# CoMargin: A System to Enhance Financial Stability

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September 19, 2011

**Abstract:** In this paper, we present a new collateral system, called CoMargin, for derivatives exchanges. CoMargin depends on both the tail risk of a given market participant and its interdependence with others participants. This collateral system aims at internalizing market interdependencies and enhancing the stability and resiliency of the financial system. CoMargin can be estimated by a model-free scenario-based methodology, backtested using formal statistical tests, and generalized to any number of market members. We show that CoMargin outperforms existing margining systems, in particular when both trading similarity and comovement among underlying assets increase. We investigate the effects on the stability of the financial system of increasing the number of market participants and of adding new derivatives to be cleared.

**JEL Classification:** G13

**Keywords:** Collateral, Counterparty Risk, Derivatives Markets, Extreme Dependence

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## 1. Introduction

Stability and resiliency are the two most important risk management goals needed to achieve a well-functioning financial market. The first goal refers to the ability of the financial system to prevent sudden disturbances or extreme fluctuations. The second refers to the capacity of the system to withstand and recover quickly from a shock without collapsing. As the major safeguards against counterparty risk in financial markets, collateral systems play a central role in achieving both objectives.

Achieving stability calls for precautionary measures. In the context of collateral requirements, a stable system balances the benefits and costs imposed on market participants from collecting guarantee funds such that the probability of financial distress in the market is not subject to fluctuations. On the other hand, achieving resiliency involves reactive measures. In our context, this implies adjusting the level of collateral by taking into account current market conditions such that the effects of shocks (i.e., economic shortfalls) on the system are minimized.

In both cases, the *allocation* of collateral requirements determines the efficiency with which these goals can be attained. Recent evidence shows, however, that commonly used collateral methods, like the Standard Portfolio Analysis of Risk (SPAN) system introduced by the Chicago Mercantile Exchange (CME) or the Value-at-Risk (VaR) system, can often fail to allocate collateral requirements effectively, particularly when the interdependence of market participants increases (Cruz Lopez, Harris and Pérignon, 2011). As a consequence of ignoring interdependencies between participants, these systems are likely to provide suboptimal stability and resiliency to the market when they are needed most.

In this paper, we propose a new margining system, called CoMargin, which explicitly considers interdependencies between participants. In this regard, CoMargin enhances the stability and resiliency of the market by providing optimal allocations that internalize these interdependencies. Our approach is model free and scenario based, so inefficiencies that arise due to distributional assumptions are minimized. In addition, we provide a formal backtesting methodology that can be used to assess the validity of our collateral system.

To provide an objective analysis, we focus on clearing houses in derivatives exchanges because these institutions concentrate a significant amount of counterparty risk in the financial system (Pirrong, 2009). However, our collateral approach and backtesting methodology is general enough to be applied to any context where counterparty risk exists and needs to be managed. Examples include, but are not limited to, banks and lending institutions, over-the-counter (OTC) securities dealers, newly-proposed swap execution facilities (SEFs), and insurance companies.

In a derivatives exchange, the clearing house conducts the clearing function, which consists on confirming, matching, and settling all trades. Clearing houses operate with a small number of members, referred as clearing firms, who are allowed to clear their own trades (i.e., proprietary trading), those of their customers, and those of non-clearing firms. Through the process of novation, the clearing house becomes the counterparty to every contract, thus guaranteeing performance and reducing the counterparty risk faced by its members. In the process of providing this service, however, the clearing house concentrates a significant amount of default risk, which is primarily managed through the use of margining systems.[2]

A clearing house margining system requires members to post funds as collateral in a margin account. These funds are used as performance bonds over a period of time, usually one day, and are designed to protect the clearing house against default and potential shortfalls. However, there are instances when clearing firms experience losses that exceed their posted collateral, leaving them with a negative balance in their margin accounts. These clearing firms have a clear incentive to delay their payments or in some cases to default. In either case, the clearing house has to honour its obligations to the members who profited from taking opposite trading positions. Therefore, it is not uncommon for the clearing house to face shortfalls that need to be covered with its own funds. This problem becomes particularly acute if two or more large clearing firms have a negative margin balance simultaneously. In this case, if the clearing members only delay their payments temporarily, the resulting shortfall may be short lived, but can significantly impact market liquidity, particularly during volatile periods. On the other hand,

[2] Other common default risk management tools include capital requirements for clearing firms, default funds, private insurance arrangements, and strict segregation between customer and house margin accounts (see Jones and Pérignon, 2011).

if the clearing members default, the shortfall is long-lived or even permanent which may trigger financial distress and may ultimately exhaust the resources of the clearing house and lead to its failure.

While clearing house failures are rare events, the cases of Paris in 1973, Kuala Lumpur in 1983 and Hong Kong in 1987 (Knott and Mills, 2002) demonstrate that these extreme scenarios are not only possible, but also very economically significant.[3] In addition, recent consolidation of clearing facilities through economic integration and mergers and acquisitions, as well as the strong pressure from governments and market participants to facilitate (or force) OTC derivatives to be cleared by central counterparties, has dramatically increased the systemic importance of these institutions (see, for instance, Acharya et al., 2009; US Congress' OTC Derivatives Market Act of 2009; US Department of Treasury, 2009; Duffie, Li, and Lubke, 2010; Duffie and Zhu, 2010).[4] Therefore, it is increasingly necessary to devise appropriate risk management systems that enhance the stability and resiliency of clearing facilities.

Current margining systems employed by derivatives exchanges, such as the CME's SPAN system, set the margin level of a derivatives portfolio based on a *coverage probability* or a target probability of a loss in excess of the posted collateral (Figlewski 1984; Booth et al. 1997; Cotter 2001).[5] However, by focusing only on individual firm portfolios, these systems ignore the fact that clearing firms sometimes face *homogenous risk exposures* that make them highly interdependent. In these cases, various clearing firms may exceed their posted margin simultaneously. As a consequence, the market experiences sudden and sometimes extreme shortfalls that undermine its stability and resiliency.

The level of *risk homogeneity* across clearing firms increases with *trade crowdedness* and *underlying asset comovement.* Trade crowdedness refers to the similarity of clearing firms' trading positions. When member portfolios are very similar, they will tend to have equivalent

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[3] Default of clearing firms are of course much more frequent. Recent examples include Refco (2005) and Lehman (2008) on the CME, and Marumura (2005) and MMG Arrows (2006) on the Japan Commodity Clearing House.
[4] The Chicago Mercantile Exchange, Intercontinental Exchange, and EUREX have each recently created clearing facilities for Credit Default Swaps.
[5] For example, Kupiec (1994) shows the empirical performance of the SPAN system for selected portfolios of S&P 500 futures and futures-options contracts and he finds that, over the period 1988-1992, the historical margin coverages exceed 99% for most considered portfolios.

exposures and returns, regardless of how underlying assets behave. Underlying asset comovement refers to underlying assets returns moving in unison. When underlying assets experience high levels of comovement, clearing firms will tend to have similar risk exposures because, regardless of their individual trading decisions, all securities in all portfolios would tend to move in the same direction.[6]

While both dimensions of risk homogeneity are related, the first one is directly influenced by the individual trading behaviour of clearing firms, and the second one is determined by aggregate market behaviour. Similar trading positions, or crowded trades, tend to arise among large clearing firms because they share a common (and superior) information set. This informational advantage leads them to pursue similar directional trades and arbitrage opportunities and to have similar hedging needs.[7] On the other hand, underlying assets tend to move in the same direction during economic slowdowns or during periods of high volatility, both of which are rarely the result of individual market participants' actions.[8]

In this paper, we depart from the traditional view of setting margin requirements based on individual member positions. Instead, we account for their interdependence by computing the margin requirement of a clearing firm conditional on one or more firms being in financial distress. By doing this, we obtain a system that allows the margin requirements of a particular member to increase when it is more likely to experience financial distress simultaneously with others.

Our method builds on the *CoVaR* concept introduced by Adrian and Brunnermeier (2010) which is defined as the VaR of the system (i.e., banking sector) conditional on a given institution being in financial distress. The core of their analysis is the so-called delta CoVaR that measures the marginal contribution of a particular institution to the overall risk in the system, i.e., the

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[6] The importance of asset comovement has been identified in previous studies. For example, in an early attempt to analyze the default risk of a clearing house, Gemmill (1994) highlights the dramatic diversification benefit from combining contracts on uncorrelated or weakly correlated assets.

[7] Much of the proprietary trading activity on derivatives exchanges consists of arbitraging futures and over-the-counter or cash markets (e.g. cash-futures arbitrage of the S&P 500 index, eurodollar-interest rate swap arbitrage, etc.).

[8] Extreme dependence and contagion across assets is discussed in Longin and Solnik (2001), Bae, Karolyi and Stulz (2003), Longstaff (2004), Poon, Rockinger and Tawn (2004), Boyson, Stahel and Stulz (2010), and Harris and Stahel (2011), among others.

difference between the VaR of the system conditional on a given institution being in distress and the VaR of the system in the median state of the institution. There are some key differences between the CoVaR and CoMargin methodologies, however. First, CoMargin is not applied to bank stock returns but to the profit-and-loss (P&L) of clearing firms. Second, as the focus of our paper is not the measurement of systemic risk, we do not consider the VaR of the system but the VaR of a given firm conditional on one or several other firms being in financial distress. Recall that by construction the aggregate P&L of all clearing firms is zero. Third, the estimation of CoMargin is much simpler than the estimation of CoVaR, which requires a quantile regression technique.

Our process starts by taking the trading positions of clearing members at the end of the trading day as given. Then, we consider a series of one-day-ahead scenarios based on changes in the price and volatility of the underlying assets. For each scenario, we mark-to-model each firm's portfolio and obtain its hypothetical P&L. Based on these hypothetical P&L calculations we compute margin requirements that minimize the probability of joint financial distress. We show that the CoMargin system enhances financial stability because it reduces the likelihood of several clearing members being in financial distress simultaneously. In addition, we also show that this method increases financial resiliency because it actively adjusts the allocation of collateral as a function of market conditions. As a result, the magnitude of the margin shortfall given simultaneous financial distress is minimized relative to other collateral systems. Both of these conditions greatly reduce systemic risk concerns.

The rest of the paper is organized as follows. In Section 2, we explain the market structure used in our model and explain how margin requirements are currently set under the SPAN system. In Section 3, we present the VaR margin requirement system. In Section 4, we introduce the CoMargin system and explain the backtesting procedure when we condition on a single firm event. Section 5 generalizes the explanation in the previous section by allowing $n$ conditioning firms. Section 6 demonstrates the relative efficiency of different margining systems using a series of controlled simulation experiments. Finally, Section 7 concludes.

## 2. Background

Consider a derivatives exchange with $N$ clearing firms and $D$ derivatives securities (futures and options) written on $U$ underlying assets. Let $w_{i,t}$ be the number of contracts in the derivatives portfolio of clearing firm $i$, for $i = 1, \ldots, N$, at the end of day $t$:

$$w_{i,t} = \begin{bmatrix} w_{1,i,t} \\ \vdots \\ w_{D,i,t} \end{bmatrix} \tag{1}$$

The *performance bond, $B_{i,t}$*, is the margin or collateral requirement imposed by the clearing house on clearing firm $i$ at the end of day $t$. This performance bond depends on the outstanding trading positions of the clearing firm at the end of day $t$, $w_{i,t}$. The *variation margin*, $V_{i,t}$, represents the aggregate P&L of clearing firm $i$ on day $t$.

In derivatives markets, margins are collected to guarantee the performance of member obligations and to guard against default. Therefore, we are interested in situations when trading losses exceed margin requirements; i.e., when $V_{i,t} < B_{i,t-1}$. In these cases, we say that firm $i$ is in *financial distress*. Identifying financial distress is important because firms in this state have an incentive to default on its positions or to delay payment on its obligations, which generates a shortfall in the system that needs to be covered by the clearing house.

The most popular margin system around the world is the Standard Portfolio Analysis of Risk (SPAN) system. This system was introduced by the CME in 1988 and is currently used by more than 50 derivatives exchanges (see Table 1). SPAN is a scenario based system that is applied on a firm by firm basis. However, it is not a comprehensive portfolio margining system. Instead, it divides the portfolio into contract families, which are defined as groups of contracts that share the same underlying asset. Thus, in a market with $U$ underlying assets, there are $U$ different contract families. SPAN sets the margin requirements for these families independently, and then the collateral level for the entire portfolio is computed by aggregating the margin requirements of all contract families according to aggregation rules set by the clearing house.

More specifically, to compute the margin level for a derivatives portfolio, the SPAN system simulates potential one-day changes in the value of each contract using sixteen scenarios that vary the value ($\Delta X$) and volatility ($\Delta\sigma_X$) of the underlying assets, as well as the time to expiration of the derivatives products. The potential price movements for each underlying asset are defined in terms of a *price range*, which is derived from historical data. In most cases, the price range is selected to cover 99% of the historical one-day price movements observed in the calibration window. A similar approach is adopted for the *volatility range*.

Every day following the market close, the clearing house applies each scenario to each of the $D$ derivatives securities traded on the exchange. The price changes of non-linear instruments, such as options, are obtained by using numerical valuation methods or option pricing models. A risk array with sixteen gain or loss values is created for each contract (i.e., each maturity and each strike price will have its own array). Using these arrays, the predicted losses across contracts are computed to find the scenario that creates the worst-case loss for the contract family as a whole. This worst-case value is then used to determine the margin requirement for that contract family.

The overall portfolio margin requirement is computed by aggregating the margin requirements of different contract families. However, since SPAN allows different futures and options months within a family to offset each other, the aggregation across contract families is adjusted with intermonth spread charges. Similarly, inter-commodity credits are given to account for inter-commodity spreads. It is important to note that the magnitude of these charges and credits is left the discretion of the margin committee of the clearing house, so they may not be consistent across commodities, market conditions or clearing houses. Therefore, the actual coverage probability of the SPAN system may not be consistent across time or markets.

As an illustration, we display in Figures 1 and 2 the daily SPAN margins and P&L for all sixty nine clearing firms in the CME between January 1 and December 31, 2001. More precisely, these figures correspond to the house trading account (i.e., proprietary trading) of these clearing firms, and as such, do not reflect customer trading. The most striking feature of the data is the segmentation of the market between extremely large (i.e., systemically-important) clearing firms and smaller ones. The top-10 largest clearing firms (the brokerage units of Morgan Stanley, Goldman Sachs, Credit Suisse, etc.) account on average for approximately 80% of all collateral collected. We see in Figure 1 that the posted margin for a single firm can be close to $3 billion and the daily trading gain or loss can exceed $1 billion.

Figure 2 displays the ratio of the daily profit-and-loss ($V_{i,t}$) and SPAN margins ($B_{i,t}$) for all clearing firms. This graph illustrates two important features of the SPAN margins. First, SPAN margins are frequently exceeded by the trading loss on that day. In our sample, there were 30 days (of 251) in which a clearing member's margin account had a negative balance (i.e., it ended up *under-water*); and 14 different clearing firms experienced at least one margin-exceeding loss event. Second, margin deficiencies tend to cluster. This feature of the data is particularly salient (and troublesome from a risk management perspective) for the largest clearing firms. The ten most extreme losses as a proportion of posted collateral ($V_{i,t}/B_{i,t-1}$) that affected the ten largest clearing firms occurred on two different trading days.

## 3. VaR Margin

### 3.1. Concept

VaR is defined as a lower quantile of a bank's P&L distribution. It is the standard measure used to assess aggregate market risk exposure for banks (Berkowitz and O'Brien, 2002; Berkowitz, Christoffersen and Pelletier, 2011) and it is also used by banking regulators to set capital requirements (Jorion, 2007). In addition, VaR can be used to set margins on a derivatives exchange. In this case, the margin requirement corresponds to a given quantile of a clearing firm's one-day-ahead P&L distribution. Like the SPAN system, the VaR collateral system is applied on a firm by firm basis using a scenario analysis (Cruz Lopez, Harris and Pérignon, 2011).

In this paper, we consider a series of $S$ scenarios based on potential one-day-ahead changes in the value and volatility of the underlying assets, as well as, in the time to expiration of the derivative securities. The one-day-ahead returns of the underlying assets are obtained from random realizations of a multivariate distribution with $U$ dimensions. A similar approach is followed to obtain the one-day-ahead changes in the volatility. The shape and the parameters of the distributions are defined by the risk manager, such that the variance-covariance matrix assigned to the distribution of the underlying asset returns is consistent with the expected one-day-ahead changes in the volatility. For example, a risk manager may select normal distributions to construct both, the underlying asset return and volatility scenarios. In this case, if the expected one-day-ahead change in the volatility of the underlying asset returns is $E[\Delta\sigma_X]$, then that value is used as the mean parameter of the distribution of the volatility changes, and $\sigma_X + E[\Delta\sigma_X]$ is used as the standard deviation parameter of the distribution of underlying asset returns; where $\sigma_X$ is the current volatility value.

For each of the $S$ scenarios, we evaluate each clearing firm's entire portfolio (i.e., we "mark-to-model" its positions) and compute the associated hypothetical P&L or variation margin, denoted $V_{i,t+1}$. Thus, for each clearing member and each date $t$, we obtain a simulated sample of $V_{i,t+1}$ denoted $\left\{v_{i,t+1}^s\right\}_{s=1}^S$.

**Definition 1:** The VaR margin, $B_i$, corresponds to the $\alpha\%$ quantile of all simulated P&L across all considered scenarios:

$$\Pr\left(V_{i,t+1} \leq -B_{i,t}\right) = \alpha \tag{2}$$

The VaR margin method has several advantages over the SPAN system. First, since VaR margins correspond to quantiles in the simulated P&L distribution, they are more robust (or less sensitive to simulation design) than SPAN margins, which correspond to the minimum value of the simulated P&L distribution. Second, unlike SPAN margins, VaR margins can be validated ex-post using a formal backtesting methodology.

### 3.2. Estimation

Given the simulated path $\left\{v_{i,t+1}^s\right\}_{s=1}^S$, the VaR collateral requirement can be estimated as follows:

$$\hat{B}_{i,t} = percentile\left(\left\{v_{i,t+1}^s\right\}_{s=1}^S, 100\alpha\right) \tag{3}$$

Compared to market risk VaR (Jorion, 2007), the estimation of VaR margin is much simpler. In general, the quantile of the return at time $t$, needed for market risk VaR, cannot be estimated without making some strong assumptions about the underlying distribution. Specifically, since there is only one P&L observation on each date, we usually assume that the P&Ls are independently and identically distributed over time. Under these assumptions, the unconditional VaR is stationary and it can be estimated from the historical path of past P&Ls. This idea captures the fundamental principle of the *historical simulation approach*, broadly used by financial institutions for market risk VaR estimations. In addition, the estimation of a conditional VaR also requires some particular assumptions regarding the dynamics of the P&L quantiles. For instance, the CaviaR approach proposed by Engle and Manganelli (2004), assumes an autoregressive process for the P&L quantile.

In our context, however, the situation is quite different because we have $S$ simulated observations of the P&L at time $t$. This is an ideal situation from an econometric point of view because the quantile of the P&L distribution can be directly estimated without making any assumptions regarding its behavior over time. Thus, the empirical quantile based on the $S$ simulated observations $v_{i,t}^{s}$ (equation 3) is a consistent estimate of the VaR when $S$ tends to infinity.

### 3.3. Backtesting

For each clearing member $i$ considered independently, we can test the validity of the VaR margining system by testing the following null hypothesis:

$$H_0 \colon \Pr\left(V_{i,t+1} \leq -B_{i,t}\right) = \alpha \tag{4}$$

In this case, a standard VaR backtesting approach based on the violations of the dichotomic process $\mathbf{I}\left(V_{i,t+1} \leq -B_{i,t}\right)$ can be applied, where $\mathbf{I}(A)$ is an indicator function equal to 1 if $A$ is true and 0 otherwise.

Let us consider the path of historical P&L, denoted $\{v_{i,t+1}\}_{t=1}^{T}$ and the corresponding violations $\mathbf{I}(v_{i,t+1} \leq -B_{i,t})$.[9] A simple LR test statistic can be defined as follows (Kupiec, 1995; Christoffersen, 2009):

$$LR_i = -2\ln[(1-\alpha)^{T-N_i}\alpha^{N_i}] + 2\ln\left[\left(1-\frac{N_i}{T}\right)^{T-N_i}\frac{N_i}{T}^{N_i}\right] \tag{5}$$

where $N_i = \sum_{t=1}^{T}\mathbf{I}(v_{i,t+1} \leq -B_{i,t})$ denotes the total number of past violations observed for the $i^{th}$ member and its historical sequence of margin requirements $\{B_{i,t}\}_{t=1}^{T}$. Under the null, this statistic has a chi-square $\chi^2(1)$ distribution.

## 4. CoMargin

### 4.1. Concept

As it was discussed in the previous sections, VaR and SPAN collateral systems only focus on *firm specific risk;* that is, the unconditional probability of financial distress of each individual member. By adopting either system, the clearing house guards itself from unique or independent financial distress occurrences, but it leaves itself exposed to simultaneous distress events. These events, however, tend to be more economically significant than individual distress situations because they place a more substantial burden on the resources of the clearing house, which may exhaust its funds and eventually default.

Consider the VaR margin requirements for firms $i$ and $j$. Their probability of *joint financial distress* is given by:

$$\begin{aligned} &\Pr[(V_{i,t+1} \leq -B_{i,t}) \cap (V_{j,t+1} \leq -B_{j,t})] \\ &= \Pr(V_{i,t+1} \leq -B_{i,t} | V_{j,t+1} \leq -B_{j,t}) \times \Pr(V_{j,t+1} \leq -B_{j,t}) \end{aligned} \tag{6}$$

[9] The historical path has to be distinguished from the simulated paths, $\{v_{i,t+1}^{s}\}_{s=1}^{S}$, obained from the $S$ scenarios used to estimate the VaR collateral requirement.

Equation 6 shows that joint financial distress events tend to happen more frequently not only when *firm specific risk* increases (i.e., $\Pr\left(V_{j,t+1} \leq -B_{j,t}\right)$ increases), but also when *risk homogeneity* increases (i.e., $\Pr\left(V_{i,t+1} \leq -B_{i,t} | V_{j,t+1} \leq -B_{j,t}\right)$ increases). In the first case, firms are more likely to experience losses that exceed their collateral levels in all situations. In the second case, firms are more likely to experience these losses when other firms are in financial distress, either because firms hold similar positions (i.e., *trade crowdedness* is high) or because underlying assets have a tendency to move together (i.e., *underlying asset comovement* is high). However, VaR and SPAN systems completely disregard risk homogeneity and its potential effect on financial distress and market stability. In the case of the VaR system, risk managers only target unconditional distress probabilities by setting a coverage level for each clearing member. In the case of the SPAN system, risk managers do not have direct control over the unconditional distress probabilities, so the clearing house is left even more vulnerable to simultaneous distress occurrences.

Now, consider a *fully orthogonal market*; that is, a market that has firms with orthogonal trading positions and orthogonal underlying asset returns. In this case, firms have orthogonal risk exposures and their probabilities of financial distress are independent. Therefore,

$$\Pr\left(V_{i,t+1} \leq -B_{i,t} | V_{j,t+1} \leq -B_{j,t}\right) = \alpha \tag{7}$$

and

$$\Pr\left[\left(V_{i,t+1} \leq -B_{i,t}\right) \cap \left(V_{j,t+1} \leq -B_{j,t}\right)\right] = \alpha^2 \tag{8}$$

Equation 8 shows that a fully orthogonal market minimizes the probability of joint financial distress across clearing members. Given a common coverage probability, $\alpha$, a *fully orthogonal market* provides the best possible level of market stability, regardless of the collateral system being adopted by the clearing house. Therefore, a fully orthogonal market can be seen as a conceptual construct that provides a common benchmark for all margin systems.

With this in mind and in the spirit of the CoVaR measure of Adrian and Brunnermeier (2010), we propose a new collateral system, called CoMargin, which enhances financial stability by taking into account the risk homogeneity of clearing firms. Our starting point is the framework used to estimate VaR collateral requirements, which was described in the previous section. Once we establish the $S$ scenarios for each underlying asset, we jointly evaluate the portfolios of firms $i$ and $j$ and compute their associated hypothetical P&Ls or variation margins, $V_{i,t+1}$ and $V_{j,t+1}$ respectively, such that for each date $t$, we obtain a panel of simulated P&Ls, denoted $\{v_{i,t+1}^s, v_{j,t+1}^s\}_{s=1}^S$. Thus, the CoMargin of firm $i$, denoted $B_t^{i|j}$, conditional on the realisation of an event affecting firm $j$ is:

$$\Pr\left(V_{i,t+1} \leq -B_t^{i|j} | \mathbf{C}(V_{j,t+1})\right) = \alpha \tag{9}$$

The conditioning event that we consider in this case is an *extreme loss* in the portfolio of firm $j$, which is defined as a loss that exceeds its $\alpha\%$ VaR, or equivalently, a loss that exceeds its VaR margin; i.e., $\mathbf{C}(V_{j,t+1}) = V_{j,t+1} \leq -B_{j,t}$.

**Definition 2:** The CoMargin, $B^{i|j}$, corresponds to the $\alpha\%$ conditional quantile of their jointly simulated P&L across all considered scenarios:

$$\Pr(V_{i,t+1} \leq -B_t^{i|j} | V_{j,t+1} \leq -B_{j,t}) = \alpha \tag{10}$$

Through Bayes theorem we know that:

$$\Pr(V_{i,t+1} \leq -B_t^{i|j} | V_{j,t+1} \leq -B_{j,t}) = \frac{\Pr\left[\left(V_{i,t+1} \leq -B_t^{i|j}\right) \cap \left(V_{j,t+1} \leq -B_{j,t}\right)\right]}{\Pr(V_{j,t+1} \leq -B_{j,t})} \tag{11}$$

where the numerator represents the joint probability of $i$ exceeding its CoMargin requirement and $j$ experiencing an *extreme loss*. From Definitions 1 and 2, we can see that the CoMargin of firm $i$ is defined as the margin level $B_t^{i|j}$ such that:

$$\Pr\left[\left(V_{i,t+1} \leq -B_t^{i|j}\right) \cap \left(V_{j,t+1} \leq -B_{j,t}\right)\right] = \alpha^2 \tag{12}$$

Notice from equation 10 that the CoMargin system starts by defining a threshold level or *extreme loss* as the $\alpha\%$ VaR of the P&L of a conditioning firm $j$. This threshold, which corresponds to that firm's VaR margin, accounts for *firm specific risk* in the CoMargin calculation. *Risk homogeneity* is then incorporated by directly targeting the conditional probability of financial distress of firm $i$, such that it behaves *as if* the market was *fully orthogonal* when firm $j$ experiences an *extreme loss*. This means that when the market is indeed fully orthogonal, the CoMargin and VaR collateral systems are equivalent and produce the same margin requirements. When the market is not fully orthogonal, any differences between the collateral requirements of these two systems can be attributed to *risk homogeneity*. Thus, the CoMargin of firm $i$, $B_t^{i|j}$, can be interpreted as the margin level that guarantees that firm $i$ remains solvent at an optimal level when firm $j$ experiences an *extreme loss*. The optimal level of solvency corresponds to that seen in a *fully orthogonal market*, in the sense that, given that firm $j$ experiences an *extreme loss*, firm $i$ will have enough funds in its margin account to cover its potential losses $1-\alpha^2\%$ of the time. Therefore, the CoMargin system greatly enhances financial stability.

### 4.2. Estimation

Given the simulated path $\left\{v_{i,t+1}^s, v_{j,t+1}^s\right\}_{s=1}^{S}$, conditional on $B_t^{i|j}$, a simple estimate of the joint probability $\Pr\left[\left(V_{i,t+1} \leq -B_t^{i|j}\right) \cap \left(V_{j,t+1} \leq -B_{j,t}\right)\right]$, denoted $P_t^{i,j}$, is given by:

$$\hat{P}_t^{i,j} = \frac{1}{S}\sum_{s=1}^{S} \mathbf{I}\left(v_{i,t+1}^s \leq -B_t^{i|j}\right) \times \mathbf{I}\left(v_{j,t+1}^s \leq -B_{j,t}\right) \tag{13}$$

where $v_{i,t+1}^s$ and $v_{j,t+1}^s$ correspond to the $s^{th}$ simulated P&L of firms $i$ and $j$, respectively. Given this result, we can now estimate $B_t^{i|j}$. For each time $t$ and for each firm $i$, we look for the value $B_t^{i|j}$, such that the distance $\hat{P}_t^{i,j} - \alpha^2$ is minimized:

$$\hat{B}_t^{i|j} = \arg\min_{\left\{B_t^{i|j}\right\}} \left(\hat{P}_t^{i,j} - \alpha^2\right)^2 \tag{14}$$

Thus, for each firm $i$, we end up with a time series of CoMargin requirements $\left\{\hat{B}_t^{i|j}\right\}_{t=1}^{T}$ for which confidence bounds can be bootstrapped.

### 4.3. Backtesting

As it is the case with VaR margin, CoMargin allows us to test the null hypothesis of an individual member exceeding its margin requirement (equation 4). More importantly, however, is the fact that we can also test the conditional probability of financial distress defined by the CoMargin of firm $i$, $B_t^{i|j}$. The null hypothesis in this case becomes:

$$H_0: \Pr\left(V_{i,t+1} \leq -B_t^{i|j} | V_{j,t+1} \leq -B_{j,t}\right) = \alpha \tag{15}$$

Since the null implies that $E\left[\mathbf{I}\left(V_{i,t+1} \leq -B_t^{i|j}\right) \times \mathbf{I}\left(V_{j,t+1} \leq -B_{j,t}\right)\right] = \alpha$, then a simple LR test can also be used to assess the conditional probability of financial distress by using the historical paths of the P&Ls for both members $i$ and $j$; i.e., $\left\{v_{i,t+1}\right\}_{t=1}^{T}$ and $\left\{v_{j,t+1}\right\}_{t=1}^{T}$. The corresponding LR test statistic, denoted $LR_{i|j}$ takes the same form as $LR_i$:

$$LR_{i|j} = -2\ln[(1-\alpha)^{T-N_{i|j}}\alpha^{N_{i|j}}] + 2\ln\left[\left(1 - \frac{N_{i|j}}{T}\right)^{T-N_{i|j}} \frac{N_{i|j}}{T}^{N_{i|j}}\right] \tag{16}$$

except that in this case $N_{i|j}$ denotes the total number of joint past violations observed for both members $i$ and $j$; that is, $N_{i|j} = \sum_{t=1}^{T} \mathbf{I}\left(v_{i,t+1} \leq -B_t^{i|j}\right) \times \mathbf{I}\left(v_{j,t+1} \leq -B_{j,t}\right)$.

### 4.4. Extension to $n$ Conditioning Firms

Consider now that the conditioning event depends on two firms denoted $j$ and $k$. In this case, the CoMargin of firm $i$, denoted by $B_t^{i|j,k}$, is defined as follows:

$$\Pr\left(V_{i,t+1} \leq -B_t^{i|j,k} | \mathbf{C}\left(V_{j,t+1}, V_{k,t+1}\right)\right) = \alpha \tag{17}$$

$$\frac{\Pr\left[\left(V_{i,t+1} \leq -B_t^{i|j,k}\right) \cap \mathbf{C}\left(V_{j,t+1}, V_{k,t+1}\right)\right]}{\Pr\left[\mathbf{C}\left(V_{j,t+1}, V_{k,t+1}\right)\right]} = \alpha \tag{18}$$

The conditioning event that we consider is either firm $j$ or firm $k$, or both, being in financial distress; i.e., $\mathbf{C}\left(V_{j,t+1}, V_{k,t+1}\right) = V_{j,t+1} \leq -B_{j,t} \; or \; V_{k,t+1} \leq -B_{k,t}$. In this case, the probability of the conditioning event is equal to $2\alpha$ only if the financial distress events of firms $j$ and $k$ are mutually exclusive. In the general case, we have:

$$\begin{aligned}
\Pr\left[\mathbf{C}\left(V_{j,t+1}, V_{k,t+1}\right)\right] &= \Pr\left[\left(V_{j,t+1} \leq -B_{j,t}\right) \text{ or } \left(V_{k,t+1} \leq -B_{k,t}\right)\right] \\
&= \Pr\left(V_{j,t+1} \leq -B_{j,t}\right) + \Pr\left(V_{k,t+1} \leq -B_{k,t}\right) \\
&\quad -\Pr\left[\left(V_{j,t+1} \leq -B_{j,t}\right) \cap \left(V_{k,t+1} \leq -B_{k,t}\right)\right] \\
&= 2\alpha - \Pr\left[\left(V_{j,t+1} \leq -B_{j,t}\right) \cap \left(V_{k,t+1} \leq -B_{k,t}\right)\right]
\end{aligned} \tag{19}$$

Hence, CoMargin $B_t^{i|j,k}$ satisfies the following condition:

$$\frac{\Pr\left[\left(V_{i,t+1} \leq -B_t^{i|j,k}\right) \cap \mathbf{C}\left(V_{j,t+1}, V_{k,t+1}\right)\right]}{2\alpha - \Pr\left[\left(V_{j,t+1} \leq -B_{j,t}\right) \cap \left(V_{k,t+1} \leq -B_{k,t}\right)\right]} = \alpha \tag{20}$$

Given this result, we proceed to estimate CoMargin $B_t^{i|j,k}$. First, notice that the probability $\Pr\left[\left(V_{j,t+1} \leq -B_{j,t}\right) \cap \left(V_{k,t+1} \leq -B_{k,t}\right)\right]$, denoted $P_t^{j,k}$, does not depend on the CoMargin level $B_t^{i|j,k}$; thus, it can simply be estimated by:

$$\hat{P}_t^{j,k} = \frac{1}{S} \sum_{s=1}^{S} \mathbf{I}\left(v_{j,t+1}^s \leq -B_{j,t}\right) \times \mathbf{I}\left(v_{k,t+1}^s \leq -B_{k,t}\right) \tag{21}$$

Second, conditional on $B_t^{i|j,k}$, the joint probability in the numerator of equation 18, denoted $P_t^{i,j,k}$, becomes:

$$\begin{aligned}
P_t^{i,j,k} &= \Pr\left[\left(V_{i,t+1} \leq -B_t^{i|j,k}\right) \cap \mathbf{C}\left(V_{j,t+1}, V_{k,t+1}\right)\right] \\
&= \Pr\left[\left(V_{i,t+1} \leq -B_t^{i|j,k}\right) \cap \left[\left(V_{j,t+1} \leq -B_{j,t}\right) \text{ or } \left(V_{k,t+1} \leq -B_{k,t}\right)\right]\right]
\end{aligned} \tag{22}$$

$$= \Pr\big[\big(V_{i,t+1} \leq -B_t^{i|j,k}\big) \cap \big(V_{j,t+1} \leq -B_{j,t}\big)$$

$$\text{or } \big(V_{i,t+1} \leq -B_t^{i|j,k}\big) \cap \big(V_{k,t+1} \leq -B_{k,t}\big)\big]$$

$$= \Pr\big[\big(V_{i,t+1} \leq -B_t^{i|j,k}\big) \cap \big(V_{j,t+1} \leq -B_{j,t}\big)\big]$$

$$+\Pr\big[\big(V_{i,t+1} \leq -B_t^{i|j,k}\big) \cap \big(V_{k,t+1} \leq -B_{k,t}\big)\big]$$

$$-\Pr\big[\big(V_{i,t+1} \leq -B_t^{i|j,k}\big) \cap \big(V_{j,t+1} \leq -B_{j,t}\big) \cap \big(V_{k,t+1} \leq -B_{k,t}\big)\big]$$

Thus, a simple estimator of this probability is given by:

$$\begin{aligned} \hat{P}_t^{i,j,k} &= \frac{1}{S}\sum_{s=1}^{S} \mathbf{I}\big(v_{i,t+1}^s \leq -B_t^{i|j,k}\big) \times \mathbf{I}\big(v_{j,t+1}^s \leq -B_{j,t}\big) \\ &+ \frac{1}{S}\sum_{s=1}^{S} \mathbf{I}\big(v_{i,t+1}^s \leq -B_t^{i|j,k}\big) \times \mathbf{I}\big(v_{k,t+1}^s \leq -B_{k,t}\big) \\ &- \frac{1}{S}\sum_{s=1}^{S} \mathbf{I}\big(v_{i,t+1}^s \leq -B_t^{i|j,k}\big) \times \mathbf{I}\big(v_{j,t+1}^s \leq -B_{j,t}\big) \times \mathbf{I}\big(v_{k,t+1}^s \leq -B_{k,t}\big) \end{aligned} \quad (23)$$

and the CoMargin $B_t^{i|j,k}$ can be estimated by:

$$\hat{B}_t^{i|j,k} = \arg\min_{\left\{B_t^{i|j,k}\right\}} \left(\frac{\hat{P}_t^{i,j,k}}{2\alpha - \hat{P}_t^{j,k}} - \alpha\right)^2 \quad (24)$$

Following a similar argument, CoMargin can be generalized to $n$ conditioning firms, with $n < N - 1$. In this case, the conditioning event is that at least one of the $n$ clearing members is in financial distress. Thus, the definition of CoMargin becomes:

$$\frac{\Pr\big[\big(V_{i,t+1} \leq -B_t^{i|n}\big) \cap \mathbf{C}\big(V_{1,t+1}, \ldots, V_{n,t+1}\big)\big]}{\Pr\big[\mathbf{C}\big(V_{1,t+1}, \ldots, V_{n,t+1}\big)\big]} = \alpha \quad (25)$$

where the probability to observe the conditioning event is:

$$\Pr\big[\mathbf{C}\big(V_{1,t+1}, \ldots, V_{n,t+1}\big)\big] = \Pr\big[\big(V_{1,t+1} \leq -B_{1,t}\big) \; or \ldots or \; \big(V_{n,t+1} \leq -B_{n,t}\big)\big] \quad (26)$$

Using Poincaré's formula for the probability of the union of events, we can see that:

$$\begin{aligned}\Pr\left[\mathbf{C}\left(V_{1,t+1},..,V_{n,t+1}\right)\right] &= \sum_{\boldsymbol{j=1}}^{\boldsymbol{n}} \Pr\left[\left(V_{j,t+1} \leq -B_{j,t}\right)\right] \\ &- \underbrace{\sum_{1\leq j_1<j_2\leq n}^{\boldsymbol{n}} \Pr\left[\left(V_{j_1,t+1} \leq -B_{j_1,t}\right) \cap \left(V_{j_2,t+1} \leq -B_{j_2,t}\right)\right]}_{\boldsymbol{2\ events}} \\ &+ \underbrace{\sum_{1\leq j_1<j_2<j_{3\leq n}}^{\boldsymbol{n}} \Pr\left[\left(V_{j_1,t+1} \leq -B_{j_1,t}\right) \cap \left(V_{j_2,t+1} \leq -B_{j_2,t}\right) \cap \left(V_{j_3,t+1} \leq -B_{j_3,t}\right)\right]}_{\boldsymbol{3\ events}} \\ &\dots \underbrace{+(-1)^{\boldsymbol{n-1}} \Pr\left[\left(V_{1,t+1} \leq -B_{1,t}\right) \cap \dots \cap \left(V_{n,t+1} \leq -B_{n,t}\right)\right]}_{\boldsymbol{n\ events}}\end{aligned} \tag{27}$$

Thus, the probability of the conditioning event can be rewritten as follows:

$$\Pr\left[\mathbf{C}\left(V_{1,t+1},..,V_{n,t+1}\right)\right] = n\alpha - P_t^n \tag{28}$$

where $P_t^n$ denotes the sum of the probabilities of all common events (for two events, three events, etc.). An estimator of this value, $\hat{P}_t^n$, can be obtained from the simulated path $\left\{V_{1,t+1}^s,..,V_{n,t+1}^s\right\}_{s=1}^S$ using a generalisation of equation 21. When the financial distress events of the conditioning firms are mutually exclusive, however, the probability of the conditioning events simplifies to $n\alpha$. Therefore, an estimator of the CoMargin of firm $i$ conditional on $n$ clearing firms, $B_t^{i|n}$, is the solution of the program:

$$\hat{B}_t^{i|n} = \arg\min_{\left\{B_t^{i|n}\right\}} \left(\frac{\hat{P}_t^{i,n}}{n\alpha - \hat{P}_t^n} - \alpha\right)^2 \tag{29}$$

where $\hat{P}_t^{i,n}$ denotes the estimator of $\Pr\left[\left(V_{i,t+1} \leq -B_t^{i|n}\right) \cap \mathbf{C}\left(V_{1,t+1},..,V_{n,t+1}\right)\right]$, which is obtained by generalizing equation 23 conditional on $B_t^{i|n}$.

## 5. Joint Margin

### 5.1. Concept

In the previous section we defined CoMargin as the collateral system that ensures that a firm remains solvent at an optimal level, when one or more firms experience an *extreme loss* that exceeds the $\alpha\%$ VaR of their P&L. Unlike VaR margin, however, CoMargin does not target a specific coverage level or probability of financial distress, so when the CoMargin system is implemented on all firms in the market, the probabilities of individual and joint financial distress cannot be directly controlled by the risk managers.

In this section we propose a new and alternative collateral system, called Joint Margin, which also aims at enhancing market stability. However, in this case, we directly target the probability of joint financial distress across clearing firms. We do this by ensuring that the probability of joint financial distress corresponds to what we would observe in a *fully orthogonal market* (see equation 8). We adopt this approach because as we explained in the previous section, given a common coverage probability, $\alpha$, a *fully orthogonal market* provides the best possible level of market stability.

Consider two firms $i$ and $j$. As in the previous section, our starting point is the framework used to estimate VaR margins. After the $S$ scenarios for each underlying asset have been set, we jointly evaluate the portfolios of these firms and compute their associated hypothetical P&Ls or variation margins, $V_{i,t+1}$ and $V_{j,t+1}$. Thus, for each date $t$, we obtain a panel of simulated P&Ls, denoted $\left\{v_{i,t+1}^{s}, v_{j,t+1}^{s}\right\}_{s=1}^{S}$. The Joint Margin approach consist on simultaneously defining the margin requirements of firms $i$ and $j$, $B_{i,t}^{i,j}$ and $B_{j,t}^{i,j}$, by finding a common quantile level, $\gamma_{t}^{i,j}$, in their simulated P&L distributions, such that the probability of *joint* financial distress equals the target level $\alpha^{2}$, which is selected by the risk manager.

**Definition 3:** The Joint Margin of firms $i$ and $j$, denoted $B_{i,t}^{i,j}$ and $B_{j,t}^{i,j}$ respectively, correspond to the $\gamma_t^{i,j}$% quantile of their jointly simulated P&L across all considered scenarios, such that

$$\Pr\left[\left(V_{i,t+1} \leq -B_{i,t}^{i,j}\right) \cap \left(V_{j,t+1} \leq -B_{j,t}^{i,j}\right)\right] = \alpha^2 \tag{30}$$

Notice from equation 30 that since the Joint Margin system endogenously determines $\gamma_t^{i,j}$, the level of *firm specific risk* is consistent across firms:

$$\Pr\left(V_{i,t+1} \leq -B_{i,t}^{i,j}\right) = \Pr\left(V_{j,t+1} \leq -B_{j,t}^{i,j}\right) = \gamma_t^{i,j} \tag{31}$$

In addition, from Bayes theorem we know that:

$$\begin{aligned} &\Pr\left[\left(V_{i,t+1} \leq -B_t^{i,j}\right) \cap \left(V_{j,t+1} \leq -B_t^{j,i}\right)\right] \\ &= \Pr\left(V_{i,t+1} \leq -B_t^{i,j} | V_{j,t+1} \leq -B_t^{j,i}\right) \times \gamma_t^{i,j} = \alpha^2 \end{aligned} \tag{32}$$

Equation 32 shows that by directly targeting the joint probability of financial distress, the Joint Margin system accounts for *risk homogeneity*. More specifically, notice that when the market is fully orthogonal, $\Pr\left(V_{i,t+1} \leq -B_t^{i,j} | V_{j,t+1} \leq -B_t^{j,i}\right) = \alpha$, and as a result $\gamma_t^{i,j} = \alpha$ and the Joint Margin and VaR collateral systems yield the same margin requirements. When the market is not fully orthogonal, the Joint Margin collateral requirements diverge from those of the VaR system to account for *risk homogeneity*. For example, when homogeneous risks increase (i.e., $\Pr\left(V_{i,t+1} \leq -B_t^{i,j} | V_{j,t+1} \leq -B_t^{j,i}\right)$ increases), the Joint Margin system collects more collateral from both firms $i$ and $j$. This adjustment reflects the fact that these firms will tend to be in financial distress at the same time more often. Therefore, they jointly represent a higher risk to the clearing house, and as a consequence, they are required to offset this risk by posting more margin. However, notice that both firms increase their collateral amounts proportionally and consistently, in the sense that their collateral requirements correspond to the same quantile level $\gamma_t^{i,j}$ of their simulated P&L distribution. In the extreme case of perfect risk homogeneity, which can happen if both firms hold equivalent portfolios or if all assets in the market have perfect comovement, the Joint Margin system requires both firms to post a collateral amount equivalent to the $\gamma_t^{i,j} = \alpha^2$ quantile of their simulated P&Ls. Thus, even though the financial distress events of firms $i$ and $j$ might not be independent, their Joint Margin ensures that their joint distress probability corresponds exactly to the target selected by the risk manager. Therefore, the Joint Margin system enhances market stability by giving more systemic control to risk managers.

### 5.2. Estimation

Given the simulated path $\left\{v_{i,t+1}^s, v_{j,t+1}^s\right\}_{s=1}^S$, conditional on $B_{i,t}^{i,j}$ and $B_{j,t}^{i,j}$, a simple estimate of the joint probability $\Pr\left[\left(V_{i,t+1} \leq -B_{i,t}^{i,j}\right) \cap \left(V_{j,t+1} \leq -B_{j,t}^{i,j}\right)\right]$, denoted $Q_t^{i,j}$, is given by:

$$\hat{Q}_t^{i,j} = \frac{1}{S}\sum_{s=1}^{S} \mathbf{I}\left(v_{i,t+1}^s \leq -B_{i,t}^{i,j}\right) \times \mathbf{I}\left(v_{j,t+1}^s \leq -B_{j,t}^{i,j}\right) \tag{33}$$

where $v_{i,t+1}^s$ and $v_{j,t+1}^s$ correspond to the $s^{th}$ simulated P&L of firms $i$ and $j$, respectively. Given this result we can now obtain an estimate of $\gamma_t^{i,j}$. For each time $t$ and for each pair of firms $i$ and $j$, we look for the value $\gamma_t^{i,j}$, such that the distance $\hat{Q}_t^{i,j} - \alpha^2$ is minimized:

$$\hat{\gamma}_t^{i,j} = \arg\min_{\left\{\gamma_t^{i,j}\right\}} \left(\hat{Q}_t^{i,j} - \alpha^2\right)^2 \tag{34}$$

Thus, for each pair of firms $i$ and $j$, we obtain

$$\begin{aligned} \hat{B}_{i,t}^{i,j} &= percentile\left(\left\{v_{i,t+1}^s\right\}_{s=1}^S, 100\hat{\gamma}_t^{i,j}\right) \\ \hat{B}_{j,t}^{i,j} &= percentile\left(\left\{v_{j,t+1}^s\right\}_{s=1}^S, 100\hat{\gamma}_t^{i,j}\right) \end{aligned} \tag{35}$$

In addition, we obtain a time series of Joint Margin requirements, $\{\hat{B}_{i,t}^{i,j}\}_{t=1}^{T}$ and $\{\hat{B}_{j,t}^{i,j}\}_{t=1}^{T}$, for which confidence bounds can be bootstrapped.

### 5.3. Backtesting

The Joint Margin system allows us to test the joint probability of financial distress defined by the Joint Margin requirements of firms $i$ and $j$, $B_{i,t}^{i,j}$ and $B_{j,t}^{i,j}$. The null hypothesis in this case becomes:

$$H_0\colon \Pr\left[\left(V_{i,t+1} \leq -B_{i,t}^{i,j}\right) \cap \left(V_{j,t+1} \leq -B_{j,t}^{i,j}\right)\right] = \alpha^2 \tag{36}$$

Since the null implies that $E\left[\mathbf{I}\left(V_{i,t+1} \leq -B_{i,t}^{i,j}\right) \times \mathbf{I}\left(V_{j,t+1} \leq -B_{j,t}^{i,j}\right)\right] = \alpha^2$, then a simple LR test can be used to assess the conditional probability of financial distress based on the historical paths of the P&Ls of both firms $i$ and $j$; i.e., $\{v_{i,t+1}\}_{t=1}^{T}$ and $\{v_{j,t+1}\}_{t=1}^{T}$. The corresponding LR test statistic, denoted $LR_{i,j}$ takes the same form as $LR_i$:

$$LR_{i,j} = -2\ln[(1-\alpha^2)^{T-N_{i,j}}\alpha^{2N_{i,j}}] + 2\ln\left[\left(1-\frac{N_{i,j}}{T}\right)^{T-N_{i,j}} \frac{N_{i,j}}{T}^{N_{i,j}}\right] \tag{16}$$

except that in this case $N_{i,j}$ denotes the total number of joint past violations observed for both members $i$ and $j$; that is, $N_{i,j} = \sum_{t=1}^{T} \mathbf{I}\left(v_{i,t+1} \leq -B_{i,t}^{i,j}\right) \times \mathbf{I}\left(v_{j,t+1} \leq -B_{j,t}^{i,j}\right)$.

### 5.4. Extension to $n$ Simultaneous Events

Definition 3 is based on two simultaneous distress events; those of firms $i$ and $j$. We can easily generalize this framework by extending it to $n \leq N$ simultaneous events, each of them representing the financial distress of an individual firm. The Joint Margin of $n$ firms, denoted $B_{i,t}^{i,n}$ for $i = 1, \dots, n$, corresponds to the $\gamma_t^{i,n}$% quantile of the jointly simulated P&L across all common scenarios, such that

$$\Pr\left[\left(V_{1,t+1} \leq -B_{1,t}^{1,n}\right) \cap \ldots \cap \left(V_{n,t+1} \leq -B_{n,t}^{n,n}\right)\right] = \alpha^n \tag{18}$$

In this case, we obtain the simulated path $\left\{v_{1,t+1}^s, \ldots, v_{n,t+1}^s\right\}_{s=1}^S$. Conditional on $B_{i,t}^{i,n}$ for $i = 1, \ldots, n$, an estimate of the joint probability $\Pr\left[\left(V_{1,t+1} \leq -B_{1,t}^{1,n}\right) \cap \ldots \cap \left(V_{n,t+1} \leq -B_{n,t}^{n,n}\right)\right]$, denoted $Q_t^{i,n}$, is given by:

$$\hat{Q}_t^{i,n} = \frac{1}{S}\sum_{s=1}^{S}\prod_{\boldsymbol{i=1}}^{\boldsymbol{n}} \mathbf{I}\left(v_{i,t+1}^s \leq -B_{i,t}^{i,n}\right) \tag{19}$$

Therefore, for each time $t$ and for each set of $n$ firms, we look for the value $\gamma_t^{i,n}$, such that the distance $\hat{Q}_t^{i,n} - \alpha^n$ is minimized:

$$\gamma_t^{i,n} = \arg\min_{\left\{\gamma_t^{i,n}\right\}} \left(\hat{Q}_t^{i,n} - \alpha^n\right)^2 \tag{21}$$

For each of the $n$ firms, we obtain

$$\hat{B}_{i,t}^{i,n} = percentile\left(\left\{v_{i,t+1}^s\right\}_{s=1}^S, 100\gamma_t^{i,n}\right) \tag{20}$$

And as it was the case before, for each of the $n$ firms, we obtain a time series of Joint Margin requirements, $\left\{B_{i,t}^{i,n}\right\}_{t=1}^T$, for which confidence bounds can be bootstrapped.

The backtesting procedure in this case is based on the following null hypothesis:

$$H_0\text{: } \Pr\left[\left(V_{1,t+1} \leq -B_{1,t}^{1,n}\right) \cap \ldots \cap \left(V_{n,t+1} \leq -B_{n,t}^{n,n}\right)\right] = \alpha^n \tag{22}$$

Since the null implies that $E\left[\prod_{\boldsymbol{i=1}}^{\boldsymbol{n}} \mathbf{I}\left(V_{i,t+1} \leq -B_{i,t}^{i,n}\right)\right] = \alpha^n$, the corresponding LR test statistic, $LR_{1,n}$, which is based on the historical P&L paths for the $n$ firms; i.e., $\left\{v_{i,t+1}\right\}_{t=1}^T$, becomes

$$LR_{1,n} = -2\ln\left[(1-\alpha^n)^{T-N_{1,n}}\alpha^{nN_{1,n}}\right] + 2\ln\left[\left(1-\frac{N_{1,n}}{T}\right)^{T-N_{1,n}} \frac{N_{1,n}}{T}^{N_{1,n}}\right] \tag{23}$$

Except that in this case $N_{1,n}$ denotes the total number of joint past violations observed for the $n$ firms; that is, $N_{1,n} = \sum_{t=1}^{T} \prod_{i=1}^{n} \mathbf{I}\left(v_{i,t+1} \leq -B_{i,t}^{i,n}\right)$.

## 6. Conclusion

In this paper, we have presented a new collateral system, called CoMargin, for derivatives exchanges. CoMargin depends on both the tail risk of a given market participant and the interdependence between this participant and others participants. CoMargin can be estimated by a model-free scenario-based methodology, backtested using formal statistical tests, and generalized to any number of market members.

In the subsequent draft of this paper, we will conduct a large-scale simulation to compare the performance of CoMargin with that of other margining systems. A particularly important situation is that where both the level of trading similarity and comovement among underlying asset increase. We will also investigate the effects on the stability of the financial system of increasing the number of market participants and of adding new derivatives to be cleared.

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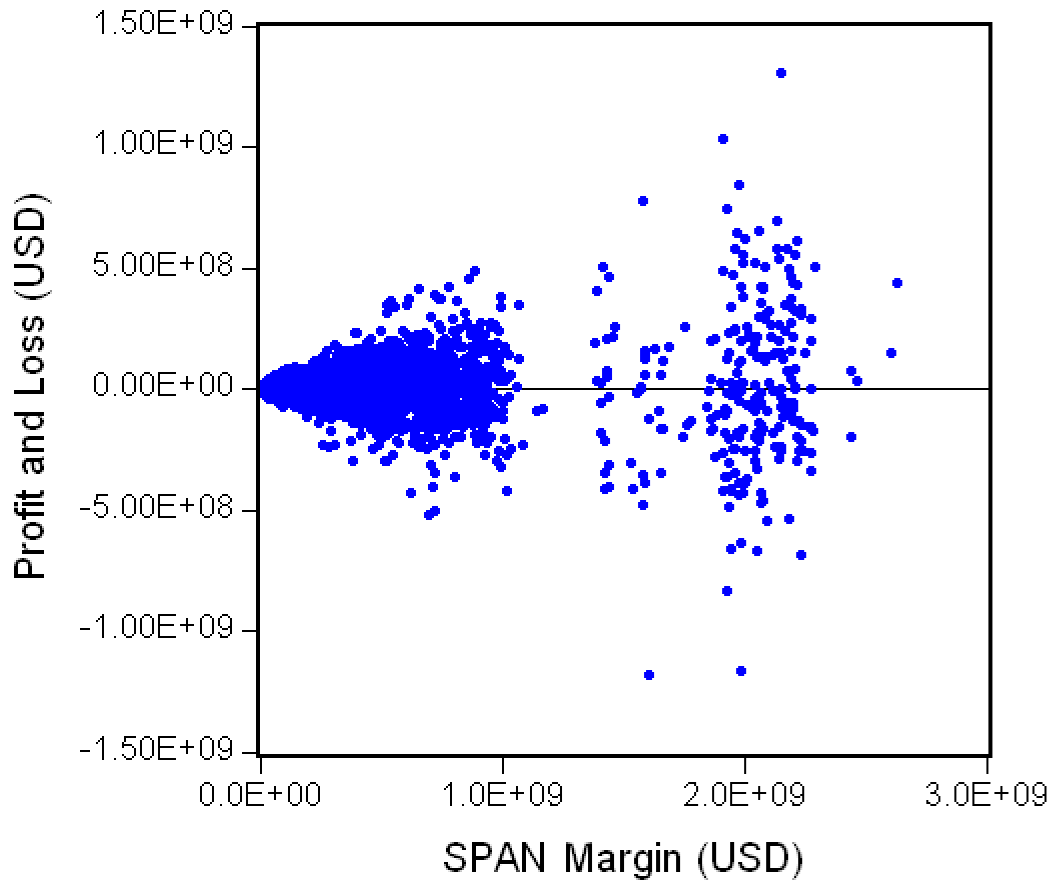
## Table 1: Major Clearinghouses

| Clearinghouses | CME | Eurex | LCH.Clearnet | Nymex | OCC |
|---|---|---|---|---|---|
| Markets cleared | CME, CBOT | Eurex | Euronext.liffe, ICE, LME, Powernext | Nymex, Comex | AMEX, CBOE, |
| Number of clearing firms | 86 | 90 | 77 | 40 | 120 |
| Average daily volume | 9 million | 6 million | 4 million | 1 million | 10 million |
| Average daily turnover | $3,000 billion | $550 billion | $2,500 billion | N/A | $10 billion |
| Margining system | SPAN | SPAN | SPAN | SPAN | SPAN |
| Aggregate margin | $47 billion | $39 billion | N/A | N/A | $77 billion |
| Default fund | $1 billion | $0.9 billion | $2.6 billion | $0.2 billion | $2.9 billion |
| Default insurance | - | - | $0.4 billion | $0.1 billion | - |
| Other guarantees | Membership value and assessment power | Deutsche Boerse, SWX | - | Protection scheme for retail customers | - |
| Total default protection | $53 billion | $41 billion | N/A | N/A | $80 billion |

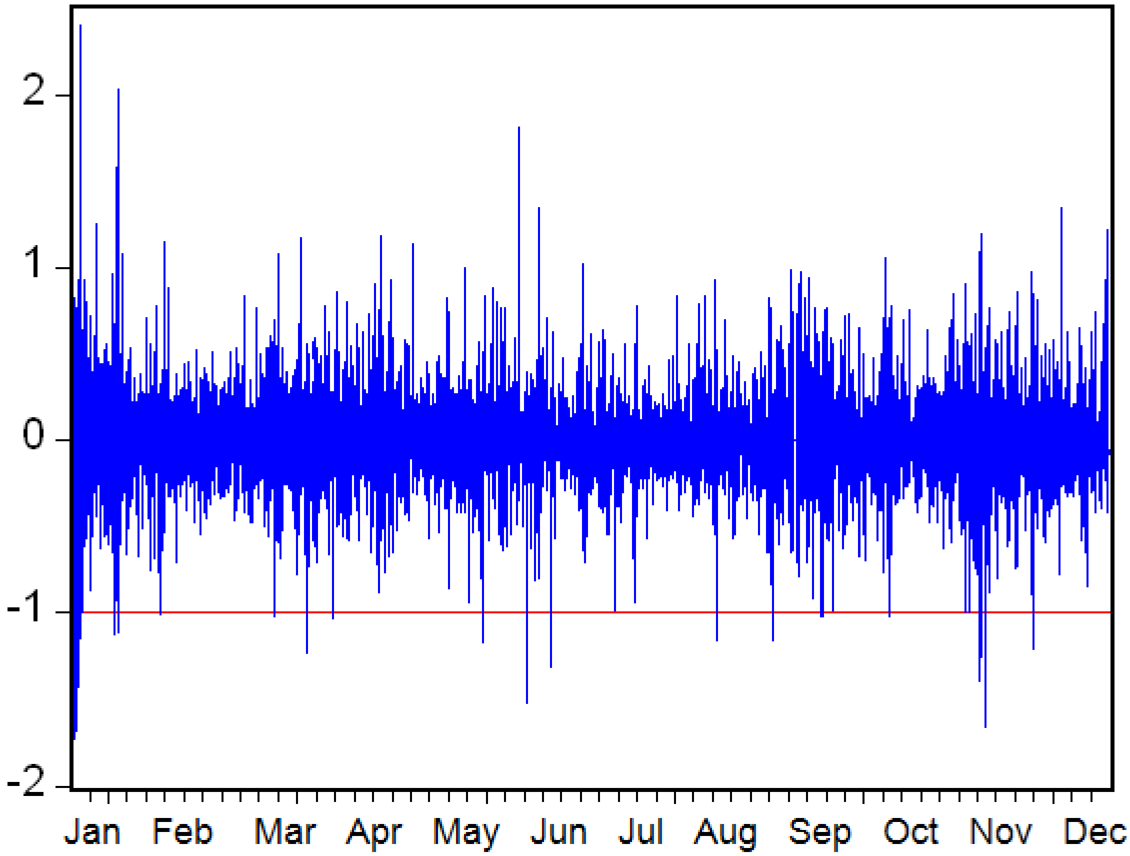
Notes: This table presents some descriptive statistics about the major clearinghouses in the world. We list the major derivatives markets that they clear, the number of clearing firms, the average daily volume in million of contracts, the average daily turnover (notional value), the margining system they use, the total aggregate margin or collateral collected from clearing firms for both customer trading and proprietary trading, the size of the default fund, the policy limit of the default insurance (if any), any other protections against default, as well as the total default protection, i.e. margin + default fund + default insurance + other guarantees. Source: Clearinghouses websites and annual reports (as of 2007).

**Figure 1: Daily Profit and Loss and Margins of CME Clearing Firms**




Notes: This figure displays a scatter plot of daily profit-and-loss and SPAN margins for all clearing firms of the Chicago Mercantile Exchange (CME). These figures correspond to the house trading account (i.e., proprietary trading) of the 69 clearing firms, and as such does not reflect customer trading. The sample period covers January 1, 2001 to December 31, 2001 and the total number of observations is 13,701.

**Figure 2: Ratio of the Daily P&L and Margin**




Notes: This figure displays the ratio of the daily profit-and-loss and SPAN margins for all clearing firms of the Chicago Mercantile Exchange (CME). A value less than -1 indicates that the daily trading loss of the clearing firm exceeded its posted margins on that day. These figures correspond to the house trading account (i.e., proprietary trading) of the clearing firms, and as such does not reflect customer trading. The sample period covers January 1, 2001 to December 31, 2001. On each sample day, we stack all clearing members' ratios. For instance, the first 69 observations are the daily variation margins of the 69 clearing members posting margins on January 1, 2001. The following 69 observations are the daily variation margins of the 69 clearing members posting margins on January 2, 2001, and so on.