag{33}$$

where the last equality follows from (7). Condition (33) is exactly $\tilde{\mathbf{z}} \stackrel{d}{=}_a \tilde{\mathbf{y}}$.

On the other hand, assume that (8) holds. By defining $\phi(\mathbf{z}|\mathbf{x}) = f_{\tilde{\mathbf{z}}}(\mathbf{z}|\mathbf{x})$, we have

$$f_{\tilde{\mathbf{y}}}(\mathbf{y}|a) = f_{\tilde{\mathbf{z}}}(\mathbf{y}|a) = \int f_{\tilde{\mathbf{z}}}(\mathbf{y}|\mathbf{x}, a) f(\mathbf{x}|a) d\mathbf{x} = \int f_{\tilde{\mathbf{z}}}(\mathbf{y}|\mathbf{x}) f(\mathbf{x}|a) d\mathbf{x} = \int \phi(\mathbf{y}|\mathbf{x}) f(\mathbf{x}|a) d\mathbf{x}, \tag{34}$$

where the first equality follows from the first part of (8) and the second-last equality follows from the second part of (8). (34) is exactly (7). This completes the proof. □

Proof of Proposition 2

Proof. The proof follows from Jewitt (2007), in which the principal's cost function (6) is transformed to a concave function of the likelihood ratio of the information system. Since all likelihood ratios have zero means, a mean-preserving spread condition on the likelihood ratio is both necessary and sufficient for ranking information systems.

If the first-order approach is valid, (2) can be replaced by the relaxed incentive compatibility constraint

$$\int u(s_{\tilde{\mathbf{x}}}(\mathbf{x}))f_{\tilde{\mathbf{x}}a}(\mathbf{x}|a)d\mathbf{x} - c'(a) = 0. \tag{35}$$

We can further relax (35) to

$$\int u(s_{\tilde{\mathbf{x}}}(\mathbf{x}))f_{\tilde{\mathbf{x}}a}(\mathbf{x}|a)d\mathbf{x} - c'(a) \geq 0. \tag{36}$$

This relaxation will be justified at the end of the proof. The resulting doubly relaxed program is (6) subject to (1) and (36). Following Grossman and Hart (1983), we define $n_{\tilde{\mathbf{x}}}(\mathbf{x}) = u(s_{\tilde{\mathbf{x}}}(\mathbf{x}))$ and $m(\cdot) = u^{-1}(\cdot)$. Therefore $m(\cdot)$ is increasing and convex, and $m(n_{\tilde{\mathbf{x}}}(\mathbf{x})) = s_{\tilde{\mathbf{x}}}(\mathbf{x})$. By changing the argument variable from $s_{\tilde{\mathbf{x}}}$ to $n_{\tilde{\mathbf{x}}}$, the doubly relaxed program can be rewritten as

$$\begin{aligned} C_{\tilde{\mathbf{x}}}(a) \equiv \min_{n_{\tilde{\mathbf{x}}}} & \int m(n_{\tilde{\mathbf{x}}}(\mathbf{x}))f_{\tilde{\mathbf{x}}}(\mathbf{x}|a)d\mathbf{x} \\ s.t. \quad & \int n_{\tilde{\mathbf{x}}}(\mathbf{x})f_{\tilde{\mathbf{x}}}(\mathbf{x}|a)d\mathbf{x} - c(a) \geq 0 \\ & \int n_{\tilde{\mathbf{x}}}(\mathbf{x})f_{\tilde{\mathbf{x}}a}(\mathbf{x}|a)d\mathbf{x} - c'(a) \geq 0. \end{aligned}$$

The lagrangian of the above program is

$$\begin{aligned} \mathcal{L}_{\tilde{\mathbf{x}}}(n_{\tilde{\mathbf{x}}}, \lambda, \mu, a) \equiv & \int m(n_{\tilde{\mathbf{x}}}(\mathbf{x}))f_{\tilde{\mathbf{x}}}(\mathbf{x}|a)d\mathbf{x} + \lambda \left[c(a) - \int n_{\tilde{\mathbf{x}}}(\mathbf{x})f_{\tilde{\mathbf{x}}}(\mathbf{x}|a)d\mathbf{x}\right] \\ & + \mu \left[c'(a) - \int n_{\tilde{\mathbf{x}}}(\mathbf{x})f_{\tilde{\mathbf{x}}a}(\mathbf{x}|a)d\mathbf{x}\right] \\ = & \lambda c(a) + \mu c'(a) - \int \left\{\left[\lambda + \mu L_{\tilde{\mathbf{x}}}(\mathbf{x}|a)\right] n_{\tilde{\mathbf{x}}}(\mathbf{x}) - m(n_{\tilde{\mathbf{x}}}(\mathbf{x}))\right\} f_{\tilde{\mathbf{x}}}(\mathbf{x}|a)d\mathbf{x}, \end{aligned}$$

where $\lambda \geq 0$ and $\mu \geq 0$ are lagrangian multipliers. According to Theorem 1 on page 224 of

Luenberger (1969), the solution to the program is at the saddle point of the lagrangian. In particular, $C_{\tilde{\mathbf{x}}}(a) = \max_{\lambda \geq 0, \mu \geq 0} \inf_{n_{\tilde{\mathbf{x}}}} \mathcal{L}_{\tilde{\mathbf{x}}}(n_{\tilde{\mathbf{x}}}, \lambda, \mu, a)$. By substituting $\mathcal{L}_{\tilde{\mathbf{x}}}$ in, we have

$$\begin{aligned} C_{\tilde{\mathbf{x}}}(a) &= \max_{\lambda \geq 0, \mu \geq 0} \inf_{n_{\tilde{\mathbf{x}}}} \left\{ \lambda c(a) + \mu c'(a) - \int \left\{ \left[\lambda + \mu L_{\tilde{\mathbf{x}}}(\mathbf{x}|a)\right] n_{\tilde{\mathbf{x}}}(\mathbf{x}) - m(n_{\tilde{\mathbf{x}}}(\mathbf{x})) \right\} f_{\tilde{\mathbf{x}}}(\mathbf{x}|a) d\mathbf{x} \right\} \\ &= \max_{\lambda \geq 0, \mu \geq 0} \lambda c(a) + \mu c'(a) - \int \sup_{n_{\tilde{\mathbf{x}}}} \left\{ \left[\lambda + \mu L_{\tilde{\mathbf{x}}}(\mathbf{x}|a)\right] n_{\tilde{\mathbf{x}}}(\mathbf{x}) - m(n_{\tilde{\mathbf{x}}}(\mathbf{x})) \right\} f_{\tilde{\mathbf{x}}}(\mathbf{x}|a) d\mathbf{x}. \end{aligned}$$

The term in the integral on the right-hand side of the last equality can be taken as a profit-maximization problem. More specifically, for each value of $\mathbf{x}$, $\lambda + \mu L_{\tilde{\mathbf{x}}}(\mathbf{x}|a)$ can be taken as the price, $n_{\tilde{\mathbf{x}}}(\mathbf{x})$ the adjustable quantity of production, and $m(n_{\tilde{\mathbf{x}}}(\mathbf{x}))$ the increasing and convex cost function of $n_{\tilde{\mathbf{x}}}(\mathbf{x})$. By conjugate duality, the maximum value function is an increasing and convex profit function $\pi\left(\lambda + \mu L_{\tilde{\mathbf{x}}}(\mathbf{x}|a)\right)$. Therefore, we have

$$C_{\tilde{\mathbf{x}}}(a) = \max_{\lambda \geq 0, \mu \geq 0} \lambda c(a) + \mu c'(a) - \int \pi\left(\lambda + \mu L_{\tilde{\mathbf{x}}}(\mathbf{x}|a)\right) f_{\tilde{\mathbf{x}}}(\mathbf{x}|a) d\mathbf{x}.$$

By properly choosing the cost function $c(\cdot)$ we can support any values of $\lambda \geq 0$ and $\mu \geq 0$, holding a constant. Therefore, $\tilde{\mathbf{x}}$ is preferred to $\tilde{\mathbf{y}}$, i.e., $C_{\tilde{\mathbf{x}}}(a) \leq C_{\tilde{\mathbf{y}}}(a)$, if and only if for all values of $\lambda \geq 0$ and $\mu \geq 0$,

$$\int \pi\left(\lambda + \mu L_{\tilde{\mathbf{x}}}(\mathbf{x}|a)\right) f_{\tilde{\mathbf{x}}}(\mathbf{x}|a) d\mathbf{x} \geq \int \pi\left(\lambda + \mu L_{\tilde{\mathbf{y}}}(\mathbf{y}|a)\right) f_{\tilde{\mathbf{y}}}(\mathbf{y}|a) d\mathbf{y}. \tag{37}$$

Note that π could be any increasing and convex function. This is because its conjugate function $m(\cdot)$ spans all the increasing and convex functions, and there is a one-to-one correspondence between conjugate functions. Since π spans all the increasing and convex functions, then according to Definition 4, (37) holds if and only if the distribution of $L_{\tilde{\mathbf{x}}}(\tilde{\mathbf{x}}|a)$ is a mean-preserving spread of that of $L_{\tilde{\mathbf{y}}}(\tilde{\mathbf{y}}|a)$, or equivalently, if and only if (9) holds.

To justify the replacement of (35) by (36), it suffices to prove that $\mu > 0$, or equivalently,

$\mu \neq 0$. The first-order condition of the lagrangian in $n_{\tilde{\mathbf{x}}}$ is

$$m'(n_{\tilde{\mathbf{x}}}(\mathbf{x})) = \lambda + \mu L_{\tilde{\mathbf{x}}}(\mathbf{x}|a).$$

If $\mu = 0$ as assumed for contradiction, then the above equality becomes $m'(n_{\tilde{\mathbf{x}}}(\mathbf{x})) = \lambda$, suggesting that the optimal contract $n_{\tilde{\mathbf{x}}}$ is constant, which fails to induce any positive effort. Therefore, we must have $\mu > 0$, and (35) can be replaced by (36).

The equivalence between (9) and (10) follows directly from the discussion after Definition 4. □

Proof of Proposition 3

Proof. We prove the proposition by showing that for the optimal contract $s_{\tilde{\mathbf{x}}}(\mathbf{x})$, there exists a small additional variation $\Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, b)$ that is Pareto improving.

Without loss of generality, we assume discrete actions, i.e., $(a_1, a_2, ..., a_I)$. The principal's problem based on the information system $\tilde{\mathbf{x}}$ can be written as

$$\max_{s_{\tilde{\mathbf{x}}}, a_k \in \{a_1, ... a_I\}} \iint v(b - s_{\tilde{\mathbf{x}}}(\mathbf{x})) f_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, b|a_k) db\, d\mathbf{x}$$

$$\text{(38)} \qquad s.t. \quad \int u(s_{\tilde{\mathbf{x}}}(\mathbf{x})) f_{\tilde{\mathbf{x}}}(\mathbf{x}|a_k) d\mathbf{x} - c(a_k) \geq 0,$$

$$\text{(39)} \qquad \int u(s_{\tilde{\mathbf{x}}}(\mathbf{x})) f_{\tilde{\mathbf{x}}}(\mathbf{x}|a_k) d\mathbf{x} - c(a_k) \geq \int u(s_{\tilde{\mathbf{x}}}(\mathbf{x})) f_{\tilde{\mathbf{x}}}(\mathbf{x}|a_i) d\mathbf{x} - c(a_i), \quad \forall i \neq k.$$

Let $(s_{\tilde{\mathbf{x}}}, a_k)$ be a solution to the above agency problem, and let λ and $\{\mu_i\}_{i \neq k}$ be the multipliers for the agent's participation and incentive compatibility constraints (38) and (39), respectively. $\lambda > 0$ and $\mu_i \geq 0$, $\forall i$. Fix $\mathbf{x}$ for a moment. If $f_{\tilde{b}}(b|\mathbf{x}, a)$ exists and is not a point mass, as assumed in the proposition, then the principal's and the agent's marginal returns ΔV and ΔU—conditional on $\mathbf{x}$, from a small additional variation $\Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, b)$ in the compensation rule—can be written as[8]

[8]For the mathematical technique used for deriving these formula, refer to proposition 9.6.1 in Luenberger (1969). See also Holmström (1979).

$$
\begin{aligned}
\Delta V = & -\int v'(b - s_{\tilde{\mathbf{x}}}(\mathbf{x}))\Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, b) f_{\tilde{b}}(b|\mathbf{x}, a_k)db \\
& + \lambda u'(s_{\tilde{\mathbf{x}}}(\mathbf{x})) \int \Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, b) f_{\tilde{b}}(b|\mathbf{x}, a_k)db \\
& + \sum_{i \neq k} \mu_i u'(s_{\tilde{\mathbf{x}}}(\mathbf{x})) \int \Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, b) \left[f_{\tilde{b}}(b|\mathbf{x}, a_k) - f_{\tilde{b}}(b|\mathbf{x}, a_i) \right] db \\
= & -\int v'(b - s_{\tilde{\mathbf{x}}}(\mathbf{x}))\Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, b) f_{\tilde{b}}(b|\mathbf{x}, a_k)db \\
& + \left(\lambda + \sum_{i \neq k} \mu_i \right) u'(s_{\tilde{\mathbf{x}}}(\mathbf{x})) \int \Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, b) f_{\tilde{b}}(b|\mathbf{x}, a_k)db \\
& - u'(s_{\tilde{\mathbf{x}}}(\mathbf{x})) \sum_{i \neq k} \mu_i \int \Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, b) f_{\tilde{b}}(b|\mathbf{x}, a_i)db \\
\Delta U = & u'(s_{\tilde{\mathbf{x}}}(\mathbf{x})) \int \Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, b) f_{\tilde{b}}(b|\mathbf{x}, a_k)db.
\end{aligned}
$$

Let $\Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, b)$ take only J values. In particular, assume that for a set B_j in the range of $\tilde{b}$, $\Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, b) = \Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, B_j)$ for all $b \in B_j$. Then

$$
\begin{aligned}
\Delta V = & -\sum_{j=1}^{J} \Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, B_j) \int_{B_j} v'(b - s_{\tilde{\mathbf{x}}}(\mathbf{x})) f_{\tilde{b}}(b|\mathbf{x}, a_k)db \\
& + \left(\lambda + \sum_{i \neq k} \mu_i \right) u'(s_{\tilde{\mathbf{x}}}(\mathbf{x})) \sum_{j=1}^{J} \Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, B_j) f_{\tilde{b}}(B_j|\mathbf{x}, a_k) \\
& - u'(s_{\tilde{\mathbf{x}}}(\mathbf{x})) \sum_{i \neq k} \mu_i \sum_{j=1}^{J} \Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, B_j) f_{\tilde{b}}(B_j|\mathbf{x}, a_i),
\end{aligned} \tag{40}
$$

$$
\Delta U = u'(s_{\tilde{\mathbf{x}}}(\mathbf{x})) \sum_{j=1}^{J} \Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, B_j) f_{\tilde{b}}(B_j|\mathbf{x}, a_k), \tag{41}
$$

where $f_{\tilde{b}}(B_j|\mathbf{x}, a_i) = \int_{B_j} f_{\tilde{b}}(b|\mathbf{x}, a_i)db$, $\forall i, j$. If there exists a variation $\{\Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, B_j)\}_{j=1\dots J}$

such that

$$\sum_{j=1}^{J} \Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, B_j) f_{\tilde{b}}(B_j|\mathbf{x}, a_k) = 0, \tag{42}$$

$$\sum_{j=1}^{J} \Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, B_j) f_{\tilde{b}}(B_j|\mathbf{x}, a_i) \leq 0, \quad \forall i \neq k \quad \text{and} \tag{43}$$

$$\sum_{j=1}^{J} \Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, B_j) \int_{B_j} v'(b - s_{\tilde{\mathbf{x}}}(\mathbf{x})) f_{\tilde{b}}(b|\mathbf{x}, a_k) db < 0, \tag{44}$$

then by substituting (42) - (44) into (40) and (41), we have $\Delta V > 0$ and $\Delta U = 0$. That is, the variation $\{\Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, B_j)\}_{j=1...J}$ makes the principal better off and the agent no worse off.

Therefore, we are left to prove that a solution $\{\Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, B_j)\}_{j=1...J}$ exists for the system of linear inequalities, (42) - (44). A solution exists if two conditions hold. First, the number of variables is no less than the number of inequalities. Second, there is no contradiction among inequalities. The first condition is satisfied if $J \geq I + 1$. As to the second condition, the only potential contradiction is between (42) and (44). In particular, if there is a constant K such that

$$\begin{bmatrix} \int_{B_1} v'(b - s_{\tilde{\mathbf{x}}}(\mathbf{x})) f_{\tilde{b}}(b|\mathbf{x}, a_k) db \\ \int_{B_2} v'(b - s_{\tilde{\mathbf{x}}}(\mathbf{x})) f_{\tilde{b}}(b|\mathbf{x}, a_k) db \\ \vdots \\ \int_{B_J} v'(b - s_{\tilde{\mathbf{x}}}(\mathbf{x})) f_{\tilde{b}}(b|\mathbf{x}, a_k) db \end{bmatrix} = K * \begin{bmatrix} f_{\tilde{b}}(B_1|\mathbf{x}, a_k) \\ f_{\tilde{b}}(B_2|\mathbf{x}, a_k) \\ \vdots \\ f_{\tilde{b}}(B_J|\mathbf{x}, a_k) \end{bmatrix}, \tag{45}$$

then we have

$$\sum_{j=1}^{J} \Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, B_j) \int_{B_j} v'(b - s_{\tilde{\mathbf{x}}}(\mathbf{x})) f_{\tilde{b}}(b|\mathbf{x}, a_k) db = K * \sum_{j=1}^{J} \Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, B_j) f_{\tilde{b}}(B_j|\mathbf{x}, a_k) = 0,$$

where the first equality follows from (45) and the the last equality follows from (42). Clearly, the above equality contradicts (44). However, we can prove that if the principal is strictly risk averse as assumed in the proposition, (45) does not hold: substituting $f_{\tilde{b}}(B_j|\mathbf{x}, a_k) = \int_{B_j} f_{\tilde{b}}(b|\mathbf{x}, a_k)db$ into (45) gives

$$\begin{bmatrix} \int_{B_1} v'(b - s_{\tilde{\mathbf{x}}}(\mathbf{x}))f_{\tilde{b}}(b|\mathbf{x}, a_k)db \\ \int_{B_2} v'(b - s_{\tilde{\mathbf{x}}}(\mathbf{x}))f_{\tilde{b}}(b|\mathbf{x}, a_k)db \\ \vdots \\ \int_{B_J} v'(b - s_{\tilde{\mathbf{x}}}(\mathbf{x}))f_{\tilde{b}}(b|\mathbf{x}, a_k)db \end{bmatrix} = \begin{bmatrix} \int_{B_1} Kf_{\tilde{b}}(b|\mathbf{x}, a_k)db \\ \int_{B_2} Kf_{\tilde{b}}(b|\mathbf{x}, a_k)db \\ \vdots \\ \int_{B_J} Kf_{\tilde{b}}(b|\mathbf{x}, a_k)db \end{bmatrix}.$$

If the principal is strictly risk averse, then $v'(b - s_{\tilde{\mathbf{x}}}(\mathbf{x}))$ is monotonic decreasing in b, whereas K is constant. Therefore, (45) does not hold. In sum, we have proven that there is no contradiction among (42) - (44) if the principal is strictly risk averse. Therefore, a solution $\{\Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, B_j)\}_{j=1...J}$ where $J \geq I + 1$ always exists for (42) - (44), or equivalently, there exists a variation $\{\Delta s_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, B_j)\}_{j=1...J}$ that makes the principal better off and the agent no worse off at $\mathbf{x}$.

The procedure can be repeated for a set of $\mathbf{x}$ values with positive measures, making the principal strictly better off and the agent no worse off at almost all values of $\mathbf{x}$. Therefore, if the principal is risk averse, $(\tilde{\mathbf{x}}, \tilde{b})$ is strictly preferred to $\tilde{\mathbf{x}}$. □

Proof of Proposition 5

Proof. The proof is similar to that of Proposition 2. In particular, following the same procedure (which we omit here), we can show that $(\tilde{\mathbf{x}}, b)$ is preferred to $(\tilde{\mathbf{y}}, b)$ regardless of the distribution of $\tilde{b}$, if and only if for all values of $\lambda \geq 0$ and $\mu \geq 0$,

(46)
$$\iint \pi\left(\lambda + \mu L_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, b|a)\right) f_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, b|a)d\mathbf{x}db \geq \iint \pi\left(\lambda + \mu L_{(\tilde{\mathbf{y}},\tilde{b})}(\mathbf{y}, b|a)\right) f_{(\tilde{\mathbf{y}},\tilde{b})}(\mathbf{y}, b|a)d\mathbf{y}db.$$

It is straightforward to derive that $L_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x}, b|a) \equiv \frac{f_{(\tilde{\mathbf{x}},\tilde{b})_a}(\mathbf{x},b|a)}{f_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x},b|a)} = \frac{f_{\tilde{b}_a}(b|a)}{f_{\tilde{b}}(b|a)} + \frac{f_{\tilde{\mathbf{x}}_a}(\mathbf{x}|b,a)}{f_{\tilde{\mathbf{x}}}(\mathbf{x}|b,a)} \equiv L_{\tilde{b}}(b|a)+$

$L_{\tilde{\mathbf{x}}}(\mathbf{x}|b,a)$. Substituting the last equality into (46) gives

$$\int \pi\left(\lambda + \mu L_{\tilde{b}}(b|a) + \mu L_{\tilde{\mathbf{x}}}(\mathbf{x}|b,a)\right) f_{\tilde{\mathbf{x}}}(\mathbf{x}|b,a)d\mathbf{x} \geq \int \pi\left(\lambda + \mu L_{\tilde{b}}(b|a) + \mu L_{\tilde{\mathbf{y}}}(\mathbf{y}|b,a)\right) f_{\tilde{\mathbf{y}}}(\mathbf{y}|b,a)d\mathbf{y}.$$

Note that π spans all the increasing and convex functions. Therefore, the above inequality holds if and only if the distribution of $L_{\tilde{\mathbf{x}}}(\mathbf{x}|b,a)$ is a mean-preserving spread of that of $L_{\tilde{\mathbf{y}}}(\mathbf{y}|b,a)$, given a and b. Or equivalently, if and only if (13) holds. □

Proof of Proposition 6

Proof. The propensity follows directly from (14). □

Proof of Proposition 7

Proof. We first prove that if Condition (18) holds, then (17) is equivalent to

$$\int \phi(\mathbf{y}|\mathbf{x},a) f_{\tilde{\mathbf{x}}_a}(\mathbf{x}|a)d\mathbf{x} = f_{\tilde{\mathbf{y}}_a}(\mathbf{y}|a). \tag{47}$$

Integrating both sides of (18) with respect to $F_{\tilde{b}}(b|a)$ gives

$$\int \phi(\mathbf{y}|\mathbf{x},a) f_{\tilde{\mathbf{x}}}(\mathbf{x}|a)d\mathbf{x} = f_{\tilde{\mathbf{y}}}(\mathbf{y}|a).$$

Taking derivative with respect to a on both sides of the above equality gives

$$\int \phi_a(\mathbf{y}|\mathbf{x},a) f_{\tilde{\mathbf{x}}}(\mathbf{x}|a)d\mathbf{x} + \int \phi(\mathbf{y}|\mathbf{x},a) f_{\tilde{\mathbf{x}}_a}(\mathbf{x}|a)d\mathbf{x} = f_{\tilde{\mathbf{y}}_a}(\mathbf{y}|a).$$

By substituting (17) into the above equation, we get (47).

Therefore, it suffices to prove that if (47) and (18) hold, then $\tilde{\mathbf{x}}$ is weakly more valuable at a than $\tilde{\mathbf{y}}$. The strategy of proof is as follows. Let $s_{\tilde{\mathbf{y}}}$ be the optimal compensation schedule for a given action a. We show that there exists a compensation schedule $s_{\tilde{\mathbf{x}}}$ contingent on $\mathbf{x}$, such that $s_{\tilde{\mathbf{x}}}$ satisfies both the participation and the relaxed incentive compatibility constraints at a, and the principal gets higher expected utility from $s_{\tilde{\mathbf{x}}}$ than from $s_{\tilde{\mathbf{y}}}$.

We construct $s_{\tilde{\mathbf{x}}}$ as follows

$$u(s_{\tilde{\mathbf{x}}}(\mathbf{x})) = \int u(s_{\tilde{\mathbf{y}}}(\mathbf{y}))\phi(\mathbf{y}|\mathbf{x}, a)d\mathbf{y}. \tag{48}$$

To show that $s_{\tilde{\mathbf{x}}}$ satisfies the participation constraint (1), we have

$$\begin{aligned} E[u(s_{\tilde{\mathbf{x}}})|a] = \int u(s_{\tilde{\mathbf{x}}}(\mathbf{x}))f_{\tilde{\mathbf{x}}}(\mathbf{x}|a)dx &= \int \left[\int u(s_{\tilde{\mathbf{y}}}(\mathbf{y}))\phi(\mathbf{y}|\mathbf{x}, a)d\mathbf{y}\right] f_{\tilde{\mathbf{x}}}(\mathbf{x}|a)d\mathbf{x} \\ &= \int u(s_{\tilde{\mathbf{y}}}(\mathbf{y}))\left[\int \phi(\mathbf{y}|\mathbf{x}, a)f_{\tilde{\mathbf{x}}}(\mathbf{x}|a)d\mathbf{x}\right] d\mathbf{y} \\ &= \int u(s_{\tilde{\mathbf{y}}}(\mathbf{y}))f_{\tilde{\mathbf{y}}}(\mathbf{y}|a)d\mathbf{y} \\ &= E[u(s_{\tilde{\mathbf{y}}})|a] \geq c(a_{\tilde{\mathbf{y}}}), \end{aligned}$$

where the first equality follows from (48), the third equality follows from (18), and the inequality is the participation constraint (1) for $\tilde{\mathbf{y}}$.

To show that $s_{\tilde{\mathbf{x}}}$ satisfies the relaxed incentive compatibility constraint (3), we have

$$\begin{aligned} \int u(s_{\tilde{\mathbf{x}}}(\mathbf{x}))f_{\tilde{\mathbf{x}}_a}(\mathbf{x}|a_{\tilde{\mathbf{y}}})d\mathbf{x} &= \int \left[\int u(s_{\tilde{\mathbf{y}}}(\mathbf{y}))\phi(\mathbf{y}|\mathbf{x}, a_{\tilde{\mathbf{y}}})d\mathbf{y}\right] f_{\tilde{\mathbf{x}}_a}(\mathbf{x}|a_{\tilde{\mathbf{y}}})d\mathbf{x} \\ &= \int u(s_{\tilde{\mathbf{y}}}(\mathbf{y}))\left[\int \phi(\mathbf{y}|\mathbf{x}, a_{\tilde{\mathbf{y}}})f_{\tilde{\mathbf{x}}_a}(\mathbf{x}|a_{\tilde{\mathbf{y}}})d\mathbf{x}\right] d\mathbf{y} \\ &= \int u(s_{\tilde{\mathbf{y}}}(\mathbf{y}))f_{\tilde{\mathbf{y}}_a}(\mathbf{y}|a_{\tilde{\mathbf{y}}})d\mathbf{y} \\ &= c'(a_{\tilde{\mathbf{y}}}), \end{aligned}$$

where the first equality follows from (48), the third equality from (47), and the last equality is the first-order necessary condition for the incentive compatibility constraint (2) for $\tilde{\mathbf{y}}$.

We have proven that $s_{\tilde{\mathbf{x}}}$ satisfies the participation and relaxed incentive compatibility constraints at a. Then we are left to prove that the principal prefers $s_{\tilde{\mathbf{x}}}$ to $s_{\tilde{\mathbf{y}}}$.

By Jensen's inequality and the concavity of $u(\cdot)$, (48) implies that

$$s_{\tilde{\mathbf{x}}}(\mathbf{x}) \leq \int s_{\tilde{\mathbf{y}}}(\mathbf{y})\phi(\mathbf{y}|\mathbf{x},a)d\mathbf{y}. \tag{49}$$

Then we have

$$\begin{aligned}
V(\tilde{\mathbf{x}}) &= \iint v(b - s_{\tilde{\mathbf{x}}}(\mathbf{x}))f_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x},b|a)d\mathbf{x}\,db \\
&\geq \iint v\Big(b - \int s_{\tilde{\mathbf{y}}}(\mathbf{y})\phi(\mathbf{y}|\mathbf{x},a)d\mathbf{y}\Big)f_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x},b|a)d\mathbf{x}\,db \\
&\geq \iint \left[\int v(b - s_{\tilde{\mathbf{y}}}(\mathbf{y}))\phi(\mathbf{y}|\mathbf{x},a)d\mathbf{y}\right] f_{\tilde{\mathbf{x}}}(\mathbf{x}|b,a)f_{\tilde{b}}(b|a)d\mathbf{x}\,db \\
&= \iint v(b - s_{\tilde{\mathbf{y}}}(\mathbf{y}))\left[\int \phi(\mathbf{y}|\mathbf{x},a)f_{\tilde{\mathbf{x}}}(\mathbf{x}|b,a)d\mathbf{x}\right] f_{\tilde{b}}(b|a)d\mathbf{y}\,db \\
&= \iint v(b - s_{\tilde{\mathbf{y}}}(\mathbf{y}))f_{\tilde{\mathbf{y}}}(\mathbf{y}|b,a)f_{\tilde{b}}(b|a)db\,d\mathbf{y} \\
&= \iint v(b - s_{\tilde{\mathbf{y}}}(\mathbf{y}))f_{(\tilde{\mathbf{y}},\tilde{b})}(\mathbf{y},b|a)d\mathbf{y}\,db = V(\tilde{\mathbf{y}}),
\end{aligned}$$

where the first inequality follows from (49), the second inequality follows from Jensen's inequality and the concavity of $v(\cdot)$, and the third equality follows from Condition (18).

The equivalence between the two sets of conditions can be easily proven as follows. If there exists an artificial conditional density function $\phi(\mathbf{y}|\mathbf{x},a)$ such that (17) and (18) hold, we can define $\tilde{\mathbf{z}}$ in such a way that $f_{\tilde{\mathbf{z}}}(\mathbf{z}|\mathbf{x},a,b) = \phi(\mathbf{z}|\mathbf{x},a)$. Then (19) and (20) are satisfied. On the other hand, if (19) and (20) hold, we define $\phi(\mathbf{y}|\mathbf{x},a) = f_{\tilde{\mathbf{z}}}(\mathbf{y}|\mathbf{x},a)$. Then (17) and (18) hold.This completes the proof. □

Proof of Lemma 1

Proof. We first prove that (19) is equivalent to

$$f_{\tilde{\mathbf{z}}_a}(\mathbf{z}|a) = \int f_{\tilde{\mathbf{z}}}(\mathbf{z}|\mathbf{x},a)f_{\tilde{\mathbf{x}}_a}(\mathbf{x}|a)d\mathbf{x}, \tag{50}$$

and then prove that (50) is equivalent to (21).

First, taking a derivative with respect to a on both sides of the formula $f_{\tilde{\mathbf{z}}}(\mathbf{z}|a) = \int f_{\tilde{\mathbf{z}}}(\mathbf{z}|\mathbf{x},a) f_{\tilde{\mathbf{x}}}(\mathbf{x}|a) d\mathbf{x}$ gives

$$f_{\tilde{\mathbf{z}}a}(\mathbf{z}|a) = \int f_{\tilde{\mathbf{z}}a}(\mathbf{z}|\mathbf{x},a) f_{\tilde{\mathbf{x}}}(\mathbf{x}|a) d\mathbf{x} + \int f_{\tilde{\mathbf{z}}}(\mathbf{z}|\mathbf{x},a) f_{\tilde{\mathbf{x}}a}(\mathbf{x}|a) d\mathbf{x}.$$

Substituting (19) into the above equation gives (21). We have thereby proven that (19) is equivalent to (50).

It then suffices to prove that (50) is equivalent to (21). Note that

$$\begin{aligned} E\left[L_{\tilde{\mathbf{x}}}\left(\tilde{\mathbf{x}}|a\right)|\mathbf{z},a\right] &= \int \frac{f_{\tilde{\mathbf{x}}a}(\tilde{\mathbf{x}}|a)}{f_{\tilde{\mathbf{x}}}(\tilde{\mathbf{x}}|a)} f_{\tilde{\mathbf{x}}}(\mathbf{x}|\mathbf{z},a) d\mathbf{x} \\ &= \int \frac{f_{\tilde{\mathbf{x}}a}(\tilde{\mathbf{x}}|a)}{f_{\tilde{\mathbf{x}}}(\tilde{\mathbf{x}}|a)} \frac{f_{(\tilde{\mathbf{x}},\tilde{\mathbf{z}})}(\mathbf{x},\mathbf{z}|a)}{f_{\tilde{\mathbf{z}}}(\tilde{\mathbf{z}}|a)} d\mathbf{x} \\ &= \frac{\int f_{\tilde{\mathbf{x}}a}(\tilde{\mathbf{x}}|a) f_{\tilde{\mathbf{z}}}(\mathbf{z}|\mathbf{x},a) d\mathbf{x}}{f_{\tilde{\mathbf{z}}}(\tilde{\mathbf{z}}|a)}. \end{aligned}$$

Therefore, (21) holds if and only if

$$\frac{\int f_{\tilde{\mathbf{x}}a}(\tilde{\mathbf{x}}|a) f_{\tilde{\mathbf{z}}}(\mathbf{z}|\mathbf{x},a) d\mathbf{x}}{f_{\tilde{\mathbf{z}}}(\tilde{\mathbf{z}}|a)} = L_{\tilde{\mathbf{z}}}\left(\mathbf{z}|a\right) \equiv \frac{f_{\tilde{\mathbf{z}}a}(\tilde{\mathbf{z}}|a)}{f_{\tilde{\mathbf{z}}}(\tilde{\mathbf{z}}|a)},$$

or equivalently, $\int f_{\tilde{\mathbf{x}}a}(\tilde{\mathbf{x}}|a) f_{\tilde{\mathbf{z}}}(\mathbf{z}|\mathbf{x},a) d\mathbf{x} = f_{\tilde{\mathbf{z}}a}(\tilde{\mathbf{z}}|a)$, which is exactly (50). This completes the proof. □

Proof of Proposition 8

Proof. The proof is composed of two parts. We first prove the sufficiency of (22) by showing that if (22) holds, then there exists a strictly risk-averse principal, for whom a small variation $\Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{y})$ to the optimal contract $s_{\tilde{\mathbf{x}}}(\mathbf{x})$ is strictly Pareto improving. We then prove the necessity of (22) by showing that if (22) does not hold, then for an arbitrary contract $s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}$, there exists a contract $s_{\tilde{\mathbf{x}}}$ that weakly Pareto dominates $s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}$.

Proof of the sufficiency of (22)*:* Let $s_{\tilde{\mathbf{x}}}$ be the optimal contract contingent on $\tilde{\mathbf{x}}$. Condition (22) holds if and only if at least one of the equations in the following formula does not hold,

$$f_{\tilde{\mathbf{y}}}(\mathbf{y}|\mathbf{x}) = f_{\tilde{\mathbf{y}}}(\mathbf{y}|\mathbf{x}, a) = f_{\tilde{\mathbf{y}}}(\mathbf{y}|\mathbf{x}, a, b). \tag{51}$$

The first equation in (51) is equivalent to $f_{\tilde{\mathbf{y}}_a}(\mathbf{y}|\mathbf{x}, a) = 0$. By substituting this result into the following equation,

$$L_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x}, \mathbf{y}|a) \equiv \frac{f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})_a}(\mathbf{x}, \mathbf{y}|a)}{f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x}, \mathbf{y}|a)} = \frac{f_{\tilde{\mathbf{y}}_a}(\mathbf{y}|\mathbf{x}, a)}{f_{\tilde{\mathbf{y}}}(\mathbf{y}|\mathbf{x}, a)} + \frac{f_{\tilde{\mathbf{x}}a}(\mathbf{x}|a)}{f_{\tilde{\mathbf{x}}}(\mathbf{x}|a)},$$

we have

$$L_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x}, \mathbf{y}|a) = \frac{f_{\tilde{\mathbf{x}}a}(\mathbf{x}|a)}{f_{\tilde{\mathbf{x}}}(\mathbf{x}|a)} \equiv L_{\tilde{\mathbf{x}}}(\mathbf{x}|a). \tag{52}$$

The second equation in (51) says that given $\mathbf{x}$ and a, $\tilde{\mathbf{y}}$ and $\tilde{b}$ are independent, or equivalently

$$f_{\tilde{b}}(b|\mathbf{x}, \mathbf{y}, a) = f_{\tilde{b}}(b|\mathbf{x}, a). \tag{53}$$

In sum, (22) holds if and only if (52) or (53) does not hold.

Next we derive the principal's and the agent's marginal return from a small additional variation $\Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}$ to the optimal contract $s_{\tilde{\mathbf{x}}}$ at a given $\mathbf{x}$. Let λ and μ be the multipliers for the agent's participation constraint and incentive compatibility constraint, respectively.

The principal's and the agent's marginal returns ΔV and ΔU at a given $\mathbf{x}$ are[9]

$$
\begin{aligned}
\text{(54)} \qquad \Delta V = & -\iint v'(b - s_{\tilde{\mathbf{x}}}(\mathbf{x}))\Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{y}) f_{(\tilde{b},\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(b,\mathbf{x},\mathbf{y}|a)\,db\,d\mathbf{y} \\
& + \lambda u'(s_{\tilde{\mathbf{x}}}(\mathbf{x}))\int \Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{y}) f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{y}|a)\,d\mathbf{y} \\
& + \mu u'(s_{\tilde{\mathbf{x}}}(\mathbf{x}))\int \Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{y}) f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})_a}(\mathbf{x},\mathbf{y}|a)\,d\mathbf{y}. \\
\text{(55)} \qquad \Delta U = & \, u'(s_{\tilde{\mathbf{x}}}(\mathbf{x}))\int \Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{y}) f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{y}|a)\,d\mathbf{y}.
\end{aligned}
$$

Let $\Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{y})$ take only two values. In particular, assume that for a set $\mathbf{Y}$ in the range of $\tilde{\mathbf{y}}$, $\Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{y}) = \Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}) > 0$ for all $\mathbf{y} \in \mathbf{Y}$ and $\Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{y}) = \Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}^c)$ for all $\mathbf{y} \in \mathbf{Y}^c$, and

$$
\text{(56)} \qquad \Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}) f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}|a) + \Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}^c) f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}^c|a) = 0,
$$

where $f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}|a) = \int_{\mathbf{Y}} f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{y}|a)\,d\mathbf{y}$ and correspondingly for $f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}^c|a)$. Therefore, the first term on the right-hand side of (54) can be written as

[9] For the mathematical technique used for deriving these formulas, refer to proposition 9.6.1 in Luenberger (1969). See also Holmström (1979).

(57)

$$
\begin{aligned}
&\iint v'(b - s_{\tilde{\mathbf{x}}}(\mathbf{x}))\Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{y}) f_{(\tilde{b},\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(b,\mathbf{x},\mathbf{y}|a)\,db\,d\mathbf{y}\\
=&\int v'(b - s_{\tilde{\mathbf{x}}}(\mathbf{x}))\left[\int \Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{y}) f_{(\tilde{b},\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(b,\mathbf{x},\mathbf{y}|a)\,d\mathbf{y}\right] db\\
=&\int v'(b - s_{\tilde{\mathbf{x}}}(\mathbf{x}))\left[\Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y})\int_{\mathbf{Y}} f_{(\tilde{b},\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(b,\mathbf{x},\mathbf{y}|a)\,d\mathbf{y} + \Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}^c)\int_{\mathbf{Y}^c} f_{(\tilde{b},\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(b,\mathbf{x},\mathbf{y}|a)\,d\mathbf{y}\right] db\\
=&\int v'(b - s_{\tilde{\mathbf{x}}}(\mathbf{x}))\left[\Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}) f_{(\tilde{b},\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(b,\mathbf{x},\mathbf{Y}|a) + \Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}^c) f_{(\tilde{b},\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(b,\mathbf{x},\mathbf{Y}^c|a)\right] db,\\
=&\int v'(b - s_{\tilde{\mathbf{x}}}(\mathbf{x}))\left[\Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}) f_{(\tilde{b},\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(b,\mathbf{x},\mathbf{Y}|a) - \frac{\Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}) f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}|a)}{f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}^c|a)} f_{(\tilde{b},\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(b,\mathbf{x},\mathbf{Y}^c|a)\right] db\\
=&\Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}) f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}|a)\int v'(b - s_{\tilde{\mathbf{x}}}(\mathbf{x}))\left[\frac{f_{(\tilde{b},\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(b,\mathbf{x},\mathbf{Y}|a)}{f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}|a)} - \frac{f_{(\tilde{b},\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(b,\mathbf{x},\mathbf{Y}^c|a)}{f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}^c|a)}\right] db\\
=&\Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}) f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}|a)\int v'(b - s_{\tilde{\mathbf{x}}}(\mathbf{x}))\left[f_{\tilde{b}}(b|\mathbf{x},\mathbf{Y},a) - f_{\tilde{b}}(b|\mathbf{x},\mathbf{Y}^c,a)\right] db,
\end{aligned}
$$

where $f_{(\tilde{b},\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(b,\mathbf{x},\mathbf{Y}|a) = \int_{\mathbf{Y}} f_{(\tilde{b},\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(b,\mathbf{x},\mathbf{y}|a)d\mathbf{y}$ and correspondingly for $f_{(\tilde{b},\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(b,\mathbf{x},\mathbf{Y}^c|a)$. The fourth equality follows from (56). Similarly, the integral in the second term on the right-hand side of (54) can be written as

(58)

$$
\begin{aligned}
&\int \Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{y}) f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{y}|a)\,d\mathbf{y}\\
=&\Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}) f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}|a) + \Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}^c) f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}^c|a)\\
=&0,
\end{aligned}
$$

where the last equality follows from (56). Let $f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})_a}(\mathbf{x},\mathbf{Y}|a) = \int_{\mathbf{Y}} f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})_a}(\mathbf{x},\mathbf{y}|a)d\mathbf{y}$, and correspondingly for $f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})_a}(\mathbf{x},\mathbf{Y}^c|a)$. The integral in the third term on the right-hand side of

(54) can be written as

$$
\begin{aligned}
(59) \qquad & \int \Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{y}) f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})_a}(\mathbf{x},\mathbf{y}|a) d\mathbf{y} \\
=&\Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}) \int_{\mathbf{Y}} f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})_a}(\mathbf{x},\mathbf{y}|a) d\mathbf{y} + \Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}^c) \int_{\mathbf{Y}^c} f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})_a}(\mathbf{x},\mathbf{y}|a) d\mathbf{y} \\
=&\Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}) f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})_a}(\mathbf{x},\mathbf{Y}|a) + \Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}^c) f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})_a}(\mathbf{x},\mathbf{Y}^c|a) \\
=&\Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}) f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})_a}(\mathbf{x},\mathbf{Y}|a) - \frac{\Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}) f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}|a)}{f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}^c|a)} f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})_a}(\mathbf{x},\mathbf{Y}^c|a) \\
=&\Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}) f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}|a) \left[\frac{f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})_a}(\mathbf{x},\mathbf{Y}|a)}{f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}|a)} - \frac{f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})_a}(\mathbf{x},\mathbf{Y}^c|a)}{f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}^c|a)} \right], \\
=&\Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}) f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}|a) \left[L_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}|a) - L_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}^c|a) \right],
\end{aligned}
$$

where the third equality follows from (56).

By substituting (57), (58), and (59) into (54) and (55), one gets

$$
\Delta V = \Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}) f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}|a) \begin{bmatrix} \mu\, u'(s_{\tilde{\mathbf{x}}}(\mathbf{x})) \left(L_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}|a) - L_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}^c|a) \right) - \\ \int v'(b - s_{\tilde{\mathbf{x}}}(\mathbf{x})) \Big(f_{\tilde{b}}(b|\mathbf{x},\mathbf{Y},a) - f_{\tilde{b}}(b|\mathbf{x},\mathbf{Y}^c,a) \Big) db \end{bmatrix}.
$$

$$
\Delta U = 0.
$$

Therefore, the sufficiency of (22) can be proven by showing that if (52) or (53) does not hold, then there exists a set $\mathbf{Y}$ and a strictly concave v such that ΔV is strictly positive.

First, if (52) does not hold, then $L_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{y}|a)$ varies with $\mathbf{y}$. Therefore, we can choose the set $\mathbf{Y}$ such that

$$
L_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}|a) - L_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}^c|a) > 0.
$$

There also exists a strictly concave $v(\cdot)$ such that $\mu > 0$. This is because $\mu > 0$ when $v''(\cdot) = 0$, i.e., μ is positive when the principal is risk neutral (see Jewitt, 1988). Then according to the envelope theorem in Milgrom and Segal (2002), there exists a strictly concave $v(\cdot)$, which is close enough to risk neutrality (for instance, if $v(c) = 1 - e^{-\alpha c}$ with $\alpha \to 0$,) so that $\mu > 0$ as well. In sum, there exists a set $\mathbf{Y}$ and a strictly concave v, such that

$$\mu\, u'(s_{\tilde{\mathbf{x}}}(\mathbf{x}))\left[L_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}|a) - L_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{Y}^c|a)\right] > 0. \tag{60}$$

Second, assume that (53) does not hold. For any set $\mathbf{Y}$ in the range of $\tilde{\mathbf{y}}$, and for any set B in the range of $\tilde{b}$, we always have

$$\begin{aligned}
\int f_{\tilde{b}}(b|\mathbf{x},\mathbf{Y},a)db &= \int_B f_{\tilde{b}}(b|\mathbf{x},\mathbf{Y},a)db + \int_{B^c} f_{\tilde{b}}(b|\mathbf{x},\mathbf{Y},a)db = 1,\\
\int f_{\tilde{b}}(b|\mathbf{x},\mathbf{Y}^c,a)db &= \int_B f_{\tilde{b}}(b|\mathbf{x},\mathbf{Y}^c,a)db + \int_{B^c} f_{\tilde{b}}(b|\mathbf{x},\mathbf{Y}^c,a)db = 1.
\end{aligned}$$

Therefore, there exists a set $\mathbf{Y}$ in the range of $\tilde{\mathbf{y}}$ and a set B in the range of $\tilde{b}$ with $B < B^c$, i.e., $b_1 \in B$ and $b_2 \in B^c$ imply $b_1 < b_2$, such that

$$\begin{aligned}
&\int_B \left[f_{\tilde{b}}(b|\mathbf{x},\mathbf{Y},a) - f_{\tilde{b}}(b|\mathbf{x},\mathbf{Y}^c,a)\right]db = -\epsilon < 0.\\
&\int_{B^c} \left[f_{\tilde{b}}(b|\mathbf{x},\mathbf{Y},a) - f_{\tilde{b}}(b|\mathbf{x},\mathbf{Y}^c,a)\right]db = \epsilon > 0.
\end{aligned}$$

Then we can choose a concave v such that $v'(b - s_{\tilde{\mathbf{x}}}(\mathbf{x})) = v_1$ if $b \in B$ and $v'(b - s_{\tilde{\mathbf{x}}}(\mathbf{x})) = v_2$ if $b \in B^c$. Because v' is non-increasing and $B < B^c$, then $v_1 > v_2$. Therefore,

$$\begin{aligned}
&\int v'(b - s_{\tilde{\mathbf{x}}}(\mathbf{x}))\left[f_{\tilde{b}}(b|\mathbf{x},\mathbf{Y},a) - f_{\tilde{b}}(b|\mathbf{x},\mathbf{Y}^c,a)\right]db\\
=&v_1 \int_B \left[f_{\tilde{b}}(b|\mathbf{x},\mathbf{Y},a) - f_{\tilde{b}}(b|\mathbf{x},\mathbf{Y}^c,a)\right]db + v_2 \int_{B^c} \left[f_{\tilde{b}}(b|\mathbf{x},\mathbf{Y},a) - f_{\tilde{b}}(b|\mathbf{x},\mathbf{Y}^c,a)\right]db\\
=&(v_2 - v_1)\epsilon < 0.
\end{aligned} \tag{61}$$

Substituting (60) and (61) into ΔV, we get $\Delta V > 0$. The procedure can be repeated for a set of $\mathbf{x}$ values with positive measures. Therefore, we have shown that if (22) holds, then by properly choosing a strictly concave utility v and a set $\mathbf{Y}$ in the range of $\tilde{\mathbf{y}}$, the principal is strictly better off with the additive variation $\Delta s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{y})$, without hurting the agent.

Proof of the necessity of (22)*:* Assume that (22) does not hold, i.e,

$$f_{\tilde{\mathbf{y}}}(\mathbf{y}|\mathbf{x},a,b) = f_{\tilde{\mathbf{y}}}(\mathbf{y}|\mathbf{x}), \quad \text{for almost all } (\mathbf{x},\mathbf{y},b). \tag{62}$$

We prove that for any $s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}$, there exists an $s_{\tilde{\mathbf{x}}}$ that is weakly more valuable than $s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}$.

Define $s_{\tilde{\mathbf{x}}}$ as follows

$$u(s_{\tilde{\mathbf{x}}}(\mathbf{x})) = \int u(s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{y}))f_{\tilde{\mathbf{y}}}(\mathbf{y}|\mathbf{x})d\mathbf{y}. \tag{63}$$

Then, by (62) and (63), we get

$$\begin{aligned} E[u(s_{\tilde{\mathbf{x}}}(\tilde{\mathbf{x}})|a)] &\equiv \int u(s_{\tilde{\mathbf{x}}}(\mathbf{x}))f_{\tilde{\mathbf{x}}}(\mathbf{x}|a)d\mathbf{x} \\ &= \iint u(s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{y}))f_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{y}|a)d\mathbf{x}d\mathbf{y} \equiv E[u(s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\tilde{\mathbf{x}},\tilde{\mathbf{y}})|a)], \quad \forall a. \end{aligned}$$

Thus $s_{\tilde{\mathbf{x}}}$ results in the same action and welfare for the agent as $s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}$ does.

By Jensen's inequality, (63) also implies $\int s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{y})f(\mathbf{y}|\mathbf{x})d\mathbf{y} \geq s(\mathbf{x})$, or equivalently

$$v\left(b - \int s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{y})f_{\tilde{\mathbf{y}}}(\mathbf{y}|\mathbf{x})d\mathbf{y}\right) \leq v(b - s_{\tilde{\mathbf{x}}}(\mathbf{x})).$$

Applying Jensen's inequality again on the above inequality gives

$$\int v(b - s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{y}))f_{\tilde{\mathbf{y}}}(\mathbf{y}|\mathbf{x})d\mathbf{y} \leq v(b - s_{\tilde{\mathbf{x}}}(\mathbf{x})).$$

Substituting for $f_{\tilde{\mathbf{y}}}(\mathbf{y}|\mathbf{x}) = f_{\tilde{\mathbf{y}}}(\mathbf{y}|\mathbf{x},a,b)$ and then taking the integral with respect to $F_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x},b|a)$ gives

$$\begin{aligned} V((\tilde{\mathbf{x}},\tilde{\mathbf{y}}),a) &\equiv \iiint v(b - s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(\mathbf{x},\mathbf{y}))f_{(\tilde{b},\tilde{\mathbf{x}},\tilde{\mathbf{y}})}(b,\mathbf{x},\mathbf{y}|a)dbd\mathbf{x}d\mathbf{y} \\ &\leq \iint v(b - s_{\tilde{\mathbf{x}}}(\mathbf{x}))f_{(\tilde{\mathbf{x}},\tilde{b})}(\mathbf{x},b|a)d\mathbf{x}\,db \equiv V(\tilde{\mathbf{x}},a). \end{aligned}$$

That is, the principal is at least as well off with $s_{\tilde{\mathbf{x}}}$ as with $s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}$. Thus $s_{\tilde{\mathbf{x}}}$ is weakly more valuable than $s_{(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}$. This completes the proof.

□

Proof of Lemma 2

Proof. First define

$$h(\mathbf{x};\mathbf{E}) = \begin{cases} 0 & \text{if } \mathbf{x} \notin \mathbf{E}, \\ 1 & \text{if } \mathbf{x} \in \mathbf{E}. \end{cases}$$

Then $h(\mathbf{x};\mathbf{E})$ is nondecreasing in $\mathbf{x}$ when $\mathbf{E}$ is an increasing set. If the transformation $\varphi^T(a)$ is nondecreasing in a for any nondecreasing function $\varphi(\mathbf{x})$, $\text{Prob}(\tilde{\mathbf{x}} \in \mathbf{E}|a) = h^T(a;\mathbf{E})$ is nondecreasing in a. This proves the "if" part.

To prove the "only if" part, assume NISP and let $\varphi(\cdot)$ be nondecreasing. For $\alpha < \beta$, define

$$\varphi_{[\alpha,\beta]}(\mathbf{x}) = \max\left(\alpha, \min(\beta, \varphi(\mathbf{x}))\right).$$

Therefore, $\varphi_{[\alpha,\beta]}(\mathbf{x})$ equals $\varphi(\mathbf{x})$ when $\alpha \leq \varphi(\mathbf{x}) \leq \beta$, but equals α when $\varphi(\mathbf{x}) < \alpha$ and equals β when $\varphi(\mathbf{x}) > \beta$. Note that $\varphi_{[\alpha,\beta]}(\cdot)$ can be approximated uniformly by the sum

$$\varphi^N_{[\alpha,\beta]}(\mathbf{x}) = \alpha + \sum_{i=1}^{N} \left[(\beta-\alpha)/N\right] h(\mathbf{x};\mathbf{E}_i), \tag{64}$$

where the sets $\mathbf{E}_i = \{\mathbf{x} : \varphi(\mathbf{x}) \geq \alpha + (\beta-\alpha)(i/N)\}$ are increasing sets. In addition, the transformation $\varphi^T(a) = E[\varphi(\tilde{\mathbf{x}})|a]$ is linear and continuous under the uniform norm. Thus, applying this transformation to (64) indicates that $\varphi^T_{[\alpha,\beta]}(a)$ is approximated uniformly by

$$\left(\varphi^N_{[\alpha,\beta]}\right)^T(a) = \alpha + \sum_{i=1}^{N} \left[(\beta-\alpha)/N\right] h^T(a;\mathbf{E}_i),$$

where $h^T(a;\mathbf{E}_i) = \int h(\mathbf{x};\mathbf{E}_i) f(\mathbf{x}|a) d\mathbf{x} = \text{Prob}(\tilde{\mathbf{x}} \in \mathbf{E}_i|a)$ is nondecreasing in a by NISP. Therefore, $\left(\varphi^N_{[\alpha,\beta]}\right)^T(a)$ is nondecreasing in a for all N, so is $\varphi^T_{[\alpha,\beta]}(a)$. Finally, letting $\alpha \to$

$-\infty$ and $\beta \to \infty$, and using the monotone convergence theorem shows that $\varphi^T_{[-\infty,\infty]}(a) = \varphi^T(a)$ is nondecreasing in a. $\square$

Proof of Proposition 9

Proof. The Kuhn-Tucker conditions for the doubly relaxed program are:[10]

(65)

$$\frac{\int v'(b - s^{**}(\mathbf{x})) f(b|\mathbf{x}, a^{**}) db}{u'(s^{**}(\mathbf{x}))} = \lambda^{**} + \mu^{**} \frac{f_a}{f}(\mathbf{x}|a^{**}).$$

$$\iint v(b - s^{**}(\mathbf{x})) f_a(b, \mathbf{x}|a^{**}) db d\mathbf{x} + \lambda^{**} \Big[\int u(s^{**}(\mathbf{x})) f_a(\mathbf{x}|a^{**}) d\mathbf{x} - c'(a^{**}) \Big]$$
$$+ \mu^{**} \Big[\int u(s^{**}(\mathbf{x})) f_{aa}(\mathbf{x}|a^{**}) d\mathbf{x} - c''(a^{**}) \Big] = 0.$$

$$\int u(s^{**}(\mathbf{x})) f(\mathbf{x}|a^{**}) d\mathbf{x} - c(a^{**}) \geq 0, \quad \lambda^{**} \geq 0, \quad \lambda^{**} \Big[\int u(s^{**}(\mathbf{x})) f(\mathbf{x}|a^{**}) d\mathbf{x} - c(a^{**}) \Big] = 0.$$

(66)

$$\int u(s^{**}(\mathbf{x})) f_a(\mathbf{x}|a^{**}) d\mathbf{x} - c'(a^{**}) \geq 0, \quad \mu^{**} \geq 0, \quad \mu^{**} \Big[\int u(s^{**}(\mathbf{x})) f_a(\mathbf{x}|a^{**}) d\mathbf{x} - c'(a^{**}) \Big] = 0.$$

To prove $\mu^* \geq 0$, it suffices to prove that $\mu^* = \mu^{**}$, which follows directly from proving that the first inequality in (66) takes strict equality, i.e., $\int u(s^{**}(\mathbf{x})) f_a(\mathbf{x}|a^{**}) d\mathbf{x} - c'(a^{**}) = 0$.

We prove $\int u(s^{**}(\mathbf{x})) f_a(\mathbf{x}|a^{**}) d\mathbf{x} - c'(a^{**}) = 0$ by the method of contradiction. If $\int u(s^{**}(\mathbf{x})) f_a(\mathbf{x}|a^{**}) d\mathbf{x} - c_a(a^{**}) > 0$, as we assumed for contradiction, then we must have $\iint v(b - s^{**}(\mathbf{x})) f_a(b, \mathbf{x}|a^{**}) db\, d\mathbf{x} < 0$. Otherwise an increase of a would be Pareto improving and a^{**} would not be optimal. We can, however, prove the opposite result, namely $\iint v(b - s^{**}(\mathbf{x})) f_a(b, \mathbf{x}|a^{**}) db\, d\mathbf{x} \geq 0$. This yields our contradiction.

[10] The derivation of the Kuhn-Tucker conditions follows the standard procedure and is available from the author upon request.

First, using $f_A(b, \mathbf{x}|a^{**}) = f_a(b|\mathbf{x}, a^{**})f(\mathbf{x}|a^{**}) + f(b|\mathbf{x}, a^{**})f_a(\mathbf{x}|a^{**})$ gives

(67)

$$
\begin{aligned}
&\iint v(b - s^{**}(\mathbf{x}))f_a(b, \mathbf{x}|a^{**})db\, d\mathbf{x} \\
=&\iint v(b - s^{**}(\mathbf{x}))\left[f_a(b|\mathbf{x}, a^{**})f(\mathbf{x}|a^{**}) + f(b|\mathbf{x}, a^{**})f_a(\mathbf{x}|a^{**})\right] db\, d\mathbf{x} \\
=&\int \left[\int v(b - s^{**}(\mathbf{x}))f_a(b|\mathbf{x}, a^{**})db\right] f(\mathbf{x}|a^{**})d\mathbf{x} + \int \left[\int v(b - s^{**}(\mathbf{x}))f(b|\mathbf{x}, a^{**})db\right] f_a(\mathbf{x}|a^{**})d\mathbf{x} \\
=&\int \left[\int v(b - s^{**}(\mathbf{x}))dF_a(b|\mathbf{x}, a^{**})\right] f(\mathbf{x}|a^{**})d\mathbf{x} + \int \left[\int v(b - s^{**}(\mathbf{x}))dF(b|\mathbf{x}, a^{**})\right] f_a(\mathbf{x}|a^{**})d\mathbf{x}.
\end{aligned}
$$

It suffices to prove that both terms after the last equality of (67) are nonnegative.

Concavity of v and the SOSD condition of $F(b|\mathbf{x}, a)$ in a imply $\int v(b - s^{**}(\mathbf{x}))\, dF_a(b|\mathbf{x}, a^{**}) \geq 0$. Therefore, the first term after the last equality of (67) is nonnegative.

As to the second term, the NISP condition on $f(\mathbf{x}|a)$ implies that the second term is nonnegative if $\int v(b - s^{**}(\mathbf{x}))dF(b|\mathbf{x}, a^{**})$ is nondecreasing in $\mathbf{x}$, that is, if

$$
\begin{aligned}
&\frac{\partial \int v(b - s^{**}(\mathbf{x}))dF(b|\mathbf{x}, a^{**})}{\partial x_i} \\
=& - s^{**}_{x_i}(\mathbf{x}) \int v'(b - s^{**}(\mathbf{x}))dF(b|\mathbf{x}, a^{**}) + \int v(b - s^{**}(\mathbf{x}))dF_{x_i}(b|\mathbf{x}, a^{**}) \geq 0, \quad \forall i = 1, 2, ..., n,
\end{aligned}
$$

which is equivalent to

$$
s^{**}_{x_i}(\mathbf{x}) \leq \frac{\int v(b - s^{**}(\mathbf{x}))dF_{x_i}(b|\mathbf{x}, a^{**})}{\int v'(b - s^{**}(\mathbf{x}))dF(b|\mathbf{x}, a^{**})}, \quad \forall i = 1, 2, ..., n. \tag{68}
$$

To prove (68), let's look at (65), which characterizes $s^{**}(\mathbf{x})$. If $\int u(s^{**}(\mathbf{x}))f_a(\mathbf{x}|a^{**})d\mathbf{x} - c_a(a^{**}) > 0$ as assumed for contradiction, then $\mu^{**} = 0$, and (65) can be rewritten as

$$
\frac{\int v'(b - s^{**}(\mathbf{x}))dF(b|\mathbf{x}, a^{**})}{u'(s^{**}(\mathbf{x}))} = \lambda^{**}. \tag{69}
$$

Multiplying both sides by $u''(s^{**}(\mathbf{x}))$ gives

$$\lambda^{**}u''(s^{**}(\mathbf{x})) = -r_A(s^{**}(\mathbf{x}))\int v'(b - s^{**}(\mathbf{x}))dF(b|\mathbf{x}, a^{**}). \tag{70}$$

Implicit differentiation of (69) in x_i gives

$$s^{**}_{x_i}(\mathbf{x}) = \frac{\int v'(b - s^{**}(\mathbf{x}))dF_{x_i}(b|\mathbf{x}, a^{**})}{\int v''(b - s^{**}(\mathbf{x}))dF(b|\mathbf{x}, a^{**}) + \lambda^{**}u''(s^{**}(\mathbf{x}))}.$$

Then substituting (70) into the above formula gives

$$s^{**}_{x_i}(\mathbf{x}) = \frac{\int v'(b - s^{**}(\mathbf{x}))dF_{x_i}(b|\mathbf{x}, a^{**})}{\int v''(b - s^{**}(\mathbf{x}))dF(b|\mathbf{x}, a^{**}) - r_A(s^{**}(\mathbf{x}))\int v'(b - s^{**}(\mathbf{x}))dF(b|\mathbf{x}, a^{**})}. \tag{71}$$

Next, (25) and (26) imply that $r_A(s^{**}(\mathbf{x})) \geq \frac{K}{\beta} \geq \frac{r_P(b-s^{**}(\mathbf{x}))}{\beta}$, or equivalently $\beta r_A(s^{**}(\mathbf{x})) \geq -\frac{v''(b-s^{**}(\mathbf{x}))}{v'(b-s^{**}(\mathbf{x}))}$. Rearranging this inequality gives

$$v''(b - s^{**}(\mathbf{x})) + \beta r_A\big(s^{**}(\mathbf{x})\big)v'(b - s^{**}(\mathbf{x})) \geq 0,$$

or equivalently, $v'(b - s^{**}(\mathbf{x})) + \beta r_A\big(s^{**}(\mathbf{x})\big)v(b - s^{**}(\mathbf{x}))$ is nondecreasing in b. This result together with the FOSD condition of $F(b|\mathbf{x}, a^{**})$ in $\mathbf{x}$ implies that

$$\int \big[v'(b - s^{**}(\mathbf{x})) + \beta r_A(s^{**}(\mathbf{x}))v(b - s^{**}(\mathbf{x}))\big]dF_{x_i}(b|\mathbf{x}, a^{**}) \geq 0,$$

or equivalently

$$r_A(s^{**}(\mathbf{x})) \geq \frac{-\int v'(b - s^{**}(\mathbf{x}))dF_{x_i}(b|\mathbf{x}, a^{**})}{\beta\int v(b - s^{**}(\mathbf{x}))dF_{x_i}(b|\mathbf{x}, a^{**})}.$$

Note that the right-hand side of the above inequality is positive, as it is a negative number over another negative number. By substituting the above inequality in (71) we get

$$s^{**}_{x_i}(\mathbf{x}) \leq \frac{1}{\frac{\int v''(b-s^{**}(\mathbf{x}))dF(b|\mathbf{x},a^{**})}{\int v'(b-s^{**}(\mathbf{x}))dF_{x_i}(b|\mathbf{x},a^{**})} + \frac{\int v'(b-s^{**}(\mathbf{x}))dF(b|\mathbf{x},a^{**})}{\beta\int v(b-s^{**}(\mathbf{x}))dF_{x_i}(b|\mathbf{x},a^{**})}}.$$

Thus the central inequality (68) will follow if

$$\frac{\int v''(b-s^{**}(\mathbf{x}))dF(b|\mathbf{x},a^{**})}{\int v'(b-s^{**}(\mathbf{x}))dF_{x_i}(b|\mathbf{x},a^{**})} \geq \left(1-\frac{1}{\beta}\right)\frac{\int v'(b-s^{**}(\mathbf{x}))dF(b|\mathbf{x},a^{**})}{\int v(b-s^{**}(\mathbf{x}))dF_{x_i}(b|\mathbf{x},a^{**})}. \tag{72}$$

Now we are left to prove that (26) implies (72).

On one hand, the first part of (26) implies that $-\frac{v''(b-s^{**}(\mathbf{x}))}{v'(b-s^{**}(\mathbf{x}))} \geq \left(1-\frac{1}{\beta}\right)K$. By multiplying both sides by $v'(b-s^{**}(\mathbf{x}))$ and then integrating with respect to $F(b|\mathbf{x},a^{**})$, one gets

$$-\int v''(b-s^{**}(\mathbf{x}))dF(b|\mathbf{x},a^{**}) \geq \left(1-\frac{1}{\beta}\right)K\int v'(b-s^{**}(\mathbf{x}))dF(b|\mathbf{x},a^{**}). \tag{73}$$

On the other hand, the second part of (26) implies that $v''(b-s^{**}(\mathbf{x}))+Kv'(b-s^{**}(\mathbf{x})) \geq 0$, which together with the FOSD condition of $F(b|\mathbf{x},a^{**})$ in $\mathbf{x}$ implies that

$$\int \left[v'(b-s^{**}(\mathbf{x}))+Kv(b-s^{**}(\mathbf{x}))\right]dF_{x_i}(b|\mathbf{x},a^{**}) \geq 0$$

or equivalently

$$-\int v'(b-s^{**}(\mathbf{x}))dF_{x_i}(b|\mathbf{x},a^{**}) \leq K\int v(b-s^{**}(\mathbf{x}))dF_{x_i}(b|\mathbf{x},a^{**}). \tag{74}$$

Due to the concavity of v in b and due to the FOSD condition of $F(b|\mathbf{x},a^{**})$ in $\mathbf{x}$, the left-hand side of (74) is positive. Clearly, (72) follows from (73) and (74). We thereby arrive at a contradiction by assuming $\int u(s^{**}(\mathbf{x}))f_a(\mathbf{x}|a^{**})d\mathbf{x}-c_a(a^{**})>0$, so it must be true that $\int u(s^{**}(\mathbf{x}))f_a(\mathbf{x}|a^{**})d\mathbf{x}-c_a(a^{**})=0$, and the relaxed and doubly relaxed programs coincide. Therefore, $\mu^*=\mu^{**}\geq 0$.

Finally, the condition that the principal has constant absolute risk aversion corresponds to the extreme case with $\beta=\infty$ and $K=r_P$, and the condition that the range of the absolute risk aversion of the principal is not higher than that of the agent corresponds to another extreme case with $\beta = 1$ and $K = \infty$. $\square$

Proof of Proposition 10

Proof. First, we need to prove that $s(\mathbf{x})$ is nondecreasing in $\mathbf{x}$. Differentiation with respect to x_i on both sides of (65) gives

$$\int v'(b - s^*(\mathbf{x}))dF_{x_i}(b|\mathbf{x}, a^*)\, u'(s^*(\mathbf{x})) - \int v''(b - s^*(\mathbf{x}))dF(b|\mathbf{x}, a^*)\, u'(s^*(\mathbf{x}))\, s^*_{x_i}(\mathbf{x})$$

$$- \int v'(b - s^*(\mathbf{x}))dF(b|\mathbf{x}, a^*)\, u''(s^*(\mathbf{x}))\, s^*_{x_i}(\mathbf{x}) = \mu^* \frac{\partial \frac{f_a}{f}(\mathbf{x}|a^*)}{\partial x_i} [u'(s^*(\mathbf{x}))]^2, \quad \forall \mathbf{x}.$$

The RHS is always nonnegative by the result that $\mu^* \geq 0$ (Proposition 9) and by the MLR condition.

As to the LHS, the first term on the LHS is nonpositive, due to the concavity of v and the FOSD condition of $F(b|\mathbf{x}, a^{**})$ in $\mathbf{x}$. Suppose for contradiction that $s^*_{\mathrm{x}_i}(\mathbf{x}) < 0$, then both the second and the third terms on the LHS would be negative. Therefore the LHS is negative, which is a contradiction to the positive RHS. Thus it must be true that $s^*_{x_i}(\mathbf{x}) \geq 0$.

Since $s^*(\mathbf{x})$ is nondecreasing in $\mathbf{x}$, so is $u(s^*(\mathbf{x}))$. Then according to Lemma 3, the CISP condition implies that $\int u(s^*(\mathbf{x})) f(\mathbf{x}|a) d\mathbf{x}$ is concave in a, and this completes the proof. $\square$

Proof of Proposition 11

Proof. According to Proposition 9, the FOSD, SOSD, and NISP conditions combined with Condition (i) or (ii) imply that $\mu^* \geq 0$. Then by the MCLR condition, the right-hand side of (65) is nondecreasing and concave in $\mathbf{x}$.

Taking a nondecreasing and concave transformation $u(\omega(\cdot\,; \mathbf{x}, a^*))$ on both sides of (65), we get

$$u(s^*(\mathbf{x})) = u\Big(\omega(\lambda^* + \mu^* \frac{f_a}{f}(\mathbf{x}|a^*); \mathbf{x}, a^*)\Big),$$

which is nondecreasing and concave in $\mathbf{x}$ by properties of composition of nondecreasing and concave functions. Then according to Lemma 4, the CICSP condition implies that

$\int u(s^*(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x}$ is concave in a, and this completes the proof. $\square$

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