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Abstract

We provide a decomposition of nominal yields into real yields, expectations of future inflation and inflation risk premiums when real bonds or inflation swaps are unavailable or unreliable due to their relative illiquidity. We combine nominal yields with surveys of inflation forecasts within a no-arbitrage model where conditional expectations are latent but spanned by the history of the observed data, analog to a GARCH model for the conditional variance. The filtering problem is numerically trivial and we conduct a battery of out-of-sample comparisons. Our favored model matches the quarterly inflation forecasts from surveys and uses the information in yields to produce the best monthly forecasts. Moreover, we restrict the distribution of the inflation Sharpe ratios to achieve economically reasonable estimates of the inflation risk premium and of the real rates. We find that the inflation risk premium (i) is positive on average, (ii) rises when the unemployment rate increases and (iii) when the level of interest rates decreases. Hence, real yields are more pro-cyclical than nominal yields due to variations of the inflation risk premiums.

JEL classification: E43, E47, G12
Bank classification: Asset pricing; Econometric and statistical methods; Interest rates; Inflation and prices

Résumé

Les auteurs proposent de décomposer les taux de rendement nominaux en trois éléments – taux réels, inflation anticipée et primes de risque d’inflation – lorsqu’il n’existe pas d’obligations ou de swaps indexés sur l’inflation ou que les données ne sont pas fiables en raison de l’illiquidité relative de ces instruments. Ils combinent les prévisions d’inflation recueillies par enquête aux rendements nominaux dans le cadre d’un modèle fondé sur l’absence d’arbitrage où les anticipations conditionnelles sont latentes mais dépendent des observations passées, à la manière de la variance conditionnelle dans un modèle GARCH. Le problème de filtrage est simple numériquement, ce qui permet aux auteurs de comparer entre elles un grand nombre de prévisions hors échantillon. Leur modèle parvient à reproduire les prévisions d’inflation trimestrielles tirées d’enquêtes, et c’est aussi celui qui fournit les meilleures prévisions mensuelles grâce à l’information extraite des taux nominaux. Les auteurs imposent en outre des restrictions à la distribution des ratios de Sharpe associés au risque d’inflation pour obtenir des estimations économiquement raisonnables de la prime de risque d’inflation et des taux réels. Ils constatent que la prime est positive en moyenne et qu’elle s’accroît quand le taux de chômage monte ou que les taux d’intérêt baissent. La procyclicité des rendements réels serait ainsi plus marquée que celle des rendements nominaux à cause de la variation des primes de risque d’inflation.

Classification JEL : E43, E47, G12
Classification de la Banque : Évaluation des actifs; Méthodes économétriques et statistiques; Taux d’intérêt; Inflation et prix

Introduction

The decomposition of market interest rates into a component attributable to the effect of inflation and a component attributable to real effects has a long history. Irving Fisher was prominent among classical economists in this regards (Fisher, 1930) and the well-known decomposition of nominal yields into real yields and compensation for inflation bears his name. In the absence of risk, the real yield can be obtained by subtracting a measure of expected inflation 'from the nominal yield. But this simplicity hides the fact that future inflation rates are difficult to forecast (see e.g. Stock and Watson 2007). Furthermore, risk adds considerable difficulty to the decomposition since the unobserved inflation risk premium contributes significant variations to nominal yields (Buraschi and Jiltsov, 2005).

The recent literature uses separate measurements for each element of Fisher's decomposition. Chernov and Mueller (2011) combine nominal yields, surveys of inflation forecasts and data from inflation-indexed bond. Similarly, Haubrich, Pennacchi, and Ritchken (2012) use nominal yields, surveys of inflation forecasts and inflation swap data (instead of real bonds). But these additional instruments are not available in many countries, at least not for the horizons of interest. Moreover, using real bond or inflation swap data relies on additional maintained hypotheses about the degree of integration between different markets and about the magnitude and variations of any liquidity premium. These assumptions may not be supported in the data (Campbell, Shiller, and Viceira 2009, Fleckenstein, Longstaff, and Lustig 2012, Pflueger and Viceira 2011, Christensen and Gillan 2011). Our main contribution is to provide an accurate decomposition of the nominal yield curve that does not rely on the availability (or reliability) of real bonds or inflation swaps. Empirically, we consider the case of Canadian interest rates but our approach has broad applicability.[1]

Our approach relies on three distinct but complementary ingredients. First, we provide a careful and parsimonious specification for the evolution of expected inflation. As Kim (2007) pointed out, inflation expectations are persistent but the inflation shocks are large and mostly transitory, something that is hard to capture within a standard macro-finance VAR(1) model. One common solution is to introduce a latent factor to represent expectations directly. But the associated filtering problem carries its own challenges. We depart from this common approach and represent the evolution of inflation and other macro and finance variables via a conditional mean model (Fiorentini and Sentana, 1998) where expected inflation is

[1] Inflation swap data are not available in Canada. Moreover, existing real bonds have very long maturities and carry a time-varying liquidity premium. The shortest maturity of real returns bonds in Canada is December 2021, making these instruments inappropriate to construct short- to medium- inflation forecasts in our sample.

persistent but where the effect of inflation innovations is transitory.[2] Instead of being purely latent, unobserved expected inflation is spanned by the history of the other state variables. The filtering problem becomes trivial and we can easily assess the inflation forecasts from the model in a wide range of out-of-sample exercises.

Second, the conditional mean specification leads to a (discrete-time) affine dynamic term structure model. We develop two types of economic restrictions on the prices of risk to achieve an accurate and sensible measure of the inflation risk premium. In the spirit of Duffee (2011), we constrain the inflation Sharpe ratio.[3] Specifically, we restrict the parameter space so that the conditional inflation Sharpe ratio lies within the interval $[-\bar{SR}, \bar{SR}]$ with 95% probability in population. This restriction is similar but simpler than the approach taken by Duffee (2011).[4] In practice, we consider a range of values for $\bar{SR}$. Our second restriction follows Chernov and Mueller (2011) and imposes that the short real rate and the real yields are not functions of inflation expectations contemporaneously. We show that this implies a connection between, on one hand, the response of the nominal short rate to expected inflation and, on the other hand, the response of the inflation risk premium to expected inflation. Therefore, restrictions on the Sharpe ratio advocated by Duffee (2011) affect estimate of the inflation coefficient in the nominal rate when combined with the exclusion of inflation from the real rate advocated by Chernov and Mueller (2011). The interaction between these two restrictions implies that imposing a very low or zero one-period inflation Sharpe ratio to obtain accurate measures of the real yields (as in Ang, Bekaert, and Wei 2008) leads to unreasonable estimates of the policy response to expected inflation.

Third, we add survey data at estimation to counter the loss in parsimony.[5] Formally, we rely on the additional assumption that survey forecasts are rational, implying that they are consistent with model inflation forecasts up to an unpredictable error term. Estimation is then based on the combination of the likelihood of survey observations with the likelihood of the model. Survey data is available for most developed countries but, on the other hand, at a frequency that is often too slow or irregular, leading to a missing variable problem for many of the existing models. This poses no difficulties in our model given the simplicity of the filter.

Empirically, we estimate a conditional mean specification where the state variables are

[2]Our specification has a VARMA representation of order (1,1).

[3]Joslin et al. (2011) discuss the role of economic restrictions in the estimation of term structure models. See also Bauer and Diez (2012) in the context of a multi-country term structure model.

[4]Duffee (2010) numerically penalizes Sharpe ratio realizations that are too high point by point within the estimation procedure.

[5]Kim and Orphanides (2012) discuss extensively the use of survey of yield forecasts to reduce sampling uncertainty and lessen the bias in estimates of persistence parameters.

the inflation rate, the unemployment rate and two term structure factors. We use data on nominal yields and survey forecasts from 1986 until 2012. Following a battery of out-of-sample forecasting exercises, we conclude that the combination of (i) a conditional mean model, (ii) estimation based on survey data, and (iii) the information content of yields produces the best out-of-sample inflation forecasts. A conditional mean model outperforms simpler VARs across a range of univariate, bivariate or multivariate specifications. Neglecting surveys affects forecast accuracy even if surveys are available only infrequently and for a subset of horizons.[6] Innovations in the level and slope of the term structure have a persistent effect on expected inflation but inflation innovations do not.[7]

Having established the accuracy of the inflation forecasts, we turn to our estimates of real yields and of inflation risk premiums. We find that the inflation risk premium increases when the unemployment rate increases and when the level of yields is low. This is consistent with risk aversion being higher in recession: losses due to unexpected inflation shocks carry a greater weight in those states. Alternatively, the response of the central bank to inflation shocks may have more adverse consequences when the unemployment rate is higher. The results imply that real yields are negatively correlated with the inflation risk premium throughout the cycle. Moreover, estimates of real yields that account for the inflation risk premium are lower than unadjusted estimates in recessionary or low-growth episodes. In other words, real yields appear more pro-cyclical than nominal yields once we adjust for the inflation risk premium.

Pennacchi (1991) and Ang, Bekaert, and Wei (2008) also combines economic restrictions and nominal yields within dynamic term structure models. Pennacchi (1991) considers an equilibrium model and uses surveys to pin down the evolution of expectations but he does not measure the inflation risk premium. Ang, Bekaert, and Wei (2008) consider an affine no-arbitrage model with regimes. They impose a zero one-period inflation risk premium to identify the level of real rates. Regime-switching models are difficult to implement and to interpret. Moreover, the zero premium assumption has far-reaching consequences on the inflation Sharpe ratios, on the variability of the inflation risk premium (across different horizons) and on the estimates of the policy response coefficients. More recently, Chun

[6] This result echoes results in Faust and Wright (2011) for the case of the US. Several models match survey accuracy at the quarterly frequency where surveys are available. This contrasts with results in Ang et al. (2007) for the US who find that survey forecasts are difficult to match but they do not use survey at estimation.

[7] In their review of the literature, Stock and Watson (2003) find no evidence that nominal yields contain marginal information for future inflation. They argue that this reflects "limitations of conventional model [...], not a fundamental absence of predictive relationships in the economy." Stock and Watson (2003), p.79. We find that combining survey data with yield data in a sufficiently rich model delivers significant predictability.

(2011) uses US survey and yield data in combination with a term-structure model but do not decompose yields. Ajello, Benzoni, and Chyruk (2012) combine measures of core, food, and energy inflation series and obtain a decomposition of yield within a term structure model. Most of these studies do not assess the inflation expectations (via out-of-sample forecast comparison) or do not constrain the inflation risk premium around economically reasonable values.[8] Kozicki and Tinsley (2001) and Kozicki and Tinsley (2006) explores the role of shifting end-points in the processes for long-horizon yields or inflation rates in the US. In Canada, Amano and Murchison (2006) show the importance of a shifting endpoint for inflation in the early 1990s during the transition toward a 2% inflation targeting regime. Ragan (1995) is an early attempt to measure anticipation of future inflation from nominal yields in Canada. However, he assumes a constant risk premium. Day and Lange (1997) provides time-series evidence that the term spread has predictive content for inflation rates over the medium-term. Fung, Mittnick, and Remolona (1999) study a joint model for the US and Canadian term structure and identify the inflation factor by assuming that it is specific to each country while the real factor is common to both countries. They do not use survey data to identify inflation expectations. Garcia and Luger (2007) estimate an equilibrium-based model where investors derive utility from consumption and an external reference level of consumption. Conditional expectation plays a central role in their model but their focus is not on inflation forecasts.

The rest of the paper is organized as follows. Section 1 discusses model specification under the historical and the risk-neutral measures. Section 2 introduces econometric and economic restrictions necessary to identify each component of the nominal yield curve. Section 3 details the data and the estimation method. Section 4 presents the results and Section 4.3 concludes. The appendix contains all proofs.

1 A Macro-Finance Conditional Mean Model

1.1 Historical Dynamics

We derive a macro-finance term structure model where the state variables combine observable macro variables, x_t, and latent yield factors, y_t (see e.g., Ang and Piazzesi (2003)). The dynamics of the state vector, $z_t' \equiv (x_t', y_t')$, is given by the dynamics of its conditional mean

[8] A large literature combines real and nominal yields within a dynamic term structure model.See, e.g., DAmico, Kim, and Wei (2008), Chen, Liu, and Cheng (2010), Christensen, Lopez, and Rudebusch (2010).

$m_t \equiv E_t[z_{t+1}]$,

$$\begin{aligned} z_{t+1} =& m_t + u_{t+1} \\ m_{t+1} =& \mu + \phi\left(m_t - \mu\right) + \psi u_{t+1}, \end{aligned} \tag{1}$$

where $u_{t+1} = \Sigma \varepsilon_{t+1}$, Σ is lower triangular and ε_{t+1} is a vector of uncorrelated standard normal innovations. Feunou and Fontaine (2012) show that the representation in Equation (1), introduced in Fiorentini and Sentana (1998) in the context of time-series models, offers several advantages in the context of term structure models, where properties of the conditional mean are the most significant economic implications.[9]

Equation (1) differs in a significant way from a standard macro-finance VAR(1) specification (see e.g., Piazzesi 2005a) but this simpler model is nested with the restriction $\psi = \phi$. In this case, the conditional expectation of z_{t+1} is a linear combination of the current values, $m_t = \mu + \phi(z_t - \mu)$. In contrast, when $\psi \neq \phi$, the effect of past information, ϕm_{t-1}, on today's conditional expectations is different than the effect of the new information embodied in the innovations $\psi u_t = \psi(z_t - m_{t-1})$. This is an essential feature of the model and the empirical results illustrate the importance of this difference.

Equation (1) corresponds to a standard VARMA(1,1). Indeed, combining the equations for z_t and m_t together yields the following equivalent representation,

$$z_{t+1} = \mu + \phi(z_t - \mu) - \theta u_t + u_{t+1}, \tag{2}$$

where,

$$\theta = \phi - \psi,$$

which shows that the VAR(1) is nested when $\theta = 0$. Equation(1) also has an extended VAR representation with several cross-equation restrictions:

$$X_t \equiv \begin{pmatrix} z_{t+1} \\ m_{t+1} \end{pmatrix} = \begin{pmatrix} \mu \\ \mu \end{pmatrix} + \begin{bmatrix} 0 & I_4 \\ 0 & \phi \end{bmatrix} \begin{pmatrix} z_t - \mu \\ m_t - \mu \end{pmatrix} + \begin{bmatrix} I_4 \\ \psi \end{bmatrix} u_{t+1},$$

where, the auto-regressive matrix and the covariance matrix of X_t are singular implying that several of the arguments made in Joslin, Singleton, and Zhu (2011) are not directly applicable.[10]

[9]The importance of specifying the process for inflation in terms of its conditional mean has been emphasized by Kim (2007) in the context of macro-finance term structure models. Piazzesi and Schneider (2006) use a similar specification in the context of a term structure model with learning. Equation 1 is also similar to the state dynamics in long-run risk models (Bansal and Yaron, 2004).

[10]In particular, the argument that a portfolio of yields, which is a linear transformation of X_t, with

1.2 Risk-Neutral Dynamics

The risk-neutral dynamics are defined via the following change of measure, ξ_t,

$$\xi_{t+1} = \frac{\exp(\lambda_t u_{t+1})}{E_t[\exp(\lambda_t u_{t+1})]} \tag{3}$$

where the prices of risk, λ_t are affine functions of the conditional mean, m_t,

$$\lambda_t \equiv (\Sigma\Sigma')^{-1}\left(m_t^Q - m_t\right) \equiv \tilde{\lambda}_0 + \tilde{\lambda}_1 m_t, \tag{4}$$

with $m_t^Q \equiv E_t^Q[z_{t+1}]$. It follows that the risk premium – the spread between m_t and m_t^Q – is given by,

$$m_t^Q - m_t = \bar{\lambda}_0 + \bar{\lambda}_1 m_t, \tag{5}$$

where $\bar{\lambda}_0 = \Sigma\Sigma'\tilde{\lambda}_0$ and $\bar{\lambda}_1 = \Sigma\Sigma'\tilde{\lambda}_1$. The change of measure, ξ_t, and the prices of risk λ_t are standard (Piazzesi 2005b) but with the difference that they functions of m_t and not z_t. The risk premium is forward-looking. Of course, it is possible to define forward-looking prices of risk in auto-regressive models, but this is nothing other than a rotation of the parameters, $\bar{\lambda}_1 E_t[z_{t+1}] = (\bar{\lambda}_1\phi)z_t$. In contrast, here the risk premium depends on the entire history of z_t via the recursion for m_t in Equation (1).

The dynamics of z_t under the risk-neutral measure, $\mathbb{Q}$, is given by:

$$\begin{aligned} z_{t+1} &= m_t^Q + u_{t+1}^Q \\ m_{t+1}^Q &= \tilde{\mu}^Q + \tilde{\phi}^Q\left(m_t^Q - \tilde{\mu}^Q\right) + \tilde{\psi}^Q u_{t+1}^Q, \end{aligned} \tag{6}$$

with parameters given in the Appendix, $u_{t+1}^Q = \Sigma\varepsilon_{t+1}^Q$, and where ε_{t+1}^Q is a vector of uncorrelated i.i.d. standard Gaussian innovations. The dynamics for z_t has the same conditional mean representation under $\mathbb{Q}$ but with shifted parameters. The parameters $\tilde{\mu}^Q$, $\tilde{\phi}^Q$ and $\tilde{\psi}^Q$ are functions of the price of risk parameters, $\bar{\lambda}_0$ and $\bar{\lambda}_1$ and of the corresponding parameters under $\mathbb{P}$. Importantly, the time-series properties of the $\mathbb{P}$-dynamics and the cross-sectional properties of the $\mathbb{Q}$-dynamics are linked. This contrasts with the standard VAR(1) model where the two dynamics are not linked unless we impose additional restrictions on the prices of risk. This has important consequences in the computation of the likelihood. In particular, two-step estimation procedures, now common for the estimation of VAR(1) term structure

dynamics defined in the text, possesses an observationally equivalent VAR(1) representation breaks down. Moreover, the covariance matrix of this portfolio is not of full rank, hence a dimension reduction technique must be applied to compute the likelihood. These difficulties do not arise if we retain the representation in Equation (1). See Feunou and Fontaine (2012) for details.

models (Joslin, Singleton, and Zhu, 2011) do not lead to consistent parameter estimates.

1.3 The Nominal and Real Yield Curves

The short-term nominal rate, i_t, is a forward-looking function of the conditional expectation, m_t,

$$i_t = \bar{i} + \delta' m_t. \tag{7}$$

Equation (7) implies that nominal yields and real yields are affine in m_t. Forward-looking rules for the short rate are discussed in Ang, Dong, and Piazzesi (2007). Moreover, forward-looking nominal yields are consistent with general equilibrium models with long-run risk (e.g., Bansal and Shaliastovich 2010, Hasseltoft 2012). The nominal yield with n-period to maturity is defined by:

$$i_t^{(n)} = -\ln E_t^Q \left[\exp \left\{ \sum_{j=0}^{n-1} (\bar{i} + \delta' m_{t+j}) \right\} \right]. \tag{8}$$

We need the dynamics of m_t under the risk-neutral measure to compute the expectation and derive term structure implications. This is given by:

$$m_{t+1} = \mu^Q + \phi^Q \left(m_t - \mu^Q \right) + \psi^Q u_{t+1}^Q, \tag{9}$$

with parameters given by,

$$\begin{aligned} \omega^Q &= \omega + \psi \bar{\lambda}_0 \\ \phi^Q &= \phi + \psi \bar{\lambda}_1 \\ \psi^Q &= \psi, \end{aligned} \tag{10}$$

and where, as above, $u_{t+1}^Q = \Sigma \varepsilon_{t+1}^Q$.[11] The solution to Equation 8 is then given by:

$$i_t^{(n)} = a_n + b_n' m_t, \tag{11}$$

with coefficients a_n and b_n given in the Appendix.

We can derive the real term structure without any further assumption. The real short rate, r_t, is given by the link between the real and the nominal stochastic discount factors,

[11] The variable m_t is *not* the conditional mean of z_t under $\mathbb{Q}$. For instance, while μ is the unconditional mean of *both* z_t and m_t under $\mathbb{P}$, the mean of m_t and z_t are different under $\mathbb{Q}$ and given by μ^Q and by $\tilde{\mu}^Q$, respectively.

$M_{t+1} \equiv e^{-i_t}\xi_{t+1} = e^{-\pi_{t+1}} M^r_{t+1}$:

$$r_t \equiv -\log\left(E_t[M^r_{t+1}]\right) = i_t - E^Q_t\left[\pi_{t+1}\right] - \frac{1}{2}\sigma^2_\pi, \tag{12}$$

which corresponds to the Fisher's decomposition but where the expectation, $E^Q_t\left[\pi_{t+1}\right]$, is taken under the risk-neutral measure and where $\sigma^2_\pi \equiv Var_t[\pi_{t+1}]$. Using Equation (5), the real short-rate can be written analogously to the nominal rate,

$$r_t = \bar{r} + \delta'_r m_t, \tag{13}$$

where we define $e'_1 = (1\ 0\ 0\ 0)$ and

$$\begin{aligned} \bar{r} &= \bar{i} - e'_1\bar{\lambda}_0 - \frac{1}{2}\sigma^2_\pi \\ \delta'_r &= \delta' - e'_1 - e'_1\bar{\lambda}_1. \end{aligned} \tag{14}$$

It follows that the real yield curve is given by:

$$\begin{aligned} r^{(n)}_t &= -\ln E^Q_t\left[\exp\left\{\sum_{j=0}^{n-1}\left(\bar{r} + \delta'_r m_{t+j}\right)\right\}\right] \\ &= a_{r,n} + b'_{r,n} m_t, \end{aligned} \tag{15}$$

with coefficients in the appendix.

Finally, the following generalized version of Fisher's decomposition of nominal yields is useful. The generalization decomposes a nominal yield into real and nominal components but account for the inflation risk premium. From Equation (7), re-write the nominal rate as:

$$i_t = \frac{1}{2}\sigma^2_\pi + r_t + m^\pi_t + irp_t, \tag{16}$$

where $irp_t \equiv E^{\mathbb{Q}}_t[\pi_{t+1}] - E^{\mathbb{P}}_t[\pi_{t+1}] = e'_1(\bar{\lambda}_0 + \bar{\lambda}_1 m_t)$ is the inflation risk premium. This decomposition can be easily extended for any maturity, n,

$$i^{(n)}_t = c^{(n)} + r^{(n)}_t + m^{\pi,(n)}_t + irp^{(n)}_t, \tag{17}$$

where the n-period ahead inflation risk premium,

$$irp^{(n)}_t = \frac{1}{n}\left(E^Q_t\left[\sum_{j=1}^{n}\pi_{t+j}\right] - E_t\left[\sum_{j=1}^{n}\pi_{t+j}\right]\right),$$

and the n-period ahead inflation expectation,

$$m_t^{\pi,(n)} = E_t \left[\frac{\sum_{j=1}^{n} \pi_{t+j}}{n} \right],$$

can be computed directly from the model and where $c^{(n)}$ is a Jensen term given in appendix.

1.4 Observable Factors

The term structure factors, $y_t = (l_t, s_t)'$ are latent. To recover these factors, we assume that the first two principal components of nominal yields, which we combined in the vector y_t^o, are measured without errors (see e.g., Chen and Scott 1993, Joslin, Singleton, and Zhu 2011). These have the usual interpretations in terms of the level and slope of the term structure. The observable factors are portfolios of yields given by:

$$y_t^o = w^o Y_t = a^o + b^o m_t,$$

where y_t^o is a 2×1 vector, Y_t the $J \times 1$ vector of observed yields, w^o is a $2 \times J$ matrix and, also, the matrix w^o depends on the data and corresponds to the principal component loadings. The coefficients a^o and b^o are given by:

$$a^o = w^o \begin{pmatrix} a_{n_1} \\ a_{n_2} \\ \vdots \\ a_{n_J} \end{pmatrix} \quad b^o = w^o \begin{pmatrix} b'_{n_1} \\ b'_{n_2} \\ \vdots \\ b'_{n_J} \end{pmatrix}.$$

We define a new state vector with observed components,

$$z_t^o = \begin{pmatrix} x_t \\ y_t^o \end{pmatrix},$$

where the dynamics of z_t^o are given by a conditional mean model analog to Equation (1),

$$\begin{aligned} z_{t+1}^o &= m_t^o + u_{t+1}^o \\ m_{t+1}^o &= \mu^o + \phi^o (m_t^o - \mu^o) + \psi^o u_{t+1}^o, \end{aligned} \tag{18}$$

with $u_{t+1}^o = \Sigma^o \varepsilon_{t+1}^o$ and where ε_{t+1}^o are i.i.d. standard normal innovations. The dynamics of the observed z_t^o is derived from the dynamics of z_t and, furthermore, the conditional mean,

m_t^o, is an affine transformation of m_t,

$$m_t^o = \gamma_0 + \gamma m_t \tag{19}$$

with parameters given in the appendix.

1.5 Filtering

The conditional mean specification offers several advantages relative to a more general specification where the stochastic evolution of m_t follows its own dynamics. One representation of this specification would introduce separate shocks for z_t and m_t in Equation 1 but would increase the complexity of the estimation problem significantly. First, the number of shocks and the size of the covariance matrix would double, increasing the number of parameters to be estimated significantly. In contrast, the conditional model used here relies on the intuitive assumption that the same stochastic sources drive the observable variables and their conditional means. Second, estimation would have to rely on a Kalman filter for the conditional means based on the measurement equations for yields, adding substantially to the complexity of the estimation problem. In contrast, filtering the conditional means from the history of the observable data is trivial and given by the second line of Equation 1 (or 18). Finally, an alternative to a Kalman filter step is to assume that several principal components of yields are measured without errors, including small components. This is unlikely to be supported in the data. On the other hand, as we show above, one can still introduce a small number of purely latent factors and rely on the assumptions that only the leading principal components are priced without error.

2 Identification and Economic Restrictions

2.1 Identification

Estimation relies on a series of identification assumptions. First, without loss of generality, we impose that $\psi_{yx} = \phi_{yx}$ and $\psi_{yy} = \phi_{yy}$ so that y_t has no moving-average component.[12] Equations (11) and (15) show that the yield factors, l_t and s_t, only affect the term structure via their conditional expectations, $m_t^y \equiv E_t[y_{t+1}]$. Moreover, these factors are latent. While yields can reveal the variations of m_t^y, we have no measurement equation to reveal variations in y_t. Heuristically, we do not have separate measurements of Σu_t^y and of ψu_t^y and, therefore, we cannot identify the components of Σ associated with y_t separately from the components of ψ associated with m_t^y. Our strategy is to fix $\psi_{yx} = \phi_{yx}$ and $\psi_{yy} = \phi_{yy}$.

[12] In Equation 2, we have that the last two lines of θ are zero.

Second, we can rotate the latent state variables without changing the probability distribution of bond yields and, therefore, not all parameters can be identified separately. However, results in Dai and Singleton (2000) imply that the identification of all the parameters can be obtained using standard assumptions that we detail in Section 2.2.3. To see why standard results apply, note that the dynamics of the factors entering the yield equation are standard under the $\mathbb{P}$ and $\mathbb{Q}$ measures and given by:

$$\begin{aligned} m_{t+1} &= \mu + \phi\left(m_t - \mu\right) + \psi u_{t+1} \\ m_{t+1} &= \mu^Q + \phi^Q\left(m_t - \mu^Q\right) + \psi^Q u_{t+1}^Q, \end{aligned}$$

respectively. Each corresponds to a VAR(1). The shifts between μ and μ^Q, between ϕ and ϕ^Q, and between ψ and ψ^Q are standard and repeated here:

$$\begin{aligned} \omega^Q &= \omega + \psi\bar{\lambda}_0 \\ \phi^Q &= \phi + \psi\bar{\lambda}_1 \\ \psi^Q &= \psi, \end{aligned} \tag{20}$$

where $\omega^Q \equiv \left(I_4 - \phi^Q\right)\mu^Q$ and $\omega \equiv \left(I_4 - \phi\right)\mu$. The parameter ψ determines the covariance of m_t shocks and remains the same under each measure as in standard gaussian models.

Third, the term structure of real yields is identified (in the econometric sense) from nominal data. The only additional parameters arising from deriving the real curve are $\bar{r}$ and δ_r, which determine the real short rate in terms of m_t. But these parameters are given from the connection between the nominal and real stochastic discount factors. Equation (14) above, repeated here for convenience,

$$\begin{aligned} \bar{r} &= \bar{i} - e_1'\bar{\lambda}_0 - \frac{1}{2}\sigma_\pi^2 \\ \delta_r' &= \delta' - e_1' - e_1'\bar{\lambda}_1. \end{aligned}$$

shows that $\bar{r}$ is fixed given estimates of $\bar{i}$, σ_π^2 and $\bar{\lambda}_0$. Similarly, δ_r is determined by estimates of δ and $\bar{\lambda}_1$. All these are identified from the nominal data using standard assumptions. Clearly, the average short rate $\bar{i}$, the volatility of inflation σ_π^2 and the nominal short rate coefficients δ can identified from the data. Equations (10) show that $\bar{\lambda}_0$ and $\bar{\lambda}_1$ are functions of the parameters that determine the times-series and cross-section of yields.

2.2 Economic Restrictions

The real curve may be identified in the econometric sense but may still be imprecisely estimated from nominal data. This problem is particularly acute in cases, like Canada, where no real bond or inflation swaps exist at the maturities of interest. Therefore, we impose a set of economic restrictions to estimate the shape of the real curve more precisely. These assumptions do not affect the model's ability to fit the nominal curve and inflation expectations.

2.2.1 Sharpe ratio restrictions

First, we restrict the Sharpe ratio associated with inflation risk to economically sensible values. These restrictions lead to plausible estimates of the real curve. Indeed, it is well known that $\bar{\lambda}_1$ and $\bar{\lambda}_0$ are estimated only imprecisely for a wide class of affine term structure models, often implying unreasonable estimates of the Sharpe ratio (Duffee, 2010) and, in our case, leading to implausible estimate of the real curve. The one-period ahead Sharpe ratio associated with inflation risk, $SR^{\pi}_{t,1}$, is defined as:

$$SR^{\pi}_{t,1} = \sigma_{\pi}^{-1}(E^{Q}_{t}[\pi_{t+1}] - E_t[\pi_{t+1}]) \tag{21}$$

and it is given by the first element of $\sigma_{\pi}^{-1}(\bar{\lambda}_0 + \bar{\lambda}_1 m_t)$. The first term in the numerator of $SR^{\pi}_{t,1}$ is the price that an investor is willing to pay to enter a contract that pays the realized inflation rate. The second term in the numerator is the investor's expectation of inflation. Hence, the numerator is the inflation premium : the premium to investors for entering a fair bet on inflation. The ratio $SR^{\pi}_{t,1}$ expresses this premium as a price per unit of risk. The mean and variance of the inflation Sharpe ratio are given by:

$$\begin{aligned} E[SR^{\pi}_{t,1}] &= \sigma_{\pi}^{-1}\left(\bar{\lambda}_{0,\pi} + \bar{\lambda}'_{1,\pi}\mu\right) \\ Var[SR^{\pi}_{t,1}] &= \sigma_{\pi}^{-2}\bar{\lambda}'_{1,\pi} Var(m_t)\bar{\lambda}_{1,\pi}. \end{aligned} \tag{22}$$

We impose that $SR^{\pi}_{t,1}$ remains within a plausible range with very high probability. Formally, we use the following restriction,

$$\underline{SR} \leq E[SR^{\pi}_{t,1}] \pm 1.96 Var[SR^{\pi}_{t,1}]^{-1/2} \leq \overline{SR}, \tag{23}$$

implying a probability of 5% that $SR^{\pi}_{t,1}$ is outside of the range $[\underline{SR}, \overline{SR}]$ in population. In practice, we vary $\underline{SR}$ and $\overline{SR}$ in 0.01 increments between 0 and ± 0.3 to assess the effect of restricting the inflation risk Sharpe ratio. These values are in line with the benchmark used by Duffee (2010) to evaluate the plausibility of model-implied Sharpe ratios for nominal bond returns in the US.[13]

2.2.2 A restriction on the short real rate

Second, we follow Chernov and Mueller (2011) and assume that real yields and the real short rate are not functions of inflation expectations. Economically, the restriction on the short rate,

$$r_t = \bar{r} + (\delta_{r,\pi}\ \delta_{r,g}\ \delta_{r,l}\ \delta_{r,s})' m_t,$$

implies a link between the policy response to expected inflation and the price of inflation risk. Equation (14) and $\delta_{r,\pi} = 0$ implies that

$$\delta_{\pi} = 1 + \bar{\lambda}_{1,\pi\pi}, \tag{24}$$

where δ_{π} is the response coefficient in the nominal short rate equation,

$$i_t = \bar{i} + (\delta_{\pi}\ \delta_{g}\ \delta_{l}\ \delta_{s})' m_t,$$

and $\bar{\lambda}_{1,\pi\pi}$ is the first element of the matrix $\bar{\lambda}_1$ The restrictions that real yields do not span inflation expectations imply that

$$b_{r\pi,n} = 0,\ \forall n > 0, \tag{25}$$

where $b_{r\pi,n}$ is the first element of $b_{r,n}$. From the recursion for $b_{r,n}$ it follows that,

$$(n+1) b_{r\pi,n+1} = n b'_{r,n} \begin{bmatrix} \phi^Q_{\pi,\pi} \\ \phi^Q_{g,\pi} \\ \phi^Q_{l,\pi} \\ \phi^Q_{s,\pi} \end{bmatrix} + \delta_{r,\pi} = 0, \quad \forall n > 1,$$

and, therefore, that $\phi^Q_{g\pi} = \phi^Q_{l\pi} = \phi^Q_{s\pi} = 0$[14] since $\delta_{r,\pi} = 0$.

Equation 24 says that the response of the monetary authority to expected inflation also determines the response of the inflation premium. The policy response to expected inflation

[13] Duffee (2010) imposes a constraint on the sample mean of $SR^{\pi}_{t,1}$ but points out that the population mean can be used instead. We use a constraint on the distribution of the inflation risk premium. This constraint implies that Duffee's constraint is satisfied in any sample. Restricting the Sharpe ratio is also consistent with Chernov and Mueller (2011) who penalize the excessive variability of the term premium.

[14] $\phi^Q \equiv \begin{pmatrix} \phi^Q_{\pi\pi} & \phi^Q_{\pi g} & \phi^Q_{\pi l} & \phi^Q_{\pi s} \\ \phi^Q_{g\pi} & \phi^Q_{gg} & \phi^Q_{gl} & \phi^Q_{gs} \\ \phi^Q_{l\pi} & \phi^Q_{lg} & \phi^Q_{ll} & \phi^Q_{ls} \\ \phi^Q_{s\pi} & \phi^Q_{sg} & \phi^Q_{sl} & \phi^Q_{ss} \end{pmatrix}$

affects the contemporaneous short rate via the compensation for inflation risk. For instance, if the policy response is more than one-for-one, and monetary policy is stabilizing with respect to inflation, then the inflation premium declines when the policy rate is lowered in response to lower inflation expectations. Therefore, the bounds on the inflation Sharpe ratio may limit the role of expected inflation in the evolution, which affects the estimate of the expected inflation response coefficient. We explore this important question in the empirical section.

2.2.3 Nelson-Siegel representation of real yields

Third, we impose that the loadings of the yield factors, y_t, correspond to the level and slope loadings in the Nelson-Siegel representation (Nelson and Siegel 1987, Christensen, Diebold, and Rudebusch 2007). Specifically, we find conditions such that real yields are given by,

$$b_{rl,n} = 1; \quad b_{rs,n} = \tfrac{1-e^{-n\lambda}}{n\lambda} m_t^s; \quad \lambda > 0. \tag{26}$$

This representation is parsimonious and widely used to model interest rates. It also directly justifies that we label l_t and s_t the level and slope factor, respectively. Following Christensen, Lopez, and Rudebusch (2010), we do not include a curvature factor to model the real curve.[15] The parameter λ controls the steepness of the slope and is estimated jointly with other parameters.

Equation 26 implies that $\phi^Q_{xy} = 0$, and

$$\delta_y = \begin{bmatrix} 1 \\ \frac{1-e^{-\lambda}}{\lambda} \end{bmatrix}; \quad \phi^Q_{yy} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\lambda} \end{bmatrix}; \quad \phi^Q_{xy} \equiv \begin{pmatrix} \phi^Q_{\pi l} & \phi^Q_{\pi s} & ; \\ \phi^Q_{gl} & \phi^Q_{gg} & \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{27}$$

Note that the Nelson-Siegel representation identifies the scale, the sign and the ordering of the latent factors via its implications for δ_y. We fix $\bar{\lambda}_0^l = \bar{\lambda}_0^s = 0$ to identify the level of the latent term structure factors.[16]

2.2.4 Summary of the restrictions

For convenience, we summarize here the restrictions discussed in this section. Parameters of the short real and nominal rates are given by:

$$\delta_r' = \begin{bmatrix} 0 & \delta_{r,g} & 1 & \frac{1-e^{-\lambda}}{\lambda} \end{bmatrix}, \tag{28}$$

[15] This is a simplification of the original NS representation. Christensen, Lopez, and Rudebusch (2010) find that a level and slope factors are sufficient to model the term structure of TIPS yields in the US.

[16] See Christensen et al. (2007) for a discussion of identification assumptions in the context of affine models with a Nelson-Siegel representation. We define $\bar{\lambda}_0' \equiv (\bar{\lambda}_{0,\pi}\ \bar{\lambda}_{0,g}\ \bar{\lambda}_{0,l}\ \bar{\lambda}_{0,s})$.

parameters of the the $\mathbb{P}$-dynamics are given by:

$$\mu = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix} \quad \phi = \begin{bmatrix} \phi_{xx} & \phi_{xy} \\ \phi_{yx} & \phi_{yy} \end{bmatrix} \quad \psi = \begin{bmatrix} \psi_{xx} & \psi_{xy} \\ \phi_{yx} & \phi_{yy} \end{bmatrix}, \tag{29}$$

and parameters of the the $\mathbb{Q}$-dynamics are given by:

$$\mu^Q = \begin{bmatrix} \mu_x^Q \\ \mu_y^Q \end{bmatrix} \quad \phi^Q = \begin{bmatrix} \phi_{xx}^Q & 0 \\ \phi_{yx}^Q & \phi_{yy}^Q \end{bmatrix} \quad \psi^Q = \begin{bmatrix} \psi_{xx} & \psi_{xy} \\ \phi_{yx}^Q & \phi_{yy}^Q \end{bmatrix}, \tag{30}$$

with ϕ_{yy}^Q given in Equation 27 and $\phi_{g\pi}^Q = \phi_{l\pi}^Q = \phi_{s\pi}^Q = 0$. The parameters under each measure are also linked via Equation 10 with the additional restrictions that $\bar{\lambda}_0^l = \bar{\lambda}_0^s = 0$.

3 Data and Estimation

3.1 Data

The sample is monthly and includes the unemployment rate, g_t, and the (total) inflation rate, $\pi = \ln \frac{p_{t+1}}{p_t}$, from January 1986 until December 2011. We use yields on zero-coupon bonds with maturities of 3, 6, 9, 12 and 18 months as well as 2, 3, 4, 7, 8, 9, 10 years over the same sample. We also use data from surveys of professional forecasters by Consensus Economics (CE). On the last month of every quarter, the survey asks for a forecast of the average of year-over-year inflation rates across all months in a given quarter. The survey covers the remaining quarters of the current calendar year, and every quarter of the following calendar year. We use the first five quarters to obtain a balanced panel since longer horizon forecasts are available only irregularly.[17]

3.2 Likelihood

3.2.1 State dynamics

First, we estimate parameters of the state dynamics based on the likelihood of z_t^o, excluding the survey data and without imposing the no-arbitrage restrictions. The conditional

[17]The inflation rate is computed from the Canadian seasonally adjusted all items consumer price index (StatCan Table 326-0020) and the unemployment is also seasonally adjusted (StatCan 282-0089). Zero-coupon yields are available from the Bank of Canada's web site. The survey is taken on the second week of each month but the inflation forecasts are only updated quarterly. StatCan releases inflation data at the end of each month and with a one-month lag. Hence, survey participants on the second week of March know the inflation rate up to the month of January.

log-likelihood of z_t^o is,

$$l\left(z_t^o|z_{t-1}^o\Xi\right) = -\frac{1}{2}\left\{2\ln(2\pi) + \ln(\det(\Sigma^o\Sigma^{o'})) + \left(z_t^o - m_{t-1}^o\right)'\left(\Sigma^o\Sigma^{o'}\right)^{-1}\left(z_t^o - m_{t-1}^o\right)\right\}, \tag{31}$$

for a given parameter vector, Ξ, where the conditional mean, m_t^o, is given by the recursion in Equation 18 with initial value $m_0^o = E[z_t^o]$.

3.2.2 Surveys of inflation forecasts

Second, we estimate the same parameters based on the joint likelihood of z_t^o and of survey data. Faust and Wright (2011) discusses the importance of using survey data to forecast inflation. Forecasting inflation may be a daunting task whenever the central bank has an explicit inflation rate target since policy actions aim to counter-act predictable deviations from the target. Nevertheless, the private sector regularly produces and publishes inflation forecasts. A quick look at the data reveals that the predictions of professional forecasters typically differ from the central bank's target: inflation still contains predictable variations.

CE reports the median forecast across respondents. We assume that the reported CE forecast is unbiased but includes a measurement error,

$$\pi_{t,h}^{CE} = E_t\left(\frac{\sum_{j=h-11}^{h}\left(\pi_{t+j} + \pi_{t+j+1} + \pi_{t+j+2}\right)}{3}\right) + \eta_{t,h}^{CE}, \tag{32}$$

where $h = 3, 6, 9, 12$, and 15, which correspond to the next five quarters. The measurement error $\eta_{t,h}^{CE}$ is i.i.d. normal with mean zero and standard deviation $\sigma_{CE,h}$. The expectation term can be computed from the model for each horizon. Equation 32 embodies the assumption that survey forecasts and the dynamics in Equation 1 are consistent. The difference between the conditional expectation, m_t, and the survey forecasts, $\pi_{t,h}^{CE}$ is unpredictable and has mean zero. Chun (2011) also estimates a model based on survey forecasts but he does not impose consistency between a model for the object of the forecasts, z_t, and the forecast itself. The log-likelihood of the survey data is given by,

$$l\left(\pi_t^{CE}|z_t^o;\Xi\right) = \sum_h\left(-\frac{1}{2}\left\{\ln\left(2\pi\sigma_{CE,h}^2\right) + \left(\frac{\eta_{t,h}^{CE}}{\sigma_{CE,h}}\right)^2\right\}\right). \tag{33}$$

Note that a comparison of results based the likelihood in Equation 31 and results based on the likelihood in Equation 32 provides an assessment of the added value from survey data.

3.2.3 *Yields cross-section*

Third, we impose the no-arbitrage restrictions and estimate parameters of the historical and of the risk-neutral dynamics jointly using all yields and survey data. The yield data include $N \leq J-2$ combinations of yields, Y_t^e, that are measured with errors. The n-th combination of yields is given by:

$$Y_{n,t}^e = w_n^e Y_t = a_n^e + b_n^e m_t^o + \eta_{n,t}^e,$$

where $\eta_{n,t}^e$ is i.i.d. normal with mean zero and standard deviation $\sigma_{e,n}$.[18]. The coefficients are given by:

$$\begin{aligned} a^e &= w^e(a + b\gamma_0) \\ b^e &= w^e b\gamma, \end{aligned}$$

and w^e is an $N \times J$ matrix and the conditional log-likelihood of $Y_{t,n}^e$ is given by

$$l\left(Y_{t,n}^e | z_t^o; \Xi\right) = \sum_n \left(-\frac{1}{2} \left\{ \ln\left(2\pi\sigma_{e,n}^2\right) + \left(\frac{\eta_{t,n}^e}{\sigma_{e,n}}\right)^2 \right\} \right). \tag{34}$$

Adding the likelihood of yields allows for the estimation of the risk-neutral parameters and delivers estimates of the inflation risk premium and real yields. We rely on the assumption that inflation expectations embodied in yields corresponds with model expectations.

3.2.4 *Combining Likelihood*

We nest the likelihoods of the data together and write:

$$L(\Xi) = \sum_{t=1}^{T} \left(\mathbb{1}_i(z_t^o) l\left(z_t^o | z_{t-1}^o; \Xi\right) + \mathbb{1}_i(\pi_t^{CE}) l\left(\pi_t^{CE} | z_t^o; \Xi\right) + \mathbb{1}_i(Y_t^e) l\left(Y_t^e | z_t^o; \Xi\right) \right) \tag{35}$$

where $z_0^o = \mu^o$. The indicator function $\mathbb{1}_i(z_t^o)$ is equal to one if model i is estimated using the likelihood of z_t^o. We estimate various models using z_t^o data (i.e., $\mathbb{1}_i(z_t^o) = 1$), some models also use survey data (i.e., $\mathbb{1}_i(\pi_t^{CE}) = 1$)and, in addition, models that impose the absence of arbitrage opportunity also use all the yield data (i.e., $\mathbb{1}_i(Y_t^e) = 1$). We have that the indicator $\mathbb{1}_i(\pi_t^{CE})$ is equal to 1 every three months since survey data is only available quarterly. In all cases, we fix parameters controlling the unconditional means of the state variables to their respective sample averages and impose the usual stationarity conditions on

[18]We set N=5, and we assume that Y_t^e is constituted of principal components 3 to 7.

the eigenvalues of ϕ. We also impose the stationarity of the macro variables under $\mathbb{Q}$. Errors in fitting z_t^o may receive a greater weight in the maximization of the likelihood. But this also depends on the relative magnitude of the variance of state innovations and the variance of survey measurement errors.

4 Decomposing Nominal Yields

We estimate the historical and risk-neutral dynamics of z_t based on the joint likelihood, $L(\Xi)$ where $-\underline{SR} = \overline{SR} = 0.20$. This is a reasonable value given existing estimates. For instance, Duffee (2011) finds that the maximum Sharpe ratio from a porfolio of US Treasury bonds and bills is 0.23 on average. Moreover, varying this parameter does not affect inflation expectations measured from the model. Figure 5 compares forecast error RMSEs at two horizons: one month and one year, across a range of values for the Sharpe ratio bounds from 0 to 0.30. We report the ratio of the forecast RMSE relative to the case where the bound is zero and the 1-month inflation risk premium is constant at zero. The 1-month RMSE decreases as we move away from a zero inflation risk premium. The maximum likelihood estimator explicitly minimizes the one-month forecast variance. In contrast, the one-year forecast RMSE remains essentially constant as we increase the bound. Overall, Figure 5 suggests that freeing the Sharpe ratio constraint leads to an over-fitting of the 1-month inflation forecast at no benefit for other horizons. Hence, we focus the analysis for the case where $\underline{SR} = \overline{SR} = 0.20$. Table 1 displays the standard deviations of survey measurement errors and of yield pricing errors for this case. The model fits the survey forecasts closely. The standard deviations range between 1.1 and 1.8 basis point annually. Similarly, the model fits the yields closely. The first two principal components of yields are priced exactly by construction and the standard deviations for the remaining five components range from 0.1 to 1.1 basis point annually.

4.1 Measuring Inflation Expectations

Figure 7 displays the time series of expected inflation at horizons of 3 months, 1 year, 2 years and 5 years. It shows the decline in inflation expectations from a level of between 3% and 4% in the late 1980s to a level around 2% in the more recent period. This is consistent with the new inflation targeting regime initiated in the 1990s. Figure 7 also shows the strong pro-cyclical behavior of expected inflation: expectations decrease markedly when the unemployment rate rises. Finally, the slope of inflation expectations across different horizons changes sign through the cycle. Short-run inflation expectations stand below long-run inflation expectations when unemployment is high, and vice-versa when unemployment is low.

A natural criteria to assess whether we obtain good measurements of expected inflation is their accuracy as inflation forecasts. Hence, we compare the out-of-sample forecast RMSEs from our preferred model with results from a battery of alternatives. We estimate each model with data until December 1991, compute inflation forecasts for horizons up to two years ahead and compute forecast errors against the realized values of inflation. Then, the estimation window is lengthened by one month, new forecasts and forecast errors are produced, and the exercise is repeated until we reach the end of the sample.

With the exception of three simple univariate alternatives, we consider a variety of models that are nested in the framework of Equation 1 and which can be estimated from the likelihood given by Equation 35. First, we estimate the following univariate models:

(i) Random walk models

- *RW1* : $E_t\left(\frac{\sum_{j=1}^{h}\pi_{t+j}}{h}\right) = \pi_t$
- *RW2* : $E_t\left(\frac{\sum_{j=1}^{h}\pi_{t+j}}{h}\right) = \frac{1}{12}\sum_{j=0}^{11}\pi_{t-j}$

(ii) Stationary models

- *AR* : $m^\pi_{t+1} = \mu^\pi + \phi_\pi(\pi_{t+1} - \mu^\pi)$
- *ARMA* : $m^\pi_{t+1} = \mu^\pi + \phi_\pi(m^\pi_t - \mu^\pi) + \psi_\pi u^\pi_{t+1}$
- *S-ARMA* : $m^\pi_{t+1} = \mu^\pi + \phi_\pi(m^\pi_t - \mu^\pi) + u^m_{t+1}$.

$RW1$ and $RW2$ are simple random walk models. The AR(1) and ARMA(1,1) models are nested in the framework of this paper, and the stochastic mean model, S-ARMA(1,1) allows for an additional shock u^m_{t+1} to the conditional mean m^π_{t+1}. We also estimate the following multivariate models combining the inflation rate and the unemployment rate, $x'_t = (\pi_t, g_t)$, but excluding the yield factors, y_t.

(iii) Inflation and Unemployment

- *VAR-U*: $m^x_{t+1} = \mu^x + \phi_x(x_{t+1} - \mu^\pi)$
- *VARMA-U*: $m^x_{t+1} = \mu^x + \phi_x(m^x_t - \mu^\pi) + \psi_x u^\pi_{t+1}$,

then we expand the system to include the yield factors,

(iv) Inflation, Unemployment, Level and Slope

· *VAR-UL and VAR-ULS* : $m_{t+1} = \mu + \phi(z_{t+1} - \mu)$

· *VARMA-UL and VARMA-ULS* : $m_{t+1} = \mu + \phi(m_t - \mu) + \psi u_t.$

Table 2 displays the relative RMSEs compared to results from the AR(1) across all models. Panel A presents results from forecasts at the quarterly frequency to compare with survey forecasts. Panel B presents monthly forecast RMSEs. The results yield several messages. First, different types of model can match the accuracy of out-of-sample *quarterly* survey forecasts. This result is consistent with the results in Faust and Wright (2011) for the US but contrasts with those of Ang, Bekaert, and Wei (2008). Second, the conditional mean model combining the macro variables and two yield factors, level and slope, delivers by far the best performance at the *monthly* frequency. This is especially true at longer horizons (see last column in Panel B).

4.1.1 The added-value of survey's information

Using survey data is essential, and this is especially true for more sophisticated models. Table 3 displays the ratio of the forecast RMSEs obtained when using survey data to the RMSEs obtained without surveys. For instance, including surveys improves forecast RMSEs of the VAR-ULS model by 22% and that of the VARMA-ULS model by 33%. Using survey data is particularly useful for when we move from multivariate VAR to VARMA models where the increase in the number of parameters is more important.

Unsurprisingly, the information from quarterly surveys improves quarterly inflation forecasts. For instance, RMSEs typically decrease by 5-10% at the 1-quarter horizon and by 15-30% at longer horizons. More importantly, using *quarterly* surveys also delivers substantial RMSE improvements for *monthly* forecasts. The evidence is particularly stark for conditional mean models and at long horizons. For instance, including surveys improves the 3-month forecast RMSE from the VARMA-ULS model by 5% but improves the 2-year ahead RMSE by 24%. This arises even if the accuracy of survey forecasts deteriorates with the horizon.

Surveys provide an effective counter-weight to the loss of parsimony associated with conditional mean models (i.e., moving from a VAR(1) to a VARMA(1,1) model). Estimates that neglect survey data suffer from substantial bias and small-sample sampling errors due to its persistence and to the presence of a volatile transitory component (Kim 2007, Kim and Orphanides 2012). Adding surveys to the measurement equations effectively lengthens the sample. The improvements can only arise from better estimates of the underlying dynamics and not from some over-fitting of the survey forecasts. There are no surveys at the 2-year horizon.

Figure 1 gives a visual impression of the importance of using surveys in the estimation. We plot the ratio of forecast RMSEs of a given model estimated using surveys over forecast RMSEs of the same model estimated without surveys. The impressive performance of the surveys-based approach across models is readily apparent.

4.1.2 The added-value of conditional mean models

Conditional mean models (Equation 1) improve inflation forecasts significantly relative to standard auto-regressive models. Table 4 displays the ratio of forecast RMSEs from each conditional mean model to the RMSEs obtained from the corresponding autoregressive model. Panel B and C displays results for monthly and quarterly results when including survey data at estimation. A clear result emerges. Conditional mean models improve upon their restricted VAR counterparts. Panel A displays the result when excluding survey information at estimation. Moving to a conditional mean model improves forecast RMSEs in some cases, but not all cases, which again suggests the importance of using survey data at estimation. But the VARMA-ULS model stands out again. It improves over the corresponding VAR model by 6% and 17% at the 1-year and 2-year horizons, respectively.

Figure 3 gives a visual impression of the importance of the moving average. For a given state vector, we plot the ratio of forecast RMSEs from VARMA over that of VAR. The impressive performance of the MA component across the dimension of the state vector is readily apparent.

4.1.3 From quarterly surveys to monthly forecasts

Quarterly results (Panel A) suggest that the SARMA model is preferable at horizons up to 2 quarters ahead and that the VARMA-UL model is preferable at longer horizons. However, both models improve only marginally relative to surveys. The most relevant question is whether the improved accuracy carries to the monthly frequency, where surveys are not available. Monthly results deliver a clear answer (Panel B). The VARMA models using yield factors deliver 10-15% RMSE improvements at horizons beyond a year, and the VARMA-ULS model, which uses both the level and the slope, eventually dominates with improvements of 20% or more at horizons of 18 months. In contrast, the univariate SARMA does not fare as well at the monthly frequency. The SARMA model is a purely latent conditional mean model that can efficiently combine the information of surveys. However, a multivariate VARMA model also uses information from the shape of the term structure to update its inflation forecasts when there are updated survey forecasts.

Figure 2 gives a visual impression of the importance of using information from unemployment rate and nominal yield curve to forecast inflation. We plot the ratio of forecast RMSEs of a given model over that of an AR(1). The impressive performance of adding information from macroeconomic and nominal yield data is readily apparent. The univariate ARMA model on inflation is dominated by a bivariate VARMA model on inflation and unemployment rate, which in turn is dominated by a VARMA model on inflation, unemployment rate and the level of the nominal yield curve. Finally the most flexible model, VARMA model on inflation, unemployment rate, level and slope of the nominal yield curve, is superior to all the other models. This, again, highlight the importance of using information from nominal yield to update our anticipation of inflation.

4.1.4 Can survey forecasts be improved?

No model systematically improves upon the accuracy of survey forecasts. Table 5 displays the ratio of each model's forecast RMSEs relative to the surveys' RMSEs. Most models perform reasonably well though. The worst performers are the AR and VAR-U model, with, at best, a 1-2% improvement as some horizons and, at worse, a 8-9% deterioration. On the other hand, a VARMA model using yield factors and estimated based on survey data matches or improves survey RMSEs. Figure 6 compares the model forecasts with the CE forecasts and with the realized values of inflation at horizons of 1, 2, 3 and 4 quarters. One-quarter ahead model-forecasts are very close to CE forecasts. Both predict realized inflation very well. The same is true for 2-quarter ahead forecasts but with a small deterioration. Model-implied and CE forecasts are still close to each others at longer horizons. This provides further confirmation that the dynamic term structure model captures inflation expectations accurately. In contrast, Ang, Bekaert, and Wei (2007) and Faust and Wright (2011) conclude that no model can match the accuracy of survey forecasts in US data.[19]

Figure 4 gives a visual impression of the CE forecasts accuracy. We plot the ratio of the forecast RMSEs of a given model over that of CE forecast RMSEs. All the models do as well as CE forecasts at the quarterly frequency but, conversely, CE forecasts are hard to outperform when available.

4.2 The Effect of Yields on Inflation and Unemployment

This Section explores the mechanism which allows our preferred specification to out-perform more parsimonious models out-of-sample. The answer lies in the information content of the yield curve. Consider the implications for the dynamics of inflation and unemployment

[19]Survey forecasts exhibit systematic errors ex-post. Ang, Bekaert, and Wei (2007) show that linear or non-linear adjustment do not improve unadjusted out-of-sample inflation forecasts.

expectations,

$$m_{x,t} = \phi_{xx}(m_{x,t-1} - \mu_x) + \phi_{xy}(m_{y,t-1} - \mu_y) + \psi_{xx}u_{x,t} + \psi_{xy}u_{y,t}. \quad (36)$$

Last period's expectations for today's values of the macro variables, $m_{x,t-1}$, and last period's expectations of today's shape of the yield curve, $m_{y,t-1}$, respectively, affect the update of inflation expectation differently than the corresponding surprise components, $u_{x,t}$ and $u_{y,t}$, respectively. This is a distinctive feature of a conditional mean model. Contrast Equation 36 with the corresponding equation in a VAR model,

$$\begin{aligned} m_{x,t} &\equiv E_t[x_{t+1}] \\ &= \phi_{xx}(x_t - \mu_x) + \phi_{xy}(y_t - \mu_y) \\ &= \phi_{xx}(m_{x,t-1} - \mu_x) + \phi_{xy}(m_{y,t-1} - \mu_y) + \phi_{xx}u_{x,t} + \phi_{xy}u_{y,t}, \end{aligned}$$

where $\phi = \psi$ and last period's expectations for today's state variables, m_{t-1}, have the same effect on the updated expectation as the innovations u_t.

But the estimates imply a remarkable difference between ψ and ϕ. Table 6 displays parameter estimates. The matrices ϕ and ψ are given in Panel (A) with standard errors in parenthesis. First, the estimate of ϕ implies that inflation expectations and unemployment expectations are persistent ($\phi_\pi = 0.84$ and $\phi_g = 0.96$). Second, the first column of ψ is insignificant and very close to zero in magnitude. In other words, the innovation in current inflation has almost no effect on the conditional expectations of inflation and of unemployment. Hence, the intuition from Kim (2007) that inflation combines a persistent conditional mean component with transitory noise carries over in our multivariate context. Third, since inflation innovations have little effect, inflation expectation updates are driven by innovations in the other variables. Their effect is given on the first line of ψ. Innovations in the unemployment rate, the level of yields and the slope of yields each have large and significant effects on inflation expectation updates. For instance, a one-standard deviation surprise increase in the level or a similar surprise decrease in the slope of the yield curve lowers inflation expectations by 0.35% and by 0.13%, respectively (all else constant). Fourth, in contrast with the effect of innovations, expected increases in the level of the curve are associated with expected increases in inflation. The initial effects of expectation changes are smaller but cumulate over time via the persistence of the level and slope factors. The effect of expected changes in the slope is insignificant.

The effects are consistent with intuition. Monetary policy shocks affect the inflation expectation downward. Yet, predictable increases in expected inflation are positively correlated with the level and slope of the yield curve. The conditional mean model separates these two effects parsimoniously. Similarly, the expected and surprise components of today's yield curve do not have the same effect on the expected unemployment rate. In particular, a surprise decrease in the slope of the yield curve is associated with a significant increase of the expected unemployment rate. On the other hand, an expected change in the slope bears almost no effect on the expected unemployment rate. This is consistent with results from Ang et al. (2006) showing that the level of interest rate is the best predictors of future real activity in the context of a terms structure model.

Finally, the standard deviations and correlations implied from the estimate of the covariance matrix $\Sigma\Sigma'$ and $\psi\Sigma\Sigma'\psi'$ are given in Panel (B) (standard deviations are given on the diagonal). Again, while the correlations between inflation innovations and other innovations are low, the correlations between updates to inflations expectations and updates to expectations of other variables are high. This highlights the role of the matrix ψ in the update of expected inflation.

4.3 Real Yields and the Inflation Risk Premium

Figure 8 displays the loadings on nominal yields. By construction, the level factor has a constant unit loading and the loadings of the slope factor decreases smoothly with maturity. The effect of macro variables on yields is most important at shorter maturities. The generalized Fisher equation,

$$i_t^{(n)} = c^{(n)} + m_t^{\pi,(n)} + r_t^{(n)} + irp_t^{(n)}, \tag{37}$$

derived in Section 1 (see Equation 17), shows that a successful decomposition of yields involves three components: a real yield, expected inflation and the inflation risk premium. The previous Section established that the model delivers an accurate measure of $m_t^{\pi,(n)}$, which in turns implies that the model delivers an accurate measure of the sum of the real yield and the inflation risk premium, $r_t^{(n)} + irp_t^{(n)}$. Heuristically, we can then subtract the Jensen term and our measure of $m_t^{\pi,(n)}$ from both sides of the decomposition,

$$i_t^{(n)} - c^{(n)} - m_t^{\pi,(n)} = r_t^{(n)} + irp_t^{(n)}. \tag{38}$$

and focus on estimates of the remaining components from the model, relying on the restrictions imposed on the prices of risk to obtain an accurate decomposition of the sum on the right hand side.

Table 7 displays summary statistics of nominal yields, real yields, inflation expectations and the inflation risk premiums computed from the model across a range of maturities between three months and five years. The average nominal yield curve is upward sloping – with an average slope close to 0.85%. But the relatively low average slope hides substantial variation over the business cycle. The average volatility is downward sloping, ranging from 3.27% to 2.56%. The average slope and volatility of the nominal curve is attributable to the level and volatility of the real curve. Moreover, their persistence is similar. Average expected inflation is mostly flat across maturities, slightly above 2% across the sample, adding little to the average slope. The inflation risk premium averages between 0.44% and 0.57% with an upward slope, adding 13 bps to the difference between the slopes of real and nominal yields.

Figure 9 displays the time-series of each component of yields – the real yield, expected inflation and inflation risk premium – for maturities of three month, one year, two years and five years. By and large, the large business cycle variations in the nominal yield curve are attributed to variations in the underlying real curve. The inflation risk premium is the least volatile component of nominal yields at short maturities and where it follows closely variations of expected inflation. The contribution of the inflation risk premium increases with the maturity and the sources of its variations also change. Figure 10 displays the contribution of each component of m_t to the variations of the 2-year inflation risk premium. At this horizon, the expected inflation and the slope factor have little effect on the inflation risk premium. Instead, the premium is high when unemployment is high or when the level of the yield curve is low– in relatively poor states of the economy.[20]

What is the source of risk behind the importance of the unemployment rate and the level factor for the variation of the longer-horizon inflation risk premium? The loadings on 1-month the inflation risk premium in Figure 10 combine the price of risk with the covariance matrix,

$$m_t^Q - m_t = (\Sigma\Sigma')(\tilde{\lambda}_0 + \tilde{\lambda}_1 m_t)$$

since the effect of each component on the inflation risk premium is determined by the product $(\Sigma\Sigma')\tilde{\lambda}_1$. A similar decomposition obtains at longer horizons but where the covariance matrix and the price of risk coefficients depend on the horizon.

Panel (A) and Panel (B) of Figure 12 decomposes the effect of unemployment and of

[20]Interestingly, the inflation risk premium has been persistently negative in the short period from the beginning of the sample until the Bank of Canada announced a new inflation targeting regime. This is consistent with an interpretation where an increase in expected inflation does not announce a tightening phase by the central bank due to an insufficient response to inflation. Exposures to inflation risk become hedges.This may also arise, for instance, if the representative agent has Epstein-Zin preferences. Inflation is always risky for a CRRA agent - inflation enters negatively in the SDF. However, higher inflation may lead to revisions of the continuation value in the Epstein-Zin SDF. In this context, changes in the sign of the inflation risk premium require changes in the covariance between current inflation and future consumption growth.

the level factor, respectively, on the 2-year inflation risk premium. Each line of Panel (A) draws the products of the effect of unemployment rate on the price of risk of a given variable with the covariance of this variable with inflation innovations across different horizons. The results show that inflation risk is the main channel behind the role of unemployment in the inflation risk premium. Across different horizons, the effects of the level factor and of the unemployment rate on the inflation risk premium work primarily through their effects on the price of inflation risk. The pricing kernel gives lower value to payoffs in states with inflation shocks, and the value decreases when expected unemployment is higher or the level of the curve is lower.

Conclusion

Our results show that the main conclusion in Faust and Wright (2011) carries over to the case of Canada : using information from CE allows even simple models to capture the predictive content from survey. We also show how to combine the conditional mean representation in Fiorentini and Sentana (1998) with survey and yield data to produce superior out-of-sample forecasts even when no updated survey is available. Finally, we develop a dynamic term structure model to decompose the yield curve into real yields, inflation expectations, and the inflation risk premium at different maturities.

We leave for future research several important extensions. First, we focus on short horizons - typically less than two years. Longer horizon forecasts must handle shifting endpoints (Kozicki and Tinsley, 2001) since the long-horizon inflation forecasts have declined in the first half of the sample. But we must also accommodate long episodes where the median long-horizon CE forecast remains constant. Second, our analysis relies on models with constant variance. Still, a large literature studies the relationship between uncertainty about inflation and future inflation. Moreover, a recent literature sees the dispersion of inflation forecasts among survey respondents as an important signal of inflation risk. Extending the model to account for time-varying uncertainty may help us identify a more fundamental role of inflation risk using yield data. Finally, we focused on inflation forecasts. But conditional forecasts of future inflation rates are not independent of forecasts of future unemployment rates (or other macro variables). Extending our approach to include CE forecasts of unemployment could help identify the dynamic relationships between economic activity, inflation and risk as perceived by bond investors.

References

Ajello, A., L. Benzoni, and O. Chyruk (2012). Core and 'crust': Consumer prices and the term structure of interest rates. Federal Reserve Bank of Chicago.

Amano, R. and S. Murchison (2006). Factor-market structure, shifting inflation targets, and the new keynesian phillips curve. In *Issues in Inflation Targeting*, pp. 89–109. Bank of Canada.

Ang, A., G. Bekaert, and M. Wei (2007). Do macro variables, asset markets, or surveys forecast inflation better? *Journal of Monetary Economics 54*(4), 1163 – 1212.

Ang, A., G. Bekaert, and M. Wei (2008). The term structure of real rates and expected inflation. *The Journal of Finance 63*(2), 797–849.

Ang, A., S. Dong, and M. Piazzesi (2007). No-arbitrage taylor rules. Working Paper 13448, National Bureau of Economic Research.

Ang, A. and M. Piazzesi (2003). A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables. *Journal of Monetary Economics 50*(4), 745 – 787.

Ang, A., M. Piazzesi, and M. Wei (2006). What does the yield curve tell us about gdp growth? *Journal of Econometrics 131*(12), 359–403.

Bansal, R. and I. Shaliastovich (2010). A long-run risks explanation of predictability puzzles in bond and currency markets. The Wharton School of Business, University of Pennsylvania.

Bansal, R. and A. Yaron (2004). Risks for the long run: A potential resolution of asset pricing puzzles. *The Journal of Finance 59*, 1481–1509.

Buraschi, A. and A. Jiltsov (2005). Inflation risk premia and the expectations hypothesis. *Journal of Financial Economics 75*(2), 429–490.

Campbell, J., R. Shiller, and L. Viceira (2009). Understanding inflation-indexed bond markets. Working Paper 15014.

Chen, R.-R. and L. Scott (1993). Maximum likelihood estimation for a multifactor equilibrium model of the term structure of interest rates. *The Journal of Fixed Income 3*, 14–31.

Chernov, M. and P. Mueller (2011). The term structure of inflation expectations. *Journal of Financial Economics*. *forthcoming*.

Christensen, J., F. Diebold, and G. Rudebusch (2007). The affine arbitrage-free class of Nelson-Siegel term structure models. Technical Report 13611, NBER.

Christensen, J. and J. M. Gillan (2011). A model-independent maximum range for the liquidity correction of tips yields. Working Paper Series 2011-16, Federal Reserve Bank of San Francisco.

Christensen, J. H. E., J. Lopez, and G. Rudebusch (2010). Inflation expectations and risk premiums in an arbitrage-free model of nominal and real bond yields. *Journal of Money, Credit and Banking 42*, 143–178.

Chun, A. L. (2011). Expectations, bond yields, and monetary policy. *Review of Financial Studies 24*(1), 208–247.

Dai, Q. and K. Singleton (2000). Specification analysis of affine term structure models. *The Journal of Finance 55*, 1943–1978.

Day, J. and R. Lange (1997). The structure of interest rates in canada: Information content about medium-term inflation. Technical Report 97-10, Bank of Canada.

Duffee, G. (2010). Sharpe ratios in term structure models. *working paper*.

Duffee, G. (2011). Information in (and not in) the term structure. *Review of Financial Studies*. *forthcoming*.

Faust, J. and J. Wright (2011). *Forecasting Inflation*. in Handbook of Economic Forecasting. Elsevier.

Feunou, B. and J. Fontaine (2012). A no-arbitrage VARMA term structure model with macroeconomic variables.

Fiorentini, G. and E. Sentana (1998). Conditional means of time series processes and time series processes for conditional means. *International Economic Review 39*(4), pp. 1101–1118.

Fisher, I. (1930). *The theory of interest*. New York: The Macmillan Co.

Fleckenstein, M., F. Longstaff, and H. Lustig (2012). The tips-treasury bond puzzle. Working paper.

Fung, B., S. Mittnick, and E. Remolona (1999). Uncovering inflation expectations and risk premiums from internationally integrated financial markets. Technical Report 99-6, Bank of Canada.

Garcia, R. and R. Luger (2007). The canadian macroeconomy and the yield curve: an equilibrium-based approach. *Canadian Journal of Economics 40*(2), 561–583.

Hasseltoft, H. (2012). Stocks, bonds, and long-run consumption risks. *47*(02), 309–332.

Haubrich, J., G. Pennacchi, and P. Ritchken (2012). Inflation expectations, real rates, and risk premia: Evidence from inflation swaps. *Review of Financial Studies 25*(5), 1588–1629.

Joslin, S., K. H. Singleton, and H. Zhu (2011). A new perspective on gaussian dynamic term structure models. *Review of Financial Studies 24*, 926–970.

Kim, D. H. (2007). Challenges in macro-finance modeling. Working Paper BIS 240, Bank of International Settlements.

Kim, D. H. and A. Orphanides (2012). Term structure estimation with survey data on interest rate forecasts. *47*(01), 241–272.

Kozicki, S. and P. Tinsley (2001). Shifting endpoints in the term structure of interest rates. *Journal of Monetary Economics 47*(3), 613 – 652.

Kozicki, S. and P. Tinsley (2006). Survey-based estimates of the term structure of expected u.s. inflation. Working Paper 2006-46, Bank of Canada.

Nelson, C. and A. Siegel (1987). Parsimonious modeling of yield curves. *The Journal of Business 60*, 473–489.

Pennacchi, G. (1991). Identifying the dynamics of real interest rates and inflation: evidence using survey data. *Review of Financial Studies 4*(1), 53–86.

Pflueger, C. E. and L. M. Viceira (2011). An empirical decomposition of risk and liquidity in nominal and inflation-indexed government bonds. Working Paper 16892, NBER.

Piazzesi, M. (2005a). *Affine term structure models*, Chapter 12. in Handbook of Financial Econometrics. Elsevier.

Piazzesi, M. (2005b). Bond yields and the federal reserve. *113*, 311–344.

Piazzesi, M. and E. Schneider (2006). Equilibrium yield curves. Working Paper 12609, NBER.

Ragan, C. (1995). Deriving agents' inflation forecasts from the term structure of interest rates. Working paper, Bank of Canada.

Stock, J. H. and M. W. Watson (2003). Forecasting output and inflation: The role of asset prices. *Journal of Economic Literature 41*(3), pp. 788–829.

Stock, J. H. and M. W. Watson (2007). Why has US inflation become harder to forecast? *Journal of Money, Credit and Banking 39*, 3–33.

A Appendix

A.1 Risk-Neutral Dynamics

The shift between m_t and m_t^Q is given by,

$$m_t^Q = \bar{\lambda}_0 + \bar{\lambda}_1 m_t, \tag{39}$$

where $\bar{\lambda}_0 = \Sigma\Sigma'\tilde{\lambda}_0$ and $\bar{\lambda}_1 = \Sigma\Sigma'\tilde{\lambda}_1 + I_4$. The dynamics of z_t under the risk-neutral measure, $\mathbb{Q}$, is given by:

$$\begin{aligned} z_{t+1} &= m_t^Q + u_{t+1}^Q \\ m_{t+1}^Q &= \tilde{\mu}^Q + \tilde{\phi}^Q \left(m_t^Q - \tilde{\mu}^Q \right) + \tilde{\psi}^Q u_{t+1}^Q, \end{aligned} \tag{40}$$

where

$$\begin{aligned} \tilde{\omega}^Q &= \left(\bar{\lambda}_1 + I_4\right) \left[\omega + (I_4 - \theta) \left(\bar{\lambda}_1 + I_4\right)^{-1} \bar{\lambda}_0 \right] \\ \tilde{\phi}^Q &= \left(\bar{\lambda}_1 + I_4\right) \left(\phi + \psi\bar{\lambda}_1\right) \left(\bar{\lambda}_1 + I_4\right)^{-1} \\ \tilde{\psi}^Q &= \psi + \bar{\lambda}_1 \psi. \end{aligned} \tag{41}$$

with $\tilde{\omega}^Q = \left(I_4 - \tilde{\phi}^Q\right) \tilde{\mu}^Q$. The dynamics of m_t under the risk-neutral measure is given by:

$$\begin{aligned} z_{t+1} &= m_t^Q + u_{t+1}^Q \\ m_{t+1} &= \mu^Q + \phi^Q \left(m_t - \mu^Q\right) + \psi^Q u_{t+1}^Q \end{aligned} \tag{42}$$

with $u_{t+1}^Q = \Sigma\varepsilon_{t+1}^Q$ and where parameters are given by:

$$\begin{aligned} \omega^Q &= \omega + \psi\bar{\lambda}_0 \\ \phi^Q &= \phi + \psi\bar{\lambda}_1 \\ \psi^Q &= \psi. \end{aligned} \tag{43}$$

with $\omega^Q \equiv \left(I_4 - \phi^Q\right) \mu^Q$ and $\omega \equiv (I_4 - \phi)\, \mu$.

A.2 Yield Coefficient Recursions - General Case

Coefficients of real yields are given by the following recursions,

$$\begin{aligned} b_{r,n+1} =& \frac{1}{n+1}\left(n\phi^{Q\prime} b_{r,n} + \delta_r\right) \\ a_{r,n+1} =& \frac{1}{n+1}\left(\bar{r}_r + n a_{r,n} + n b_{r,n}^{\prime}\left(\mu^Q - \phi^Q \mu^Q\right) - n^2 \frac{b_{r,n}^{\prime} \Sigma_m \Sigma_m^{\prime} b_{r,n}}{2}\right) \end{aligned} \tag{44}$$

for $n > 0$, with initial conditions $a_{r,0} = 0$ and $b_{r,0} = 0$, and where we defined $\Sigma_m \equiv \psi\Sigma$. Similarly, the coefficient of nominal yields are given by,

$$\begin{aligned} b_{n+1} =& \frac{1}{n+1}\left(n\phi^{Q\prime} b_n + \delta\right) \\ a_{n+1} =& \frac{1}{n+1}\left(\bar{r} + n a_n + n b_n^{\prime}\left(\mu^Q - \phi^Q \mu^Q\right) - n^2 \frac{b_n^{\prime} \Sigma_m \Sigma_m^{\prime} b_n}{2}\right) \end{aligned} \tag{45}$$

for $n > 0$, with initial conditions $a_0 = 0$ and $b_0 = 0$.

A.3 Yields Observed without Error

Consider M linear combinations of J observed nominal yields ($M \leq J$) that are observed without error, i.e.,

$$Y_t^o = w^o Y_t = a^o + b^0 m_t,$$

where Y_t^o is a 2×1 vector, Y_t is $J \times 1$ vector, w^o is a $2 \times J$ matrix. Define a new state vector with observed components,

$$z_t^o = \begin{pmatrix} x_t \\ Y_t^o \end{pmatrix},$$

then the dynamics of z_t^o is given by

$$\begin{aligned} z_{t+1}^o &= m_t^o + u_{t+1}^o \\ m_{t+1}^o &= \mu_w + \phi^o\left(m_t^o - \mu^o\right) + \psi^o u_{t+1}^o, \end{aligned}$$

with parameters given by,

$$\Sigma_o = \begin{bmatrix} e \\ b\psi \end{bmatrix} \Sigma \quad \text{and } e = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\mu^o = \begin{pmatrix} e\mu \\ a + b\mu \end{pmatrix}, \quad \psi^o = \begin{bmatrix} e \\ b\phi \end{bmatrix} \psi \begin{bmatrix} e \\ b\psi \end{bmatrix}^{-1}$$

$$\theta^o = \begin{bmatrix} e \\ b\phi \end{bmatrix} \left(\theta - \psi \begin{bmatrix} e \\ b\psi \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ b\theta \end{bmatrix} \right) \begin{bmatrix} e \\ b\phi \end{bmatrix}^{-1},$$

and $\phi^o = \psi^o + \theta^o$. Furthermore, the m_t^o is an affine transformation of m_t,

$$m_t^o = \gamma_0 + \gamma m_t$$

where,

$$\gamma_0 = \begin{pmatrix} 0_2 \\ w(a + b(\mu - \phi\mu)) \end{pmatrix}, \quad \gamma = \begin{pmatrix} e \\ b\phi \end{pmatrix}.$$

A.4 Nominal Yield Decomposition

$$i_t^{(n)} = r_t^{(n)} + E_t \left[\frac{\sum_{j=1}^{n} \pi_{t+j}}{n} \right] + E_t^Q \left[\frac{\sum_{j=1}^{n} \pi_{t+j}}{n} \right] - E_t \left[\frac{\sum_{j=1}^{n} \pi_{t+j}}{n} \right] + c_n$$

where $c_n = \frac{1}{2}\sigma_\pi^2 + \frac{1}{2n}\sum_{j=0}^{n-1} j^2 \left(b'_{r,j}\Sigma_m \Sigma'_m b_{r,j} - b'_j \Sigma_m \Sigma'_m b_j \right)$ and $\Sigma_m = \psi\Sigma$.

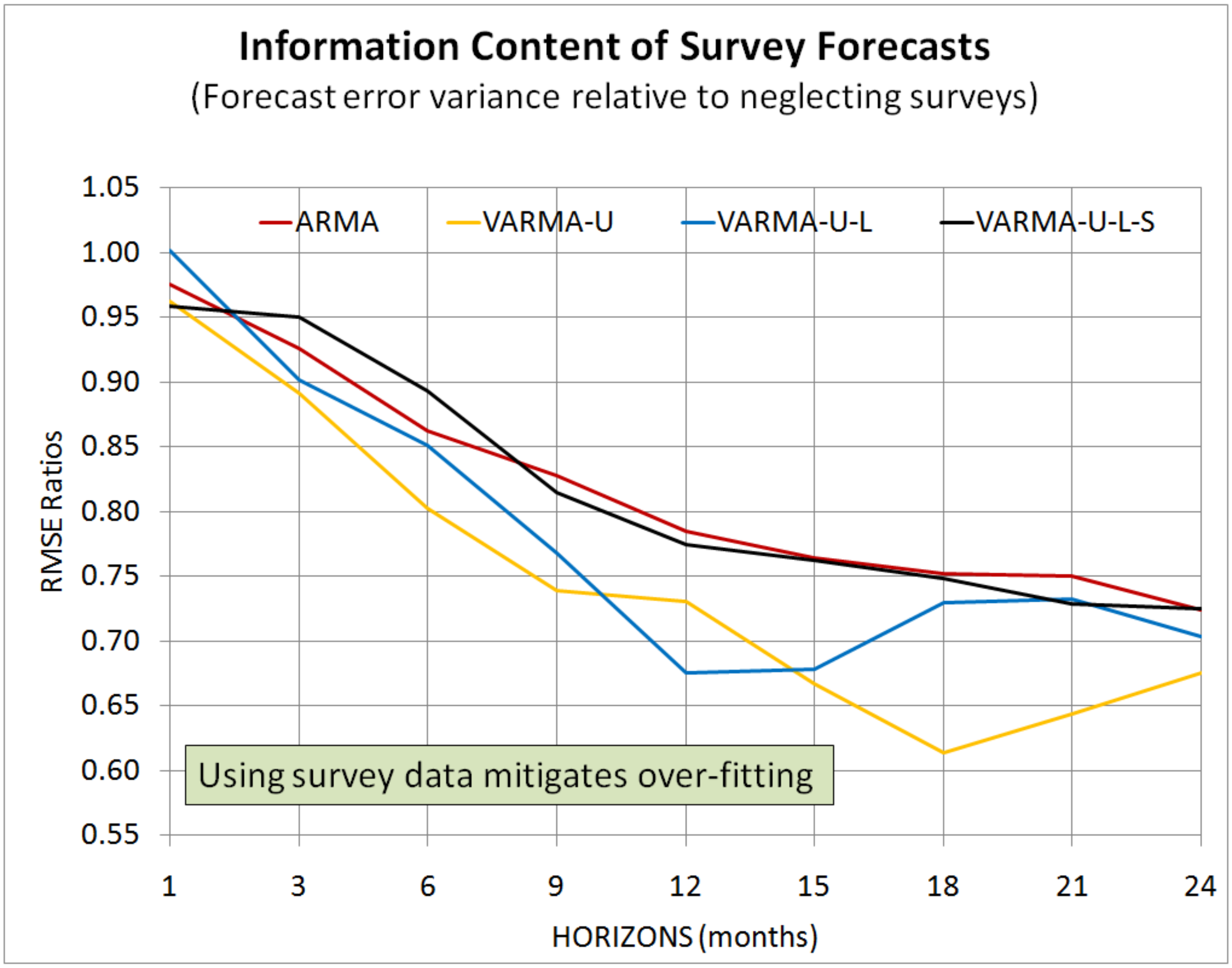


Figure 1: **Information Content of Survey Forecasts:** For each of the models and forecasting horizons we investigate, we plot the ratio of forecast RMSE when the model is estimated using surveys over that of the same model when estimated while neglecting surveys.

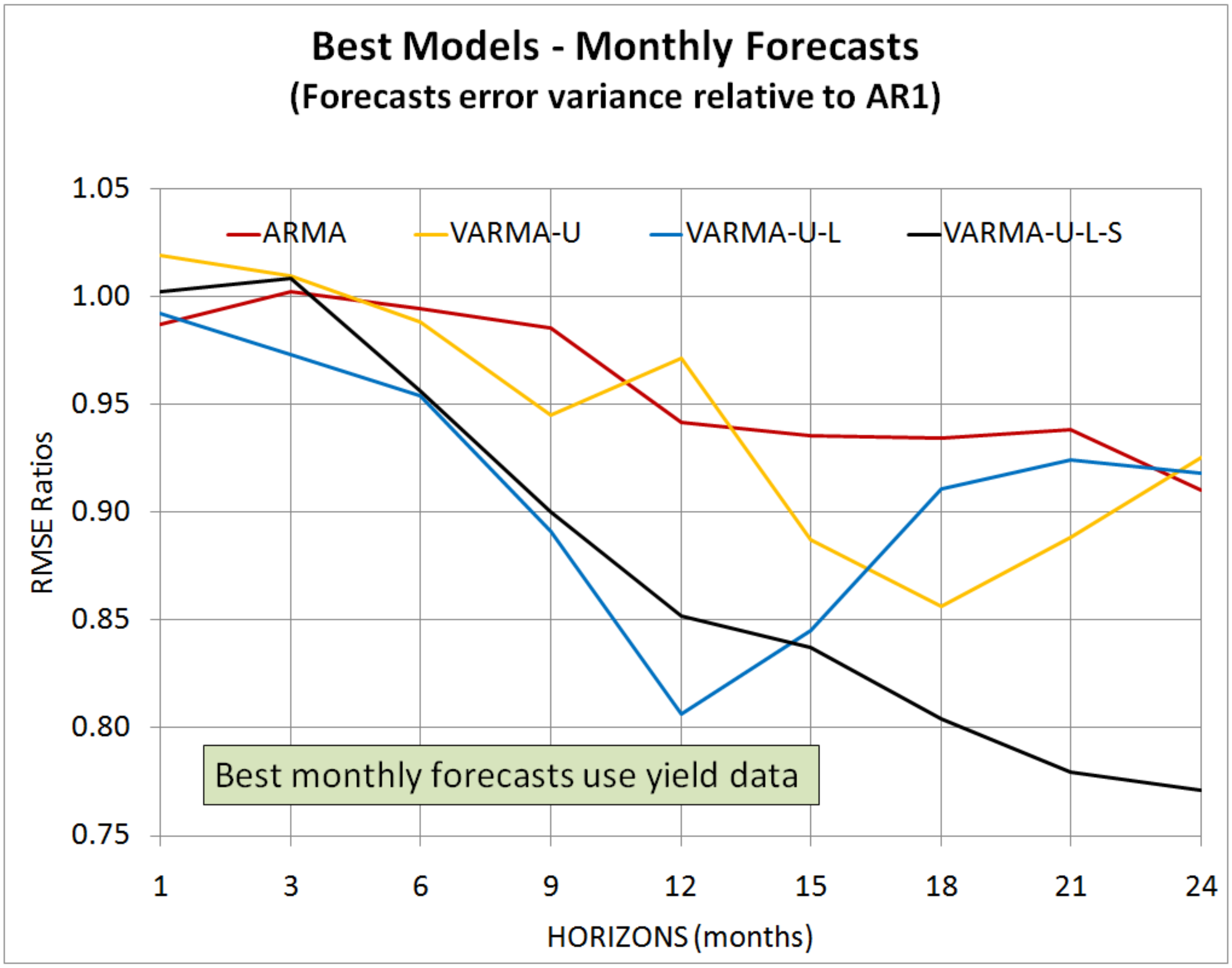


Figure 2: **Information Content of Survey Forecasts:** For each of the models and forecasting horizons we investigate, we plot the ratio of forecast RMSE over that of AR(1). All the models are estimated using surveys.

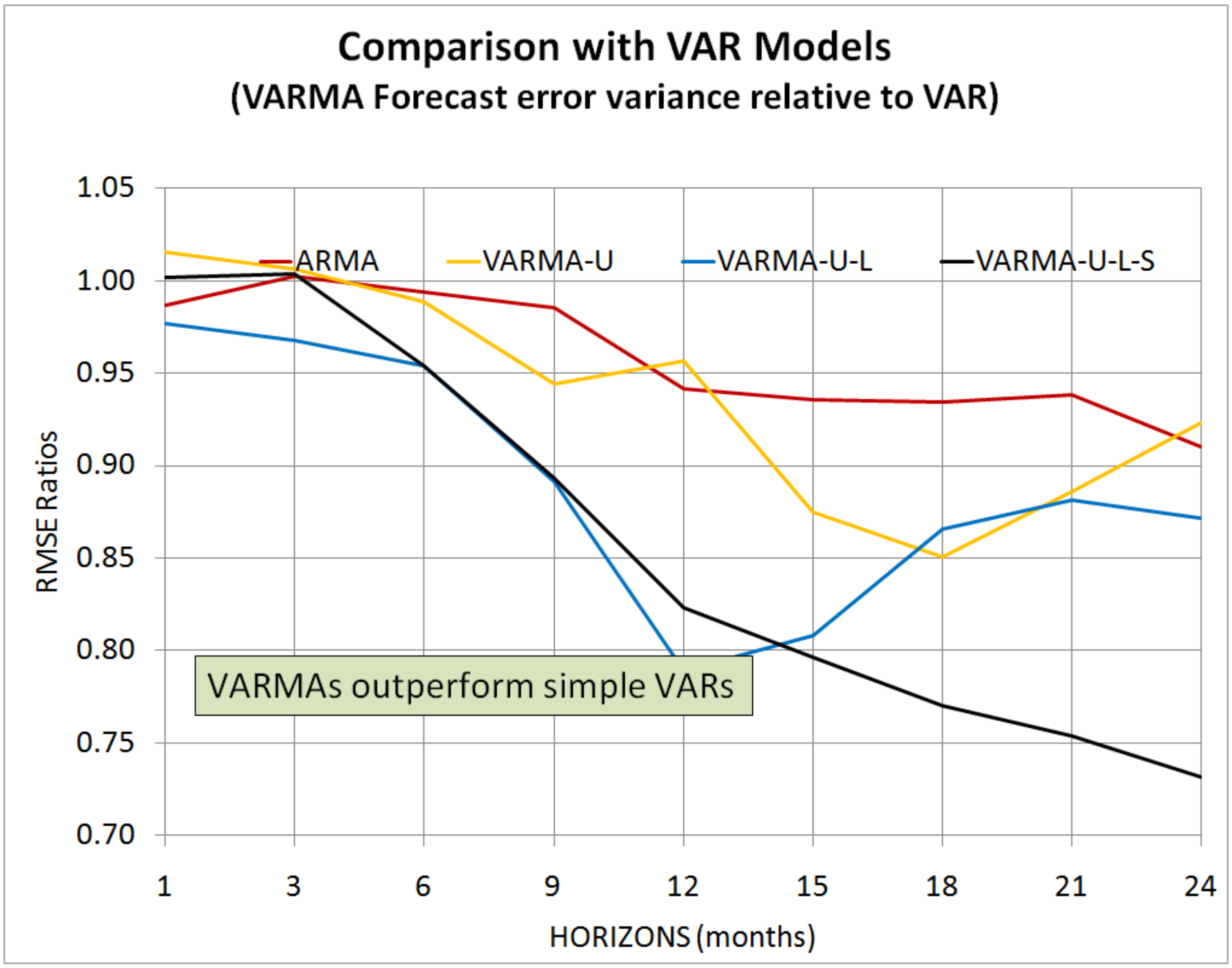


Figure 3: **VAR vs VARMA:** For each of the state vector z_t and forecasting horizons we investigate, we plot the ratio of forecast RMSE from VARMA over that of VAR. All the models are estimated using surveys.

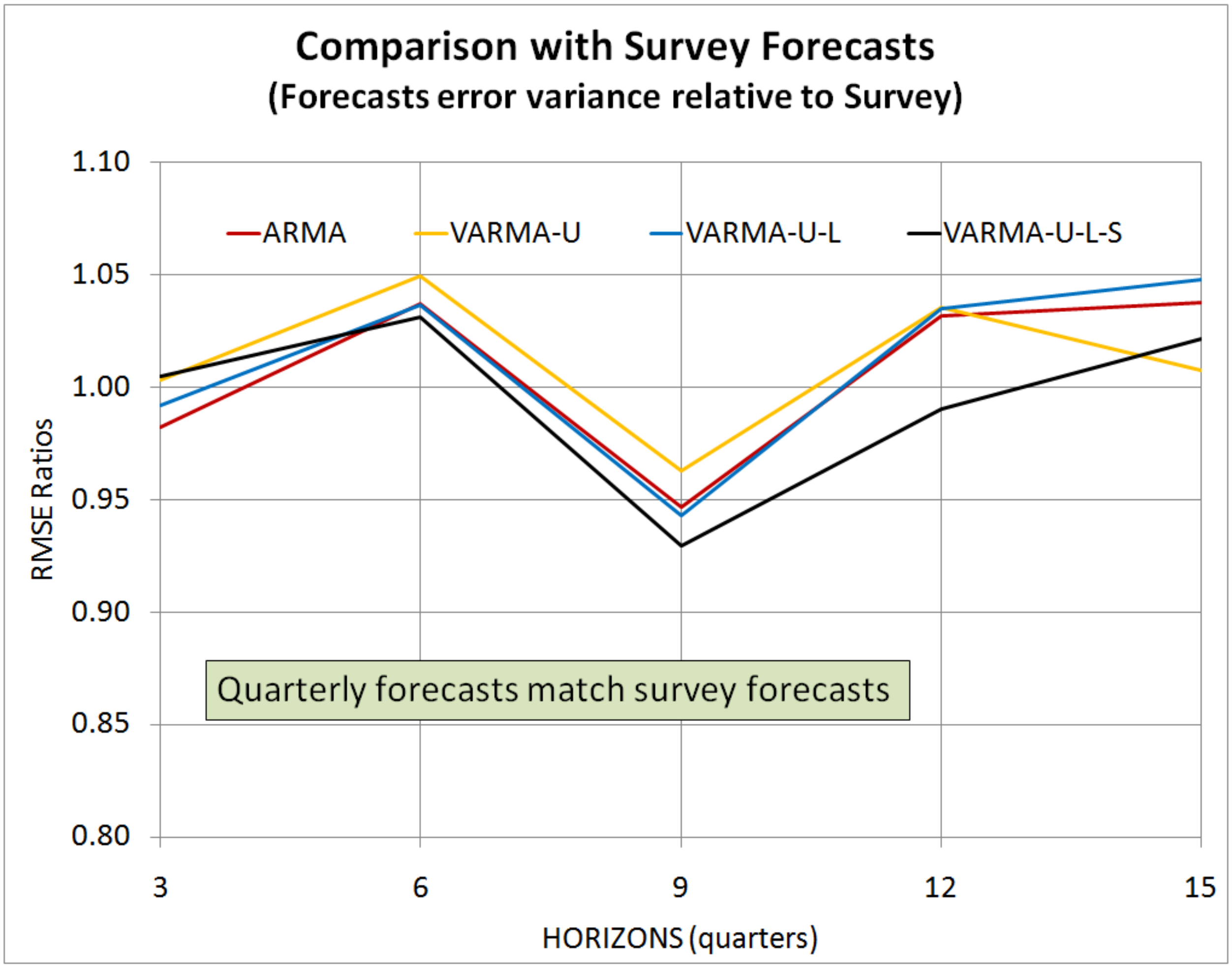


Figure 4: **Comparison with Surveys:** For each of the models and forecasting horizons we investigate, we plot the ratio of forecast RMSE over that of surveys. All the models are estimated using surveys. Because surveys data are available only quarterly, all the forecast errors are sampled quarterly prior to computing the RMSEs.

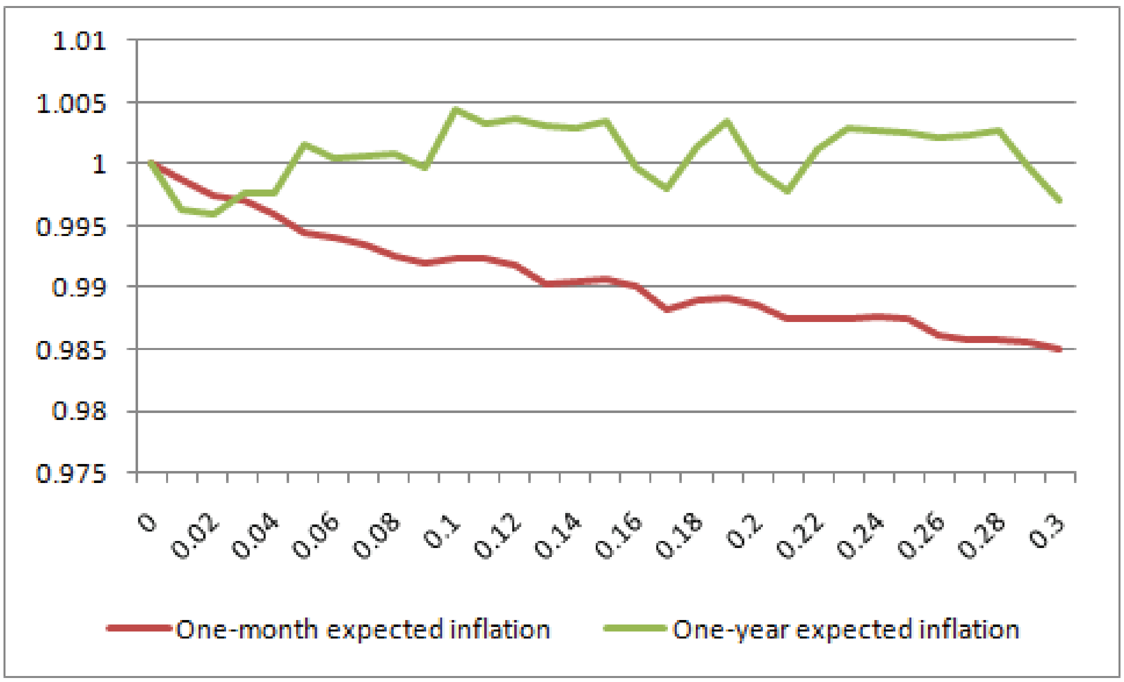


Figure 5: **Inflation Forecasts RMSEs:** For the no-arbitrage model, we plot the 1-month and 1-year monthly inflation forecast RMSEs varying the boundaries on the one-month ahead inflation Sharpe ratio distribution $-\underline{SR} = \overline{SR} = \kappa$, where κ range between 0 and 0.3.

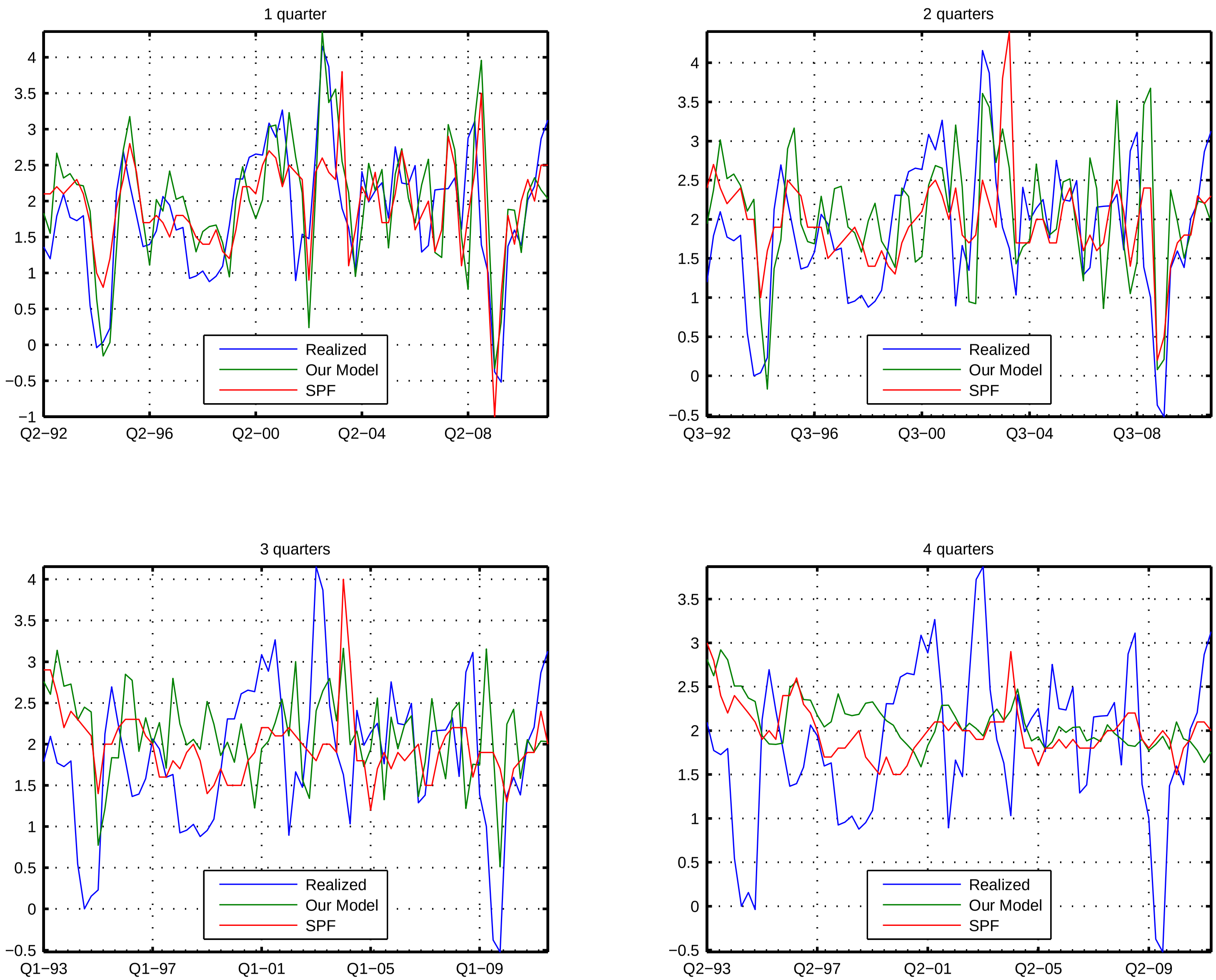


Figure 6: **Inflation and Inflation Expectations:** We plot the Quarterly inflation, survey forecasts and no-arbitrage model-implied inflation expectations, from Q1-1992 to Q2-2012, at horizons of 1, 2, 3 and 4 quarters ahead.

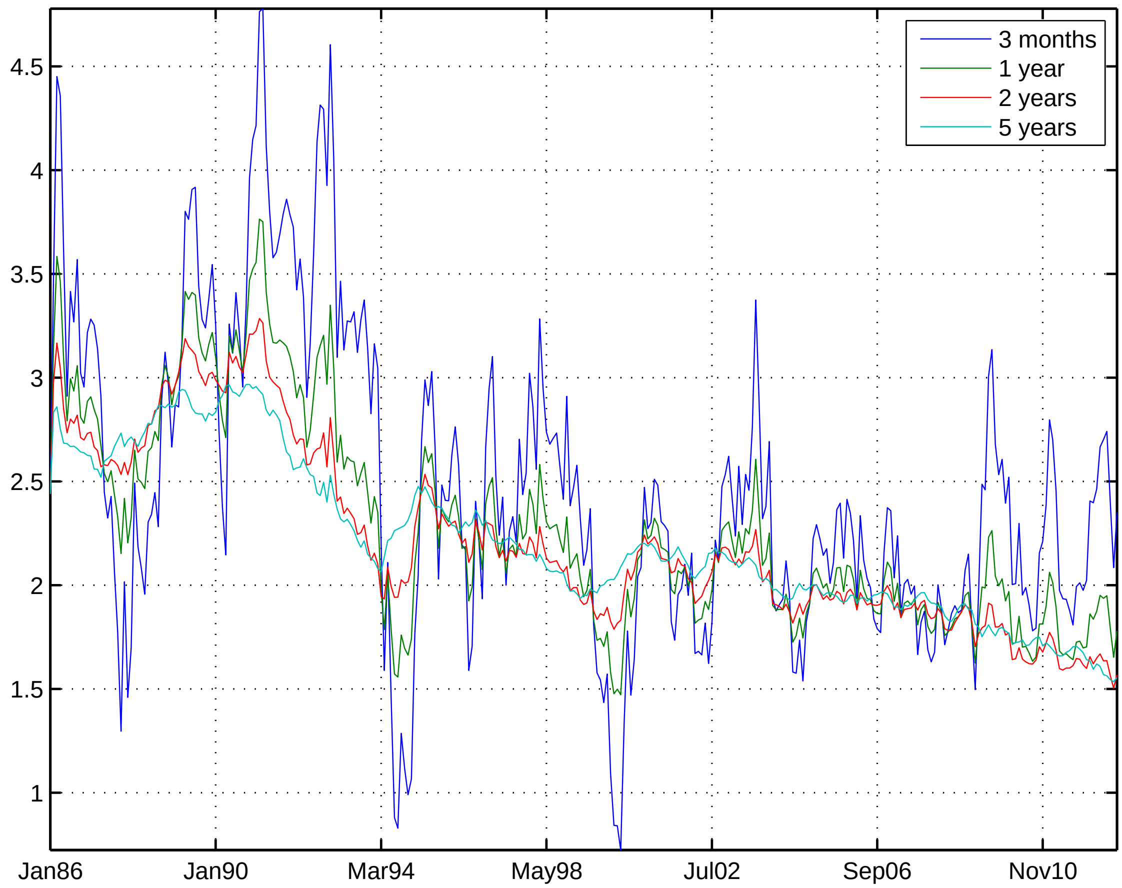


Figure 7: **Monthly Inflation Expectations:** For the no-arbitrage model, we plot the monthly expectation of average inflation, from January 1986 to May 2012, over the horizons 3 months, 1 year, 2 years and 5 years.

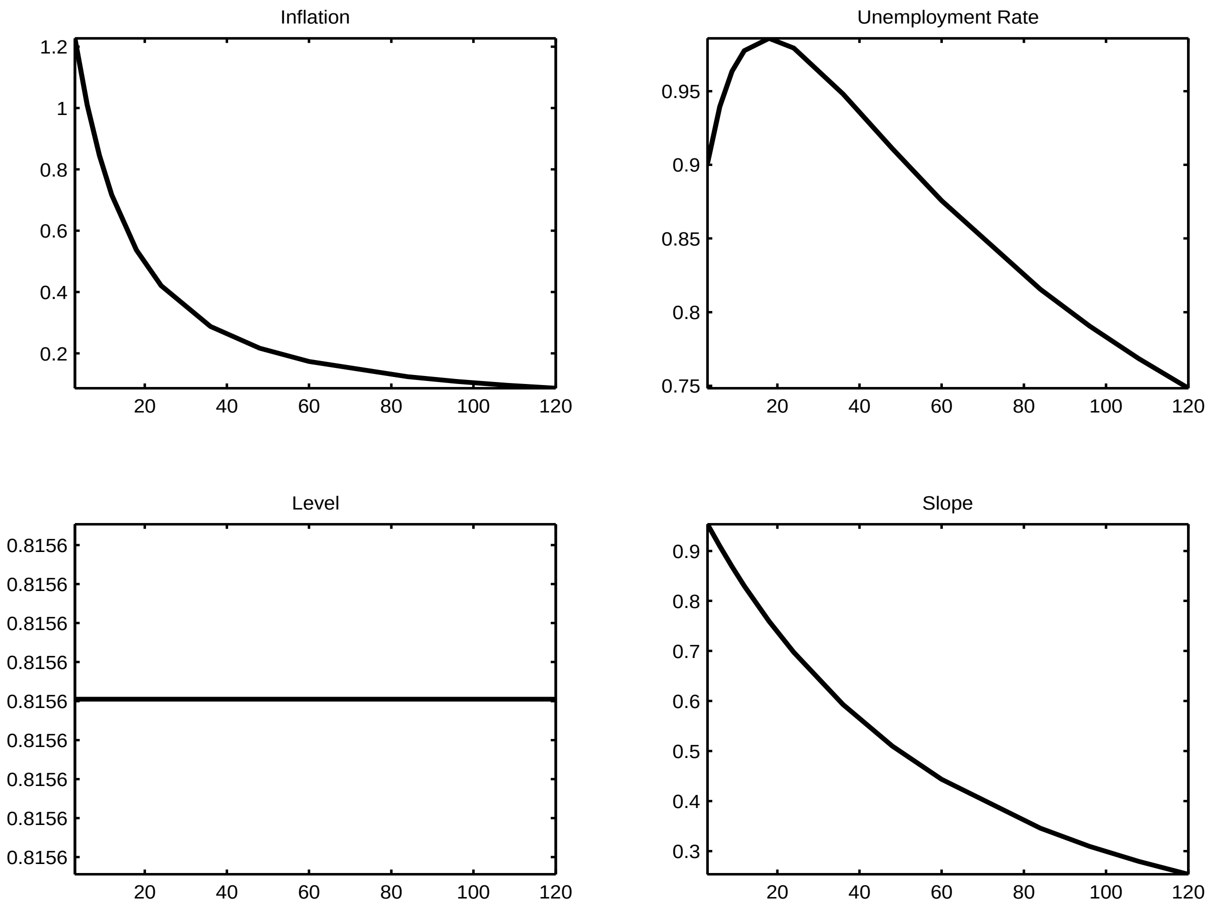


Figure 8: **Nominal Yield Loadings:** For the no-arbitrage model, we plot the nominal yield loadings on inflation expectations, unemployment expectations, the level and slope of the real curve for maturities ranging from three months to ten years.

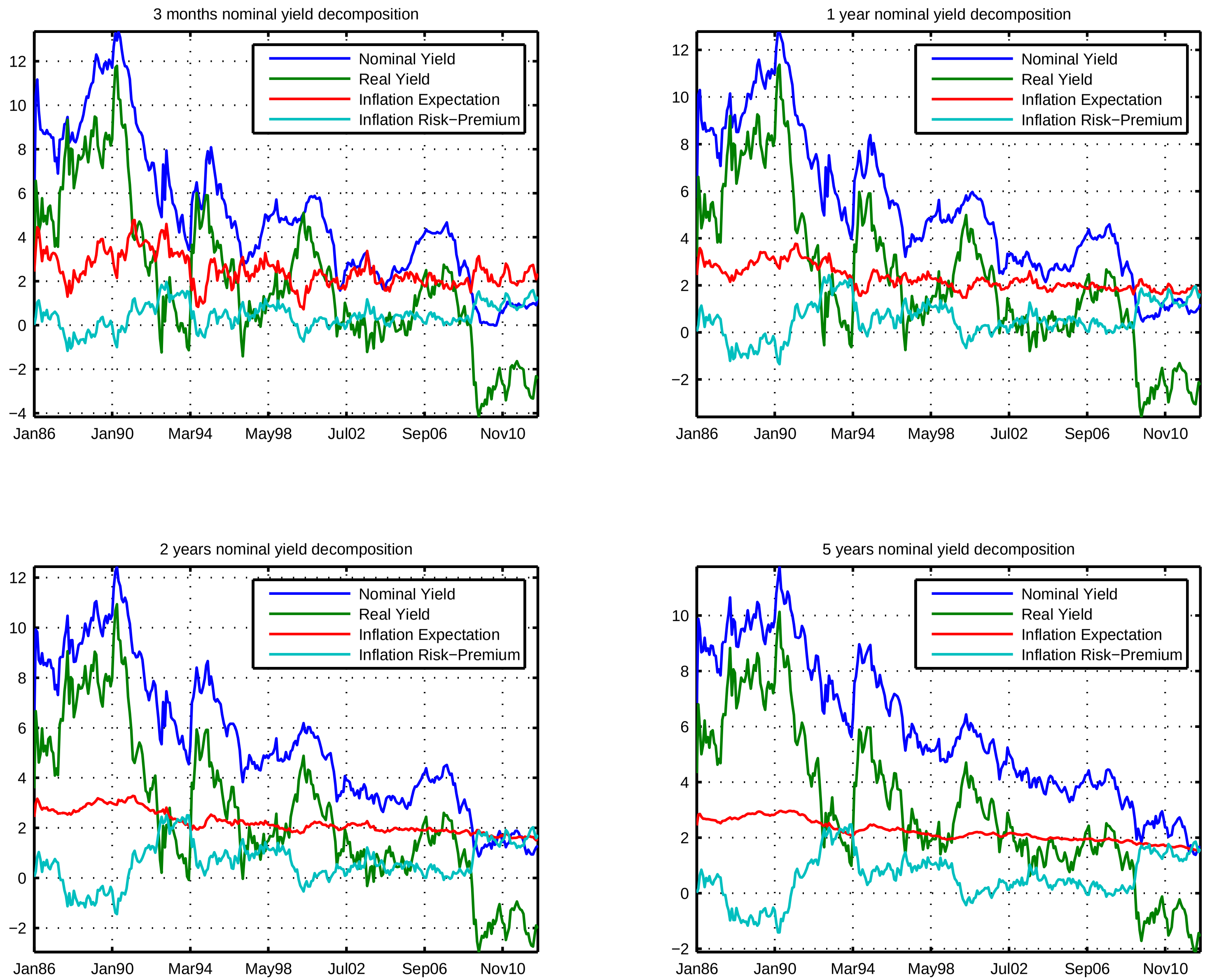


Figure 9: **Nominal Yield Decomposition:** For the no-arbitrage model, we plot the decomposition of the 3-months, 1-year, 2-years and 5-years nominal yields, from January 1986 to May 2012, into its components: a real yield, the inflation expectation and the inflation risk premium.

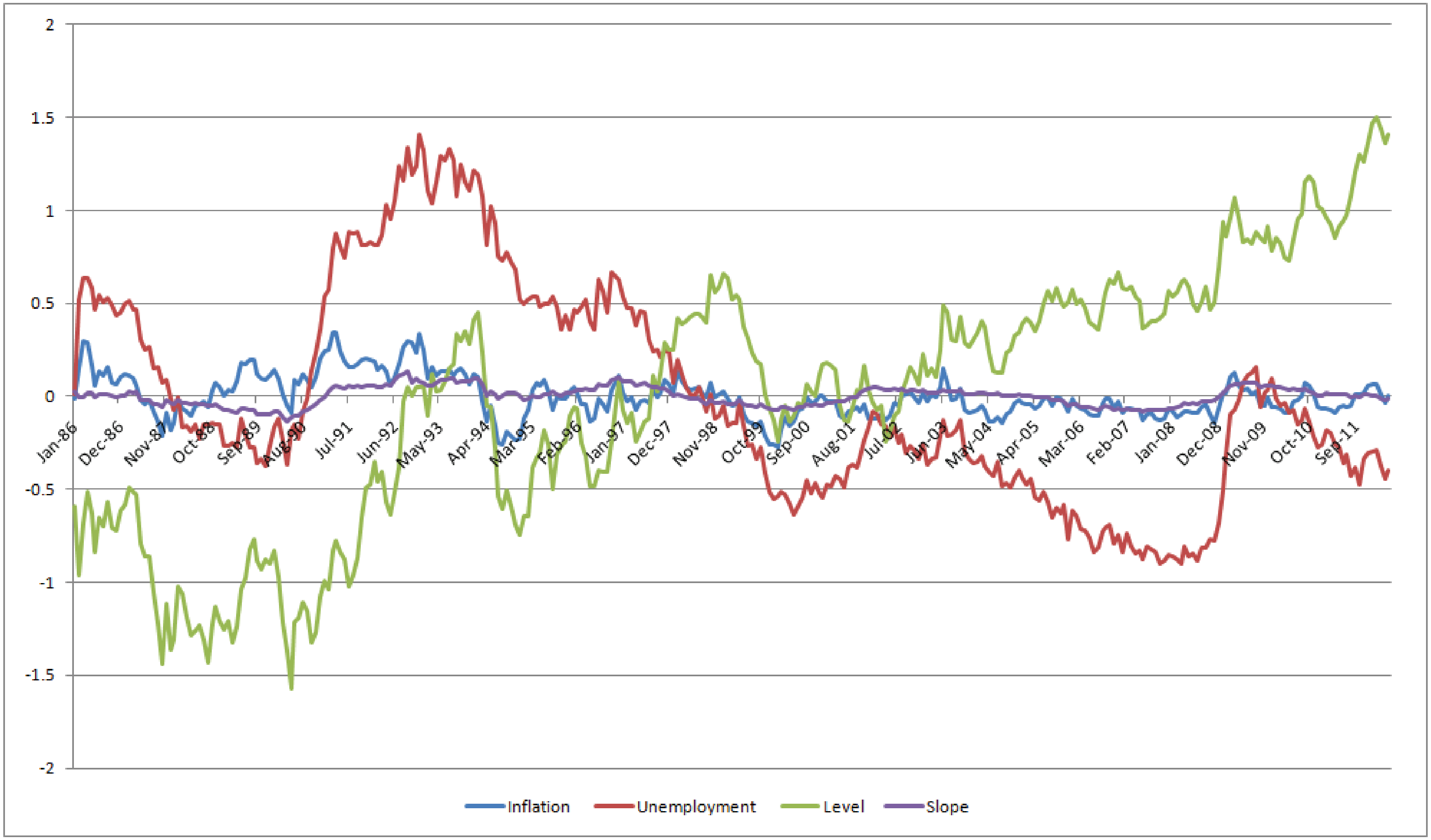


Figure 10: **Contribution to the 2-yr Inflation Risk Premium:** For the no-arbitrage model, we plot contribution of expected inflation, expected unemployment rate and of the expected level and slope factors, to the variations of the 2-yr inflation risk premium.

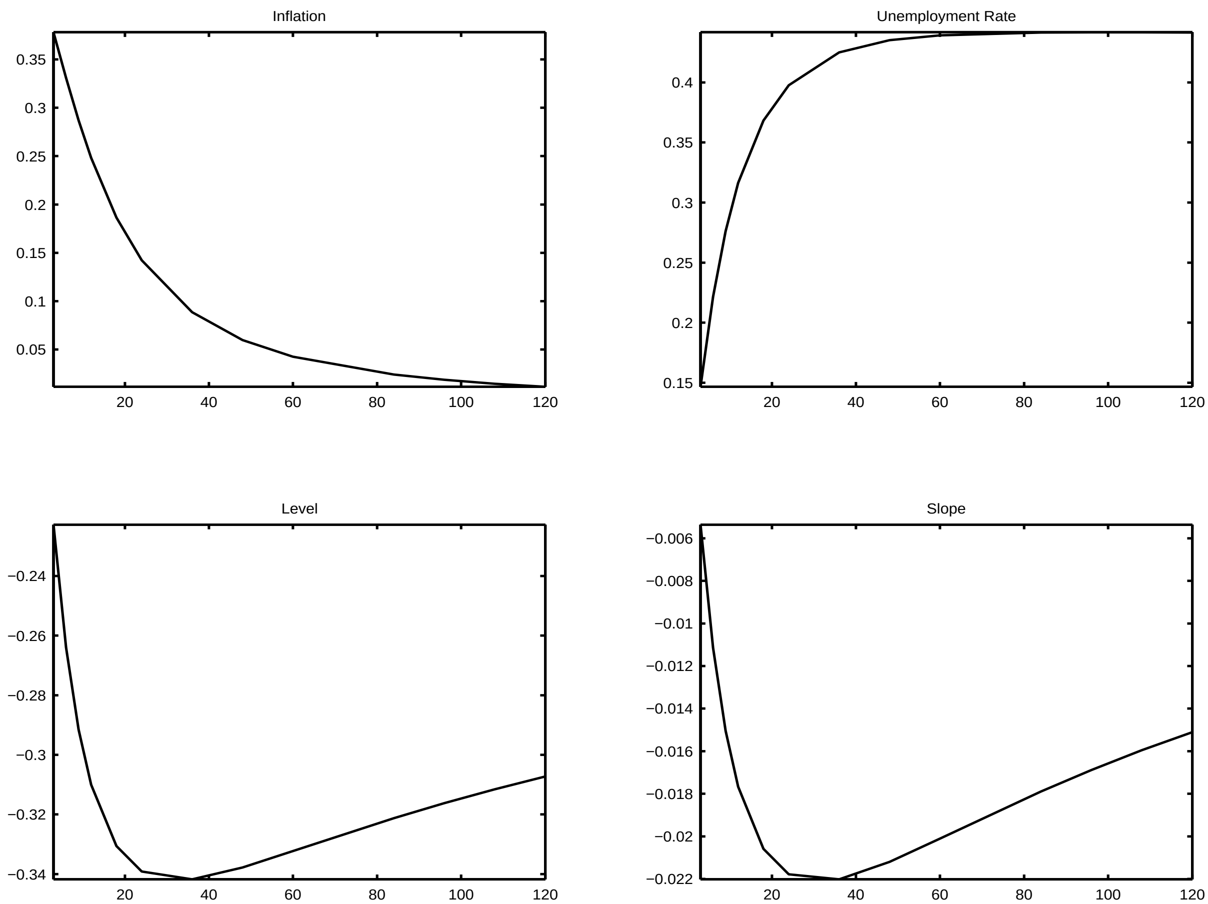
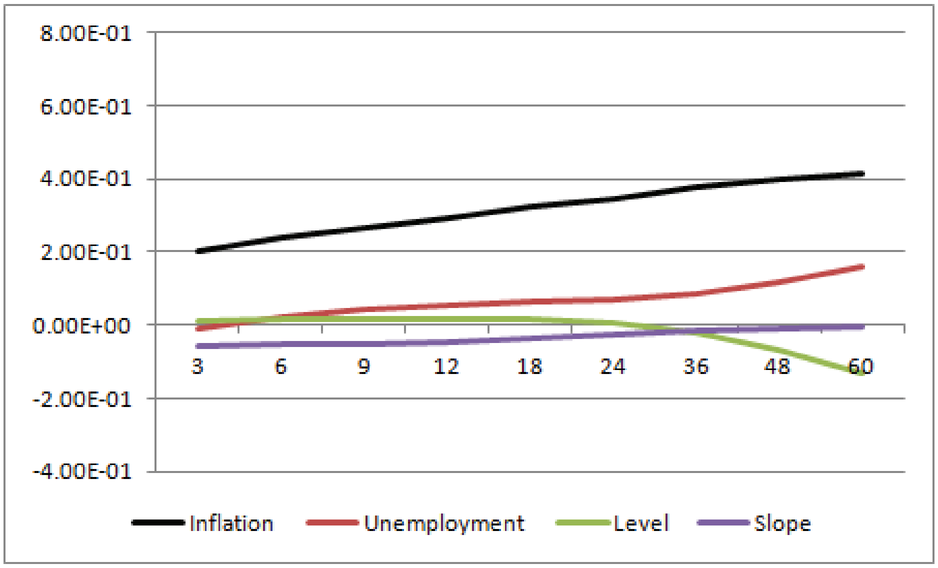
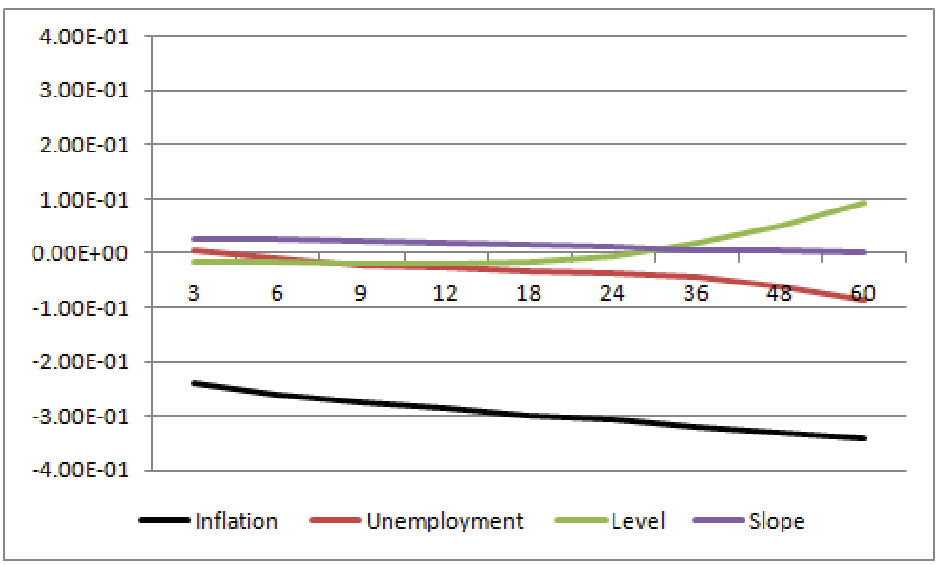


Figure 11: **Inflation Risk Premium Loadings:** For the no-arbitrage model, we plot inflation risk premium loadings on inflation expectations, unemployment expectations, the level of the real curve and the slope of the real curve for maturities ranging from three month to ten years.



(A) - Components of Unemployment Loadings



(B) - Components of Level Loadings

Figure 12: **Sources of Inflation Risk Premium:**
For the no-arbitrage model, Panel 12A draws the four components of the loadings of expected unemployment on the 2-year inflation risk premium. Panel 12B draws the four components of the loadings of the level factor on the 2-year inflation risk premium. Horizons up to 60 months.

Table 1: **Measurement Errors**

Standard deviation of measurement errors in CE inflation forecasts and yields, respectively, with asymptotic standard errors in parenthesis. CE inflation forecast inflation at quarterly horizons from 1 to 5 quarters ahead. Yield combinations measured with errors, $Y^e_{n,t}$, are the principal components of yields (except the first two measured without errors). Macro and yield data 01/1984-09/2011. Survey data Q1-1992/Q2-2011.

	SPF			Yields	
Horizon	$\sigma_{spf,h}$		$Y^e_{n,t}$	$\sigma_{e,n}$	
1	0.016	(1.31E-04)	1	0.009	(1.52E-04)
2	0.017	(1.29E-04)	2	0.004	(7.01E-05)
3	0.015	(1.60E-04)	3	0.003	(2.47E-04)
4	0.009	(2.45E-04)	4	0.002	(8.36E-06)
5	0.007	(3.23E-04)	5	0.001	(2.52E-05)

Table 2: **Out-of-sample Forecasting RMSE - 1992 to 2011**

This table compares the out-of-sample inflation forecast RMSEs from each model. Panel A presents results from quarterly forecasts. Panels B presents results from monthly forecasts. Survey data is included in the likelihood function in each case. Forecast horizons in months. Macro and yield data 01/1984-09/2011. Survey data Q1-1992/Q2-2011.

Panel (A) Quarterly forecasts

Horizon	CE	RW1	RW2	ARMA	SARMA	VAR-U	VARMA-U	VAR-UL	VARMA-UL	VAR-ULS	VARMA-ULS
3	1.01	1.96	1.13	0.99	0.95	1.01	1.01	1.00	0.97	1.00	1.01
6	0.95	2.81	1.22	0.99	0.90	1.01	1.00	0.99	0.95	0.99	0.98
9	1.02	3.71	1.31	0.96	0.95	1.01	0.98	0.96	0.89	0.97	0.95
12	0.93	4.03	1.34	0.96	0.96	1.01	0.96	0.96	0.89	0.97	0.92
15	0.92	3.80	1.21	0.95	0.94	1.00	0.92	0.96	0.92	0.95	0.94

Panel (B) Monthly forecasts

Horizon	CE	RW1	RW2	ARMA	SARMA	VAR-U	VARMA-U	VAR-UL	VARMA-UL	VAR-ULS	VARMA-ULS
1		1.28	1.02	0.99	0.99	1.00	1.02	1.02	0.99	1.00	1.00
3		1.74	1.09	1.00	1.00	1.00	1.01	1.01	0.97	1.00	1.01
6		2.36	1.16	0.99	1.00	1.00	0.99	1.00	0.95	1.00	0.96
9		2.97	1.24	0.99	0.99	1.00	0.94	1.00	0.89	1.01	0.90
12		3.20	1.30	0.94	0.97	1.02	0.97	1.02	0.81	1.03	0.85
15		3.70	1.33	0.94	0.97	1.01	0.89	1.05	0.84	1.05	0.84
18		3.87	1.34	0.93	0.95	1.01	0.86	1.05	0.91	1.04	0.80
21		4.06	1.35	0.94	0.94	1.00	0.89	1.05	0.92	1.03	0.78
24		4.11	1.35	0.91	0.92	1.00	0.93	1.05	0.92	1.05	0.77

Table 3: **The Added Value of Inflation Surveys in Out-of-sample Forecasting**

This Table compares the relative performance of each model in forecasting inflation out-of-sample. It reports the ratio of inflation forecast RMSEs obtained when including surveys in the likelihood function to the forecasting RMSEs obtained without surveys. Forecast horizons in months. Macro and yield data 01/1984-09/2011. Survey data Q1-1992/Q2-2011.

Panel (A) Monthly forecasts

Horizon	AR	ARMA	SARMA	VAR-U	VARMA-U	VAR-UL	VARMA-UL	VAR-ULS	VARMA-ULS
1	0.97	0.98	0.97	0.97	0.96	1.01	1.00	0.99	0.96
3	0.92	0.93	0.94	0.91	0.89	0.97	0.90	0.97	0.95
6	0.86	0.86	0.88	0.85	0.80	0.93	0.85	0.92	0.89
9	0.79	0.83	0.83	0.78	0.74	0.92	0.77	0.90	0.81
12	0.76	0.79	0.80	0.75	0.73	0.90	0.68	0.88	0.77
15	0.74	0.76	0.78	0.73	0.67	0.89	0.68	0.87	0.76
18	0.73	0.75	0.76	0.72	0.61	0.87	0.73	0.85	0.75
21	0.71	0.75	0.74	0.70	0.64	0.85	0.73	0.83	0.73
24	0.70	0.72	0.72	0.69	0.68	0.86	0.70	0.83	0.73

Panel (B) Quarterly forecasts

Horizon	AR	ARMA	SARMA	VAR-U	VARMA-U	VAR-UL	VARMA-UL	VAR-ULS	VARMA-ULS
3	0.87	0.88	0.82	0.84	0.80	0.97	0.88	0.96	0.89
6	0.79	0.82	0.76	0.77	0.75	0.95	0.80	0.94	0.84
9	0.70	0.74	0.73	0.68	0.65	0.91	0.68	0.92	0.78
12	0.67	0.71	0.73	0.66	0.63	0.85	0.64	0.85	0.75
15	0.69	0.72	0.74	0.69	0.65	0.81	0.66	0.80	0.80

Table 4: **The Added Value of a Moving Average in Out-of-sample Forecasting**

This Table assesses the contribution of a moving average component to forecast inflation out-of-sample. It reports the ratio of the inflation forecasts RMSEs of ARMA or VARMA relative to the RMSE obtained with the corresponding AR or VAR model. Panel A displays results obtained when excluding surveys in the likelihood function. Panel B and Panel C display RMSE ratios from monthly and quarterly forecasts, respectively, obtained when including surveys in the likelihood function. Forecast horizons in months. Macro and yield data 01/1984-09/2011. Survey data Q1-1992/Q2-2011.

Panel (A) Excluding survey data - Monthly forecasts

	ARMA	VARMA-U	VARMA-UL	VARMA-ULS
1	0.99	1.02	0.99	1.04
3	1.00	1.03	1.04	1.02
6	0.99	1.04	1.05	0.98
9	0.94	1.00	1.06	0.99
12	0.91	0.98	1.05	0.94
15	0.90	0.96	1.05	0.91
18	0.90	1.00	1.03	0.87
21	0.89	0.97	1.02	0.86
24	0.88	0.95	1.06	0.83

Panel (B) Including survey data - Monthly forecasts

	ARMA	VARMA-U	VARMA-UL	VARMA-ULS
1	0.99	1.02	0.98	1.00
3	1.00	1.01	0.97	1.00
6	0.99	0.99	0.95	0.95
9	0.99	0.94	0.89	0.89
12	0.94	0.96	0.79	0.82
15	0.94	0.87	0.81	0.80
18	0.93	0.85	0.87	0.77
21	0.94	0.89	0.88	0.75
24	0.91	0.92	0.87	0.73

Panel (C) Including survey data - Quarterly forecasts

Horizon	ARMA	VARMA-U	VARMA-UL	VARMA-ULS
3	0.99	1.00	0.97	1.02
6	0.99	0.99	0.96	0.99
9	0.96	0.97	0.92	0.97
12	0.96	0.95	0.93	0.95
15	0.95	0.92	0.95	0.98

Table 5: **Can Model Forecasts Improves Survey Forecast?**

This Table compares the performance of each model's RMSE with survey forecast RMSEs. It reports the ratio of the inflation forecast RMSEs relative to the RMSEs from survey forecasts. Survey data is included at estimation in all cases. Forecast horizons in months. Macro and yield data 01/1984-09/2011. Survey data Q1-1992/Q2-2011.

	AR	ARMA	SARMA	VAR-U	VARMA-U	VAR-UL	VARMA-UL	VAR-ULS	VARMA-ULS
3	0.99	0.98	0.95	1.00	1.00	0.99	0.96	0.99	1.00
6	1.05	1.04	0.95	1.06	1.05	1.04	1.00	1.04	1.03
9	0.98	0.95	0.93	0.99	0.96	0.94	0.87	0.95	0.93
12	1.08	1.03	1.03	1.09	1.04	1.04	0.97	1.05	0.99
15	1.09	1.04	1.03	1.09	1.01	1.05	1.00	1.04	1.02

Table 6: **Parameter Estimates**

Parameter estimates for the no-arbitrage conditional mean model. Panel A displays parameters for the historical dynamics, ϕ and ψ, respectively. Panel C displays parameters for the risk-neutral dynamics, ϕ^Q. Recall that $\psi^Q = \psi$. Panel B displays the correlation of innovations to z_t and innovations to m_t implicit in the matrices $\Sigma\Sigma'$ and $\psi\Sigma\Sigma'\psi'$, respectively. Diagonal elements correspond to the standard deviations of the corresponding innovations. Estimation is based on the joint likelihood of inflation, unemployment, yields and survey forecasts. Asymptotic standard errors in parentheses when applicable. Macro and yield data 01/1984-09/2011. Survey data Q1-1992/Q2-2011.

Panel (A) Historical dynamics

$$\phi = \begin{pmatrix} 8.40E-01 & -8.50E-03 & 4.04E-02 & 5.61E-03 \\ \scriptstyle(2.02E-02) & \scriptstyle(1.20E-02) & \scriptstyle(5.52E-03) & \scriptstyle(5.55E-03) \\ 6.95E-05 & 9.63E-01 & 2.06E-02 & 1.00E-03 \\ \scriptstyle(1.35E-02) & \scriptstyle(1.13E-02) & \scriptstyle(3.68E-03) & \scriptstyle(4.92E-03) \\ 2.50E-02 & 1.42E-02 & 9.80E-01 & -5.82E-04 \\ \scriptstyle(5.22E-03) & \scriptstyle(1.38E-02) & \scriptstyle(7.91E-03) & \scriptstyle(6.76E-03) \\ 3.84E-02 & 2.96E-02 & -3.92E-02 & 9.78E-01 \\ \scriptstyle(7.51E-03) & \scriptstyle(2.54E-02) & \scriptstyle(1.72E-02) & \scriptstyle(1.49E-02) \end{pmatrix} \quad \psi = \begin{pmatrix} 1.73E-03 & 1.28E+00 & -9.85E-01 & 2.24E-01 \\ \scriptstyle(5.24E-03) & \scriptstyle(1.48E-01) & \scriptstyle(1.00E-01) & \scriptstyle(3.29E-02) \\ -9.89E-03 & 9.08E-01 & 2.77E-02 & -8.33E-02 \\ \scriptstyle(2.23E-03) & \scriptstyle(5.99E-02) & \scriptstyle(2.51E-02) & \scriptstyle(1.93E-02) \\ 2.50E-02 & 1.42E-02 & 9.80E-01 & -5.82E-04 \\ \scriptstyle(5.22E-03) & \scriptstyle(1.38E-02) & \scriptstyle(7.91E-03) & \scriptstyle(6.76E-03) \\ 3.84E-02 & 2.96E-02 & -3.92E-02 & 9.78E-01 \\ \scriptstyle(7.51E-03) & \scriptstyle(2.54E-02) & \scriptstyle(1.72E-02) & \scriptstyle(1.49E-02) \end{pmatrix}$$

Panel (B) Standard Deviations and Correlations

$$\Sigma\Sigma' \to \begin{pmatrix} 3.58E-03 & -0.03 & -0.12 & -0.24 \\ & 1.77E-04 & -0.21 & -0.59 \\ & & 2.98E-04 & 0.54 \\ & & & 5.25E-04 \end{pmatrix} \quad \psi\Sigma\Sigma'\psi' \to \begin{pmatrix} 3.26E-04 & 0.63 & -0.81 & -0.53 \\ & 1.91E-04 & -0.25 & -0.72 \\ & & 2.98E-04 & 0.50 \\ & & & 4.94E-04 \end{pmatrix}$$

Panel (C) Risk-neutral dynamics

$$\phi^Q = \begin{pmatrix} 8.62E-01 & 4.35E-02 & 0 & 0 \\ \scriptstyle(5.85E-03) & \scriptstyle(9.34E-03) & & \\ 0 & 9.99E-01 & 0 & 0 \\ & \scriptstyle(6.03E-03) & & \\ 0 & -6.03E-04 & 1 & 0 \\ & \scriptstyle(3.48E-03) & & \\ 0 & -1.96E-02 & 0 & 9.68E-01 \\ & \scriptstyle(8.20E-03) & & \end{pmatrix}$$

Table 7: **Model Implied Summary Statistics**

Sample summary statistics of nominal yields, $y_t^{(n)}$, real yields, $i_t^{(n)}$, and inflation risk premium, $rp_{t,n}^{\pi}$, at different maturities, n, as well as inflation expectation, $m_{t,h}^{\pi}$, at the corresponding horizons, h. These quantities are computed from the no-arbitrage model estimated using the state variables, CE inflation forecasts and yields, jointly. Macro and yield data 01/1984-09/2011. Survey data Q1-1992/Q2-2011.

	3 months				6months			
	$y_t^{(n)}$	$i_t^{(n)}$	$m_{t,h}^{\pi}$	$rp_{t,n}^{\pi}$	$y_t^{(n)}$	$i_t^{(n)}$	$m_{t,h}^{\pi}$	$rp_{t,n}^{\pi}$
Mean	5.06	2.15	2.47	0.44	5.12	2.22	2.39	0.51
Median	4.57	1.67	2.36	0.40	4.67	1.68	2.25	0.48
Std	3.27	3.35	0.74	0.58	3.17	3.29	0.62	0.66
AC(1)	0.99	0.98	0.90	0.92	0.99	0.98	0.92	0.93
AC(3)	0.97	0.94	0.76	0.77	0.97	0.94	0.80	0.81
AC(6)	0.94	0.87	0.56	0.58	0.94	0.88	0.65	0.65
AC(12)	0.86	0.76	0.33	0.33	0.87	0.76	0.46	0.42
	1 year				2 year			
	$y_t^{(n)}$	$i_t^{(n)}$	$m_{t,h}^{\pi}$	$rp_{t,n}^{\pi}$	$y_t^{(n)}$	$i_t^{(n)}$	$m_{t,h}^{\pi}$	$rp_{t,n}^{\pi}$
Mean	5.23	2.35	2.29	0.58	5.44	2.57	2.22	0.62
Median	4.71	1.71	2.15	0.54	4.9	1.91	2.12	0.56
Std	3.03	3.17	0.51	0.76	2.84	2.99	0.44	0.83
AC(1)	0.99	0.98	0.96	0.95	0.99	0.98	0.99	0.96
AC(3)	0.97	0.94	0.89	0.86	0.97	0.94	0.97	0.89
AC(6)	0.94	0.88	0.81	0.72	0.94	0.89	0.94	0.78
AC(12)	0.88	0.77	0.68	0.51	0.9	0.79	0.87	0.58
	3 years				5years			
	$y_t^{(n)}$	$i_t^{(n)}$	$m_{t,h}^{\pi}$	$rp_{t,n}^{\pi}$	$y_t^{(n)}$	$i_t^{(n)}$	$m_{t,h}^{\pi}$	$rp_{t,n}^{\pi}$
Mean	5.61	2.75	2.21	0.61	5.89	3.04	2.21	0.57
Median	5.07	2.09	2.12	0.56	5.43	2.38	2.15	0.53
Std	2.71	2.85	0.41	0.84	2.56	2.67	0.37	0.82
AC(1)	0.99	0.98	0.99	0.97	0.99	0.98	0.99	0.97
AC(3)	0.97	0.95	0.98	0.91	0.97	0.95	0.99	0.92
AC(6)	0.95	0.89	0.96	0.8	0.95	0.9	0.97	0.82
AC(12)	0.91	0.81	0.91	0.61	0.92	0.83	0.93	0.63