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Abstract

This paper presents a general equilibrium model with endogenous collateral constraints to study the relationship between financial development and business cycle fluctuations in a cross-section of economies with different sizes of their financial sector. The financial sector can amplify or dampen the volatility of income by increasing or reducing the business cycle effects of technological shocks. We find a non-monotonic relationship between the volatility of income and financial development measured by total borrowing and lending. A more developed financial system unambiguously increases the income level however the volatility can rise or fall depending on the degree of financial development.

JEL classification: E32, E60
Bank classification: Credit and credit aggregates; Financial stability

Résumé

Au moyen d'un modèle d'équilibre général qui intègre des contraintes d'emprunt endogènes liées à la valeur des garanties, les auteurs étudient, à partir d'un échantillon représentatif de pays aux secteurs financiers de taille différente, la relation entre le degré de développement financier d'un pays et les variations de son cycle économique. Le secteur financier peut accentuer ou modérer la volatilité des revenus en amplifiant ou en réduisant les effets des chocs technologiques sur les cycles économiques. Les auteurs font ressortir une relation non monotone entre la volatilité des revenus et le degré de développement du système financier mesuré par le volume des prêts. Sans contredit, le niveau des revenus s'accroît dans les pays au système financier plus développé; la volatilité peut, par contre, augmenter ou diminuer selon le degré de développement financier.

Classification JEL : E32, E60
Classification de la Banque : Crédit et agrégats du crédit; Stabilité financière

1 Introduction

The financial crisis of 2007-2009 renewed the attention to the linkages between the financial sector and the real economy. As a consequence of the crisis the world experienced a large and sharp contraction of output. In this paper we analyze the relationship between financial sector development and the level and volatility of income. We ask two questions. Does a larger financial sector amplify or dampen technological shocks? Does a larger financial sector allow a faster or slower recovery?

To answer these questions we develop a model of endogenous borrowing constraints based on Kiyotaki (1998). Agents' productivity is heterogeneous and financial markets intermediate loans from the least productive agents to the most productive. Capital markets are imperfect and credit constraints arise because lenders cannot force borrowers to repay their debts unless the debts are secured with collateral. Because the durable assets of the economy serve as collateral for loans, borrowers' credit limits are affected by the prices of the collateralized assets. At the same time, the prices of these assets are affected by the size of the credit limits. The dynamic interaction between credit limits and asset prices amplifies and increases the persistence of productivity shocks thus raising the volatility of output. We generalize the model of Kiyotaki (1998) by adding an exogenous variation to the collateral constraint to capture, in a reduced form, possible cross-sectional differences in the institutional framework as well as financial innovation. We call this variation financial development as it can can ease or worsen the borrowing constraint problem.

We show that, depending on the value of the financial development parameter, the equilibrium is one of three types. First, the traditional credit constrained equilibrium type described by Kiyotaki and Moore (1997) arises with low values of financial development. The second equilibrium type occurs with larger values of the parameter, corresponding to a situation in which productive agents use all the available assets in the economy but remain credit constrained. The third equilibrium type occurs with yet larger values of the financial development parameter and corresponds to the case in which productive agents use all available assets and the financial sector allows them to become unconstrained, akin to Moll (2010).

We show a non-monotonic relationship between financial development and the volatility of income from technological shocks. When a negative shock hits, the dynamic interaction between credit limits and asset prices can amplify the effect of the shock on output and wealth. We show that the degree of amplification depends on the level of financial development and that this relationship is non-monotonic.

This non-monotonicity is due to two opposing effects of the financial development parameter. In an environment with endogenous borrowing constraints, financial development increases the sensitivity of the credit limits to changes in the value of the collateral. Thus, when a technology shock reduces the value of the collateral, the credit limits fall more in economies with a more developed financial sector. This effect amplifies the impact of the shock. On the other hand, as the economy recovers from the shock and the collateral increases in value, the credit limit increases more in economies with more financial development. This accelerates the recovery, thus increasing the value of the collateral and the credit limit at the time of the shock. This effect dampens the impact of the shock. We argue that this non-monotonic relationship is general to models with endogenous borrowing constraints.

Studying the effects of financial development on the variability of macroeconomic aggregates is important to understand the potential effects that financial regulation or innovation can have on the steady state level and variability of income. Understanding this link is the first step towards building a normative theory of the welfare costs of the excess variability of business cycles induced by financial frictions.

Related Theoretical Literature

The idea that the financial sector can be a source of economic fluctuations is not new. Early formal treatments of this idea were developed by Bernanke and Gertler (1989) and Kiyotaki and Moore (1997). These two papers started a long literature on the "financial accelerator" by which the effects of a fundamental (technological) shock can be amplified by the financial sector. These papers develop theories of endogenously determined credit limits and are useful to understand how the financial system can amplify small technological shocks and make them more persistent. In these models, credit constraints arise because creditors cannot force debtors to repay debts unless they are secured by collateral.

The recent financial crisis also spurred interest in examining the possibility of new linkages between the financial sector and the real economy. One of the new channels being explored in the literature is the informational channel through dispersed, private or heterogeneous information. In a model with heterogeneous information, La'O (2010) shows how financial frictions in the form of collateral constraints can drive business cycle fluctuations even when fundamentals remain constant in a standard real business cycle model. Another novel channel of amplification is the one explored by Brunnermeier and Sannikov (2010). In their model, amplification of shocks occurs not only through the fall in the value of collateral, but also through the increased volatility in asset prices which in turn provides incentives for households to hold precautionary cash balances and fire sales. Gertler and Kiyotaki (2010) consider a mechanism of endogenous risk taking by which lower exogenous risk leads to greater leverage that may lead to higher total risk. In their model, securitization helps share idiosyncratic risk but amplifies endogenous risk. They calibrate their model and then generate a crisis experiment to show the effects on asset prices and output of an exogenous change to capital quality. Mendoza (2010) uses a DSGE model with an endogenous collateral constraint to analyze the amplification and asymmetry in the responses to shocks. In this environment he also documents a non-monotonic relationship between the shocks and macro aggregates responses.

In terms of technique we are closest to Kiyotaki and Moore (2005) which explore the idea of financial deepening by analyzing regions of the parameter space of their model of borrowing constraints (due to limited commitment) to describe the effects of financial intermediation on output and investment. Their model predicts three regions of financial deepening: the first which is cash based and without any financial intermediation, the second in which there are sophisticated financial markets, and the third in which the economy achieves the first best allocation and there is no need for financial markets since there is full commitment.

Empirical Evidence

Regarding the direct relationship between the level of income and the development of the financial sector Beck et al. (2000) show a strong positive causal relationship between measures of financial development and size and the rate of per capita GDP growth in a large cross section of countries. Narrowing the focus to developing economies, some papers studying sudden stops have found a negative relationship between financial development and income volatility. The most likely driver of the result seems to be the financial liberalization that preceded some of the financial crises that lead to large contractions of income (Aizenman, 2004). For developed economies the picture is less clear. Although credit expansions are one of the best predictors of financial instability in long historical samples (Jorda et al., 2010), the long-term effects on output are not clear. Loayza and Ranciere (2005) argue that the apparent empirical contradiction between the positive and negative effects of the financial sector can be resolved empirically if we distinguish between short and long-term effects of financial intermediation.

Regarding the cross sectional evidence of financial development and volatility of income several papers have documented empirically this relationship. More generally these studies often look at the cross-sectional variation of the degree of international financial integration and macroeconomic volatility because their interest tends to be on international risk sharing and propagation of financial shocks. Easterly et al. (2001) find that a higher level of development of the domestic financial sector is associated with lower volatility in developing countries. They document a non monotonic relation between the volatility of income growth and the level of credit to the private sector as a share of GDP. At low levels of credit the volatility of growth rates is large, while at intermediate levels of credit its the lowest. At higher levels of credit volatility of growth somewhat picks up but still less than at the other extreme.

Kose et al. (2003) and Kose et al. (2009) document the role of financial integration and macroeconomic volatility and international risk sharing. The first paper documents that on average the volatility of output from 1960 to 1999 has decreased for a large cross-section of countries while increased financial openness is associated with increased volatility of consumption. In the follow up paper they document more specifically the relation of financial integration and reduced international risk sharing. They find that developed economies actually benefit from the integration while developing do not.

Denizer et al. (2002) analyze a similar question to ours. They find that countries with a more developed financial sector experience less acute fluctuations in real per capita output, consumption, and investment growth. In particular, their dependent variables are the standard deviation of real per capita GDP, consumption and investment growth rates. The key variable in their results is the proportion of credit provided to the private sector. Their estimates for a large cross-section of countries finds a negative relationship between the standard deviation of the growth rates of macroeconomic aggregates and private sector credit controlling for the volatility of the exchange rate, the openness of the country and other suitable controls.

In what is possibly the closest empirical investigation of our idea, Darrat et al. (2005) document the experience of the UAE in their transition to a more developed financial sector and the outcome for the volatility of business cycles. This is a time-series and not a cross-sectional investigation and therefore we cannot take their results directly to support our result. However, they document that financial development shares a statistically significant cointegration relation with growth volatility between 1973-2000 suggesting that in the UAE it is strongly related to the development of its financial sector over this period. One caveat in relating these results to our model is that the particular source of shocks, either technological or financial, matters for the dynamics of the macroeconomic aggregates and their interaction with the borrowing channel that we explore.

Next section 2 describes our model and discusses the equilibria in detail. Section 3 discusses the results of the dynamic exercises. Section 4 has some concluding remarks.

2 The Model

2.1 Overview of the Model

Our model is an extension of Kiyotaki (1998) in which durable assets play a dual role, as factors of production and as collateral for loans, with the addition of an exogenous multiplicative parameter to the borrowing constraint. We call this parameter the degree of financial development.

In the model, there are two inputs in the production of goods: physical capital and land which is in fixed supply and normalized to 1. Goods produced can be used to invest in physical capital or consumption but not to produce land. The consumption good is perishable and cannot be stored. The economy has two types of agents, productive and unproductive which are both risk averse and derive utility from consumption. Agents evolve between the two states according to a Markov process and the realization of the productivity is known at the beginning of period so agents can contract on their potential output.

Given the heterogeneity of productivity, agents engage in borrowing and lending. We interpret the total lending and borrowing activity between agents in this economy as the size of the financial sector. This activity has a friction in the form of imperfect enforcement of debt contracts since borrowers can only recover a fraction of the non-collateralized credit if entrepreneurs can walk away from their debt. As a protection, lenders demand collateral which gives rise to the borrowing constraint faced by the productive agents.

We model this by assuming that agents cannot borrow more than a constant times the value of their collateral. Our variation is to introduce a parameter multiplying the value of the collateral. We interpret this scaling parameter as a measure of conditions that ease or aggravate the borrowing constraints, for example weakness of the legal system may further aggravate the constraint. On the other hand, we allow this parameter to represent the possibility of lending on margin in which case, the value of the collateral can be smaller than the amount borrowed. This variation gives rise to three types of equilibrium depending on parameter values. We now turn to the details of the model.

2.2 Model Details

Consider a discrete-time economy with a single homogenous good, a fixed asset, and a continuum of agents which live for ever and have preferences of the form:

$$\mathbb{E}_t\left(\sum_{\tau=0}^{\infty}\beta^{\tau}\ln c_{t+\tau}\right).$$

In this economy, there are two production inputs, the fixed asset k_t, called land, which is in fixed supply and does not depreciate, and an intermediate good x_t, called investment, which fully depreciates every period.

Every agent shifts stochastically between two states, the productive and unproductive according to a Markov process. Productive agents become unproductive next period with probability δ, and unproductive agents become productive with probability $n\delta$, where $n < 1$. We use superscript i for the agent types, p for the set of agents that are productive, and u for the set of unproductive agents. The production function of both types exhibit constant returns to scale, but the total factor productivity of productive agents is higher than the one of unproductive agents:

$$y_{t+1}^{i} \equiv \alpha\left(\frac{k_t^i}{\sigma}\right)^{\sigma}\left(\frac{x_t^i}{1-\sigma}\right)^{1-\sigma} \quad \forall i\in p \tag{1}$$

$$y_{t+1}^{i} \equiv \gamma\left(\frac{k_t^i}{\sigma}\right)^{\sigma}\left(\frac{x_t^i}{1-\sigma}\right)^{1-\sigma} \quad \forall i\in u \tag{2}$$

where $\gamma < \alpha$ and $0 < \sigma < 1$.

To maximize their expected utility every agent chooses a path of consumption, land holdings, investment, output and debt subject to the technology constraints (1) and (2) and the corresponding flow-of-funds constraint:

$$c_t + x_t + q_t\left(k_t - k_{t-1}\right) = y_t + \frac{b_{t+1}}{r_t} - b_t, \tag{3}$$

for all $t > 0$, where q_t denotes the price of land, b_t is the debt repayment in period t, b_{t+1} is the new debt in period t, and r_t is the real interest rate at time t. More precisely, q_t is the price of a unit of land at time t that can be used in production during the period $t+1$.

Every period, agents face the following borrowing constraint:

$$b_{t+1} \leq \theta q_{t+1} k_t,\ \theta \in [0,\infty). \tag{4}$$

This implies that the debt repayment cannot exceed the adjusted value of the collateral at period t+1, where the adjustment is given by θ. We allow this adjustment parameter to vary in the cross-section and regard $\theta = 1$ as the benchmark value, which corresponds to the model of Kiyotaki (1998). When θ is smaller than 1, the collateral constraint is further reduced because lenders can only recover a fraction of the collateral. On the other hand, when $\theta > 1$, the borrowing constraint problem is reduced because it implies lenders can recover the whole collateral plus a fraction of the non-collateralized credit.

We interpret the parameter θ as a reduced form summary of potential factors that allow expand or constrain the borrowing and lending between productive and unproductive agents. Factors that constrain the borrowing potential are for example weaknesses in the legal system that hinder lenders from collecting the collateral from borrowers. Factors that expand the borrowing potential are for example the ability of financial institutions to lend on margin. For the case when $\theta > 1$, note that some conditions have to be met for the existence of equilibrium which we clarify below.

2.3 Equilibrium types

Depending on the value of θ and the productivity parameters α and γ, we can have one of three types of equilibrium which we describe in detail below. An equilibrium is defined in the standard way: a set of sequences of state-contingent allocations for each type of agent $\{c_t, k_t, b_t\}_{t=0}^{\infty}$ and prices $\{q_t, r_t\}_{t=0}^{\infty}$ such that i) unproductive agents maximize lifetime expected utility subject to their period budget constraints, ii) productive agents maximize lifetime expected utility subject to their budget and borrowing constraints, and iii) land and goods markets clear. Kiyotaki and Moore (2005) analyze financial deepening in a similar way by looking at the different steady state equilibria in three regions of their parameter space.

Given the logarithmic preferences, a simpler way to characterize the equilibrium is in terms of the aggregate wealth, the share of wealth that productive agents hold, the land price and the user cost of capital. We focus on describing the equilibrium using the real interest rate, whether the borrowing constraint is binding or not and the holdings of land. This characterization will allow us to provide intuition in the simulations we present in section 3. Proofs can be found in the Appendix.

2.3.1 Equilibrium Type I

Equilibrium type I occurs when the borrowing constraint is binding for all productive agents and investment is positive for all agents. In each period, both productive and unproductive agents optimally chose to consume a fraction $(1-\beta)$ of their wealth w_t:

$$c_t = (1-\beta) w_t. \tag{5}$$

where agents' wealth equals their output plus the value of the land minus the value of their debt,

$$w_t^i \equiv y_t^i + q_t k_{t-1}^i - b_t^i. \tag{6}$$

Unproductive agents. The investment-to-land ratio of unproductive agents is given by

$$\frac{x_t^i}{k_t^i} = \frac{1-\sigma}{\sigma} u_t \ \forall i \in u, \tag{7}$$

where we define $u_t \equiv q_t - q_{t+1}/r_t$ as the user cost of land. In this case (verified in the Appendix), the real interest rate is given by:

$$\begin{aligned} r_t &= \frac{1}{q_t}\left(q_{t+1} + \gamma\left(\frac{k_t^i}{\sigma}\right)^{\sigma-1}\left(\frac{x_t^i}{1-\sigma}\right)^{1-\sigma}\right) \\ &= \gamma u_t^{-\sigma}. \end{aligned} \tag{8}$$

The real interest rate is given by the opportunity cost of unproductive agents' land holdings. Intuitively, in this equilibrium since unproductive agents' investment is positive, the rate of return of lending and investing have to be equal. Therefore, the real interest rate is determined by the opportunity cost of unproductive agents. Moreover, the evolution of wealth of unproductive agents is given by:

$$w_{t+1}^i = \gamma u_t^{-\sigma} \beta w_t^i \ \forall i \in u. \tag{9}$$

This means that the wealth of unproductive agents next period is equal to their lending and investing this period, which equals a fraction β of current wealth, times the rate of return of their lending and investing, given by the equilibrium real interest rate.

Productive agents. Now we turn to the choices of productive agents. In this type of equilibrium, the rate of return on investment of productive agents exceeds the real interest rate, therefore productive agents borrow up to the credit limit given by:

$$b_{t+1}^i = \theta q_{t+1} k_t^i \ \forall i \in p. \tag{10}$$

In this equilibrium type, the optimal investment-to-land ratio of productive agents:

$$\frac{x_t^i}{k_t^i} = g\left(\omega_t\right) \ \forall i \in p, \tag{11}$$

where $g\left(\omega_t\right)$ is defined implicitly by:

$$\frac{\sigma}{1-\sigma} g\left(\omega_t\right) + q_{t+1}\frac{1-\theta}{\alpha}\left(\frac{\sigma}{1-\sigma} g\left(\omega_t\right)\right)^{\sigma} - \left(q_t - \theta\frac{q_{t+1}}{r_t}\right) = 0 \tag{12}$$

where ω_t stands for $(q_{t+1}, q_t, r_t; \alpha, \sigma, \theta)$. See the Appendix for the derivation.

We can write the investment and land holdings of productive agents as a proportion of their wealth by using the borrowing constraint (10), the flow of funds constraint (4), and the optimal consumption choice (5):

$$x_t^i + \left(\theta u_t + \left(1-\theta\right) q_{t+1}\right) k_t^i = \beta w_t^i. \tag{13}$$

Note that term in front of land is the user cost of capital plus an adjustment term which is positive for values of $\theta > 1$ which represents the additional wedge introduced by financial development. Using the previous equation and equation (11), we get the investment and land in terms of their current wealth and the user cost of capital:

$$k_t^i = \frac{1}{g(\omega_t) + (\theta u_t + (1-\theta) q_t)} \beta w_t^i, \tag{14}$$

$$x_t^i = \frac{g(\omega_t)}{g(\omega_t) + (\theta u_t + (1-\theta) q_t)} \beta w_t^i. \tag{15}$$

The evolution of wealth of productive agents is given by:

$$w_{t+1}^i = h(\omega_t) \beta w_t^i \tag{16}$$

where

$$h(\omega_t) = \alpha \left(\frac{\sigma}{1-\sigma} g(\omega_t) \right)^{-\sigma}, \tag{17}$$

and $g(\omega_t)$ is defined in equation (12). This means that the wealth of productive agents next period is equal to the non-consumed fraction of today's wealth times the function $h(\cdot)$, which is decreasing in the investment-to-land ratio of productive agents (see the Appendix).

Finally, in equilibrium the land market, the goods market, and credit markets must clear. The perfect-foresight equilibrium type I at time t given state variables (W_t, s_t, q_t) is described by a sequence of aggregate wealth W_t, productive agents' wealth ratio s_t, land prices q_t, and user cost of land u_t, $\{W_{t+1},\ s_{t+1},\ q_{t+1},\ u_t \mid t = 0,\ 1,\ 2, ..\}$, satisfying the following dynamic equations:

1. Aggregate wealth evolution:

$$\begin{aligned} W_{t+1} &\equiv \int_{i \in p} w_{t+1}^i di + \int_{i \in u} w_{t+1}^i di \\ &= (h(\omega_t) s_t + r_t (1 - s_t)) \beta W_t. \end{aligned} \tag{18}$$

   where r_t is defined in equation (8) and $h(\omega_t)$ is defined in equation (17).

2. Evolution of the productive agents' share of aggregate wealth:

$$\begin{aligned} s_{t+1} &\equiv \frac{\int_{i \in p} w_{t+1}^i di}{W_{t+1}} \\ &= \frac{(1-\delta) h(\omega_t) s_t + n\delta r_t (1 - s_t)}{h(\omega_t) s_t + r_t (1 - s_t)}. \end{aligned} \tag{19}$$

3. Land price evolution:

$$q_{t+1} = r_t (q_t - u_t). \tag{20}$$

4. Aggregate wealth W can be expressed as:

$$\beta W_t = \frac{1-\sigma}{\sigma} u_t + \frac{g(\omega_t) - \frac{1-\sigma}{\sigma} u_t}{g(\omega_t) + \left(q_t - \theta \frac{q_{t+1}}{r_t} \right)} s_t \beta W_t + q_t. \tag{21}$$

2.3.2 Equilibrium Type II: Constrained Productive Agents and Zero Investment of Unproductive Agents

The second type of equilibrium, which we call type II, is the one in which the borrowing constraint is binding for productive agents and investment is zero for unproductive agents. The equilibrium type I and II only differ in the land holdings between both types of agents since all the land is owned by productive agents in each period.

As before, when productive agents are constrained, they borrow up to the credit limit and the credit market clearing condition, $-\int_{i\in u} b_t^i di = \int_{i\in p} b_t^i di$, together with the unproductive agents' budget constraint determine the amount of land held by unproductive agents:

$$\left(q_t + \frac{1-\sigma}{\sigma}\right)\int_{i\in u} k_t^i = \beta\left(1-s_t\right)W_t - \int_{i\in p}\frac{\theta q_{t+1}k_t^i}{r_t}di, \tag{22}$$

which is a non-negative. Hence, for the equilibrium described in the previous subsection to be the prevailing one, the right hand side of equation (22) must be non-negative when the interest rate is the one defined in equation (8).

If this is not the case, then productive agents hold all the land and, using equation (22), the interest rate is equal to

$$r_t = \frac{\theta q_{t+1}}{\beta\left(1-s_t\right)W_t}. \tag{23}$$

Note importantly that this does not imply that the borrowing constraint of productive agents is not binding and that productive agents can freely choose their optimal amount of borrowing. In particular, the ratio x_t/k_t is lower than in the unconstrained equilibrium case, since productive agents cannot buy as much intermediate good as it would be optimal for them.

As before, the perfect-foresight type II equilibrium is described by a sequence of aggregate wealth W_t, and productive agents' wealth ratio s_t, land prices q_t, and user cost of capital u_t, $\{W_{t+1},\ s_{t+1},\ q_{t+1},\ u_t \mid t = 0,\ 1,\ 2, ..\}$. It satisfies the same dynamic equations of equilibrium I (18)-(21) with the difference that the interest rate is given by (23) and the definition of aggregate wealth:

$$\beta W_t = g_t(\omega_t) + q_t. \tag{24}$$

2.3.3 Equilibrium Type III: Unconstrained Productive Agents

The third equilibrium type, which we call type III, occurs when the borrowing constraint of productive agents is not binding and the non-negativity constraints of unproductive agents are binding. In this type of equilibrium productive agents can achieve a higher rate of return and the real interest rate is equal to their opportunity cost of land. If the parameters of the model and the wealth distribution are such that the borrowing constraint does not bind, the land holdings of unproductive agents is zero, therefore do not undertake any investment. For productive agents, the first order conditions are the same as the ones for unproductive agents in equilibrium type II, which imply an investment-to-land ratio of productive agents:

$$\frac{x_t^i}{k_t^i} = \frac{1-\sigma}{\sigma} u_t \; \forall i \in p, \tag{25}$$

where the real interest rate is

$$r_t = \alpha u_t^{-\sigma}. \tag{26}$$

Notice the slight difference in the equilibrium condition where the interest rate reflects the productivity level of productive agents, α.

As before, the perfect-foresight type III equilibrium is described by a sequence of aggregate wealth W_t, productive agents' wealth ratio s_t, land prices q_t, and user cost of land u_t, $\{W_{t+1},\ s_{t+1},\ q_{t+1},\ u_t \mid t = 0,\ 1,\ 2, ..\}$ which satisfy the following dynamic system of equations:

1. Aggregate wealth is equal to:
$$W_t = \frac{\alpha}{\sigma} u_{t-1}^{1-\sigma} + q_t. \tag{27}$$

2. Aggregate wealth evolution:
$$W_{t+1} = r_t \beta W_t. \tag{28}$$
where r_t is defined in (26).

3. Evolution of the productive agents' share of aggregate wealth:
$$s_{t+1} = (1-\delta)\, s_t + n\delta\,(1-s_t)\,. \tag{29}$$

2.3.4 Equilibrium types bounds

Here we examine the boundaries between the three types of equilibria. If we keep all the parameters constant except the borrowing constraint parameter θ, we can think of two equilibrium thresholds θ_t^L and θ_t^H such that

$$\text{if } \theta \begin{cases} < \theta_t^L & \text{the period-I equilibrium type is } I \\ \in \left(\theta_t^L, \theta_t^H\right) & \text{the period-I equilibrium type is } II \\ > \theta_t^H & \text{the period-I equilibrium type is } III \end{cases}, \tag{30}$$

where θ_t^L denotes the lower bound of equilibrium type II and θ_t^H denotes the upper bound of equilibrium type II.

Equilibrium type II lower bound. Equilibrium type I is feasible if the solution to equations (21) - (20) in section 2.3.1 is such that productive agents do not own all the productive inputs and, consequently, unproductive agents produce positive output. For equilibrium II to be feasible, on the other hand, it must be that unproductive agents optimally choose to lend their saving to productive agents instead of investing in land, which is the case when $r_{t,II} \geq \gamma u_{t,II}^{-\sigma}$.

Thus, when θ is such that $r_{t,II} < \gamma u_{t,II}^{-\sigma}$ equilibrium type II is unfeasible. Let θ_t^L denote the lower bound of equilibrium II, which must satisfy

$$r_{t,II}\left(\theta_t^L\right) = \gamma u_{t,II}\left(\theta_t^L\right)^{-\sigma}. \tag{31}$$

Using equations (12), (20), (23), and (24), we can rewrite (31) as

$$\frac{\theta_t^L}{1-\theta_t^L}\frac{\alpha\left[\frac{\sigma}{1-\sigma}\left[\beta W_t - q_t\right]\right]^{-\sigma}}{\left(1-s_t\right)\beta W_t}\left[\frac{1}{1-\sigma}q_t - \beta W_t\left[\left(1-s_t\right)+\frac{\sigma}{1-\sigma}\right]\right]\left[q_t - \frac{\beta}{\theta_t^L}\left(1-s_t\right)W_t\right]^{\sigma} = \gamma, \tag{32}$$

as we explain in detail in the Appendix.

Equilibrium type III lower bound. Equilibrium type III is feasible if the solution to equations (27) - (29) together with equation (20) is such that productive agents hold all the land and their borrowing constraints are not binding, so that $\int_{i\in p} b_{t+1}^i di \leq \theta q_{t+1}$. Hence, the lower bound of equilibrium type III is the one determined by the following equation:

$$\int_{i\in p} b_{t+1}^i\left(\theta_t^H\right)di = \theta q_{t+1}\left(\theta_t^H\right). \tag{33}$$

If we then use equations (3), (25), (20), and (24), we get that the lower bound of equilibrium III is equal to

$$\theta_t^H = \frac{\left(1-\sigma\right)\beta\left(1-s_t\right)W_t}{q_t - \sigma\beta W_t}, \tag{34}$$

as explained in the appendix. Note that if $\theta_t^L \leq \theta_t^H$, there is no coexistence of equilibrium types at a given point in time.

2.4 Steady State

In this section we summarize the relationship between the three types of equilibria and the level of financial development.

Let ω now represent the vector of parameters $(\beta, \alpha, \gamma, \delta, n)$. We focus on θ separately as it is the source of variation we are mostly interested in. Let $\underline{\theta}$ be the threshold between the equilibria I and II, and let $\bar{\theta}$ be the threshold between II and III. Note these thresholds may be different from the ones discussed before. Table 1 summarizes the main equilibrium quantities for each steady state. The most important is the real interest rate which for steady state type I is given by the opportunity cost of land of unproductive agents. For $\underline{\theta} < \theta < \bar{\theta}$, the real interest rate depends on the level of financial development and the share of wealth of productive agents. Finally for $\theta > \bar{\theta}$ the real interest rate is given by the opportunity cost of land of productive agents.

From equation (**??**) we can write the bounds $\underline{\theta}$ and $\bar{\theta}$:

$$\frac{\underline{\theta} q^{II}\left(\omega,\underline{\theta}\right)}{\beta\left[1-s^{II}\left(\omega,\underline{\theta}\right)\right]W^{II}\left(\omega,\underline{\theta}\right)} = \gamma\left[u^{II}\left(\omega,\underline{\theta}\right)\right]^{-\sigma} \tag{35}$$

Table 1: Description of the Steady State Equilibria

	I	II	III
Interest rate	$r = \gamma u^{-\sigma}$	$r = \frac{\theta q}{\beta(1-s)W}$	$r = \alpha u^{-\sigma}$
Land holdings	Interior solution	$K^U = 0$,$K^P = 1$	$K^U = 0$,$K^P = 1$
s	$s \in [n\delta, 1-\delta]$	$s \in (1-\delta, \bar{s})$	$s \in [\bar{s}, 1)$
θ	$[0, \underline{\theta}]$ with $\underline{\theta} > 1$	$\left(\underline{\theta}, \bar{\theta}\right)$	$\left[\bar{\theta}, \infty\right)$
Description	Constrained/Interior borrowing	Constrained/All land	Unconstrained

and

$$\frac{\bar{\theta} q^{II}\left(\omega, \bar{\theta}\right)}{\beta\left[1 - s^{II}\left(\omega, \bar{\theta}\right)\right] W^{II}\left(\omega, \bar{\theta}\right)} = \alpha\left[u^{II}\left(\omega, \bar{\theta}\right)\right]^{-\sigma}, \tag{36}$$

where $\underline{\omega}$ and $\bar{\omega}$ are parameter values consistent with each type of equilibria.

The threshold between type II and type III equilibria can be derived from the borrowing constraint. In the steady state type III, productive agents are unconstrained, therefore their borrowing is given by:

$$b_t^i \leq \theta q_{t+1} k_t^i,$$

for any t. At the same time, unproductive agents lend all their non-consumed wealth to productive agents, which implies:

$$-\int_{i \in u} b_{t+1}^i di = r_t\left(1 - s_t\right)\beta W_t,$$

and productive agents hold all the land, $\int_{i \in p} k_t^i di = 1$. Thus, in the type III steady state it must be the case that the borrowing of productive agents (which equals lending of unproductive agents by market-clearing conditions of credit markets) is smaller or equal to the total value of their collateral:

$$\int_{i \in p} b_{t+1}^i di \leq \theta q_{t+1} \int_{i \in p} k_t^i di.$$

Using the previous two conditions we can write:

$$r^{III}\left(1 - s^{III}\right)\beta W^{III} \leq \theta q^{III},$$

which implies with equality the solution for the upper threshold $\bar{\theta}$:

$$\bar{\theta} = \frac{\delta}{\delta + n\delta}\left(1 + \frac{1-\beta}{\beta\sigma}\right). \tag{37}$$

Note that $\underline{\theta}$ has no closed form solution.

3 Numerical results

3.1 Parametrization

To understand the different equilibria we plot the regions for a range of parameters using the following parameter values. For the share of land in the production function we set $\sigma = 0.5$. The discount factor

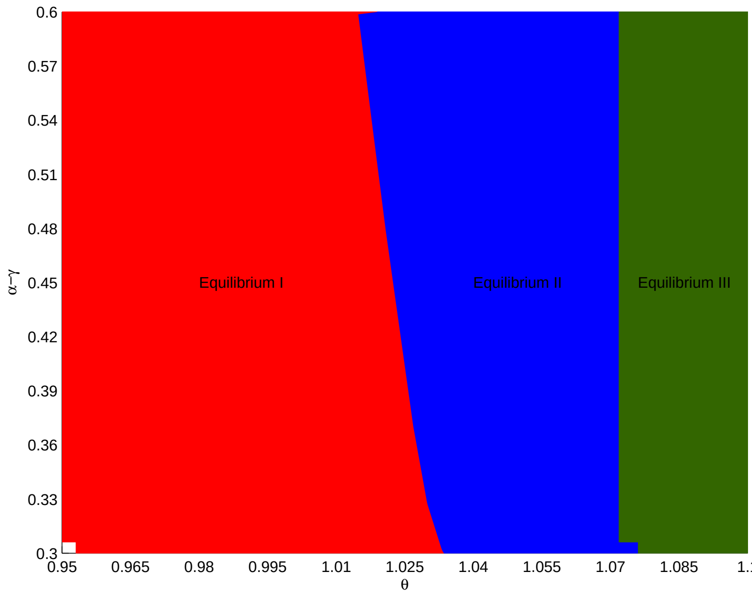

Figure 1: Regions of each type of equilibria as a function of $\alpha - \gamma$ given θ. This plot maps the corresponding combination to θ and $\alpha - \gamma$ to the type of equilibria with the rest of the parameters calibrated as in Table 2. The upper bound between regions II and III is given by equation (37) which does not depend on $\alpha - \gamma$. The boundary between regions I and II occurs at a higher level of θ when the difference between the productivity levels of each type of agent is smaller.

Table 2: Summary of the parameter values of the calibration

Parameter	Value
β	0.95
σ	0.5
δ	0.7
n	0.03

is set to $\beta = .95$. For the Markov process we set the following values: $\delta = 0.7$ for the probability that a productive agent becomes unproductive, and $n = .03$ which is the ratio of productive to unproductive agents. Note that $n\delta$ is the probability that an unproductive agent becomes productive next period and that $n\delta + \delta < 1$. Figure 1 shows the different equilibria for a range of feasible values of θ and the productivity difference between productive and unproductive agents, $\alpha - \gamma$. Notice first that the upper bound between regions II and III does not depend on productivity, $\alpha - \gamma$. However the boundary between regions I and II does depend on productivity differences and occurs at a higher level of θ when the productivity difference between each type of agent is smaller.

3.2 Dynamics

This section describes the exercise to evaluate the transition dynamics after an unexpected aggregate productivity shock. Figures 2 and 3 summarize the dynamics of different aggregate quantities after an unexpected shock of 1% to aggregate productivity for five different values of $\theta = [.75, .95, 1, 1.04, 1.12]$. These values were chosen so that given our parametrization of α and γ, the first three values of θ lie in region I, the fourth in region II and the last in region III. When the borrowing constraint parameter is below the benchmark level the computation of the dynamics are straightforward but when the parameter is above this level, the computation is more involved. For simplicity, we choose paths in which the equilibrium at every point in the transition is of the type achieved in steady state. We describe in detail the particular algorithm to compute the dynamics in the Appendix. We do not rule out other potentially convergent paths with cycles or jumps between the three types of equilibria.

We are interested in showing the non-monotonic relationship between the transition dynamics and the particular value of θ. Figure 2 shows the dynamics of the land price, q_t, and the dynamics of the real interest rate, r_t. On impact, the land price falls more when θ is larger if the equilibrium is in region I. In fact, the largest potential fall in the land price in our model occurs when the equilibria is around the threshold value $\underline{\theta}$, between equilibria I and II. For values of θ that lie in the region of equilibria III, with $\theta > \bar{\theta}$, the change in the land price on impact is smaller the larger the value of θ. This reversion of the effect on land price is consistent with our idea that financial constraints have non-monotonic effects. Recall that the lager the value of θ the steady state level of wealth is higher. Intuitively when productive agents are unconstrained, shocks to productivity reduce the value of land through the reduction in output but do not interact with the borrowing constraint through the value of their collateral.

The mechanism is the following. Productivity shocks reduce the value of the collateral resulting in a reduced level of borrowing. In the case of high level of θ borrowing is reduced more. With less borrowing, productive agents buy less land. Land is used more inefficiently. Its value is further depressed. On the other hand, a higher level of θ accelerates the recovery after a productivity shock. In other words, the higher the financial development, the more productive agents can borrow as their collateral gains value. This speeds recovery and a more efficient use of the collateral over time. The combination of these two effects provides the non-monotonicity of the dynamics with θ. A similar mechanism has been explored in a production environment by Tomura (2011).

Figure 3 shows the dynamics of the investment-to-land ratio of both types of agents, the share of wealth of productive agents and output. Output falls on impact 1% for the four paths as a response to the fall on productivity but clearly recovery occurs at different rates. For the two extreme values of θ recovery is almost at an identical pace, but recall that they converge to different absolute values of output and wealth. For the benchmark value $\theta = 1$ the recovery of output is the slowest of the four equilibria. The recovery path of output of equilibria II falls further after period 1 but recovers faster than the benchmark path. This fall is due to the reallocation of investment between productive and unproductive agents. When productive agents hold all the land but are still restricted a shock

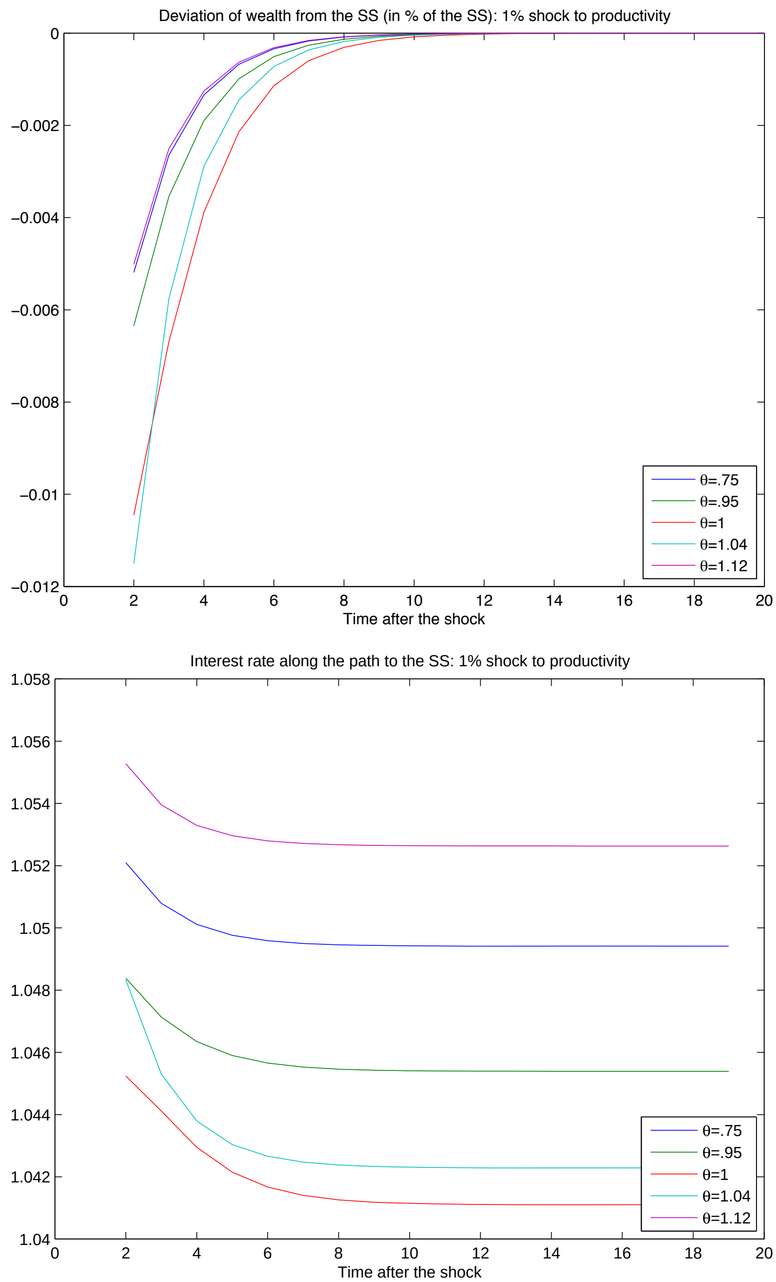

Figure 2: Dynamics towards the steady state after a 1% unexpected shock to the aggregate productivity level. This plot shows that the fall in percentage terms from the steady state is larger at impact for larger θ but the recovery is faster. Notice that the steady state value of wealth is higher for a larger value of θ. The right panel shows the level of real interest rate for the different paths. Frequency is quarterly.

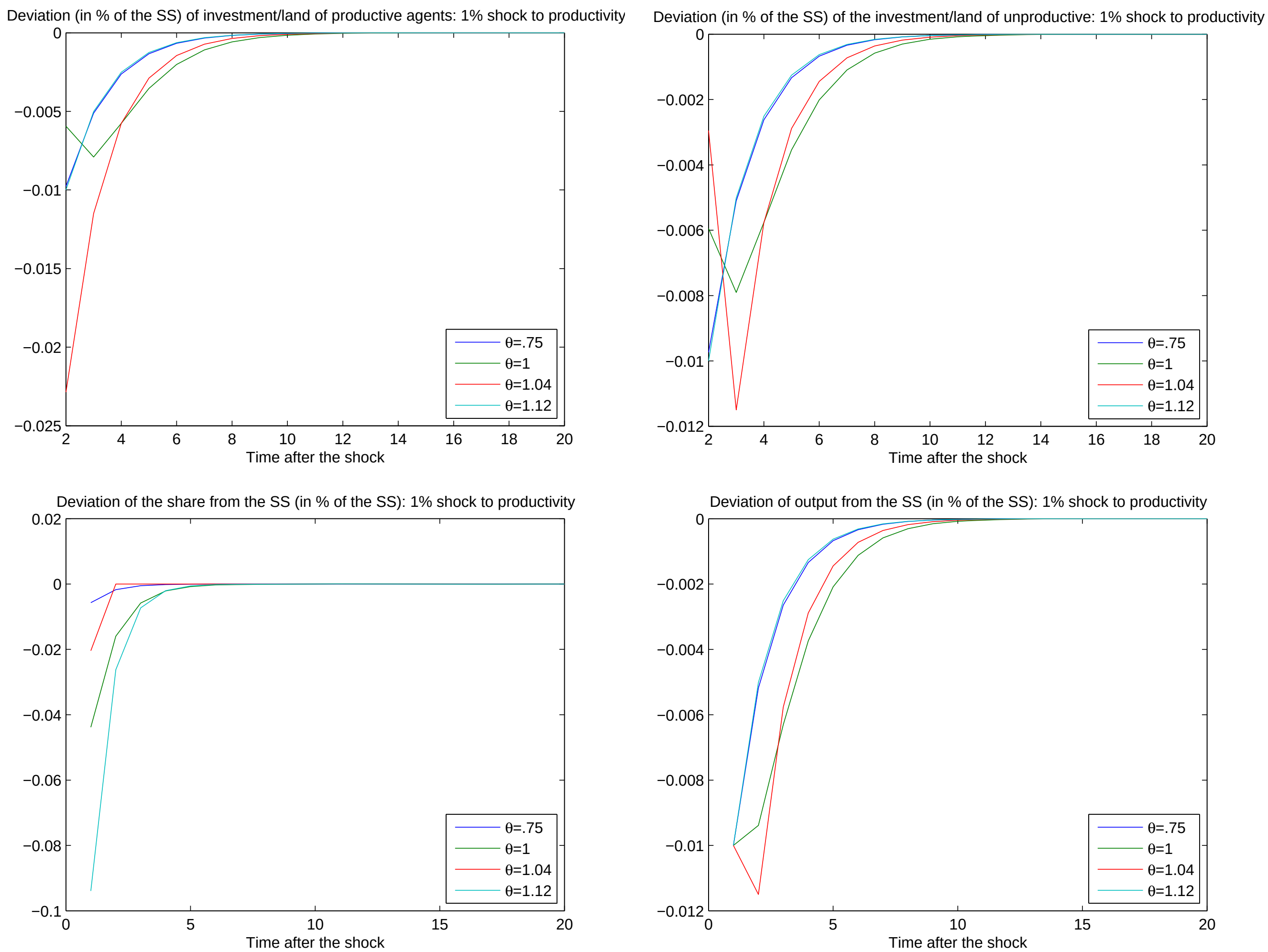

Figure 3: Transition dynamics towards the steady state of the investment-to-land ratio, share of wealth of productive agents, and total output after a 1% unexpected shock to the aggregate productivity level. Frequency is quarterly.

to the aggregate productivity has the effect of reducing the next period collateral value of land. This reallocation can be seen in the top two panels of the figure.

Our model maintains the amplification mechanism observed in other models of endogenous collateral constraints. When the equilibrium is in I, a higher θ amplifies the effect of the shock in terms of current wealth. On the other hand, when the equilibrium is in region III, a higher value of the parameter *dampens* the effect of the shock. One interpretation of this effect is the negative effects on growth from financial liberalization. Empirically, Beck et al. (2000) found that financial liberalization in economies with underdeveloped financial system usually is followed by financial crises. Developing the financial sector when it is underdeveloped amplifies the effects of real shocks in the economy. On the other hand, a highly developed financial sector dampens the effect of real shocks.

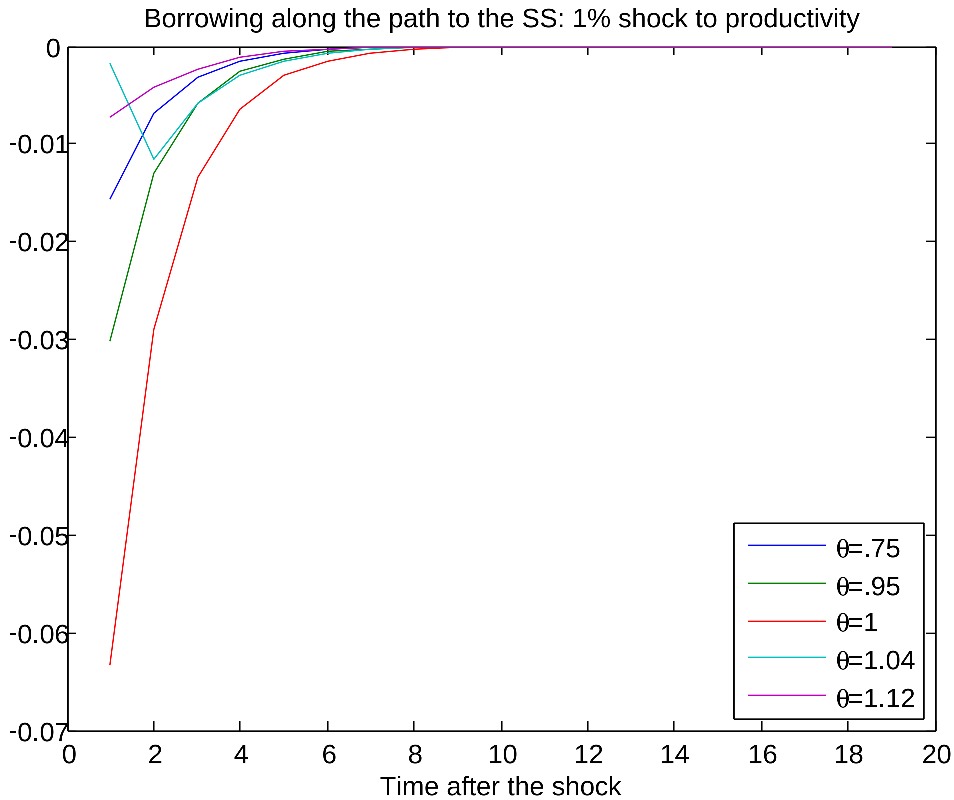

Figure 4: Transition dynamics towards the steady state of the total amount of borrowing after a 1% unexpected shock to the aggregate productivity level for different values of financial development. This plot shows that the size of the financial sector, measured by the total borrowing and lending, falls the most relative to the steady state value when $\theta = 1$ after an aggregate productivity shock. Farther from this value, the response is smaller. Frequency is quarterly.

4 Concluding remarks

This paper develops a model of endogenous borrowing constraints to analyze the effects to income volatility from the variation of the borrowing constraints. One key difference in our model from previous work with borrowing constraints is that we allow for lending to occur in equilibrium when the value of the collateral is smaller than the value of the loan. We do this by introducing a parameter of financial development in the borrowing constraint of agents that expands the borrowing capacity of productive agents. We show that there are three types of equilibria in our model corresponding to three regions of the parameter of financial development. The first is the well-known equilibrium were the economy faces aggregate borrowing constraints, reduced output and lower level of wealth for the lower levels of financial development. The second type of equilibrium in the second region of the financial development parameter, productive agents hold all the capital available yet they remain credit constrained and the economy achieves a higher steady state level of wealth than in the first one. The third region, with higher values of the parameter of financial development, the third type of equilibrium occurs in which the borrowing constraints are not binding.

With this model we analyze the volatility of income in each type of equilibrium. The financial sector can amplify the technological component of business cycle fluctuations through the effects of the endogenous borrowing constraints as in the traditional models of endogenous collateral constraints. We show in addition that the financial sector can also dampen business cycle fluctuations in excess of the technological shock when the equilibria is of the third type as in the Arrow-Debreu complete markets equilibrium. Under this interpretation the financial sector has an intermediate level of development in which it introduces excess volatility.

We parametrize the model to analyze the transition dynamics in each of the three regions in our model after a unexpected aggregate productivity shock. The well-know amplification channel of Kiyotaki and Moore (1997) remains in our model for equilibria of the first type. In the middle region of financial development the impact to output is further amplified but its recovery rate is faster than on the first type of equilibrium. Finally, in the last region, the financial sector does not amplify the shock and its rate of recovery is constrained simply by the recovery allowed by the aggregate technology.

In future work we will extend our model to allow capital accumulation to explore the tradeoff of the short-run costs of income volatility and long-run effects on income growth induced by the size of the financial sector. As argued by Barlevy (2004), the potential costs of business cycles under endogenous growth may be substantially higher than the ones computed by Lucas (1990) among others. Another related avenue we plan to explore is to analyze the relationship between the likelihood and severity of shocks with the level of the parameter θ to evaluate potential optimal choices. Finally, an important extension we will explore is to endogenize θ to recognize that changes in the level of financial development are a costly choice and to address the relationship between the the rate of growth of financial aggregates that is associated with sharp subsequent contractions in income.

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A Derivation of the equilibrium conditions

At each period, agents choose the amount of consumption good c_t, debt to be repaid next period b_{t+1}, intermediate good, x_t, and land, k_t, for next period's output to maximize their expected intertemporal utility. Their optimization is subject to the budget constraint in equation (3), the borrowing constraint in equation (4) and the non-negative production input constraints $k_t^i \geqslant 0$ and $x_t^i \geqslant 0$.[1] The agents' optimization problem is

$$\max_{\{c_t, x_t, k_t, b_{t+1}\}} \mathbb{E}_t \left(\sum_{t=0}^{\infty} \beta^t \ln c_t \right)$$

subject to

$$c_t + x_t + q_t \left(k_t - k_{t-1}\right) = y_t + \frac{b_{t+1}}{r_t} - b_t$$

$$b_{t+1} \leq \theta q_{t+1} k_t$$

$$c_t \geq 0, \; x_t \geq 0, \; k_t \geq 0.$$

Let μ_t^i and λ_t^i be the multipliers on the flow-of-funds and borrowing constraints of agent i at period t, and let η_t^i and ψ_t^i be the multipliers of the non-negative input constraints of an arbitrary agent i. The first order conditions from the utility maximization problem are the following four equations:

$$c_t^i : \; \frac{\beta^t}{c_t^i} - \mu_t^i \quad = \quad 0$$

[1] To be rigorous, consumption must also be positive but this constraint is not included in the optimization problem.

$$k_t^i: \; -\mu_t^i q_t + \mu_{t+1}^i \frac{\partial f^i\left(k_t^i, x_t^i\right)}{\partial k_t} + \mu_{t+1}^i q_{t+1} + \lambda_t^i \theta q_{t+1} + \eta_t^i = 0$$

$$x_t^i: \; -\mu_t^i + \mu_{t+1}^i \frac{\partial f^i\left(k_t^i, x_t^i\right)}{\partial x_t} + \psi_t^i = 0$$

$$b_{t+1}^i: \; \frac{\mu_t^i}{r_t} - \mu_{t+1}^i - \lambda_t^i = 0$$

Depending on the parameters of the model and the wealth distribution, the constraints may or may not be binding, so the multipliers may or may not be equal to zero. This gives rise to different types of equilibrium, as we analyze next. We say the equilibrium is type I if the borrowing constraint for productive agents is binding but the non-negative production constraints for unproductive agents are not binding; type II if the constraints are all binding, and type III if the non-negative input constraints are binding for unproductive agents but the borrowing constraint is not.

A.1 Equilibrium type I

A.1.1 Consumption policy function

For unproductive agents, using the first order conditions with respect to c_t^i and b_{t+1}^i when $\lambda_t^i = \psi_t^i = \eta_t^i = 0$, we get their the Euler equation when the equilibrium type is I:

$$c_{t+1}^i = \beta r_t c_t^i.$$

And using then the first order conditions with respect to k_t^i, x_t^i and b_{t+1}^i when $\lambda_t^i = \psi_t^i = \eta_t^i = 0$ together with equations (13), (6) and (20), we get equations (8) and (7), as well as the flow of funds of unproductive agents when the equilibrium type is I:

$$w_{t+1}^i = r_t \left(w_t^i - c_t^i\right).$$

It is then possible show that the policy function $c_t^i = (1-\beta)\, w_t^i$ is consistent with these two conditions that summarize the optimal choices of unproductive agents when the equilibrium type is I. When the equilibrium type is II or III, the Euler equation of unproductive agents is exactly the same. The flow of funds also looks identical, given that they do not hold any land or investment good, so their wealth next period is equal to their current non-consumed wealth times the interest rate paid on savings.

For productive agents, using the first order conditions with respect to c_t^i and x_t^i when $\psi_t^i = \eta_t^i = 0$, together with equation (1), we get their Euler equation when the equilibrium type is I:

$$c_{t+1}^i = \beta\alpha \left(\frac{\sigma}{1-\sigma} \frac{x_t^i}{k_t^i}\right)^{-\sigma} c_t^i.$$

And using the first order conditions with respect to k_t^i, x_t^i and b_{t+1}^i when $\psi_t^i = \eta_t^i = 0$ together with equations (4), (4) and (20) we get the flow of funds of productive agents when the equilibrium type is I:

$$w_{t+1}^i = \alpha \left(\frac{\sigma}{1-\sigma} \frac{x_t^i}{k_t^i} \right)^{-\sigma} \left(w_t^i - c_t^i \right).$$

As before, it is straightforward to show that the policy function $c_t^i = (1-\beta) w_t^i$ is consistent with these two conditions summarizing the optimal choices of productive agents in equilibrium type I. When the equilibrium is of type II, the Euler equation and flow of funds of productive agents are exactly the same as in type I, so their optimal policy function is also $c_t^i = (1-\beta) w_t^i$. When the equilibrium is of type III, productive agents are no longer constrained, so their Euler equation and flow of funds are the same as the ones of unproductive agents as well as the optimal policy function of consumption.

A.1.2 Wealth evolution of unproductive agents

Using the first order conditions with respect to k_t^i, x_t^i and b_{t+1}^i when $\lambda_t^i = \psi_t^i = \eta_t^i = 0$, we obtain equation (7) shown in the text. If we then use equation (7) together with the first order conditions with respect to x_t^i and b_{t+1}^i we get the interest rate shown in equation (8).

Using equations (2) and (7), we can rewrite next period production as:

$$y_{t+1}^i = \gamma u_t^{1-\sigma} \left(\frac{k_t^i}{\sigma} \right). \tag{38}$$

Moreover, from the budget constraint in equation (3), and using equations (5) and (6), b_{t+1}^i can be written in the following form:

$$\begin{aligned} b_{t+1}^i &= r_t c_t^i + r_t x_t^i + r_t q_t \left(k_t^i - k_{t-1}^i \right) - r_t y_t^i + r_t b_t^i \\ &= r_t w_t^i + r_t (x_t^i + q_t k_t^i - \beta w_t^i) - r_t (y_t^i + q_t k_{t-1}^i - b_t^i) \\ &= r_t (x_t^i + q_t k_t^i - \beta w_t^i). \end{aligned}$$

Then, using this last result and the evolution of q_{t+1} in equation (20), we can rewrite next period wealth as

$$\begin{aligned} w_{t+1}^i &= y_{t+1}^i + r_t (q_t - u_t) k_t^i - r_t \left(x_t^i + q_t k_t^i - \beta w_t^i \right) \\ &= y_{t+1}^i - r_t u_t k_t^i - r_t x_t^i + r_t \beta w_t^i \end{aligned}$$

Finally, using this last equation together with equation (38), the real interest rate in equation (8), the optimal investment-to-land ratio of unproductive agents in equation (7) in the next wealth equation, we get:

$$\begin{aligned} w_{t+1}^i &= \gamma u_t^{1-\sigma} \left(\frac{k_t^i}{\sigma} \right) - \gamma u_t^{1-\sigma} k_t^i - \gamma \frac{1-\sigma}{\sigma} u_t^{1-\sigma} k_t^i + \gamma u_t^{-\sigma} \beta w_t^i, \\ &= \gamma u_t^{-\sigma} \beta w_t^i, \end{aligned}$$

which is equation (9) in the text.

A.1.3 Derivation of $g(\omega_t)$

Using the first order condition with respect to k_t and b_{t+1} when $\psi_t^i = \eta_t^i = 0$, dividing by μ_t and replacing the Lagrange multiplier λ_t, we obtain:

$$\frac{\mu_{t+1}}{\mu_t}\frac{\alpha}{\sigma}\left(\frac{k_t^i}{\sigma}\right)^{\sigma-1}\left(\frac{x_t^i}{1-\sigma}\right)^{1-\sigma} = \frac{\mu_{t+1}}{\mu_t}q_{t+1}\theta + q_t - \theta\frac{q_{t+1}}{r_t}.$$

Now, replace μ_{t+1}/μ_t from the first order condition of x_t

$$\frac{x_t}{k_t}\frac{\sigma}{1-\sigma} = (1-\theta)\left[\alpha\left(\frac{k_t}{\sigma}\right)^{\sigma}\left(\frac{x_t}{1-\sigma}\right)^{-\sigma}\right]^{-1} + q_t - \theta\frac{q_{t+1}}{r_t},$$

which can be rewritten as

$$\frac{x_t}{k_t}\frac{\sigma}{1-\sigma} + \frac{1-\theta}{\alpha}q_{t+1}\left(\frac{x_t}{k_t}\frac{\sigma}{1-\sigma}\right)^{\sigma} - q_t + \theta\frac{q_{t+1}}{r_t} = 0.$$

Using our definition of the investment-to-land ratio, it yields the equation that determines the optimal investment-land ratio of productive agents:[2]

$$\frac{\sigma}{1-\sigma}g(\omega_t) + q_{t+1}\frac{1-\theta}{\alpha}\left(\frac{\sigma}{1-\sigma}g\left(\omega_t\right)\right)^{\sigma} - q_t + \theta\frac{q_{t+1}}{r_t} = 0.$$

A.1.4 Wealth evolution of productive agents

Following a similar strategy as for unproductive agents, we can get the wealth dynamics for productive agents deriving the factor demand function and using the assumption of the binding borrowing constraint.

Start by replacing the borrowing constraint in equation (4) in the budget constraint in equation (3) and the definition of wealth in equation (6) to obtain:

$$c_t^i + x_t^i + q_t k_t^i = w_t^i + \frac{\theta q_{t+1} k_t^i}{r_t}.$$

If we then replace the optimal consumption rule from equation (5) and rearrange, we obtain:

$$x_t^i + k_t^i\left(q_t + \frac{\theta q_{t+1}}{r_t}\right) = \beta w_t^i.$$

Equations (15) and (14) follow easily using the optimal investment-to-land ratio of productive agents in (11):

$$k_t^i = \frac{1}{g\left(\omega_t\right) + \left(\theta u_t + \left(1-\theta\right)q_t\right)}\beta w_t^i,$$

$$x_t^i = \frac{g\left(\omega_t\right)}{g\left(\omega_t\right) + \left(\theta u_t + \left(1-\theta\right)q_t\right)}\beta w_t^i.$$

[2]Note that a solution to this equation exists and is unique when $\theta \leq 1$ because the LHS of the previous equation strictly increases with $g(\omega_t)$.

Finally using the production function of productive agents in equation (1), we can rewrite next period output as:

$$y_{t+1}^i = \frac{\frac{\alpha}{\sigma^\sigma(1-\sigma)^{1-\sigma}} g\left(\omega\right)^{1-\sigma}}{g\left(\omega\right) + \left(\theta u_t + \left(1-\theta\right) q_t\right)} \beta w_t^i.$$

Using the last result, together with real interest rate of the benchmark equilibrium in equation (8) and the investment-to-land ratio of productive agents in equations (11) and (12):

$$\begin{aligned}
w_{t+1}^i &= y_{t+1}^i + q_{t+1} k_t^i - b_{t+1} \\
&= y_{t+1}^i + \left(1-\theta\right) q_{t+1} k_t^i \\
&= \frac{\frac{\alpha}{\sigma^\sigma(1-\sigma)^{1-\sigma}} g\left(\omega\right)^{1-\sigma}}{g\left(\omega\right) + \left(\theta u_t + \left(1-\theta\right) q_t\right)} \beta w_t^i + \frac{\left(1-\theta\right) q_{t+1}}{g\left(\omega_t\right) + \left(\theta u_t + \left(1-\theta\right) q_t\right)} \beta w_t^i \\
&= \frac{\frac{\alpha}{\sigma^\sigma(1-\sigma)^{1-\sigma}} g\left(\omega\right)^{1-\sigma} + \alpha\left(\frac{\sigma}{1-\sigma} g\left(\omega_t\right)\right)^{-\sigma} \left[-\frac{\sigma}{1-\sigma} g(\omega_t) + q_t - \theta\left(q_t - u_t\right)\right]}{g\left(\omega\right) + \left(\theta u_t + \left(1-\theta\right) q_t\right)} \beta w_t^i \\
&= \alpha\left(\frac{\sigma}{1-\sigma} g\left(\omega_t\right)\right)^{-\sigma} \frac{\frac{1}{1-\sigma} g\left(\omega\right) - \frac{\sigma}{1-\sigma} g(\omega_t) + \left(\theta u_t + \left(1-\theta\right) q_t\right)}{g\left(\omega\right) + \left(\theta u_t + \left(1-\theta\right) q_t\right)} \beta w_t^i \\
&= \alpha(\frac{\sigma}{1-\sigma} g(\omega))^{-\sigma} \beta w_t^i.
\end{aligned}$$

Hence,

$$h(\omega_t) = \alpha(\frac{\sigma}{1-\sigma} g(\omega_t))^{-\sigma}.$$

A.1.5 Aggregate wealth evolution

Let s_t be the wealth share of productive agents, i.e. $s_t = \frac{\int_{i\in p} w_t^i di}{W_t}$ and $1 - s_t = \frac{\int_{i\in u} w_t^i di}{W_t}$. Using equations (16) and (9), aggregate wealth evolution can be written as

$$\begin{aligned}
W_{t+1} &\equiv \int_{i\in p} w_{t+1}^i di + \int_{i\in u} w_{t+1}^i di \\
&= \int_{i\in p} h(\omega_t) \beta w_t^i di + \int_{i\in u} \gamma u_t^{-\sigma} \beta w_t^i di \\
&= \left(h(\omega_t) s_t + \gamma u_t^{-\sigma}\left(1 - s_t\right)\right) \beta W_t,
\end{aligned}$$

which is equation (18).

A.1.6 Evolution of productive agents' wealth share

Using equations (16) and (9) and the fact that a fraction δ of productive agents at t become unproductive at $t+1$ and a fraction $n\delta$ of unproductive agents at t become productive at $t+1$,

$$\begin{aligned} s_{t+1} &\equiv \frac{\int_{i\in p} w_{t+1}^i di}{W_{t+1}} \\ &= \frac{(1-\delta)\, h(\omega_t) s_t \beta W_t + n\delta\gamma u_t^{-\sigma}(1-s_t)\,\beta W_t}{\left(h(\omega_t)s_t + \gamma u_t^{-\sigma}(1-s_t)\right)\beta W_t} \\ &= \frac{(1-\delta)\, h(\omega_t) s_t + n\delta\gamma u_t^{-\sigma}(1-s_t)}{h(\omega_t)s_t + \gamma u_t^{-\sigma}(1-s_t)}, \end{aligned}$$

which is equation (19).

A.1.7 Evolution of land price

Using the definition of u_t, we get that

$$q_{t+1} = r_t\,(q_t - u_t)\,,$$

which gives us equation (20) once we replace r_t using equation (8):

$$q_{t+1} = \gamma u_t^{-\sigma}(q_t - u_t)\,.$$

A.1.8 Aggregate Wealth

Equation (21) is derived using the definition of aggregate wealth and the land market clearing condition $\int_{i\in u} k_t^i di + \int_{i\in p} k_t^i di = 1$, as well as equations (5), (7), (11) and (14):

$$\begin{aligned} W_t &\equiv \int_i y_t^i di + q_t \int_i k_{t-1}^i di - \int_i b_t^i di \\ &= \int_i c_t^i di + \int_i x_t^i di + q_t \int_i k_t^i di \\ &= (1-\beta)\int_i w_t^i di + \int_{i\in u} x_t^i di + \int_{i\in p} x_t^i di + q_t \\ &= (1-\beta)\, W_t + \frac{1-\sigma}{\sigma} u_t \int_{i\in u} k_t^i di + g(\omega_t)\int_{i\in p} k_t^i di + q_t \\ &= (1-\beta)\, W_t + \frac{1-\sigma}{\sigma} u_t + \left(g(\omega_t) - \frac{1-\sigma}{\sigma}u_t\right)\int_{i\in p} k_t^i di + q_t \\ &= (1-\beta)\, W_t + \frac{1-\sigma}{\sigma} u_t + \frac{g(\omega_t) - \frac{1-\sigma}{\sigma}u_t}{g(\omega_t) + (\theta u_t + (1-\theta)\, q_t)}\beta s_t W_t + q_t \end{aligned}$$

$\Rightarrow$

$$\beta W_t = \frac{1-\sigma}{\sigma} u_t + \frac{g(\omega_t) - \frac{1-\sigma}{\sigma}u_t}{g(\omega_t) + \left(q_t - \theta\frac{q_{t+1}}{r_t}\right)} s_t \beta W_t + q_t,$$

which is equation (21).

A.2 Equilibrium type II

As it was the case in equilibrium type I, the credit constraint is binding for productive agents, so their optimization conditions are the same and the evolution of their wealth is also given by equation (16):

$$w_{t+1}^i = h\left(\omega_t\right)\beta w_t^i \ \forall i \in p.$$

Unproductive agents, on the other hand, do not produce anything so their wealth evolution is determined by the equilibrium interest rate

$$w_{t+1}^i = r_t \beta w_t^i \ \forall i \in u,$$

where r_t is specified in equation (23). The aggregate wealth evolution is also given by

$$W_{t+1} \quad = \quad \left(h(\omega_t)s_t + r_t\left(1-s_t\right)\right)\beta W_t,$$

which is equation (18). Likewise the evolution of productive agents' wealth share is given by

$$s_{t+1} \quad = \quad \frac{\left(1-\delta\right)h(\omega_t)s_t + n\delta r_t\left(1-s_t\right)}{h(\omega_t)s_t + \gamma r_t\left(1-s_t\right)},$$

which is equation (19).

When the equilibrium is type II, $\int_{i\in u} k_t^i di = \int_{i\in p} x_t^i di = 0$, $\int_{i\in u} k_t^i di = 1$and $\int_{i\in p} x_t^i di = g\left(\omega_t\right)$. Thus, aggregate wealth can be written as

$$\begin{aligned} W_t &\equiv \int_i y_t^i di + q_t \int_i k_{t-1}^i di - \int_i b_t^i di \\ &= \int_i c_t^i di + \int_i x_t^i di + q_t \int_i k_t^i di \\ &= \left(1-\beta\right)\int_i w_t^i di + \int_{i\in u} x_t^i di + \int_{i\in p} x_t^i di + q_t \\ &= \left(1-\beta\right)W_t + g\left(\omega_t\right) + q_t \end{aligned}$$

$\Rightarrow$

$$\beta W_t = g\left(\omega_t\right) + q_t,$$

which is equation (24).

A.3 Equilibrium type III

When the equilibrium type is III, unproductive agents do not produce anything, as it was the case in equilibrium type II. Thus, their wealth evolution is determined by the equilibrium interest rate

$$w_{t+1}^i = r_t \beta w_t^i \ \forall i \in u,$$

where r_t is specified in equation (26).

For productive agents, on the other hand, the borrowing constraint is no longer binding, so that

$$\frac{x_t^i}{k_t^i} = \frac{1-\sigma}{\sigma} u_t \ \forall i \in p,$$

as it was the case with unproductive agents in equilibrium type I. As a result, using equations (3), (5), (6), and (20)

$$\begin{aligned}
w_{t+1}^i &\equiv y_{t+1}^i + q_{t+1} k_t^i - r_t b_{t+1}^i \\
&= y_{t+1}^i + r_t \left(q_t - u_t\right) k_t^i - r_t \left(x_t^i + q_t k_t^i - \beta w_t^i\right) \\
&= \alpha u_t^{1-\sigma} \left(\frac{k_t^i}{\sigma}\right) + -\alpha u_t^{1-\sigma} k_t^i - \alpha \frac{1-\sigma}{\sigma} u_t^{1-\sigma} k_t^i + \alpha u_t^{-\sigma} \beta w_t^i \\
&= \alpha u_t^{-\sigma} \beta w_t^i \ \forall i \in p.
\end{aligned}$$

Hence, the evolution of aggregate wealth is given by

$$\begin{aligned}
W_{t+1} &\equiv \int_{i \in p} w_{t+1}^i di + \int_{i \in u} w_{t+1}^i di \\
&= \int_{i \in p} \alpha u_t^{-\sigma} \beta w_t^i di + \int_{i \in u} \alpha u_t^{-\sigma} \beta w_t^i di \\
&= \alpha u_t^{-\sigma} \beta W_t,
\end{aligned}$$

and the evolution of productive agents' wealth share is given by

$$\begin{aligned}
s_{t+1} &\equiv \frac{\int_{i \in p} w_{t+1}^i di}{W_{t+1}} \\
&= \frac{\left(1-\delta\right) \alpha u_t^{-\sigma} s_t \beta W_t + n \delta \alpha u_t^{-\sigma} \left(1-s_t\right) \beta W_t}{\alpha u_t^{-\sigma} \beta W_t} \\
&= \left(1-\delta\right) s_t + n\delta \left(1-s_t\right).
\end{aligned}$$

When the equilibrium type is III, $\int_{i \in u} k_t^i di = \int_{i \in p} x_t^i di = 0$, $\int_{i \in u} k_t^i di = 1$ and $\int_{i \in p} x_t^i di = \frac{1-\sigma}{\sigma} u_t$. Thus, aggregate wealth can be written as

$$\begin{aligned}
W_t &\equiv \int_i y_t^i di + q_t \int_i k_{t-1}^i di - \int_i b_t^i di \\
&= \int_i c_t^i di + \int_i x_t^i di + q_t \int_i k_t^i di \\
&= \left(1-\beta\right) \int_i w_t^i di + \int_{i \in u} x_t^i di + \int_{i \in p} x_t^i di + q_t \\
&= \left(1-\beta\right) W_t + \frac{1-\sigma}{\sigma} u_t + q_t
\end{aligned}$$

which implies

$$\beta W_t = \frac{1-\sigma}{\sigma} u_t + q_t.$$

A.4 Equilibrium type bounds

A.4.1 Equilibrium type II lower bound

As explained in section 2.3.4, the equilibrium II lower bound θ_t^L is defined as $r_{t,II}\left(\theta_t^L\right) u_{t,II}\left(\theta_t^L\right)^\sigma = \gamma$. Using equations (12), (20), and (24), we get that

$$u_{t,II} = \left(q_t - \frac{\beta}{\theta}\left(1 - s_t\right) W_t\right),$$

and

$$\begin{aligned} q_{t+1} &= \alpha \frac{\left(q_t - \beta\left(1 - s_t\right) W_t - \frac{\sigma}{1-\sigma} g\left(\omega_t\right)\right)}{\left(\frac{\sigma}{1-\sigma} g\left(\omega_t\right)\right)^\sigma} \\ &= \alpha \frac{\left(q_t - \beta\left(1 - s_t\right) W_t - \frac{\sigma}{1-\sigma}\left(\beta W_t - q_t\right)\right)}{\left(\frac{\sigma}{1-\sigma}\left(\beta W_t - q_t\right)\right)^\sigma}. \end{aligned}$$

Hence, using equation (23), we can rewrite the condition that defines θ_t^L as:

$$\frac{\theta_t^L}{1 - \theta_t^L} \frac{\alpha\left(q_t - \frac{\beta}{\theta_t^L}\left(1 - s_t\right) W_t\right)^\sigma \left(q_t - \beta\left(1 - s_t\right) W_t - \frac{\sigma}{1-\sigma}\left(\beta W_t - q_t\right)\right)}{\left(\frac{\sigma}{1-\sigma}\left(\beta W_t - q_t\right)\right)^\sigma \beta\left(1 - s_t\right) W_t} = \gamma.$$

A.4.2 Equilibrium type III lower bound

When equilibrium is of type III, productive agents are no longer credit constrained, so that their optimal borrowing is lower than their credit limit $\theta q_{t+1} k_t^i$, so that

$$\int_{i \in p} b_{t+1}^i di < \theta q_{t+1}.$$

The equilibrium type III lower bound θ_t^H must then satisfy equation (33). Using equations (4) and (25), we can rewrite $b_{t+1}^i = r_t\left(\beta w_t^i + k_t^i\left(q_t + \frac{1-\sigma}{\sigma} u_t\right)\right)$, which implies that

$$\int_{i \in p} b_{t+1}^i di = r_t\left(\beta s_t W_t + q_t + \frac{1 - \sigma}{\sigma} u_t\right).$$

Then, using also equation (20) we get

$$\beta s_t W_t + q_t + \frac{1 - \sigma}{\sigma} u_t\left(\theta_t^H\right) = \theta_t^H\left(q_t - u_t\left(\theta_t^H\right)\right),$$

and using also equation (27),

$$\beta\left(1 - s_t\right) W_t = \theta\left(q_t + \frac{\sigma}{1 - \sigma}\beta W_t - \frac{\sigma}{1 - \sigma} u_t\left(\theta_t^H\right)\right)$$

$\Rightarrow$

$$\theta_t^H = \frac{(1 - \sigma)\beta\left(1 - s_t\right) W_t}{q_t - \sigma\beta W_t}.$$

A.5 Steady State

A.5.1 Type I Steady State

From equations (21) - (20) together with equation (8), we can derive the conditions that define type I steady-state variables $\left(W^I, s^I, q^I, u^I\right)$:

- Aggregate wealth:

$$W = \left[\beta - \beta \frac{g(\omega) - \frac{1-\sigma}{\sigma} u}{g(\omega) + \theta u - (1-\theta) q} s\right]^{-1} \left(\frac{1-\sigma}{\sigma} u + q\right), \tag{39}$$

$$1 = \beta \left[h(\omega) s + \gamma u^{-\sigma} (1-s)\right] \tag{40}$$

where

$$h(\omega) = \alpha \left(\frac{\sigma}{1-\sigma} g(\omega)\right)^{-\sigma}$$

and

$$\frac{\sigma}{1-\sigma} g(\omega) + \frac{(1-\theta)}{\alpha} q \left(\frac{\sigma}{1-\sigma} g(\omega)\right)^{\sigma} - \theta u - (1-\theta) q = 0.$$

- Productive agents' wealth share:

$$s = \frac{(1-\delta) h(\omega) s + n\delta\gamma u^{-\sigma} (1-s)}{h(\omega) s + \gamma u^{-\sigma} (1-s)}. \tag{41}$$

- Land price:

$$q = \frac{u}{1 - \frac{u^{\sigma}}{\gamma}}. \tag{42}$$

A.5.2 Type II Steady State

From equations (21)-(20) together with equations (24) and (23), we can derive the equations that define the type II Steady-State variables $\left(W^{II}, s^{II}, q^{II}, u^{II}\right)$:

- Aggregate wealth:

$$\beta W = g(\omega) + q, \tag{43}$$

$$1 = h(\omega) s + r(1-s) \tag{44}$$

where $h(\omega)$ and $g(\omega)$ are defined above, and

$$r = \frac{\theta q}{\beta (1-s) W}.$$

- Productive agents' wealth share:

$$s = \frac{(1-\delta) h(\omega) s + n\delta r (1-s)}{h(\omega) s + r(1-s)}. \tag{45}$$

- Land price:

$$q = u + \frac{\beta (1-s) W}{\theta}. \tag{46}$$

A.5.3 Type III Steady State

From equations (27)-(29) together with equations (20) and (26), we can derive the equations that define the type III Steady-State variables $\left(W^{III}, s^{III}, q^{III}, u^{III}\right)$:

- Aggregate wealth:
$$\beta W=\left(\frac{1-\sigma}{\sigma}+\frac{1}{1-\beta}\right)(\alpha\beta)^{\frac{1}{\sigma}}. \tag{47}$$

- Productive agents' wealth share:
$$s=\frac{n}{1+n}. \tag{48}$$

- Land price:
$$q=\frac{(\alpha\beta)^{\frac{1}{\sigma}}}{1-\beta}. \tag{49}$$

A.5.4 Steady-State types thresholds

From equation (31), the threshold $\underline{\theta}$ between type I and type II steady states is determined by:

$$r^{II}\left(\underline{\theta}\right)=\gamma u^{II}\left(\underline{\theta}\right)^{-\sigma}$$

$\Leftrightarrow$

$$\frac{\underline{\theta} q^{II}\left(\underline{\theta}\right)}{\beta\left(1-s^{II}\left(\underline{\theta}\right)\right)W^{II}\left(\underline{\theta}\right)}=\gamma u^{II}\left(\underline{\theta}\right)^{-\sigma}. \tag{50}$$

Note that $\underline{\theta}$ has no closed form solution.

From equation (34), we get that the threshold $\bar{\theta}$ between type II and type III steady states is equal to

$$r^{III}\left(\bar{\theta}\right)\left(1-s^{III}\left(\bar{\theta}\right)\right)\beta W^{III}\left(\bar{\theta}\right)=\bar{\theta} q^{III}\left(\bar{\theta}\right).$$

Using equations (47) - (49), we find that

$$\bar{\theta}=\frac{1}{1+n}\left(1+\frac{1-\beta}{\beta\sigma}\right). \tag{51}$$

B Computational details

Since we only consider equilibria where there are no exploding bubbles in the land price,

$$\lim_{t\longrightarrow\infty}\frac{q_t}{r_0 r_1 \ldots r_{t-1}}=0,$$

the equilibrium path is the one that converges to the steady state, where all variables are constant. There are three types of steady states, as described in section 2, and the parameter values determine to which one the economy converges.

B.1 Model simulation

The initial conditions of the economy are given exogenously, and at each period during the transition path there are also three possible types of equilibria, as described in section 2. The state variables at each period together with the parameter values, determine which one is the prevailing equilibrium at each period.

- Simulations steps:

1. Specify value of model parameters: $(\sigma, \gamma, \beta, \delta, n; \theta)$
2. Specify value of initial conditions $\left(Y_0^j, B_{-1}^j, K_{-1}^j\right)_{j=u,p}$
   (a) initial wealth W_0 , which depends on the initial output of agents, given by the initial stocks of land, k_{-1}^i, and intermediate good, x_{-1}^i:

$$W_0 = \underbrace{\int_{i\in p} \alpha \left(\frac{k_{-1}^i}{\sigma}\right)^{\sigma} \left(\frac{x_{-1}^i}{1-\sigma}\right)^{1-\sigma} di}_{\equiv Y_0^p} + \underbrace{\int_{i\in u} \gamma \left(\frac{k_{-1}^i}{\sigma}\right)^{\sigma} \left(\frac{x_{-1}^i}{1-\sigma}\right)^{1-\sigma} di}_{\equiv Y_0^u} + q_0$$

   (b) initial s_0, which depends on the initial outputs and initial debts:

$$s_0 = \frac{(1-\delta)\left(Y_0^p + q_0 K_{-1}^p - B_0^p\right) + n\delta\left(Y_0^u + q_0 K_{-1}^u - B_0^u\right)}{W_0}.$$

3. Solve for the value of the variables of interest (W, s, q, u) in the steady state:
   (a) Solve for the steady state thresholds $\underline{\theta}$ and $\overline{\theta}$ using equations (35) and (36) or (37). Compare them with the actual θ and determine which one is the prevailing steady state.
   (b) Solve then for the steady state value of variables (q^*, W^*, s^*, u^*) using the right system of equations:
      i. If steady state type is I, use system in subsection 2.3.1.
      ii. If steady state type is II, use system in subsection 2.3.2.
      iii. If steady state type is III, use system in subsection 2.3.3.
4. Use a forward shooting algorithm to solve for the time path of the variables of interest, $\{W_t, s_t, q_t, u_t \mid t = 0, 1, 2, ..T\}$, where T is a *large enough* number:
   (a) Guess an initial value for the variable q_0, and use it to obtain values for

$$W_0 = Y_0^u + Y_0^p + q_0,$$

$$s_0 = \frac{(1-\delta)\left(Y_0^p + q_0 K_{-1}^p - B_0^p\right) + n\delta\left(Y_0^u + q_0 K_{-1}^u - B_0^u\right)}{W_0}.$$

   (b) Solve for θ_t^L and θ_t^H.

(c) For $t = 1, 2, ...T$, use the appropriate equation system to solve for (W_t, s_t, q_t, u_t) given $(W_{t-1}, s_{t-1}, q_{t-1})$:

   i. If $\theta \leq \theta_t^L$, use the system of equations in (18) - (21).

   ii. If $\theta \in \left(\theta_t^L, \theta_t^H\right)$, use the system of equations in (22) - (24).

   iii. If $\theta \geq \theta_t^H$, use the system of equations in (28) - (29).

5. Check whether (W_t, s_t, q_t, u_t) is close enough to the steady state (W^*, s^*, q^*, u^*) for some $t \in [1, T]$.

   (a) If it is, stop the code because we found a *good enough* approximation to the solution.

   (b) If it is not, go back to step 3 trying a different initial guess for q_0.