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Abstract

As part of managing a debt portfolio, debt managers face the challenging task of choosing a strategy that minimizes the cost of debt, subject to limitations on risk. The Bank of Canada provides debt-management analysis and advice to the Government of Canada to assist in this task, with the Canadian debt-strategy model being developed to help in this regard. The authors outline the main elements of the model, which include: cost and risk measures, inflation-linked debt, optimization techniques, the framework used to model the government's funding requirement, the sensitivity of results to the choice of joint stochastic macroeconomic term-structure model, the effects of shocks to macroeconomic and term-structure variables and changes to their long-term values, and the relationship between issuance yield and issuance amount. Emphasis is placed on the degree to which changes to the formulation of model elements impact key results. The model is an important part of the decision-making process for the determination of the government's debt strategy. However, it remains one of many tools that are available to debt managers and is to be used in conjunction with the judgment of an experienced debt manager.

JEL classification: C0, G11, G17, H63
Bank classification: Debt management; Econometric and statistical methods; Financial markets; Fiscal policy

Résumé

Les gestionnaires d'un portefeuille de dette ont la difficile tâche de choisir une stratégie qui réduira au maximum le coût de la dette étant donné certaines limites de risque. La Banque du Canada fournit au gouvernement canadien des analyses et des conseils en appui à la sélection d'une telle stratégie. Les auteurs décrivent les grandes caractéristiques du modèle servant à éclairer la prise de décisions, notamment : les mesures de coût et de risque; les conséquences de l'intégration des obligations indexées sur l'inflation; les techniques d'optimisation; le cadre utilisé pour modéliser les besoins de financement du gouvernement; la sensibilité des résultats au choix du modèle stochastique qui formalise à la fois la dynamique de l'économie et celle de la structure des taux d'intérêt; l'incidence de chocs touchant les variables macroéconomiques et les variables relatives à la structure des taux d'intérêt ainsi que l'effet de modifications apportées à leurs valeurs de long terme; et la relation entre le rendement à l'émission et le montant émis. Les auteurs accordent une attention particulière à la mesure dans laquelle des changements dans la formulation des éléments du modèle influent sur les principaux résultats. Le modèle fait partie intégrante du processus décisionnel à la base de la stratégie d'emprunt de l'État. Cependant, il n'est que l'un des nombreux outils à la disposition des gestionnaires et son bon emploi doit reposer sur le jugement d'un gestionnaire expérimenté.

Classification JEL : C0, G11, G17, H63
Classification de la Banque : Gestion de la dette; Méthodes économétriques et statistiques; Marchés financiers; Politique budgétaire

1 Introduction and Overview

As part of managing a debt portfolio, debt managers face the complex task of choosing a strategy that minimizes the cost of debt, subject to certain risk limits.[1] Debt-management decisions depend on numerous factors, many of which are either unknown or are not under the control of the debt manager. These include the future behaviour of interest rates, the macroeconomy and the government's fiscal policy. Given the level of uncertainty and interrelationships between these factors, the Canadian debt-strategy model was developed to assist in debt-strategy decision making. This paper provides details on the important elements within the model and the risks inherent in these elements.

A critical question to ask when using the output of any model for the purposes of decision making is "just how reliable is it." The Canadian debt-strategy model is, of course, no exception. Unfortunately, in this situation, this question is easy to ask, but more difficult to answer. The reason for the difficulty stems from the fact that this is a large-scale model with an enormous number of moving parts.[2] To answer the question thoroughly, therefore, it is necessary to reflect upon the model at substantial length and consider its output from a variety of perspectives. The following chapters will examine key components of the model, largely on an empirical basis, to provide a deeper understanding of what results are available and how they can be used to assist in debt-strategy formulation.[3] The chapters were originally prepared as discussion notes for senior management. Although most of the data used are a few years old, we do not deem this a problem. The purpose is not to show current results or strategies, but to demonstrate how the model can be used, and to highlight its important (and potentially results-changing) concepts and components. Furthermore, the paper will not delve into issues related to the recent financial crisis. Rather, the separate chapters provide background on the model structure and help to address questions regarding the model's reliability.

Key assumptions

The heart of the model—which remains largely unchanged from the approach used in 2002/03 policy analysis and described in Bolder (2003)—is a collection of balance equations that describe the interaction of debt-service payments, debt maturities, government financing requirements and debt issuance. The element of the model under the control of the debt manager—the amount and proportion of debt to be issued in each period—is termed the *financing strategy*. A fundamental assumption of the model is that the financing strategy is constant over time. A second key assumption is that the random evolution of the Canadian macroeconomy and interest rates, which determine the risk and cost characteristics of alternative financing strategies, can be summarized by a reduced-form statistical model whose parameters are estimated from historical data. Chapter 2 describes the basic assumptions in more detail.

Cost and risk measures

A comparison of alternative financing strategies amounts to a comparison of the moments of the debt-service-charge, debt-issuance, debt-maturity and financial-requirement distributions. The question, of course, is which moments and which distributions. The answer lies in the Government of Canada's objectives of raising stable, low-cost funding while simultaneously maintaining well-functioning government securities' markets. Cost is straightforward and is generally measured as the average debt charges as a percentage of debt stock. Stability is rather more difficult. We analyze both debt-charge and budgetary-risk measures. Within those two types, we look at regular dispersion and tail-risk measures. Rollover risk measures are also discussed in this chapter.

[1] In Canada, an associated objective is to maintain a well-functioning Government of Canada securities market.

[2] To give you a sense of the scale, the current implementation of the debt-strategy model consists of more than 250 computer subroutines and has more than 100 parameters.

[3] Numerous working papers on technical aspects of the model can be found in the references. For example, Bolder and Rubin (2007) provide a detailed description and evaluation of different optimization techniques.

Clearly, the model output will be reliable only insofar as the selected measures of cost and risk adequately reflect the government's true objectives. Cost and risk measures are discussed in Chapter 3.

Inflation-linked bonds

Determining the cash flows of an inflation-linked bond is straightforward given that we explicitly model inflation in the stochastic model. The first part of Chapter 4 focuses on some subtleties between nominal and inflation-linked cash flows in a stylized example. The second part of Chapter 4 compares portfolios with varying levels of 30-year inflation-linked and 30-year nominal debt, but with otherwise identical portfolios.

Optimization module

Direct optimization of the simulation model is not feasible due to computational constraints. Consequently, we use mathematical techniques to approximate the objective function and perform the optimization on these approximations. Bolder and Rubin (2007) use these function-approximation techniques and apply them to a number of notoriously pathological mathematical functions, with good results. The bulk of the chapter focuses on the flexibility of the optimization module with respect to risk measures, and having multiple constraints. The optimization module is treated in Chapter 5.

The government's fiscal position

In the model, we assume that the government's primary balance is a function of the macroeconomy. We use the stochastic model to forecast the macroeconomy and consequently the primary balance. A fixed component (i.e., a free parameter) of the primary balance is used to target, at the beginning of the fiscal year, the government's debt paydown. As the actual macroeconomy evolves, deviations from the forecast values will occur. This structure and the corresponding parameter estimates are based on generalized rules of thumb presented in federal budgets, as described in Robbins, Torgunrud, and Matier (2007). Caution and sensitivity analysis are nonetheless required. In addition, illustrative examples are provided to highlight model results and findings from the perspective of budgetary risk. Chapter 6 provides more details on these areas.

Stochastic models

There are, at least, two principal issues related to stochastic models. The first is model risk. If one uses two alternative models estimated to the same input data, one will generally receive two alternative sets of results. Second, even with the same model, two different input datasets will yield different model parameters and hence different results. These are undesirable, but unavoidable, facts. We address each directly. We handle model risk by the use of not one, but five, alternative state-of-the-art stochastic models. As well, various time periods have been tested and sensitivity analysis has been performed on key yield-curve parameters, with the portfolio allocations remaining extremely stable in this analysis. A more formal comparison of the stochastic models is provided in Chapter 7.

Macroeconomy

Understanding the interaction of macroeconomic and term-structure variables is essential to understanding the results of the model. Perhaps more important is the sensitivity of the model to macroeconomic and financial shocks, and to different model parameters and steady-state values. Given the depth of the analysis required, one model (Nelson-Siegel) will be analyzed and tested in this regard. Such analysis is required to understand the reasoning behind the results, and to test their robustness (and ultimately their practical importance) to different model parameters and specifications. Furthermore, the impacts of shocks to, and altering the long-term values of, key variables is analyzed. This analysis is provided in Chapter 8.

Issuance-penalty functions

Although the government has discretion in terms of what it can issue at prevailing market rates, it is impractical, and incorrect, to suggest that this is limitless. We address this issue by constructing an issuance-penalty function that alters the funding cost for excessively large or small levels of issuance. Given historic issuance patterns, the function must be subjectively determined. This is not ideal, but we feel it is preferable to ignoring the issue completely. However, due to this subjectivity, understanding the implications of different specifications and having consensus among policy-makers is essential. More information on the issuance-penalty function is provided in Chapter 9.

2 The Basics of the Model

This chapter begins with the basic structure of the model. It is organized into three parts. First, we begin with a brief definition of the key aspects of the debt manager's challenge. Second, we describe the model by focusing on the inputs, the key computations and the outputs. Along the way, we summarize some of the principal assumptions of the model. It should be noted that we have not completely rebuilt the model. A number of aspects remain from the original model described in Bolder (2003), and, to a further extent, Bolder (2008). As such, while describing the model as a whole, we will indicate those aspects of the model that are new.

2.1 The debt manager's challenge

The role of a debt manager is to select a financing strategy that meets a set of policy objectives. Government debt managers, however, face a substantial amount of uncertainty, over which they have relatively little control. Clearly, they do not control future interest rate and macroeconomic outcomes, nor do they exert any direct influence over the government's primary surplus or fiscal policy. They do, however, control the government's financing choices, or what we term a financing strategy.[4] In short, a government debt manager's job is quite challenging. Selecting a financing strategy, in the face of substantial uncertainty, to meet a complex set of policy objectives is a challenging task. As a consequence of this complexity, it is helpful to include a mathematical model as a component in the decision-making process.

2.2 The model

In any model, there is a tension between complication and simplification. The model attempts to balance these challenges in describing the following five aspects of the debt manager's challenge:

1. the financing strategy;
2. uncertainty about the future;
3. the mechanics of government debt and fiscal management;
4. interactions and feedbacks between key variables;
5. the set of policy objectives.

Our description of the model consecutively addresses each of these aspects to provide an overview of the general approach.

2.2.1 The financing strategy

The first, and perhaps most central, component of the debt-strategy model is the financing strategy. To describe it completely, one must specify the amount of treasury bill issuance—at 3-, 6- and 12-month tenors—and the amount of both nominal and inflation-linked coupon bond issuance—at 2-, 3-, 5-, 10- and 30-year tenors.[5] Moreover, the relative mix of issuance can vary through time on an annual or even quarterly basis. This degree of complexity is excessive for long-term strategic analysis, because it is very difficult to compare such highly complex financing strategies. Consequently, we make the following simplifying assumption, which is *unchanged* from the original model implementation.

[4]In the Canadian context, this amounts to a determination of the relative mix of nominal versus inflation-linked debt and the maturity composition of the debt stock.

[5]Currently, the Government of Canada issues only 30-year inflation-linked bonds, known as Real Return Bonds (RRBs). As well, most examples used in this paper are from 2007 and have eight financing instruments, excluding the 3-year nominal bond that was reintroduced in 2008/09.

Assumption 1. *A financing strategy is summarized as a set of issuance weights—summing to one—describing the proportion of new issuance in each of the available financing instruments. The financing strategy is assumed to be constant through time.*

This foundation of the model permits a mathematically precise and succinct definition of a government financing strategy.[6] It also reflects the reality that the Canadian government does not dramatically alter its financing strategy from one year to the next. It is a foundational aspect of the model, because our central objective is to understand how key aspects of the debt manager's challenge—such as the size of the debt, the cost of the debt, the volatility of debt costs, the volatility of the government's funding requirement and the amount of issuance—react to different financing strategy choices.

2.2.2 Introducing uncertainty

If we knew, with complete certainty, the future path of the Canadian economy and interest rates, it would be relatively straightforward to determine how alternative financing strategies would impact the previously mentioned elements of the government's debt strategy. Since this is not the case, we need an approach that permits us to incorporate future macroeconomic and interest rate uncertainty in an organized manner. This leads us to the next assumption.

Assumption 2. *The random evolution of the Canadian macroeconomy and interest rates can be summarized by a reduced-form statistical model whose parameters are estimated from historical data.*[7]

The statistical model is the second key aspect and a critical input into the stochastic-simulation model. This new component of the model is *not* a single model, but rather a collection of approaches. Alternative statistical models were implemented and evaluated in an effort to understand this important input. As a consequence, we are in a position to use more than one approach in our analysis.[8] Each of the statistical models summarizes the uncertainty that policy-makers face in making debt-strategy decisions. In all models, the macroeconomy is described by the output gap, inflation and a monetary policy rate. Interest rates are assumed to depend on these macroeconomic quantities, as well as a collection of term-structure-related variables. The explicit inclusion of inflation permits, for the first time, the consideration of inflation-linked debt in a comprehensive manner.

Figure 1 provides an illustrative summary of the evolution of inflation, the output gap, the monetary policy rate and interest rates associated with 10,000 simulations from the statistical model. In other words, our statistical model describes how key aspects of the debt manager's challenge move randomly through time.

2.2.3 Debt and fiscal mechanics

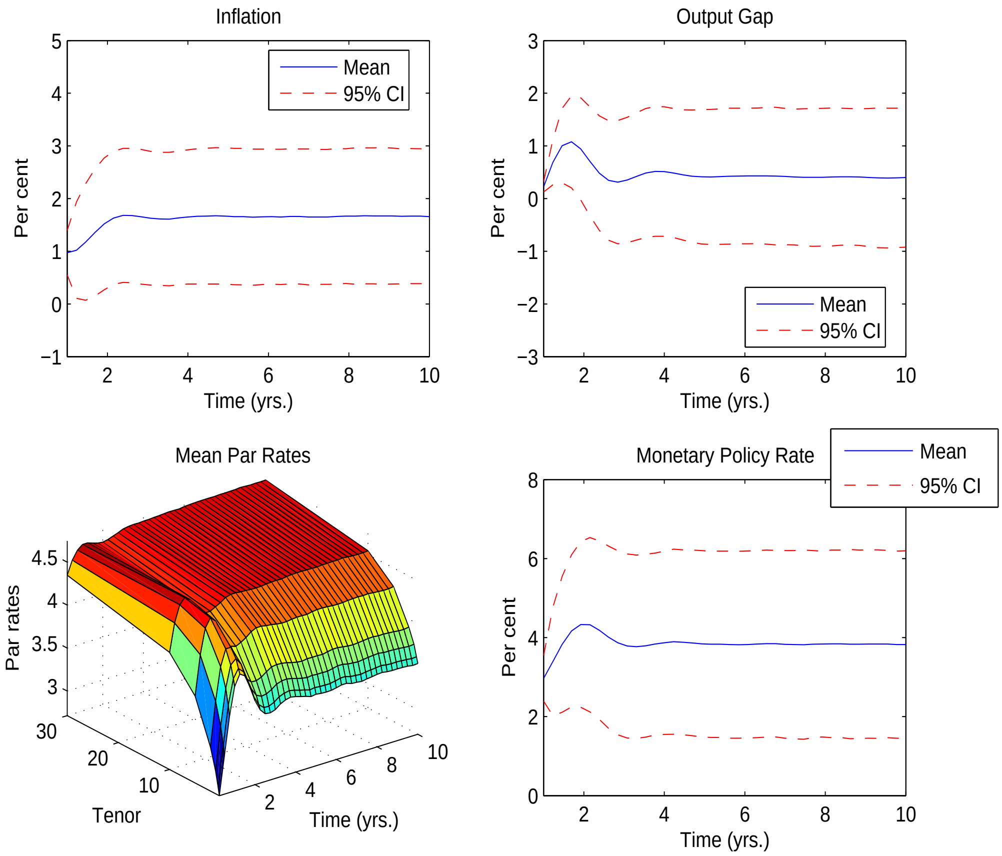
How exactly are these statistical inputs used to provide insight into how a given financing strategy influences key debt-strategy indicators? This brings us to the third component of the stochastic-simulation model: a description of the mechanics of government debt and fiscal management. Put more simply, a large part of the stochastic-simulation model is a collection of mathematical expressions that describe how the debt stock matures, how the government's funding requirement is computed, how the maturing debt and new funding requirement are refinanced, how debt charges are computed, and how these outcomes impact the size and composition of the debt stock. For a given financing strategy and a single realization of future macroeconomic and interest rate outcomes, each of these quantities can be computed.

[6] The model can also be used for more detailed analysis where highly specialized financing strategies are provided. This is useful for the determination of annual bond programs once a decision on the strategic direction has been taken.

[7] A sequence of Bank of Canada working papers describes the structure of the statistical model in substantial detail. See Bolder (2001, 2002, 2006) and Bolder and Liu (2007).

[8] While our analysis has pointed to one preferred model, the ability to use alternative approaches will help guard against model risk in our policy recommendations. See Chapter 7 for further discussion.

Figure 1: **The Inputs**: This figure illustrates the principal inputs to the model including inflation, the output gap, the monetary policy rate and the term structure of interest rates. These inputs are randomly generated from a statistical description of the Canadian macroeconomy and interest rates. Observe that approximately four years are required for the different variables to reach their long-run averages. The starting point for this example is 31 March 2006. Note that the variables converge to their long-term historical means. Current practice is to set the long-term mean of inflation to 2 per cent and the long-term mean of the output gap to 0.



Since this is the heart of the model, it merits more description. We need three inputs to run the model. First, we require the existing federal debt stock: the amount of treasury bills, nominal bonds and inflation-linked bonds. Second, we need a sequence of future macroeconomic and interest rate outcomes from the statistical model. Finally, we need a financing strategy. Armed with these three items, we can proceed. From the debt stock, we determine a sequence of known maturities into the future. In the first period, we compute the government's funding requirement (i.e., surplus or deficit), which will depend on the state of the macroeconomy in that period. Adding in the maturing debt, from previous periods, provides us with the amount of debt that must be issued in the first period. The financing strategy determines how this amount will be issued and the implications for the debt stock. Once the amount and composition of issuance is determined, we proceed to compute the debt charges for the first period, which will depend on current and past interest rates. In the second period, this sequence of steps is repeated, although it is slightly more complicated, since the results of the second step will depend on what was done in the first step. This sequence of steps is repeated, in an iterative fashion, for each period across the simulation horizon. Figure 2 provides a schematic overview of the simulation algorithm.

Thus far, we have described only the computation of debt quantities for a single realization of the statistical model. Clearly, this provides little or no insight into the uncertainty faced by the debt manager. The solution

Figure 2: **The Stochastic-Simulation Algorithm**: This figure describes, in a heuristic manner, the basic steps in the stochastic-simulation algorithm employed by the debt-strategy model.

`begin algorithm`

1. Select a financing strategy

    2. Generate a set of random outcomes

        3. Select a time step

            √ Compute maturing debt
            √ Compute funding requirement
            √ Issue new debt
            √ Compute debt charges
            √ Compute debt stock

        Repeat third step for each of the quarterly time steps

    Generate another set of random outcomes and iterate on step 3.

Select another financing strategy and repeat steps 2 and 3.

4. Save and review results

`end algorithm`

Step 1: repeat for each financing strategy

Step 2: repeat for each randomly generated scenario

Step 3: repeat for each quarterly time step

to this problem is to repeat the previous analysis, for the same fixed financing strategy, many thousands of times. In this way, one constructs a statistical distribution of debt quantities associated with a given financing strategy. This technique is termed stochastic, or Monte Carlo, simulation. Comparisons among different financing strategies, therefore, essentially amount to comparisons between different aspects of these distributions.

To underscore these ideas, we perform a simple, illustrative example of the debt-strategy stochastic-simulation model. We take the actual debt portfolio as of 31 March 2006 and apply a financing strategy composed of equal amounts in each of the available financing instruments.[9] Figure 3 provides an overview—across the 10,000 simulations summarized in Figure 1—of the debt charges, the funding requirement, the size of the debt stock and bond/treasury bill issuance over the course of a 10-year simulation horizon. Observe that each quantity is surrounded by a 95 per cent confidence interval that describes the uncertainty around their future evolution. The model can, of course, provide much more detailed information about these specific aspects of the debt manager's challenge.[10] The idea of Figure 3, however, is merely to provide some flavour for the principal model outputs. The important point is that the model provides, for any choice of financing strategy, a rich description of the key elements of the government's debt strategy and a quantification of their relative uncertainty.

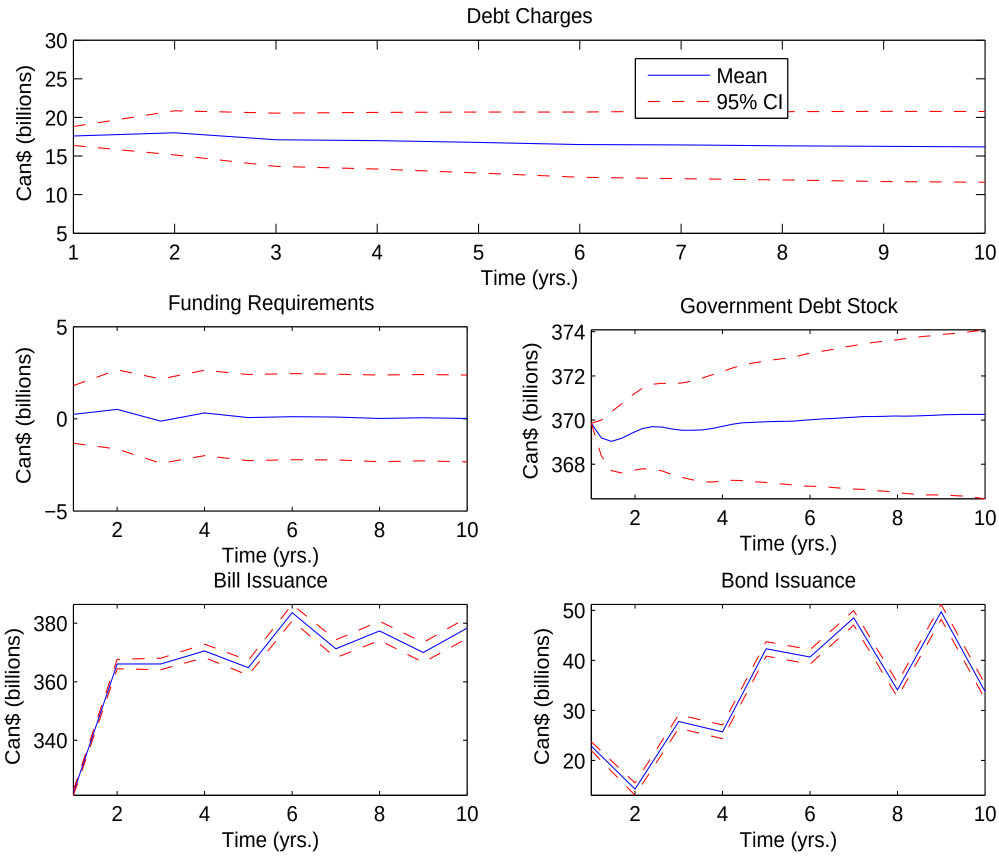
There are a number of details that make the mechanics of government debt and fiscal management rather complex. In particular, how do we handle reopenings? A new bond is typically issued at, or close to, par. Subsequent reopenings of this bond all share the same maturity date and coupon; consequently, the price received by the government may either be par, a premium or a discount, depending on prevailing market conditions. Dealing with premiums or discounts, however, requires amortization of these amounts through time, leading to increased model complexity and increased computational effort. Our solution, which has *not* changed in the new implementation of the model, is summarized in the following assumption.

Assumption 3. *We assume that a reopened bond is actually a collection of bonds that share a single maturity date, but not a single coupon rate. If a bond is to be reopened five times, for example, then five separate bonds*

[9] This is the so-called $\frac{1}{N}$ approach.

[10] In particular, the model computes portfolio-summary measures such as the refixing share of debt, the average term to maturity and the duration. The model also includes a number of different measures of cost and risk associated with a given strategy.

Figure 3: **Some Key Outputs**: This figure illustrates the evolution of a number of summary measures of the portfolio over the next 10 years including the debt charges, funding requirements and debt stock in addition to treasury bill and bond issuance. For each quantity, the expected value and a 95 per cent confidence interval is shown.



will be issued with the same maturity date.[11] *Each bond, however, has a different coupon that ensures the bond is issued at par. This implies that the premium or discount is handled through the time-varying coupon and is automatically amortized over the life of the bond. Each bond in the model, therefore, is essentially a portfolio of separate coupon bonds, all sharing the same maturity date with different coupon rates. This assumption also applies to the previous version of the model.*

2.2.4 Interactions and feedbacks

Another complication is associated with what we previously described as interactions and feedbacks between key variables. The government's funding requirement, for example, depends on a number of elements. It is related to the government's fiscal policy, the choice of financing strategy and the evolution of the macroeconomy. Given the importance of this aspect of the model, we make the following set of *new* assumptions regarding the government's funding requirement.

Assumption 4. *We assume that the government's funding requirement is equivalent to the surplus-deficit position of the government. We abstract from the non-cash items that generally contribute to a difference between the cash-based funding requirement and the government's ultimate budgetary position. We further assume a specific relationship between the government's primary balance and the macroeconomy. Using this relationship,*

[11]In the model, the user specifies the number of reopenings.

the funding requirement is subsequently computed as the government's primary balance less government debt charges. Finally, we make assumptions about the government's treatment of a given surplus or deficit position. This amounts to a fiscal rule for the government that describes how they react to deviations from expected deficit or surplus positions.

Yet another feedback relates to the fact that we do not place any *a priori* bounds on the financing strategies; in other words, the evolution of the macroeconomy does not depend on fiscal policy.[12] We could, for example, consider a financing strategy entirely composed of 3-month treasury bills. With a debt stock over Can$500 billion, this would amount to over Can$2.0 trillion of annual treasury bill issuance. This is clearly an extreme financing strategy, which would result in biweekly 3-month treasury bill issuance over $70 billion, several times higher than the historical maximum. For this reason, the model includes a price adjustment for excessively large or small issuance amounts. This adjustment is described in the following assumption.

Assumption 5. *We assume that if issuance falls within "normal" ranges, then it can be issued at prevailing market prices described by the statistical model. If issuance falls below or above these target ranges, then the financing cost generally increases.*[13] *This amounts to a penalty function that penalizes excessively small or large issuance in a given financing instrument.*[14]

2.2.5 Policy objectives

The final aspect of the debt manager's challenge relates to the set of policy objectives of the government. Traditionally, debt management has focused on attempting to find a trade-off between the level of debt charges and debt-charge volatility.[15] Understanding this trade-off has been the focus of much debt-management research in past years. This is consistent with the historical fact that debt strategy has been conducted fairly independently of fiscal policy.[16] However, there has been an increasing appreciation that debt-charge volatility is partially important, insofar as it leads to an associated increase in budgetary volatility. The selection of a portfolio that minimizes budgetary volatility—while also considering the level and volatility of debt costs—could potentially permit a greater degree of flexibility in fiscal policy. That is, greater certainty would allow for a smoother tax profile and a larger proportion of permanent, as opposed to temporary, expenditure initiatives.[17] However, the extent to which budgetary volatility can be meaningfully impacted by the choice of financing strategy remains an open question.[18] Recently, the importance of having "reasonable" debt rollover amounts has become more apparent. A smooth maturity profile will provide insulation against adverse financing conditions, and help minimize disruptions from large single-day maturities.

How are these policy objectives incorporated into the analysis? There are three steps. First, one must define a set of policy objectives for the debt-strategy decision. Second, one must determine what measure, from the stochastic-simulation model, best describes attainment of each policy objective. Finally, one considers a wide range of financing strategies and selects the specific strategy that best achieves one's policy-objective-related measures. Let us consider a simple example where the government wishes only to minimize the cost of

[12] Fiscal policy is assumed to depend on the macroeconomy, but we make the simplifying assumption that the macroeconomy does not depend on fiscal policy.

[13] Note that, for some instruments, such as 3-month treasury bills and inflation-linked bonds, the borrowing costs could actually fall as issuance decreases. This interesting issue is addressed in Chapter 9.

[14] Determining the parameters for this penalty function is far from obvious. The idea is not to be precisely correct, but rather generally reasonable. Currently, therefore, we use conservative values determined through discussion and consensus. See Chapter 9 for a more detailed discussion of these issuance-penalty functions.

[15] The generally upward-sloping nature of the yield curve implies that, on average, nominal short-term debt is less expensive. Since nominal short-term interest rates are more volatile than their long-term counterparts, one typically has to be prepared to accept higher uncertainty for lower nominal debt charges. This relationship is less obvious when considering inflation-linked debt.

[16] This was not, of course, a deliberate choice, but rather a simplifying assumption.

[17] A tax-smoothing objective for debt management is prevalent in the academic literature; for example, in Barro (1999).

[18] Of course, this is directly related to the relative proportion of debt costs in the budget. That is, for a given budget size, the larger the debt stock (and therefore nominal debt cost), the more impact debt strategy can have on budgetary volatility.

debt issuance. In this case, a reasonable policy-objective measure would be the average debt charges over the simulation horizon. We could then proceed to examine a large set of financing strategies and select the financing strategy with the smallest average debt charges over the simulation horizon.

Clearly, the set of policy objectives are much richer than suggested by our simple example. This suggests that finding the specific financing strategy that best meets a set of policy objectives can be quite challenging. For this reason, an optimization module was developed that permits one to find the financing strategy that provides the best fit to a given set of policy objectives.[19] Moreover, this module permits a greater degree of flexibility in defining the specific forms of the government policy objectives. One may, for example, wish to minimize the cost of debt issuance with constraints on the volatility of debt-service charges, the amount issued in various financing instruments and the highest quarterly rollover in the simulation. While one can use the stochastic-simulation model without the optimization module, it is a useful tool given the complexity of the debt manager's challenge.

2.3 Conclusion

The chapter has provided a brief, non-technical overview of the debt-strategy stochastic simulation model. In the interest of brevity, we have focused primarily on key inputs, outputs and assumptions. This chapter is intended to set the stage for the subsequent discussion that will delve more deeply into the key aspects and assumptions of the model.

[19]The model is *one* tool available to debt managers. In practice, an optimal financing strategy will not be implemented exactly, since some considerations cannot be fully incorporated into the model.

3 Measuring Cost and Risk

In this chapter, we address the measurement of cost and risk in the specific context of the debt-strategy model, so that we can proceed to consider their respective trade-offs. Since cost is simply negative return, we can think of risk-cost trade-offs in a manner analogous to the ubiquitous risk-return trade-off found in finance theory.

This chapter is organized into three parts. We begin with an overview of cost and risk as it pertains to the debt manager's challenge. There are a number of different dimensions to the debt manager's challenge and, as such, no one, *single* measure encompasses all of these important dimensions. Therefore, it is necessary to use a combination of measures to evaluate the extent to which different financing strategies meet our policy objectives.

The charts and tables in this chapter, as elsewhere in the paper, are not current and are presented for illustrative purposes only. They are intended to show what measures are available and how they could be used in debt-strategy analysis and decision making.

In the first section of this chapter, we introduce the principal dimensions of cost and risk. In the second section, we will introduce and review a collection of specific measures, talk about how they are computed, and provide some illustrative examples. The final section provides some concluding remarks.

3.1 Defining cost and risk

What measures should we use to characterize the cost and risk of the government's debt portfolio? Perhaps a good place to begin is to consider the publicly stated objectives of the Canadian government with respect to this portfolio:

> Our debt-management objective is to raise stable, low-cost funding for the government and to maintain a well-functioning market for government securities.

This high-level statement provides some useful direction. In particular, the terms "low-cost" and "stable" are important, even if they do not tell us exactly how to compute the cost and risk of our debt portfolio. The rest of this section, and indeed the rest of the chapter, will discuss a number of different possible candidate measures to help us identify "low-cost" and "stable" financing strategies.

3.1.1 Cost

In basic portfolio theory, one generally measures return as intermediate cash flows plus the percentage change in an asset's value over some period. The risk component is then typically described as the variance of this return. Indeed, the foundations of portfolio theory are constructed with these two simple statistical objects. One determines the optimal portfolio weights by simply minimizing the return variance subject to an expected return target.[20]

Fundamental to this approach is the idea that the investor stands ready to liquidate his or her portfolio at any moment. This quite reasonable assumption makes it difficult for us to use the cost and risk analogue of simple return and return variance. The reason is that the government's domestic debt portfolio is—and, it is safe to assume, will remain—a buy-and-hold portfolio.[21] This implies that measuring cost as the percentage change in the liability portfolio's value is not particularly useful. Moreover, the variance of changes in the market value of the liability portfolio is also of limited usefulness.

Taking cost and risk definitions directly from portfolio theory, therefore, will not work. One reasonable alternative approach to measuring one's funding cost, in a buy-and-hold setting, would be to compute the debt-

[20] It can, of course, get more complicated than this, but the basic idea remains the same.

[21] The government does engage in a regular program of bond repurchases, which would seem to contradict this buy-and-hold assumption. The bond repurchase program, however, is operated in a cost- and risk-neutral fashion to enhance primary and secondary market liquidity, and to reduce large single-day maturity amounts by operating in a specific sector.

service charges associated with a given liability portfolio.[22] The notion of cost can be represented in absolute terms, say billions of Canadian dollars, or in relative terms as a percentage of the outstanding debt stock. Low cost could then be translated into a liability portfolio with *low* debt-service charges.

There are, of course, a few complications. First, we have described the debt-service charges in nominal terms. Since fiscal planning, government budgets and public announcements generally occur in nominal terms, it would be hard to eliminate the use of a nominal perspective. It may, nevertheless, be interesting, and useful, to also consider the government's debt-service costs in real terms. Second, we examine the debt portfolio over a multiple-year horizon. As such, it is important to consider not only the debt charges in the next period, but also for each of the future periods over the analysis horizon. When we turn to examine specific measures, we will identify a few methods for dealing with these issues.

3.1.2 A traditional notion of risk

What about risk? If we define cost as debt-service charges, then it would be natural to define risk as the inherent uncertainty, or dispersion, of these debt charges. How might we measure this uncertainty? We could use, as in basic portfolio theory, the variance or standard deviation of the annual debt-service charges. Stability would then be interpreted to imply stable debt-service charges. Such a measure would provide a useful description of normal variation in the government's financing costs.

Risk may also be concerned with the occurrences of relatively rare and undesirable events. In recent years, debt managers have been inspired by the risk-management literature to consider percentile measures of the debt-service-charge distribution. In particular, it is common to compute what is termed cost-at-risk (CaR)—which is analogous to the well-known risk-management concept, value-at-risk—to describe the maximum debt charges for a given period with a given probability. There are a variety of flavours of CaR, which we will discuss in greater detail in the next section. All of these measures, however, attempt to describe the tail of the debt-service charge distribution.

The two preceding notions of risk are what statisticians generally term conditional on the initial starting point. In other words, if we consider the variance of debt-service charges in the fifth year of the simulation horizon, we compute this variance from the perspective of the first period. The implication is that, as we move further out in time, uncertainty will naturally increase. Clearly, we expect to observe a wider range of interest rate and macroeconomic outcomes, for example, over the next five years as compared to the next year. The variance of debt charges in the second period will, therefore, almost certainly be larger than first-year debt-charge variance. Similarly, fifth-year variance will almost certainly be greater than fourth-year variance.[23]

While this is not necessarily problematic, the debt manager may also be interested in a slightly different notion of risk. Specifically, the debt manager may be interested in knowing the fifth-year debt-charge variance conditional on what happened in the fourth year, as opposed to the first year. Debt charges may have been quite high, or quite low, in the fourth year. What may be important to the debt manager is, taking the previous year's outcome into account, the amount of uncertainty regarding debt charges in the coming year. We attempt to address this notion by computing what we term conditional debt-charge volatility.

3.1.3 Another notion of risk

Stability can mean a number of things. As we have seen, the traditional definition of stability relates to uncertainty associated with government debt-service charges. The logic behind this choice is that highly volatile debt-service charges are undesirable, because they have the potential to spill over into budgetary volatility. If

[22]This would include the implied interest on treasury bills, the coupon payment for nominal bonds, and the combination of a real coupon and inflation uplift for the inflation-linked bonds.

[23]There is, of course, a limit to this increasing uncertainty, since policy tends to be stabilizing and keeps the economy on a fairly balanced growth path. Over a 10-year time horizon, however, we observe this general increase in uncertainty.

budgetary uncertainty is the primary concern, however, then it seems reasonable to look directly at budgetary outcomes.[24] Stability, from this alternative perspective, would be interpreted as a stable budgetary outcome.

The debt model has an explicit description of the dynamics of the government's primary balance, the government debt charges, and a fiscal rule describing how fiscal authorities deal with deficits and surpluses. Consequently, the debt model permits a relatively detailed description of the government's budgetary position. Thus, we can compute measures of the uncertainty associated with the budget. These could include—in a manner directly analogous to the debt-service-charge measures described above—the variance of the government's budget, different flavours of budget-at-risk, and the conditional volatility of the government's budgetary balance. It should nonetheless be stressed that nothing precludes simultaneous examination of cost- and budget-based risk measures.

3.1.4 Yet another notion of risk

Another less technical, albeit quite important, risk dimension is rollover risk. This is the amount of debt coming to maturity on a given day or in a particular quarter or year. A large amount of debt maturing on a single day can be complicated and costly to manage for the government and debt holders, and may create distortions in the short-term money market. Meanwhile, a large amount of debt maturing in a given quarter can lead to refinancing in potentially adverse financing conditions. Therefore, it is typically viewed as desirable to have a debt portfolio with a relatively smooth maturity profile.

Excessive refinancing amounts in a given quarter or year will show up in the various CaR measures. Nonetheless, it is useful to explicitly consider this notion. One quite useful, commonly used measure is the refixing share of debt. This measure describes the amount of debt that will be refinanced (repriced) during the coming 12-month period, as a percentage of the total debt stock.[25] A complement to this measure would be to examine the average maturity profile of the debt stock over the simulation horizon. Clearly, this is a bit more complicated, but will provide more detailed information. Avoiding large single-day maturities is done outside the model, since the model's quarterly time horizon cannot provide a complete analysis. To mitigate this risk, the number of such days where bonds mature during the course of the year can be increased.

3.2 Illustrating cost and risk

Perhaps the best way to understand the various cost and risk measures of the government's debt portfolio, after the previous high-level discussion, is to look at some specific numerical examples. Each example will be illustrated starting from the actual Government of Canada federal debt portfolio as of 31 March 2006, over a 10-year time horizon, with 10,000 stochastic simulations, and a financing strategy consisting of equal amounts in each financing instrument.[26] We should note that the specific values described in the examples are not the key focus of this discussion. Instead, the examples demonstrate the various dimensions of risk and cost that can be examined within the context of the debt-strategy model. What is important is how these risk and cost measures compare among alternative financing strategies, and their sensitivity to key model assumptions.

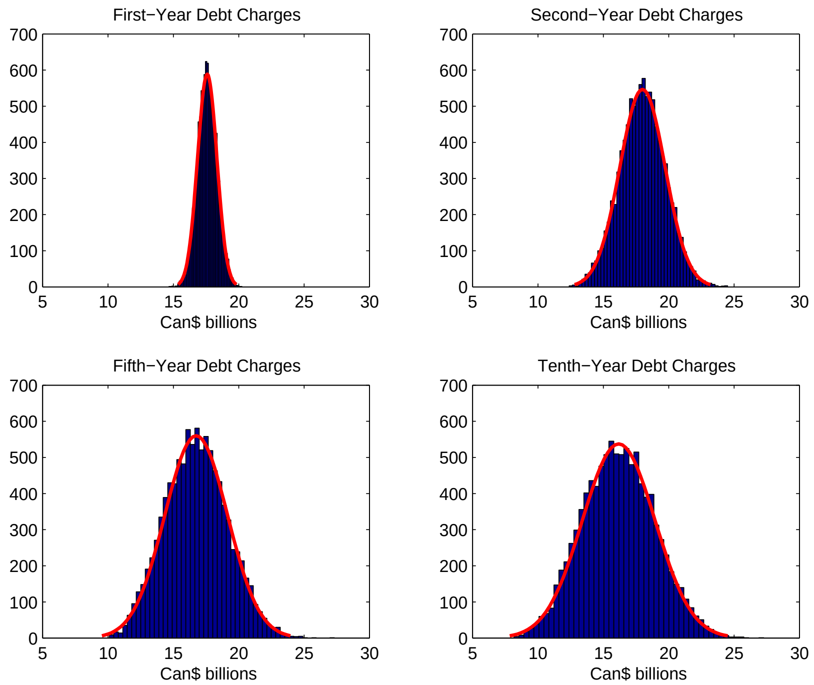
The first, and perhaps most natural in our context, measure of cost is the annual debt-service charges associated with the portfolio. Figure 4 outlines the debt-service-charge distributions associated with, for the purposes of brevity, the first, second, fifth and tenth years of the stochastic simulation. These distributions provide a fairly rich description of the cost and risk associated with this financing strategy across a 10-year time horizon. More specifically, for any given period, one can compute any summary statistic of the debt-service-

[24] Recall that in the previous chapter we defined, for the purposes of our model, the budgetary outcome (i.e., surplus or deficit) as the primary balance less debt charges.

[25] An associated measure is the refixing share of debt as a percentage of output (GDP), which provides added context to the overall debt environment.

[26] This is the same illustrative example used in Chapter 2.

Figure 4: **Absolute Debt-Charge Distributions**: This figure illustrates the debt-charge distributions—at 1-, 2-, 5- and 10-year horizons—associated with a financing strategy including equal amounts in each financing instrument. Observe how the dispersion in the debt-charge outcomes increases with the time horizon. This arises because the uncertainty about future outcomes increases over time. The red line represents a normal density function.



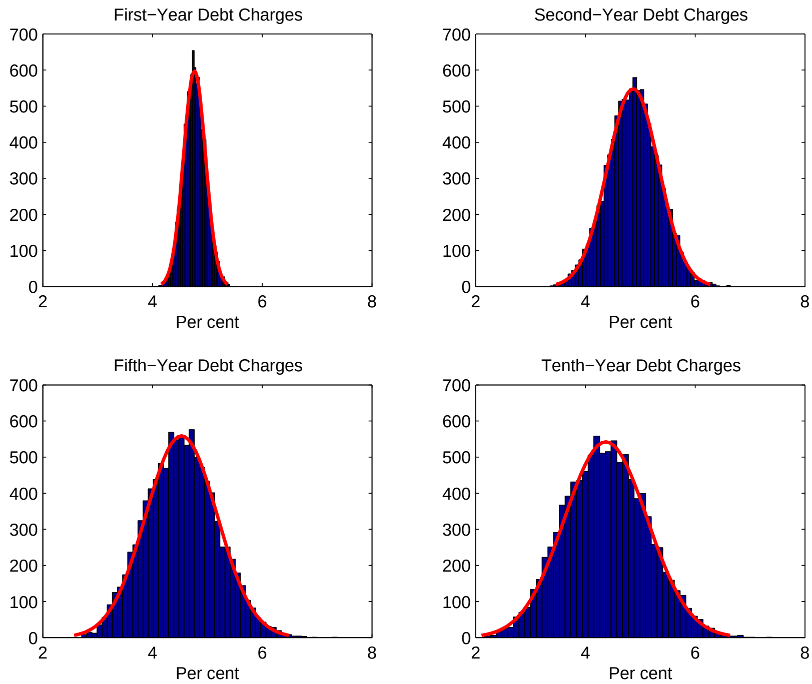
charge distribution. This could include, for example, the mean, the standard deviation or an order statistic such as the 95th percentile.

Observe that although the mean of each distribution is quite similar—it varies around Can$16-17 billion—the dispersion of the debt-service-charge outcomes steadily widens as we increase the time horizon. Indeed, the standard deviation in the first year is approximately Can$0.75 billion, while it increases to Can$2.8 billion in the tenth year. This is a clear example of how the uncertainty in debt-service-charge outcomes increases with time.

Figure 5 provides a slight variation on Figure 4 by demonstrating the debt-service charges as a proportion of the contemporaneous government debt stock. This can be useful when comparing debt-service charges across periods where the size of the debt portfolio is expected to increase or decrease over time. It may also be useful for general interpretation. It may, for example, be difficult to know whether average debt service charges, for a given financing strategy, of Can$19 billion are excessive or reasonable. An alternative, although equivalent, representation of government debt charges as 5.4 per cent of the debt stock might be more insightful. Note, however, that since Figure 5 is merely a transformation of Figure 4, the distributions also widen as we increase the time horizon.

The distributions outlined in Figures 4 and 5 have two dimensions: debt-service charges and time. Although this is a natural consequence of the structure of the problem, it can, as previously mentioned, be conceptually difficult to deal with the time dimension. One way to deal with this issue is to eliminate the time dimension. A simple way to do this is to consider the *total* debt charges over the simulation horizon. Figure 6 demonstrates the distribution of total debt charges. In the first quadrant, we illustrate the distribution of total undiscounted debt-service charges over a 10-year time horizon. The average is approximately Can$168 billion with a standard deviation of about Can$15 billion. This approach places an equal weight on debt charges occurring in each period. Since it may not be desirable to treat next year's debt charges in the same way as debt charges occurring

Figure 5: **Relative Debt-Charge Distributions**: This figure illustrates the relative debt-charge distributions—at 1-, 2-, 5- and 10-year horizons—associated with a financing strategy including equal amounts in each financing instrument. In this case, the debt charges are represented as a percentage of the total debt stock at that point in time. Again, we observe how the dispersion in the debt-charge outcomes increases with the time horizon.



10 years from now, we may wish to apply a discount factor to future debt-service charges. The complexity of discounting cash flows, however, stems from the selection of the discount factor. Indeed, the remaining quadrants of Figure 6 demonstrate the distribution of total discounted debt-service charges using flat 3 per cent, 5 per cent and 7 per cent discount rates, respectively.

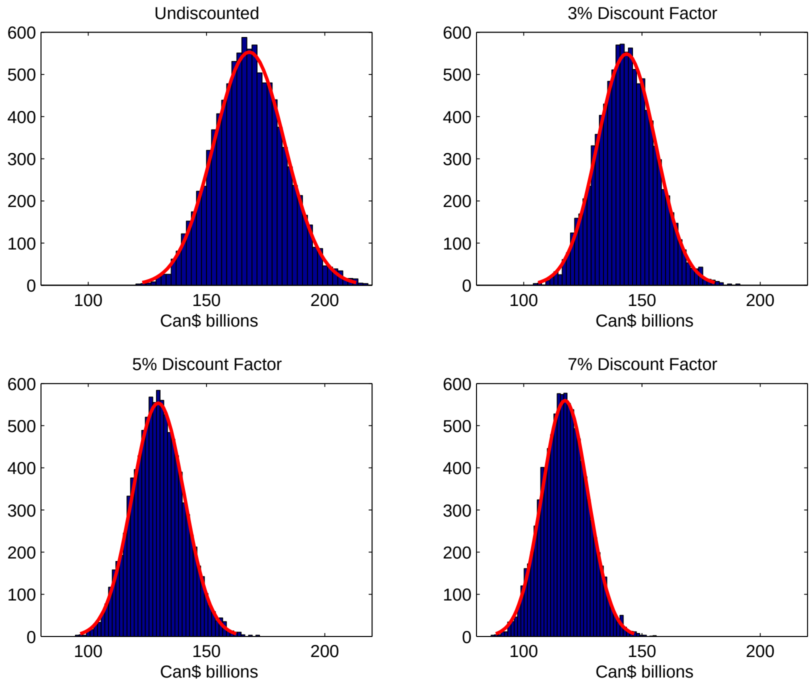
The discounting of future cash flows has two consequences. First, it reduces the absolute value of the total government debt charges by reducing the weight on future debt-service charges. Second, it leads to a decrease in the variance of the total debt-charge distributions. The undiscounted total debt charges have a standard deviation of approximately Can\$15 billion, while the total debt charges discounted at 7 per cent have a standard deviation of slightly less than Can\$10 billion. The reason is that, reducing the weight on the more volatile future debt charges, we actually decrease overall volatility. Clearly, selecting a discount factor is important when considering total debt charges.[27]

The mean and standard deviation of the debt-charge distributions outlined in Figures 4 to 6 provide a reasonable description of the normal cost and risk characteristics of a financing strategy. Debt managers are also interested in extreme outcomes; consequently, it makes sense to examine the tails of the previously illustrated distributions. Figure 7 outlines a number of different flavours of CaR. The first quadrant plots the 10,000 debt-charge sample paths from the stochastic simulation.

Absolute CaR is the worst-case expected debt-service charge with a certain degree of probability. Another way to think of absolute CaR is as a percentile measure; the 95 per cent CaR states that 95 per cent of the time, the government will not pay more than this amount in debt-service charges. The second quadrant of Figure 7 provides a summary of the absolute CaR for 90 per cent, 95 per cent and 99 per cent probabilities, respectively.

[27]One could argue, of course, that, because of its arbitrary nature, we should forego the discounting of cash flows and merely consider the simple sum.

Figure 6: **Total Debt-Charge Distributions**: This figure illustrates the total debt-charge distributions associated with a financing strategy including equal amounts in each financing instrument. We provide the distribution of a simple sum of debt charges over the 10-year simulation horizon as well as the distribution of summed and discounted debt charges for alternative discount factors.



Clearly, the higher the associated probability, the larger the CaR value. This is because we are moving further out the right tail of the distribution.

Relative CaR is simply the difference between absolute CaR and the mean of the distribution. The idea here is that, if one uses the mean of the debt-charge distribution for planning, then relative CaR tells you that with, say, 95 per cent probability, one's worst-case outcome will not exceed one's planned outcome by more than this amount. If one's objective is to minimize budgetary shocks, such a measure could be quite useful in budgetary planning.

Conversely, tail CaR states how bad things can go, if they go bad. More specifically, a 95 per cent tail CaR states the average debt charges assuming that debt charges are greater than or equal to the 95th percentile. In other words, this is the average debt charge in the tail of the distribution. This measure is useful when the distribution has a small probability of extremely negative outcomes.[28] Both relative and tail CaR are illustrated in Figure 7 for 90 per cent, 95 per cent and 99 per cent probabilities, respectively.

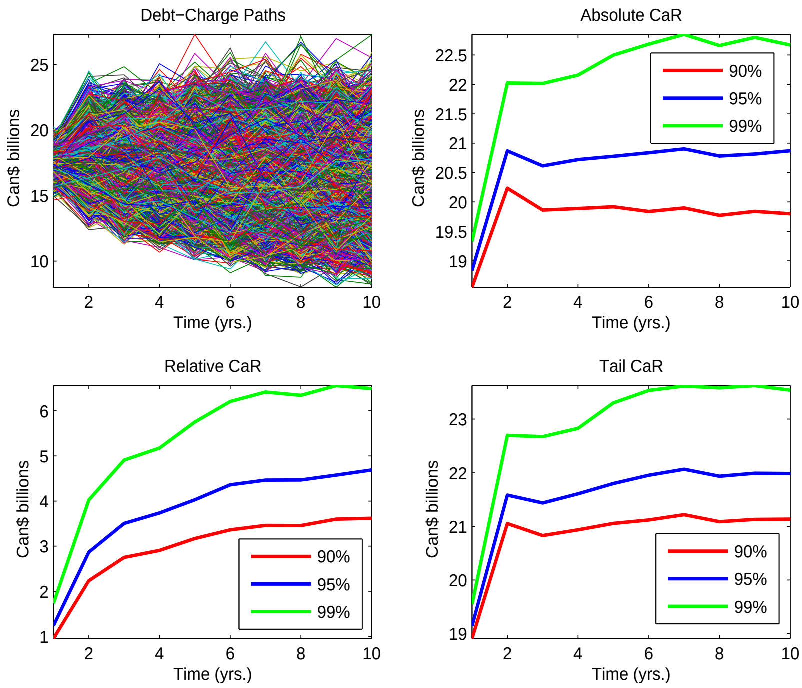
As noted earlier, there are potential advantages associated with defining risk in terms of budgetary outcomes in addition to government debt-service charges. Figure 8 illustrates the unadjusted funding-requirement distributions associated with the first, second, fifth and tenth year of the stochastic simulation. This is a funding-requirement analogue of the debt-charge distributions outlined in Figure 4. The unadjusted funding requirement represents the size of the government's deficit or surplus position before the application of the model's fiscal rule.[29]

We observe that the mean funding requirement, across all periods, is approximately zero. We also observe that the standard deviation of the government's funding requirement increases over time, albeit more slowly

[28]Statisticians refer to such distributions as fat-tailed or *leptokurtotic*.

[29]A fiscal rule defines how the government reacts to deviations from expected fiscal positions.

Figure 7: **Flavours of Cost-at-Risk**: This figure illustrates the evolution of debt charges as well as absolute, relative and tail cost-at-risk for a financing strategy consisting of equal weights in each financing instrument.



than the debt-service charges. Both of these observations are a consequence of the specific parameterization of the funding-requirement process.

What is more important to draw from Figure 8 is the fact that, armed with these funding-requirement distributions, we can proceed to compute the same mean, standard deviation and percentile measures in the same manner as with the debt-service-charge distributions. We could, for example, construct a 95 per cent budget-at-risk measure that would describe, with 95 per cent probability, the worst-case budgetary outcome that the government could experience in a given period for a particular financing strategy.[30] Such a risk measure could be quite useful in comparing the relative advantages and disadvantages of various financing strategies.

Table 1: **Conditional Volatility**: This table provides—in the context of the equally weighted financing strategy with 10,000 simulations—the conditional volatility of the government debt charges and the government's funding requirement, which we treat as a proxy for the government's surplus/deficit position. Chapter 6 describes in greater detail how the government's fiscal position is determined.

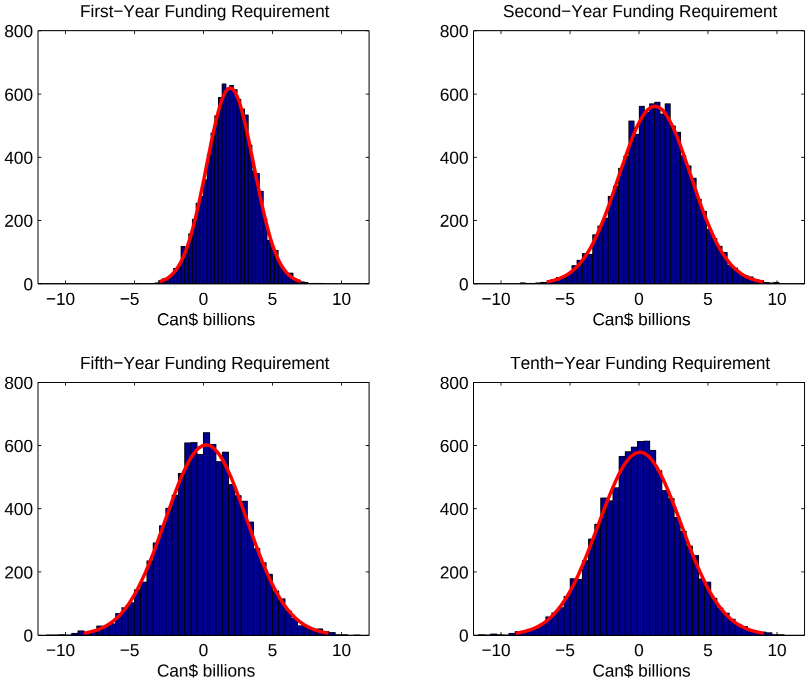
Debt Quantity	Conditional Volatility
Debt charges	1.73
Funding Requirements	2.92

The next set of risk measures relates to the previously discussed notion of conditionality. The conditional debt volatility is computed as the standard deviation of the residuals of a regression of debt charges in period t run against debt charges from period $t-1$.[31] This is a simple statistical technique for conditioning the debt charges in

[30] We can also compute interesting measures such as the probability that the government has a deficit of less than 1 billion dollars in a given period.

[31] More specifically, we run a simple first-order autoregression of the form $y_t = \alpha + \beta y_{t-1} + \epsilon_t$. The actual regression equation,

Figure 8: **Unadjusted Funding-Requirement Distributions**: This figure illustrates the unadjusted funding-requirement distributions—at 1-, 2-, 5- and 10-year horizons—associated with a financing strategy with equal amounts in each financing instrument. The unadjusted funding requirement represents the size of the government's deficit or surplus position before the application of the fiscal rule that (partially) offsets deviations from the expected deficit or surplus position. Again, we observe how the funding-requirement outcomes increase with the time horizon.



the current period on the previous period's debt charges. The standard deviation of the regression errors captures the residual uncertainty in this relationship; i.e., the volatility that is left when one conditions on the current state of the world. This residual uncertainty is another potential dimension of risk to consider in comparing financing strategies. Moreover, by providing a single value, it essentially eliminates the time dimension.

Table 1 illustrates the conditional volatility of debt charges of Can$1.73 billion. What does this mean? It means that, even when one takes into account the current debt charges, the standard deviation of the next period's debt charges is Can$1.73 billion. In other words, the residual debt-charge uncertainty over a 1-year period, when taking into account the previous year's outcomes, is Can$1.73 billion. This, of course, applies across all periods in the stochastic simulation.

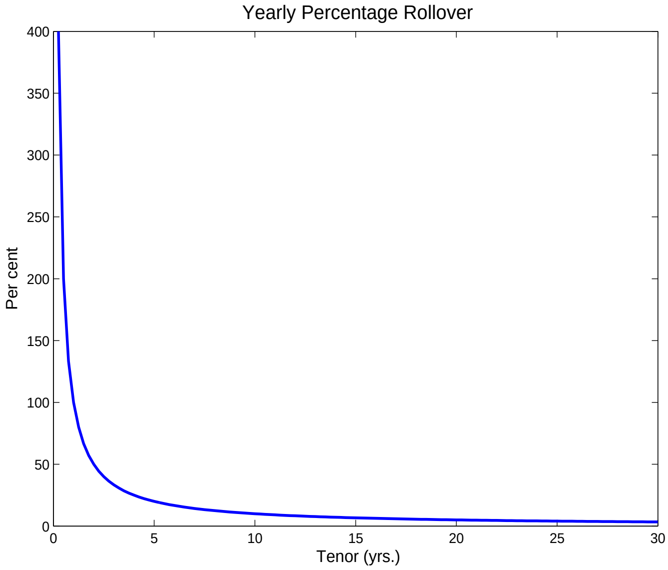
The conditional volatility of the government's funding requirement from Figure 8 is Can$2.92 billion. Again, this is residual budgetary uncertainty taking into account current conditions—which we can consider a measure of the fiscal uncertainty being faced. A government may wish, depending on its relative expense, to consider financing strategies with slightly higher debt-service costs that act to simultaneously lower the conditional volatility associated with the funding requirement.

Next we consdier rollover risk. For single-day maturity risk, one intuitively understands that the more days in a year that have bonds maturing, the lower the amount that matures on any given day. More specifically, doubling the number of maturity dates will halve the average amount maturing on any given day.[32] Again, intuitively, one understands that as the term to maturity of a debt instrument increases, the amount rolling over

however, is stacked so as to incorporate all of the simulation outcomes. The conditional volatility is then estimated as $\sqrt{\frac{1}{T-2}\sum_{t=1}^{T}\epsilon_t^2}$.

[32] Of course, in practice, maturing amounts will not be spread evenly, but the amount maturing on any single day will decrease significantly.

Figure 9: **Yearly Rollover Amounts**: This figure illustrates the percentage of the debt that would, on average, rollover every year if all debt was issued at the tenor on the horizontal axis. For example, 3-month bills must be refinanced four times per year, so their annual rollover percentage is 400 per cent.



in a given time period decreases. Figure 9 illustrates this, with an obvious bend occurring in the graph in the 1- to 5-year area.

3.3 Conclusion

This chapter has provided a brief, non-technical, example-based overview of a wide range of alternative measures of the cost and risk associated with a given financing strategy. Two principal ideas are addressed in this discussion. First, the government faces a variety of fundamental risk and cost trade-offs in selecting a financing strategy for its domestic debt portfolio. Based on these definitions, therefore, we can proceed to select a financing strategy that minimizes the cost of the portfolio subject to constraints on its risk. This is the fundamental lesson from portfolio theory.

The second point is that both cost and risk have a variety of dimensions. Cost can be computed in absolute, relative, summed or discounted terms. Risk can be related to the dispersion of debt-service charges and/or funding-requirement outcomes as well as to financing risk. Moreover, it can be considered on a period-by-period basis or across time using a notion of conditionality. Table 2 provides a brief summary of the various risk measures introduced during this discussion. The complexity of the risk and cost dimensions reflects, in part, the underlying complexity of the debt manager's challenge. For this reason, we should not consider the preceding discussion of risk and cost measures as exhaustive, but rather as some measures that can, and should, be considered in debt-strategy analysis.

Table 2: **Risk and Cost Measures**: This table attempts to organize the collection of suggested risk and cost measures, described in the text, into a logical structure. The idea is to highlight the various dimensions of the debt manager's challenge and a number of alternative measures for describing each dimension in a modelling context.

Principal Dimension	**Secondary Dimension**	**Measure**
Cost	Annual perspective	√ Absolute debt charges √ Relative (percentage) debt charges
	Aggregate perspective	√ Total (summed) undiscounted debt charges √ Total (summed) discounted debt charges
Risk	Cost-related risk	√ Standard deviation of debt charges √ Absolute, relative and tail *cost-at-risk* √ Conditional debt-charge volatility
	Budget-related risk	√ Standard deviation of budgetary outcomes √ Absolute, relative and tail *budget-at-risk* √ Conditional budgetary volatility
	Refinancing risk	√ Refixing share of debt √ Daily and quarterly rollover √ Maturity profile

4 Introducing Inflation-Linked Bonds

This chapter focuses on the implications of incorporating inflation-linked debt into the debt-strategy model.

As of 31 March 2011, approximately 6 per cent (or about Can$30.8 billion in real terms, Can$37.7 billion inflation adjusted) of the government's marketable debt stock consisted of inflation-linked debt.[33] Introduced in 1991, RRBs have been issued regularly for almost 20 years.

The debt-strategy model is sufficiently flexible to consider the relative advantages and disadvantages of nominal and inflation-linked debt. This chapter provides an overview of the principal issues involved in introducing inflation-linked debt into the debt-strategy analysis.

The chapter is divided into three sections. We begin by performing a simple comparison of a single inflation-linked bond relative to a single nominal bond: the aim is to highlight some of the modelling issues that arise in this context and to generalize them for consideration in the debt-strategy model. We then examine an illustrative example, in the context of the full-blown debt-strategy model, of alternative financing strategies with different proportions of inflation-linked debt. Finally, we make some concluding remarks.

4.1 A simple example

There are a few subtleties involved in comparing nominal and inflation-linked bonds. Before beginning a comparison of different financing strategies involving various proportions of nominal and inflation-linked debt, therefore, it is useful to begin in a simpler setting. Let us consider two 30-year bonds: one nominal bond with a 4 per cent coupon, and an inflation-linked bond with a 2 per cent real coupon. Assume that inflation is constant through time at 2 per cent.

We begin our comparison of these two bonds by examining their relative debt charges. The nominal bond, of course, pays an equal amount in each period. A coupon of 4 per cent and a notional value of, say, Can$100 leads to an annual payment of Can$4. The inflation-linked coupon bond payment, however, in nominal terms, increases at the inflation rate of 2 per cent per year. The top-left quadrant of Figure 10 demonstrates these relative debt charges. Observe that the nominal debt charges dominate their inflation-linked counterparts for every period throughout the 30-year tenor of the bonds.

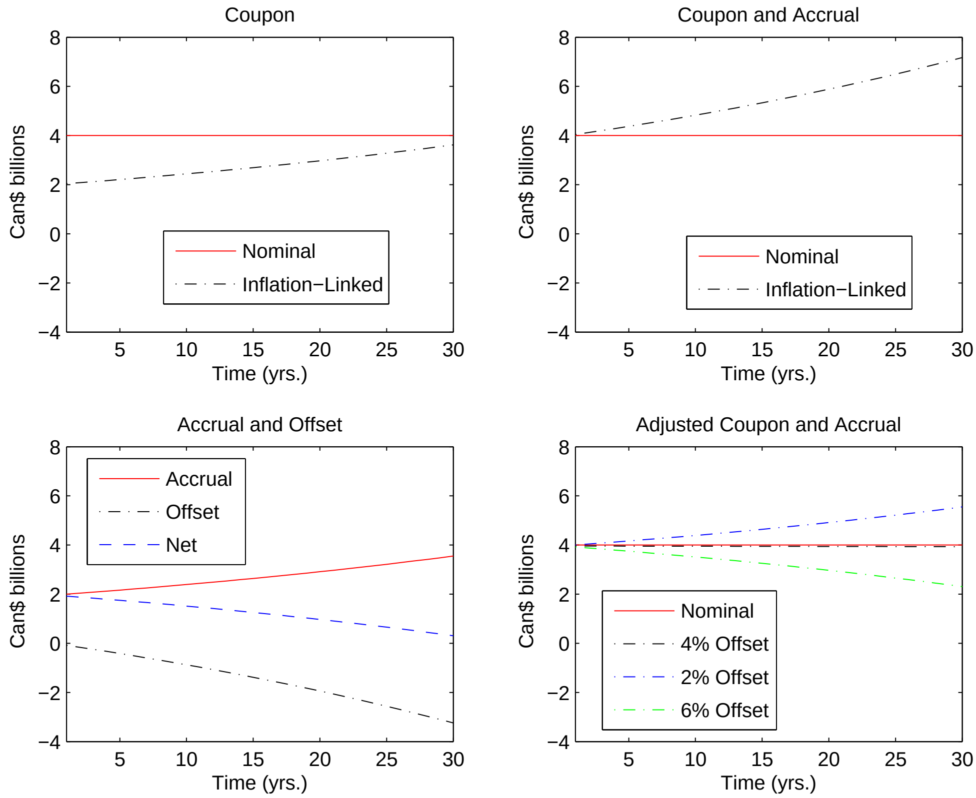
The principal payment of an inflation-linked bond, however, is adjusted upward for the accumulated inflation over its life. At 2 per cent inflation, therefore, the final principal payment of the inflation-linked bond would be Can$181.14. Clearly, this is a cost to the government and must be considered. In other words, an examination of only the inflation-linked coupon, without inclusion of the principal compensation, would distort the analysis. When we compute and accrue the compounded 2 per cent growth in the inflation-linked principal, we generate an increasing annual cost for the government. The top-right quadrant of Figure 10 illustrates the sum of the inflation-linked coupon and principal uplift relative to the nominal bond's debt charges. Observe that in this case, the inflation-linked debt charges dominate the nominal debt charges over the 30-year horizon.

There is, however, more to the story. The accrual of the inflation-linked principal uplift is necessary, but we cannot lose sight of the fact that it is a non-cash expense. The actual payment of the principal uplift occurs at maturity. Conceptually, therefore, we can consider these non-cash expenses as a type of sinking fund. These expenses must be offset by the interest compensation that could have been earned on these funds. In other words, even if no cash payment is made, by realizing a non-cash expense many periods before it is actually required, it is necessary to discount this cash flow to its present value. Not making this adjustment will lead one to conclude that inflation-linked bonds are more expensive relative to their nominal equivalent.

The bottom-left quadrant of Figure 10 illustrates the annual accrual of the 2 per cent compounded increase in the principal of the inflation-linked bond. It also illustrates the return—assumed to be 4 per cent—on the non-cash accruals. We term this the *offset.* Combining the accrual and offset generates the net debt charges

[33] In Canada, as described earlier in the paper, inflation-linked debt is known as Real Return Bonds (RRBs). RRBs are long-term instruments, with issuance beginning at 30+ years to maturity.

Figure 10: **Relative Nominal and Inflation-Linked Cash Flows**: This figure illustrates the cash flows for a 30-year nominal bond with a 4 per cent coupon and a 30-year inflation-linked bond with a 2 per cent real coupon. In all states, for simplicity, inflation is assumed to be 2 per cent. The top-left quadrant illustrates the relative cash flows associated with each bond. The top-right quadrant adds in an accrual for the increase in the principal associated with the inflation-linked bond. Since the accrual is a non-cash item, the bottom-left quadrant demonstrates the effect of adjusting these non-cash accruals (i.e., offset) for their investment return, which is assumed to be 4 per cent. The final, bottom-right, quadrant displays the total debt charges for various assumptions associated with the return on the non-cash accruals. Please note that all graphs are in nominal terms.



associated with the inflation uplift. In the bottom-right quadrant of Figure 10, we compare the nominal against the adjusted inflation-linked debt charges. Observe that when the rate earned on the non-cash accruals is assumed to be 4 per cent, there is no longer any visible difference between the relative debt charges associated with our nominal and inflation-linked bonds. If, however, the rate earned on the non-cash accruals is less than 4 per cent, then the inflation-linked debt charges exceeds those of the nominal bond. Conversely, if the non-cash accrual return rate exceeds 4 per cent, the inflation-linked debt charges are less than their nominal equivalent.

From the previous example, two things are clear. First, it is necessary to accrue the inflation-related principal compensation for inflation-linked bonds. Second, an adjustment is required to account for the fact that the accruals are *not* cash payments. For this reason, an accrual account is included in the debt-strategy model.

4.2 A more complex example

We next compare inflation-linked and nominal financing instruments in the context of the debt-strategy model. This illustrative example takes the form of a simple experiment. We begin with a portfolio that consists entirely of nominal bonds; the relative portfolio weights of this portfolio are very close to those in the portfolio at 31 March 2006, although all of the 30-year inflation-linked stock is reallocated to 30-year nominal bonds. The second financing strategy involves a transfer of one-half of 30-year nominal bond stock into 30-year inflation-linked debt. The third financing strategy replaces all 30-year nominal debt with inflation-linked bonds of equivalent tenor. These three financing strategies are summarized in Table 3.

Table 3: **Financing Strategies**: This table outlines three alternative financing strategies involving nominal and inflation-linked bonds of alternative tenors.

Debt Type	**Tenor**	**Financing Strategies**		
		0% Real	**17.5% Real**	**35% Real**
Nominal	3 month	19%	19%	19%
	6 month	6%	6%	6%
	1 year	6%	6%	6%
	2 year	5%	5%	5%
	5 year	6%	6%	6%
	10 year	22%	22%	22%
	30 year	35%	17%	0%
Real	30 year	0%	17%	35%

Clearly, such abrupt changes in the government's debt portfolio *cannot* be achieved in a short period of time. We can think of this exercise, therefore, as a thought experiment involving the comparison of three portfolios—and their associated financing strategies—that differ only in their relative proportions of nominal and inflation-linked debt. By varying the proportions of 30-year nominal debt and inflation-linked securities sharing the same tenors, the basic maturity structure of the portfolio is preserved. This permits, in our view, a fair characterization of the risk and cost differences between portfolios with an increasing share of inflation-linked debt.

We compare these three alternative financing strategies using 10,000 randomly generated realizations of output, inflation, the monetary policy rate and the term structure of interest rates. The average financing rates, and their associated standard deviations, are presented in Table 4. These financing rates are essentially the coupon rates associated with issuing inflation-linked or nominal bonds at a price of par.

Table 4 shows that the average 30-year nominal par coupon is 5.27 per cent. This compares to a mean real coupon of 3.07 per cent for the 30-year inflation-linked bonds. If we adjust these real coupons for the average inflation of 1.98 per cent, we arrive at an approximate average cost for the 30-year inflation-bonds of 5.05 per cent. This is approximately 20 basis points less than the corresponding coupon associated with 30-year nominal borrowing. Based on the analysis in the previous section, therefore, we should expect to observe only slight cost differences for the financing strategies that include more inflation-linked debt.[34]

Figure 11, in fact, illustrates the annual debt charges for each of three financing strategies across the 10-year simulation horizon in Canadian-dollar and percentage terms. Observe that the financing strategy with the largest proportion of inflation-linked debt demonstrates slightly higher expected debt charges; this effect is related to the issuance-penalty function described in Chapter 9. The 17.5 per cent allocation to inflation-linked debt, however, leads to, relative to the nominal-debt-dominated strategy, lower debt-service charges on the order of a few basis points. We also observe that the two financing strategies with inflation-linked debt exhibit lower debt-charge standard deviation across almost all periods in the simulation horizon.

Figure 12 demonstrates how the greater dispersion of debt-charge outcomes in the nominal-bond-based financing strategy translates into slightly higher relative cost-at-risk values. This is particularly true—given the higher expected debt charges of the 35 per cent inflation-linked financing strategy—on a relative basis. This appears to be a diversification benefit. Real par coupon rates are slightly more variable than nominal par coupon rates and the entire inflation-indexed portfolio is repriced every period, while only the 30-year nominal

[34] As well, the calculation does not include an issuance penalty. The issuance-penalty function is described in Chapter 9.

Table 4: **Financing Rates**: This table outlines mean and standard deviation of the randomly generated par coupons associated with nominal and inflation-linked debt in the model. It also includes the mean and standard deviation of realized-inflation outcomes stemming from the stochastic model. The historical data period used is 1994 to 2007.

Debt Type	**Tenor**	**Financing Cost**	
		Mean Coupon	**Standard Deviation**
Nominal	3 month	3.85	1.29
	2 year	4.26	1.15
	10 year	5.04	1.10
	30 year	5.27	1.08
Real	10 year	2.12	0.78
	30 year	3.07	1.10
Inflation	Annualized	1.98	0.26

debt coming to maturity, and any issuance to cover the financial requirement, is repriced in a given quarter.[35] Therefore, inflation-indexed bonds alone are more variable, but they appear to contribute to lower variability in debt charges in a portfolio with nominal bonds.

Figure 12 also illustrates the budget-at-risk (BaR) values for our three financing strategies. Similar to the debt-charge setting, the financing strategies with inflation-linked debt appear to be less risky than the 35 per cent nominal 30-year portfolio. These modest differences apply on both an absolute and a relative basis. The reason for these phenomena appear to be found in the bottom-left quadrant of Figure 13, which describes the average correlation between the government's primary balance and the debt-service charges. As we increase the proportion of inflation-linked debt, the magnitude of this correlation increases. The variance of the government's budget balance is equal to the sum of the variance of the primary balance and the debt charges less twice their covariance.[36] As such, the larger the covariance between the primary balance and debt charges, ceteris paribus, the lower the variance of the government's budgetary balance. We suggest that the larger positive correlation between the debt charges and the primary balance, for the financing strategies containing 30-year inflation-linked debt, is responsible for the superior BaR numbers, compared to the financing strategy containing only 30-year nominal bonds, in Figure 12.[37]

Why do we observe a higher correlation between the primary balance and debt charges as we increase the proportion of inflation-linked debt in the portfolio? We believe that the answer relates to the structure of the primary balance and the associated debt charges. The debt charges represent a weighted average of current and past financing decisions. A financing strategy dominated by long-term issuance places more weight on past financing decisions (and consequently past interest and inflation rates), whereas a short-term financing strategy leads to a larger weight on current interest and inflation rates. The primary balance, conversely, depends principally on contemporaneous changes in output and inflation. The three financing strategies analyzed in this chapter have, by their very construction, the same amount of short- and long-term debt. As a consequence, the differences in correlation do *not* directly depend on the relative weights of interest rates through time.

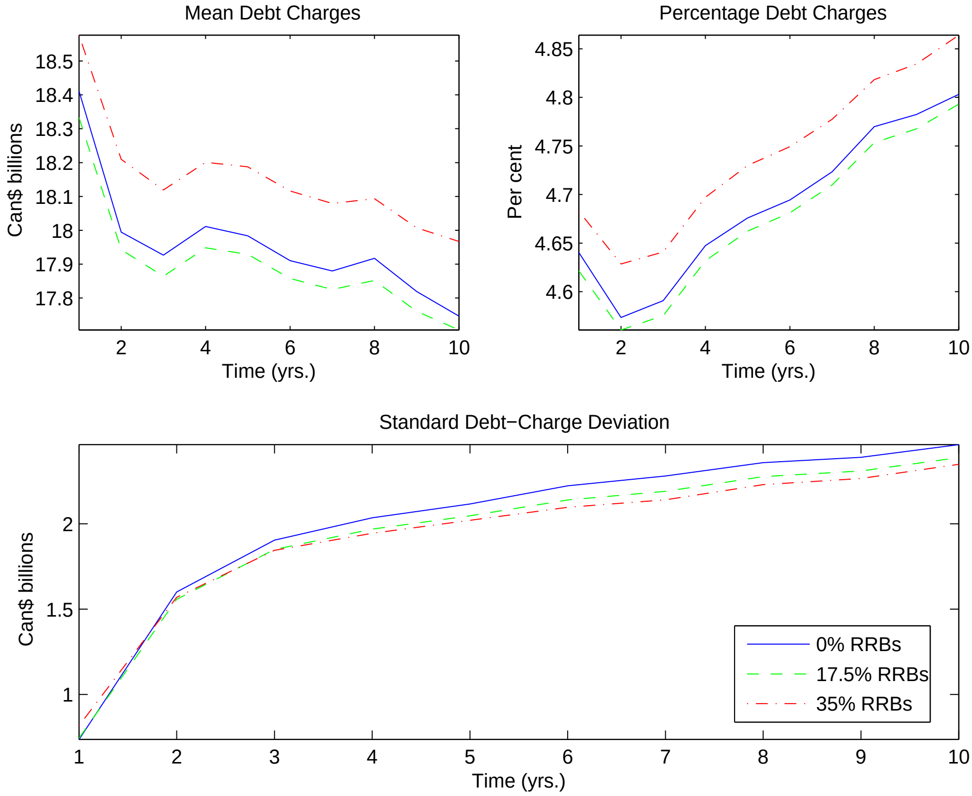
The difference relates, however, to the composition of the debt charges. With a financing strategy involving only nominal bonds, the debt charges consist entirely of coupon payments determined either in the past or the current period. When we add inflation-linked bonds, the debt charges consist of nominal and real coupons determined in current and past periods plus the uplift in the inflation-linked principal, which is a function of

[35] In fact, a 100 per cent portfolio of 30-year RRBs has a conditional debt-charge volatility of around 2, versus about 0.5 for a 100 per cent portfolio of 30-year nominal bonds.

[36] The fiscal rule implemented in the model is described in Chapter 6.

[37] Clearly, there is some offset in the other parts of budgetary volatility. The debt charges of the 35 per cent inflation-indexed portfolio have a higher correlation with the primary balance, but similar budgetary-risk outcomes, compared to the 17.5 per cent inflation-indexed portfolio.

Figure 11: **Mean and Standard Deviation Government Debt Charges**: This figure illustrates the expected debt charges and standard deviation associated with three alternative financing strategies: no inflation-linked bonds, the replacement of one-half of the nominal 30-year nominal bonds with inflation-linked bonds, and the entire replacement of 30-year nominal with inflation-linked bonds. The mean debt charges are presented, for each year in the 10-year simulation horizon, in absolute and relative terms. We also present the standard deviation of these debt charges.



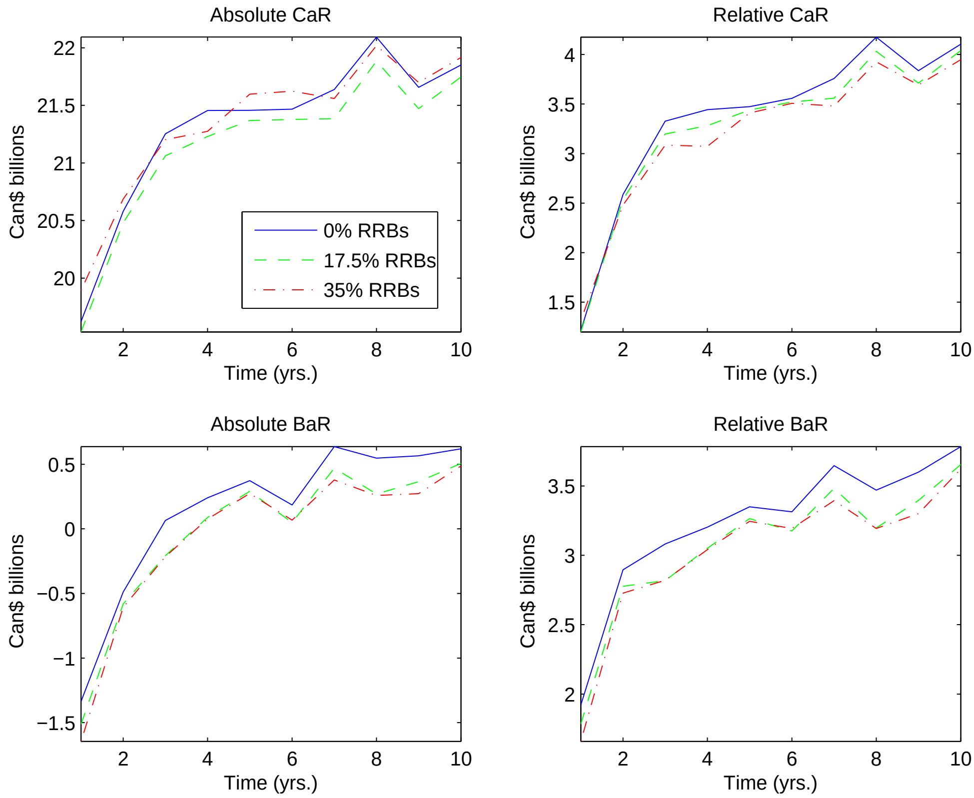
contemporaneous inflation. While these three elements generally move together, they are *not* perfectly correlated. As such, movements in the contemporaneous component of debt charges for a financing strategy that includes inflation-linked bonds are more complex than for an equivalent strategy that consists only of nominal bonds. Moreover, the inflation-linked principal compensation has a much larger contemporaneous component than its nominal equivalent. We argue that the addition of the relatively more contemporaneous component to the inflation-linked debt charges acts to modestly increase its overall correlation with the primary balance.

We performed some sensitivity analysis around these results in the form of using different starting points for the simulation of the random outcomes. The reported results use interest rates, inflation, output and monetary conditions from 2007 as the starting point. We also considered using the long-term mean values from the estimation and the average historical values as our starting points. The consequence of using different starting values is alternative paths for interest rate and macroeconomic variables. Our sensitivity analysis indicates that, while the general pattern of the results remains the same, the magnitude of the differences in the financing strategies depends on the starting point for the simulation.

Analysis has been performed on the correlation structure of debt charges using 100 per cent portfolios in each instrument. It has shown that inflation-indexed 30-year bonds have extremely low correlations with nominal instruments, relative to nominal instruments' correlations with each other.[38] This provides further evidence of

[38] It is unclear how much of this relates to incorporating the present value of principal uplifts in costs as they occur. As discussed, in reality this is the incurrence of a cost, but not the payment of one. It is therefore, within the model, a non-cash item in the

Figure 12: **Cost- and Budget-at-Risk Measures**: This figure again illustrates various risk measures associated with three alternative financing strategies: no inflation-linked bonds, the replacement of one-half of the 30-year nominal bonds with inflation-linked bonds, and the entire replacement of 30-year nominal with inflation-linked bonds. The 95 per cent absolute cost-at-risk (CaR) and budget-at-risk (BaR)—over the 10-year simulation horizon–is presented for each of the strategies. The 95 per cent relative CaR and BaR are also computed.



the diversification benefit of holding some inflation-indexed debt in the government's debt portfolio.

As a final point, it is important to highlight that the above thought experiment and discussion centre on the characteristics of a portfolio that includes inflation-indexed bonds. However, caution is warranted. It does not describe their role in an *optimal* portfolio at a given risk level, or at a desired cost-risk trade-off.[39]

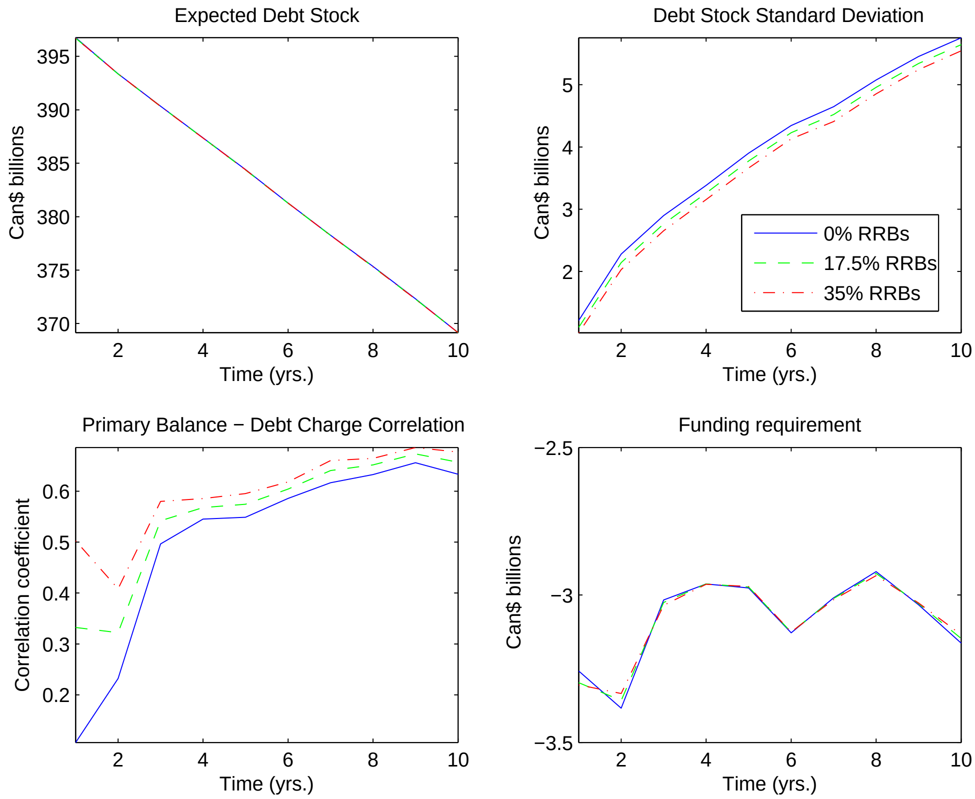
4.3 Conclusion

This chapter has introduced inflation-linked bonds into the analysis of the government's debt strategy. A thought experiment involving alternative financing strategies with different proportions of inflation-linked debt has led to three preliminary conclusions. First, using stochastic outcomes estimated to Canadian data from 1994 to 2007 implies relatively modest differences in the expected cost of these financing strategies. Second, the volatility of the debt charges appears to be somewhat lower for financing strategies with inflation-linked debt; the reason appears to be a diversification benefit from having both nominal and inflation-indexed debt in the portfolio. This result also extends to budgetary volatility. In that context, it seems to be also driven by the positive correlation between debt charges and the primary balance among financing strategies with inflation-indexed bonds. The model is capable of adding shorter inflation-indexed maturities, and evaluating instruments that link payments to other economic indicators.

budget.

[39]Optimization is discussed in Chapter 5.

Figure 13: **Debt Stock and Funding Measures**: This figure again illustrates various debt-stock and funding-requirement measures associated with three alternative financing strategies: no inflation-linked bonds, the replacement of one half of the 30-year nominal bonds with inflation-linked bonds, and the entire replacement of 30-year nominal with inflation-linked bonds. For each financing strategy, the expected debt stock, its standard deviation, the correlation between debt charges and the primary balance, and finally the expected funding requirement are presented.



5 The Optimization Module

This chapter explores the optimization tool developed specifically for application to the debt-strategy problem. As we have seen in the preceding chapters, stochastic-simulation models provide substantial information on a given financing strategy. Indeed, they permit the detailed comparison of two or more alternative financing strategies on a wide variety of dimensions. One shortcoming of this approach, when we rely only on the simulation engine, is that the model does not provide any insight into the *optimal* debt strategy that should be followed by the government. That is, we can compare two or more strategies in great detail, but there is nothing to say that any of the considered strategies is optimal. This is not to say, however, that stochastic-simulation models are incapable of providing insight into a government's optimal debt strategy. More structure is required. This chapter discusses this structure in the form of the recently developed optimization module that was built to supplement our stochastic-simulation model.

Two substantial challenges must be overcome to use stochastic-simulation models in an optimization context. First, one must handle the difficulties associated with optimizing in a computationally expensive setting. This is essentially a mathematical challenge that will not—beyond a brief high-level description—be addressed in this paper.[40] The second challenge is that we require precision about what exactly is meant by the idea of an *optimal* debt strategy. This is far from straightforward. Even experienced debt managers can disagree on the specific risk constraints that should apply to the government's debt strategy. Even in the event of an agreement on the type of constraint, there may not be consensus on the level of the constraint.

Irrespective of the complexities of explicitly describing the government's objective function, the discipline of characterizing one's objective function is, to our mind, quite valuable. Moreover, examining the sensitivity of the portfolio to different types of constraints and different constraint levels can be very illuminating. The second challenge, therefore, forms the backbone of this discussion paper. It should, however, be stressed that this paper does not seek to suggest the optimal debt strategy for the government. Instead, these are illustrative and exploratory results intended to provide a sample of what can be done with the optimization tool.

A note of caution should be given regarding the results shown. They are from 2007 and are for illustrative purposes only, to show the types of analysis that can be accomplished with the model.[41]

The chapter is organized into two parts. In the first part, we engage in a high-level discussion of the optimization algorithm to motivate and build intuition about the tool. The second part considers the minimization of debt charges subject to a reasonably wide range of constraints. We consider two broadly defined constraint categories. In the first group, the constraints relate to notions of interest rate and budgetary risk. In the second group, the constraints are related to portfolio-summary measures (i.e., average term to maturity and the refixing share of debt) and treasury bill and bond issuance. For each constraint, we examine how the portfolio weights evolve for different constraint levels, as well as the trade-off between expected cost and the constraint. The current portfolio is also indicated to provide some context to our analysis.

5.1 Background

Optimizing in a stochastic-simulation setting is a high-dimensional, non-linear and computationally intensive mathematical problem. It is the computational expense that is the real challenge. The reason is simple—simulation is a powerful statistical technique, but it is slow. The evaluation of a single financing strategy with 10,000 randomly generated outcomes can require more than one half hour of computation. Most non-linear optimization algorithms work iteratively and require repeated evaluation of one's objective function—in our case, this means repeatedly rerunning the simulation. Depending on the number of instruments, this can mean literally thousands of simulations. Put simply, solving a single optimization problem can take weeks. This is unacceptably slow and makes direct optimization impractical.

[40] A detailed discussion of the mathematics of the optimization algorithm is provided in Bolder and Rubin (2007).

[41] Some differences from today are the Can$400 billion debt stock and the absence of the recently reintroduced 3-year maturity.

If there was universal consensus on the form of the government's debt-strategy objective function, then we might be content to merely let an optimizer run over a number of weeks to obtain the final response. This, however, is not the case. We would like to run numerous optimizations with alternative constraints and constraint levels. We also want to tweak the model assumptions and understand the sensitivity of the optimum to these changes. Each optimization, and the corresponding comparison with our intuition, has the potential to help us learn more about the debt manager's challenge. In short, we desire a fast and efficient optimization algorithm.

Fortunately, recent work has helped lead to the development of an optimization module for solving this mathematical challenge. The basic idea behind the optimization module is quite simple. First, we cannot, by virtue of the incredible computational expense, perform the optimization directly on the simulation engine. As a consequence, we instead approximate a given debt-strategy objective function. We then proceed to perform the optimization algorithm on this approximation. In other words, we run a fixed number of simulations—say, one or two thousand—for different financing strategies. We can consider this work as a form of overhead, which generally requires a few days of computation. Using this fixed number of simulations, we construct an approximation of the simulation for *any* financing strategy. Any feature of the simulation can thus be approximated in an extremely rapid manner. Once we have incurred the overhead, a given optimization can be performed in a number of seconds.

This is the basic idea behind the optimization engine. Clearly, there is a substantial amount of mathematical detail behind this component. The interested reader can consult the references for more discussion, although it is not necessary for the use of the optimization policy tool.

5.2 Alternative optimization constraints

There are a virually limitless number of possible specifications for the government's debt-management objective function. Some direction, therefore, is required. Once again, it makes good sense to consider the publicly stated objectives of the Canadian government with respect to this portfolio:

> The main debt-management objective is to raise stable, low-cost funding to meet the operational needs of the government. An associated objective is to maintain a well-functioning market for government securities.

In Chapter 3, we examined a number of approaches for translating these notions of *low cost* and *stability* into more formal risk and cost measures.

In this analysis, we use the average annual debt charges as a percentage of the total debt stock over the 10-year simulation horizon as our measure of cost. The advantage of this approach is that it avoids the need to discount the debt stock and is relatively robust to different debt-stock levels. It is also attractive because of its relative ease of computation.[42]

The actual optimization problem used to generate the results in this section is quite straightforward. We assume that the government wishes to minimize the percentage cost of the debt over the next 10 years—this is our definition of *low cost*—subject to a single constraint. We vary the choice of constraint and its level and examine the results. Each of these constraints can loosely be considered as an alternative definition of *stability*. A verbal description of the optimization problem, therefore, following from the guidance provided by our debt-management guidelines, would look something like,

$$\min_{\substack{\text{Portfolio}\\\text{weights}}} \ \textbf{Expected Debt Cost}, \tag{1}$$

$$\text{subject to:}$$

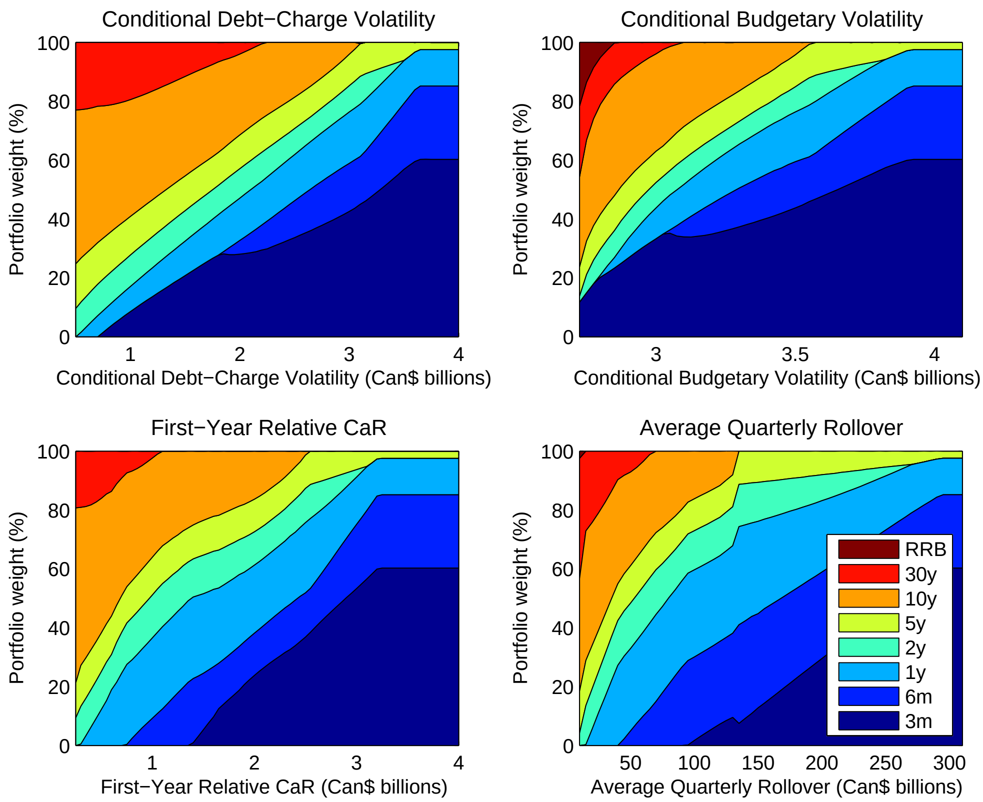
$$\textbf{Risk} \leq \ \textbf{Predetermined Level}.$$

[42]It should be stressed, however, that the following results are generally robust to the choice of cost measure and where relevant, and within a reasonable range, to the discount factor.

Risk, of course, is the mirror image of stability. We can either require that the risk be *less* than some predetermined level, or demand that the stability be *greater* than another predetermined level.

Let us examine the application of this notion to our debt-strategy model. We consider 2,000 alternative financing strategies—with 10,000 random realizations of our stochastic model—including eight financing instruments. The data used to estimate the stochastic model run from 1994 to February 2007.[43] The financing strategies include 3-, 6- and 12-month treasury bills; 2-, 5-, 10- and 30-year nominal bonds; and 30-year inflation-linked bonds. In each case, we assume an annual debt paydown target of Can$3 billion and follow the description of the government's fiscal situation provided in Chapter 6. These 2,000 financing strategies serve as the dataset for training, or fitting, our approximation functions. Once the approximation functions are trained, we can consider a wide range of possible government objective functions.

Figure 14: **Risk-Constrained Optimizations**: This figure illustrates how the optimal portfolio weights evolve for different fixed risk-constraint values. In each case, the optimizer seeks to minimize the expected average percentage debt charges over the 10-year simulation horizon. Each of the four graphs represents an alternative risk constraint. The horizontal axis represents alternative levels of the constraint. The first graph, for example, examines how the portfolio weights change as we increase the constraint on annual conditional debt-charge volatility from Can$500 million to Can$4 billion. The other three graphs demonstrate the corresponding results by using conditional budgetary volatility, the first-year relative CaR and average quarterly rollover, respectively.



[43] There are a variety of possible stochastic models from which we can choose. In this analysis we use the Diebold and Li (2003) approach applied to the Nelson and Siegel (1987) model as described in Bolder (2006). We have forced the long-term inflationary mean to be 2 per cent and the long-term output gap to be zero.

5.2.1 Risk constraints

Armed with these results, each of the four graphs in Figure 14 represents an alternative risk constraint. The horizontal axis represents alternative levels of the constraint, while the vertical axis denotes the portfolio weights. A specific example of the results found in Figure 14 would have the following form,[44]

$$\min_{\substack{\text{Portfolio}\\ \text{weights}}} \textbf{Expected Average Percentage Debt-Service Charges}, \tag{3}$$

subject to:

$$\textbf{Conditional Cost Volatility} \leq \textbf{Can\$1 billion}.$$

In words, the optimizer attempts to find the set of portfolio weights that provide the lowest possible total debt charges while simultaneously respecting the constraint of conditional debt-charge volatility less than Can$1 billion.

We are not, however, restricted to merely constraining conditional debt-charge volatility to Can$1 billion; we can examine a wide range of possible constraint levels. The top-left graphic in Figure 14, in fact, provides the optimal financing strategies (i.e., portfolio weights) for conditional debt-charge volatility ranging from Can$500 million to Can$4 billion. The easiest way to read these figures is to work from left to right on the horizontal axis. At the lowest level of the conditional debt-charge volatility constraint (i.e., Can$500 million), the optimal portfolio weights involve about 23 per cent 30-year, 52 per cent 10-year, 15 per cent 5-year and 10 per cent 2-year nominal bonds. As the conditional debt-charge volatility constraint relaxes, the 30- and 10-year bonds are replaced with 2-year bonds and treasury bills. Beyond a constraint of approximately Can$3.5 billion, the portfolio weights settle down at over 95 per cent treasury bills, with the majority allocated to 3- and 6-month tenors, as well as a small allocation to 5-year bonds. Once the constraint no longer binds, we have the lowest-cost portfolio, which is the same for all risk constraints.[45]

The changing portfolio weights in Figure 14 are interesting, but they do not tell us how much debt-service charges are reduced as the constraint is eased. This information is provided in Figure 15. We see, for example, in the top-left quadrant of Figure 15, that as the conditional debt-charge volatility constraint is eased—from Can$500 million to Can$3 billion—the expected cost falls by approximately 100 basis points.[46]

The top-right quadrant of Figure 14 illustrates the conditional budgetary volatility. For lower levels of the risk constraint, the optimal portfolio allocation involves a substantial proportion of inflation-linked debt, 10- and 30-year nominal debt, and treasury bills. As the risk constraint is relaxed, however, the optimal portfolio weights shift to include significant proportions of 10-year bonds and treasury bills. Beyond approximately Can$3.75 billion, however, the portfolio involves 5 per cent 5-year bonds with the remaining 95 per cent in treasury bills—the majority of this, about 60 per cent, is allocated to the 3-month treasury bill. Figure 15 shows that there is also approximately 100 basis points of cost reduction as one eases the conditional budgetary volatility constraint from Can$2.7 to Can$3.75 billion.

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[44] In mathematical notation, this problem is stated as follows,

$$\min_{\theta \in \Theta} \frac{1}{T} \sum_{t=1}^{T} \left(\frac{1}{\Xi_t} \right) \cdot \mathbb{E} \left(\tilde{c} \left(t-1, t, \theta, \tilde{X}(t,\omega) \right) \middle| \mathcal{F}_0 \right), \tag{2}$$

subject to:

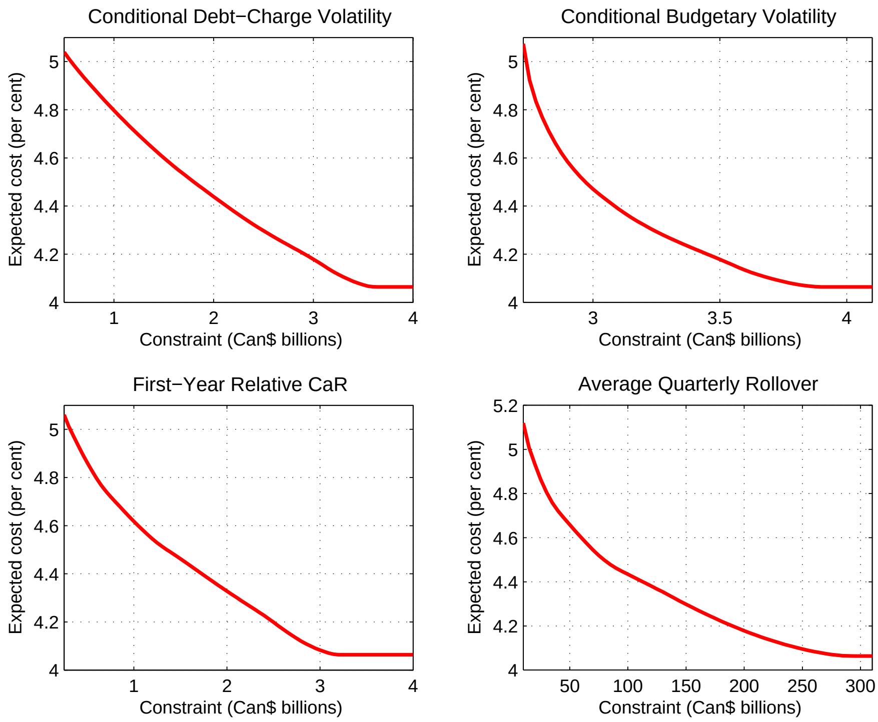
$$\tilde{\sigma} \left(t-1, t, \theta, \tilde{X}(t,\omega) \middle| \mathcal{F}_{t-1} \right) \leq \delta.$$

where c denotes the debt-servicing charges over the interval $[0,t]$ for fixed $\theta \in \Theta$ and financing strategy $\omega \in \Omega$, $\tilde{X}$ denotes the discretized state variable vector, $\tilde{\sigma}$ is the conditional debt-charge volatility, Ξ_t is the nominal value of the government debt stock at time t, and δ is Can$1 billion.

[45] This occurs when loosening the constraint leads to no change in the portfolio, and therefore no further cost reduction.

[46] For a debt stock of Can$400 billion, this represents approximately Can$4 billion of annual cost reduction.

Figure 15: **Expected Cost of Risk-Constrained Optimizations**: This figure illustrates the optimal risk-cost trade-off for different fixed risk-constraint values. These results are described in Figure 14. The underlying figures illustrate the expected cost, as a percentage of the debt stock, for the optimal portfolios associated with each given constraint level.



The bottom-left quadrant of Figure 14 involves the application of a constraint on the relative cost-at-risk (CaR) in the first year of the simulation. This constraint applies over the range Can$250 million to Can$4 billion. Once again, for tight constraint levels, the allocation is predominately composed of long-tenor nominal bonds with modest amounts of 2- and 5-year nominal bonds and zero treasury bill allocation. Beyond a constraint of approximately Can$2.5 billion, the optimal portfolio weights involve treasury bills and 2- and 5-year bonds. Figure 15 illustrates that there are approximately 100 basis points of cost reduction in easing the first-year relative CaR constraint from Can$250 million to Can$3 billion. Beyond this point, the constraint no longer binds.

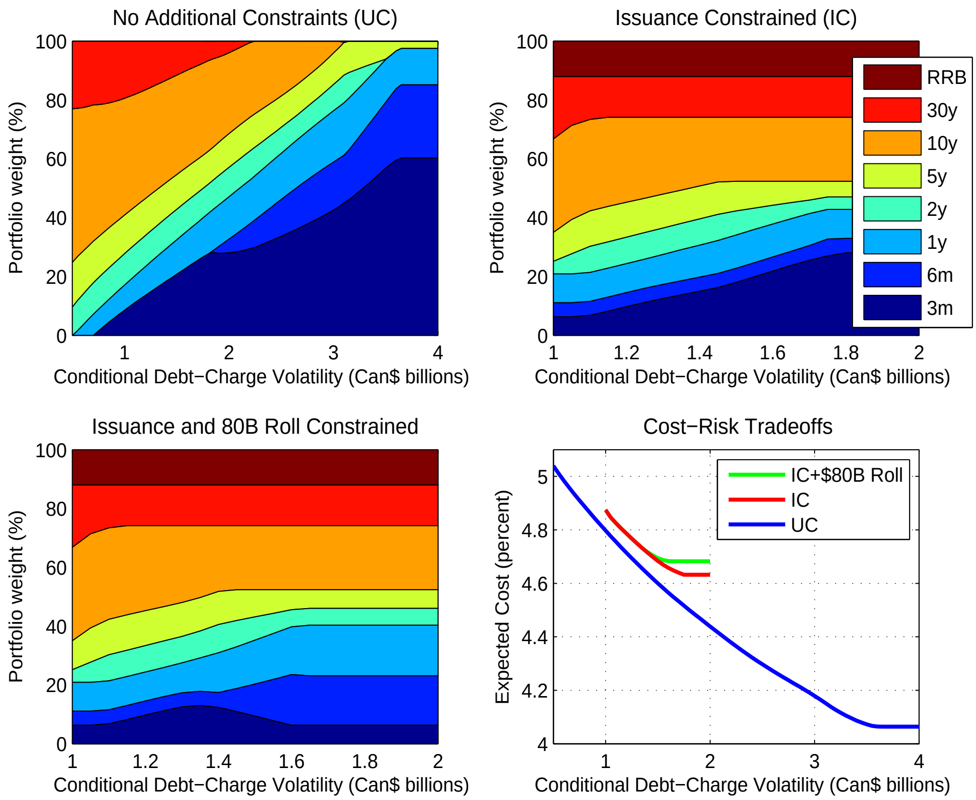
The bottom-right quadrant in Figure 14 describes the final constraint on the average quarterly rollover amount. The constraint ranges from Can$10 billion to Can$300 billion. Not surprisingly, low-risk levels see large allocations to 10-year, and to a lesser extent 30-year, bonds. As the constraint is loosened, there is a gradual increase in the treasury bill allocation, starting with the 1-year treasury bill. Moreover, the cost of the portfolio, as evidenced in Figure 15, falls by about 100 basis points as we ease this constraint.

Figures 14 and 15 illustrate how the optimal portfolio weights and expected costs evolve for different risk constraints at alternative constraint levels. While the results vary as one changes the constraint, there is a general trend toward an increase in the amount of longer-tenor instruments as one desires to reduce risk. Moreover, in all cases, this reduction in risk is accompanied by a commensurate increase in expected debt-service charges.

5.2.2 Adding more constraints

This section outlines the use of two additional constraints. The first one is having a minimum issuance amount (in terms of portfolio weight) for each maturity. This restricts the optimization to only those portfolios that have a portfolio weight above each instrument's constrained level.[47] The second is a maximum average quarterly rollover constraint. While a constraint on rollover was used earlier as the main risk constraint, it can also be added to other optimizations to ensure that the level of rollover remains at an acceptable level. Both constraints can be used at the same time, or each separately.[48]

Figure 16: **Adding Issuance and Rollover Constraints**: This figure compares how the optimal portfolio weights and expected costs evolve when issuance and rollover constraints are added to the conditional debt-charge volatility optimization. The first three graphs show the optimal portfolio weights for the no additional constraints (same as the top-left graph in Figure 14), issuance constrained, and issuance and Can$80 billion rollover constrained optimizations. In each case, the optimizer seeks to minimize the expected average percentage debt charges over the 10-year simulation horizon. The final graph (bottom right) shows the expected costs for each of the three optimizations.



There are several points to emphasize in Figure 16. First, the range of constraint levels is significantly reduced (from Can$0.5 billion to Can$4 billion, to Can$1 billion to Can$2 billion) in the issuance-constrained and issuance and rollover constrained (top-right and bottom-left) graphs. Debt-charge volatility cannot be reduced below Can$1 billion, likely because of forced issuance in treasury bills, and perhaps also due to forced issuance in RRBs. The inability to increase debt-charge volatility beyond Can$2 billion appears mainly due to the forced issuance in the 30-year nominal and 30-year RRB maturities (and, to a lesser extent, the 10-year maturity), which are either not present or present at a reduced level in the top-left graph with no additional constraints.[49] The forced long-term issuance also contributes to higher expected cost, as can be seen from the

[47] This level will differ across instruments, and is updated regularly to reflect market conditions.

[48] There are numerous combinations of constraints that could be used. For example, one could run an optimization with rollover as the main risk constraint, with a maximum allowable value on conditional debt-charge volatility and/or conditional budgetary volatility.

[49] RRBs appear to have a high enough impact on debt-charge volatility to infringe on risk reduction, while also having a small enough impact on debt-charge volatility to prevent reaching increasing risk levels.

fact that the red and green lines are above the blue line in the bottom-right graph. Clearly, imposing issuance and rollover constraints is restrictive in terms of reachable conditional debt-charge volatility levels, and leads to higher expected costs.

Second, the issuance constrained and issuance and rollover-constrained optimizations give the same results up to a conditional debt-charge volatility level of about Can$1.4 billion. This can be seen by the fact that the red and green lines overlap in the bottom-right graph in Figure 16. This occurs because the rollover constraint does not bind until the Can$1.4 billion constraint level. In other words, the issuance-constrained optimal portfolios until that point have an average quarterly rollover level under Can$80 billion, and so adding the rollover constraint has no effect on them.

Third, in terms of portfolio allocation, the addition of the Can$80 billion rollover constraint causes a reduction in the 3-month treasury bill weight, starting at about a Can$1.4 billion conditional debt-charge volatility level, with it being replaced with 6-month and 1-year treasury bills.

5.2.3 Portfolio-summary measures and primary issuance constraints

This section examines constraints that are motivated by notions of portfolio composition and debt issuance. Practically, debt managers are interested not only in notions of risk stemming from the moments of debt-charge and budgetary distributions. Constraints also arise in other forms. In particular, portfolio-summary measures are of interest to debt managers for internal and external communication. We consider two common measures: the refixing share of debt and the average term to maturity (ATM).

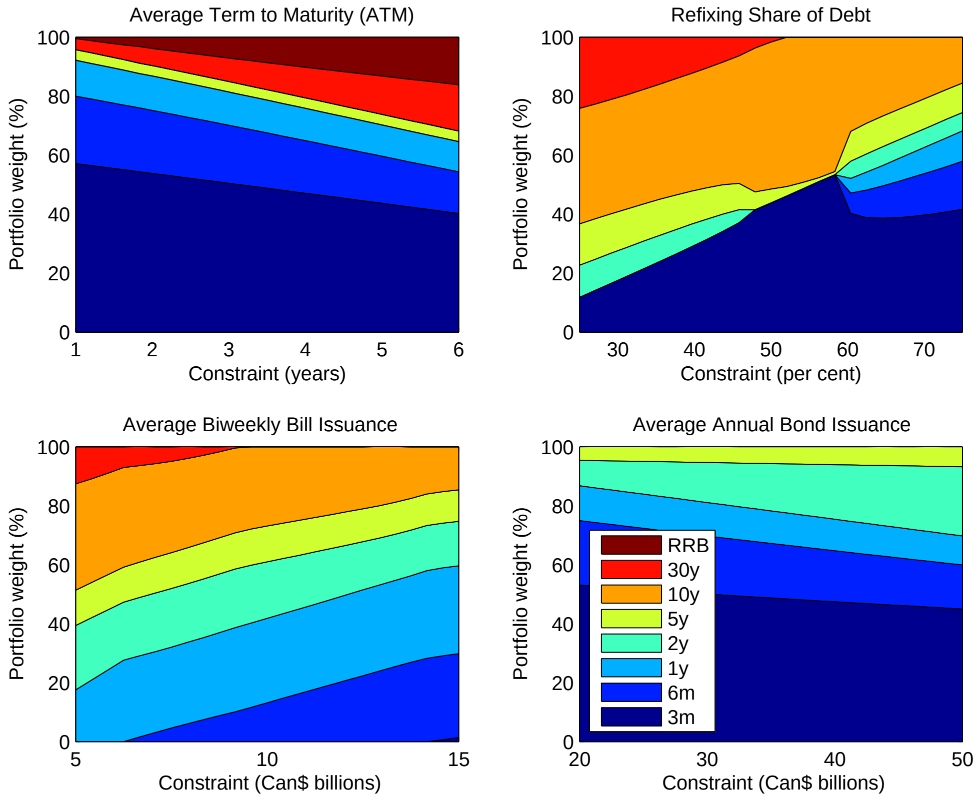
A second example is primary issuance. A fundamental principal of debt management is that concentration of issuance in specific market segments permits reduction of liquidity premia that creates a corresponding cost savings for debt issuers. To this end, it also makes sense to consider the application of constraints to primary issuance. Again, we consider two potential measures: average biweekly treasury bill issuance and average annual bond issuance.

Figure 17 illustrates the portfolio weights—in a manner analogous to Figure 14—for these four alternative portfolio-summary and issuance constraints. The top-left quadrant focuses on the ATM with constraint levels ranging from one to six years. Note that the direction of this constraint is opposite to those illustrated in Figure 14. This is evidenced in Figure 18, which demonstrates how the cost of the portfolio is increasing in the ATM. For a target ATM of one year, we observe an optimal portfolio consisting almost entirely of treasury bills. There is a small allocation to different bond sectors to assist in keeping up the ATM. As we increase the ATM target, however, we see that the proportion of treasury bills decreases and that the proportion of long-term debt grows. We suspect that the reason for the 30-year debt is that the marginal impact on the ATM is greater, per unit of cost, relative to the use of 5- or 10-year bonds. Figure 18 indicates that the expected cost increases by approximately 20 basis points.

The top-right quadrant of Figure 17 describes the portfolio weights for the refixing share of debt constraint over the range from 25 per cent to 75 per cent. We should stress that, as we have not explicitly considered retail, non-market, and foreign debt, these fixed-debt ratio figures will *not* correspond to the official figures. Observe that, for a refixing share of debt constraint level of 25 per cent, the portfolio is principally composed of 10-year and 30-year bonds, with smaller, but significant, allocations to 3-month treasury bills, and 2-year and 5-year bonds. As the refixing share of debt constraint increases to 75 per cent, the optimal portfolio weights shift to more 3-month (and eventually 6-month and 12-month) treasury bills and less long-term bonds. The reason for the large amount of 3-month bills relative to other bill maturities is that they have the lowest cost and all bill maturities have a refixing share of debt of one, since they are refinanced every year. Finally, moving from a refixing share of debt level of 25 per cent to 75 per cent—as shown in Figure 18—decreases expected debt-service charges by approximately 60 basis points.

We next consider the issuance constraints. We decided to impose constraints on average biweekly treasury bill issuance because this is a familiar and pertinent quantity for Canadian debt managers. The bottom-left

Figure 17: **Portfolio-Measure and Issuance-Constrained Optimizations**: This figure illustrates how the optimal portfolio weights evolve for different fixed portfolio-summary measure and issuance-constraint values. In each case, the optimizer seeks to minimize the expected average percentage debt charges over the 10-year simulation horizon. Each of the four graphs, as in Figure 14, represents an alternative constraint. The horizontal axis represents alternative levels of the constraint. The first graph, for example, examines how the portfolio weights change as we increase the constraint on the minimum average term to maturity on the portfolio from one to six years. The other three graphs demonstrate the corresponding results by using the maximum refixing share of debt, the maximum amount of biweekly treasury bill issuance and the minimum annual bond program, respectively.

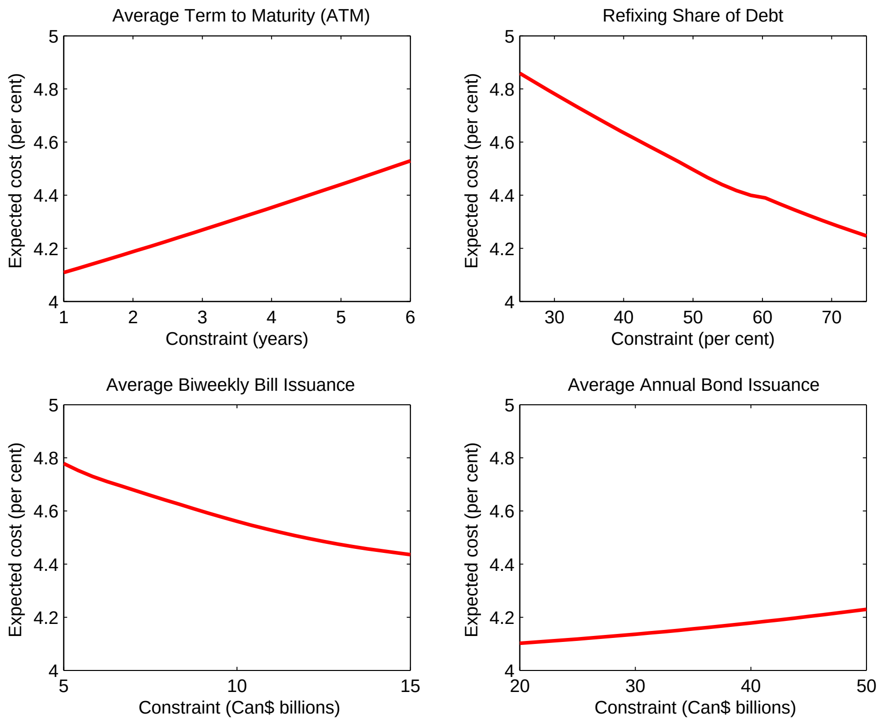


quadrant of Figure 17 illustrates the optimal portfolio weights for this constraint with values from Can$5 to 15 billion.[50] Interestingly, it appears relatively straightforward to accommodate increased treasury bill issuance. In particular, across the range of constraint levels the allocation to treasury bills increases from about 20 per cent to 50 per cent. Meanwhile, 10-year and 30-year allocations decline as the constraint level increases. The 2-year and 5-year allocations, however, remain fairly stable across all constraint levels. Finally, the cost savings of moving from Can$5 to 15 billion of average biweekly treasury bill issuance (see Figure 18) is only about 40 basis points.

The final issuance measure is annual gross bond issuance. The optimal portfolio weights, from Can$20 to 50 billion, are provided in the bottom-right quadrant of Figure 17. It is relatively easy to increase the amount of bond issuance. Creating Can$50 billion of gross bond issuance—in the absence of any other constraints—mostly requires simply increasing the weight of 2-year bonds to just below 25 per cent. With a debt stock of Can$400 billion, this implies a 2-year stock of just under Can$100 billion. Since about half of the 2-year bonds are refinanced each year, this yields, when adding a small allocation to 5-year bonds, a Can$50-billion bond program—moreover, the savings are only about 10 basis points for the 30-billion decrease in average annual

[50] In 2006, average bi-weekly treasury bill issuance was approximately Can$10 billion and ranged from Can$7.5 billion to Can$12 billion.

Figure 18: **Expected Cost of Portfolio-Measure and Issuance-Constrained Optimizations**: This figure illustrates the optimal risk-cost trade-off for different fixed portfolio-measure and issuance-constraint values. In each case, the optimizer seeks to minimize the expected average percentage debt charges over the 10-year simulation horizon. These results are described in Figure 17. These figures illustrate the expected cost, in percentage terms, associated with each of these optimal portfolios.



bond issuance. This would suggest that constraints on issuance in specific bond tenors, as found in Figure 16, are more meaningful than an aggregate annual bond-issuance constraint.

5.3 Conclusion

The objective of this chapter was to illustrate the role that the optimization tool can play in our debt-strategy analysis. Clearly, this paper merely touches on a subset of what can be accomplished with the optimization module. We briefly examined the simultaneous impact of multiple constraints, and this can be extended further. There is no reason, for example, why we could not impose simultaneous constraints on, say, conditional debt-charge volatility, the refixing share of debt and average biweekly treasury bill issuance.

The larger question, however, is which risk, portfolio-summary and issuance constraints does one wish to impose on the portfolio? Moreover, what are reasonable levels for these constraints? The answers to these questions are essential to effectively using an optimization tool.

6 Modelling the Government's Fiscal Policy

This chapter reviews the framework used to model the government's funding requirement. A principal advantage associated with a joint model of the macroeconomy and the term structure of interest rates is the ability to use information about inflation, output and interest rates to determine the funding requirement of the government in a comprehensive manner. Following years of budgetary surpluses, the events of the past few years have demonstrated that Canadian government revenues and expenditures are not immune to macroeconomic and financial conditions. Moreover, primary-balance volatility can be expensive for the government even when it does not contribute to a budgetary deficit. Such volatility in the government's primary balance can, for example, lead to changes in taxation and expenditures that are economically suboptimal.

The chapter is organized into two parts. In the first part, we discuss the fiscal manager's challenge. This includes the intuition and mathematical assumptions regarding the government's primary balance and how we assume that the fiscal authorities will react to surplus and deficit positions. In other words, we outline the key modelling assumptions and relate them to the tasks facing a fiscal manager. This is math-intensive and can be browsed or skipped over without detracting from subsequent sections. In the second part of this chapter, we provide some illustrative examples to enhance understanding of the critical components of the financing-requirement process and its implications for the government's debt portfolio.

6.1 The fiscal manager's challenge

Before turning to the details, it is useful to clarify some terminology. We use the term *funding requirement* to describe the amount of funds required, before considering the existing debt stock, by the government in a given period.[51] This is reminiscent of the government's surplus or deficit position. Practically speaking, however, this cash-based notion amounts to the government's primary balance less debt charges. In the real world, the funding requirement almost always differs from the actual surplus or deficit position due to non-cash accounting factors. Since dealing with such idiosyncratic non-cash items is difficult in a modelling setting, we ignore these details and assume that the funding requirement is the primary balance less debt charges. We use the term *financial requirement* to refer to the amount that the government must borrow in a given period. This amounts to the funding requirement plus any maturing debt that must be refinanced.

6.1.1 Intuition

In the model, incorporating the fiscal side begins by forecasting government revenues, expenditures and debt charges. This is, of course, an extremely complex, highly dimensional process that depends on taxation policy, government-expenditure programs and the composition of the debt stock. An extremely high level of detail, however, is neither desirable nor necessary for our model. Instead, we simply assume that government revenues and expenditures are functions of the general macroeconomy: that is, output, inflation and the change in the level of short-term interest rates. Forecasting government revenues and expenditures, therefore, reduces to an exercise in forecasting macroeconomic outputs. Given that we explicitly model the stochastic evolution of the Canadian macroeconomy, this is a straightforward task. Forecasting debt-service charges is also possible given the highly detailed information regarding the government debt stock available, at each time step, in the debt-strategy model. Nevertheless, as forecasting debt charges involves some degree of interest rate forecasting, there also remains a non-trivial degree of uncertainty in these forecasts.

At the beginning of each year, therefore, we produce four consecutive quarterly forecasts of government revenue, expenditures and debt-service charges. What is now required is some degree of flexibility, a lever if you will, that permits us to, in expectation, attain a given target amount of funding requirements. The lever used in our analysis is the fixed component of government expenditure. We use the deficit or surplus forecasts and

[51] Note that the funding requirement can be positive (i.e., a source of funds) or negative (i.e., a requirement of funds).

adjust the fixed part of government expenditures to attain, in expectation, the government's fiscal target. From a modelling perspective, this is a simple exercise in algebra. From a conceptual perspective, it amounts to the fiscal manager pre-committing to a given set of government expenditures, conditioned on future expectations of other revenue and expenditures, to obtain a fiscal goal (e.g., a target financial source for the paydown of government debt).

In reality, as in the model, the forecasted government expenditures, revenues and debt-service charges will not be realized exactly as expected, which implies that, in any given period, the fiscal target may or may not be attained. In the model, as in reality, the fiscal manager must live with this uncertainty.[52] The addition of this explicit description of the government's fiscal policy will be to see to what extent different financing strategies can help (or hinder) this budgetary uncertainty.

6.1.2 The mathematics

In this section, we provide a mathematical description of the intuitive discussion provided in the previous section. To begin, we need a general expression for the government's funding requirement. Let $F(t), R(t), E(t)$ and $\Delta(t)$ denote the funding requirement, government revenue, non-debt-related expenditures and debt charges over the interval $[t, t+1]$. The funding requirement has the following form,

$$\begin{aligned} F_t &= R_t - (E_t + \Delta_t), \\ &= R_t - E_t - \Delta_t. \end{aligned} \tag{4}$$

This usual identity holds that the government's funding requirement is the total revenues less total expenditures. Let us now make two assumptions. First, we assume that revenues and expenditures are functions of the state-variable vector stemming from our joint macroeconomic term-structure model.[53] Second, we assume that government debt charges are a function of historical and current interest rates. This permits us to modify equation (4) as follows,

$$F_t(X_t) = R_t(X_t) - E_t(X_t) - \Delta_t(\mathcal{Z}_t), \tag{5}$$

where $\mathcal{Z}_t$ denotes the collection of current and historical interest rates necessary to compute government debt charges over the interval, $[t, t+1]$.

Robbins, Torgunrud, and Matier (2007) propose the following two equations to describe the annual growth of government revenues (i.e., $\{R_t, t = 0, 1, 2...\}$),

$$\ln\left(\frac{R_t}{R_{t-1}}\right) = 1.02y_t - 0.34y_{t-1} + 1.05\pi_t + 0.19(r_t - r_{t-1}) + \epsilon_t, \tag{6}$$

where, y_t denotes real GDP, π_t denotes the rate of inflation, and r_t represents the monetary policy rate.[54] Similarly, the annual growth of government expenditures (i.e., $\{E_t, t = 0, 1, 2...\}$),

$$\ln\left(\frac{E_t}{E_{t-1}}\right) = c - 0.27y_t + 0.32\pi_t + \nu_t, \tag{7}$$

[52] In reality, the fiscal manager can exercise some limited control throughout the course of the year in response to macroeconomic surprises; such reactions, however, are generally complex and idiosyncratic and are thus ignored from a modelling perspective.

[53] The details of the stochastic model will be addressed in Chapter 7. For the moment, all one needs to keep in mind is that, in our model, the state of the world is fully described by a vector of financial and macroeconomic state variables termed X_t. Knowledge of this state vector permits us to determine interest rates and the state of the economy (i.e., output, inflation and monetary policy).

[54] In the model proposed by Robbins, Torgunrud, and Matier (2007), π_t denotes GDP inflation. In our model, however, we use the CPI since it forms the basis for the pricing of inflation-linked bonds. These series are quite highly correlated and, as such, this should not create much difficulty.

in the Canadian economy. The innovations to both growth equations are assumed to be Gaussian, with

$$\epsilon_t \sim \mathcal{N}\left(0, \sigma_R^2\right), \qquad (8)$$
$$\nu_t \sim \mathcal{N}\left(0, \sigma_E^2\right).$$

These two equations will form the basis of our model. We would first like to slightly simplify and adjust their mathematical form. First, let us denote the macroeconomic state variables as,

$$\tilde{X}_t^T = \begin{bmatrix} y_t & y_{t-1} & \pi_t & (r_t - r_{t-1}) \end{bmatrix} \qquad (9)$$

This permits us to redefine equation (6) as,

$$\ln\left(\frac{R_t}{R_{t-1}}\right) = \beta_R^T \tilde{X}_t + \epsilon_t, \qquad (10)$$

where,

$$\beta_R^T = \begin{bmatrix} 1.02 & -0.34 & 1.05 & 0.19 \end{bmatrix}. \qquad (11)$$

The corresponding form for the government expenditures is,

$$\ln\left(\frac{E_t}{E_{t-1}}\right) = c + \beta_E^T \tilde{X}_t + \nu_t, \qquad (12)$$

where,

$$\beta_E^T = \begin{bmatrix} -0.27 & 0 & 0.32 & 0 \end{bmatrix}. \qquad (13)$$

Two additional points arise. First, the debt-strategy model operates on a quarterly rather than an annual basis. This implies that it will be necessary to make an adjustment to these annual growth rates for the model. Second, we will require levels instead of growth rates to actually target a specific amount of debt paydown in the debt-strategy model. We make both of these adjustments in the simplest possible manner. First, we merely divide the growth rate by a factor of four to make it a quarterly quantity. Second, we use the following approximation to transform these growth rates into levels,

$$\ln\left(\frac{x_t}{x_{t-1}}\right) \approx \frac{x_t - x_{t-1}}{x_{t-1}}. \qquad (14)$$

This approximation is not exact, but it is reasonably close. Moreover, it eases some of the latter calculations when we need to compute the expectations of future government revenues and expenditures. Applying these two ideas to equation (10) yields,

$$\frac{R_t - R_{t-1}}{R_{t-1}} = \frac{1}{4}\beta_R^T \tilde{X}_t + \frac{1}{4}\epsilon_t, \qquad (15)$$
$$R_t = R_{t-1} + R_{t-1}\left(\frac{1}{4}\beta_R^T \tilde{X}_t + \frac{1}{4}\epsilon_t\right),$$
$$R_t = \left(1 + \beta_R^T \tilde{X}_t^q\right) R_{t-1} + \tilde{\epsilon}_t,$$

where,

$$\tilde{\epsilon}_t \sim \mathcal{N}\left(0, \left(\frac{R_{t-1}}{4}\right)^2 \sigma_R^2\right), \qquad (16)$$

and $\tilde{X}_t^q$ is the quarterly adjusted macroeconomic state variables. A similar calculation for the government expenditures in equation (12) yields,

$$E_t = \frac{c}{4}E_{t-1} + \left(1 + \beta_E^T \tilde{X}_t^q\right) E_{t-1} + \tilde{\nu}_t, \tag{17}$$

where,

$$\tilde{\nu}_t \sim \mathcal{N}\left(0, \left(\frac{E_{t-1}}{4}\right)^2 \sigma_E^2\right). \tag{18}$$

Inspection of equations (15) and (17) reveals that the only free variable is c. This variable is the fixed component of the expenses. The government attempts to meet its debt-payment target in expectation, by forecasting future economic variables and debt charges. In principle, it can adjust taxation and spending to meet this debt-paydown target. In this setting, however, the only degree of freedom is the free parameter, c.

Therefore, at time $t-1$, c is selected to meet, in expectation, the government's fiscal target (let us call this τ), over the next four (i.e., κ) quarters. That is, we wish to solve:

$$\tau = \sum_{k=0}^{\kappa-1} \mathbb{E}\left(F_{t+k} \mid \mathcal{F}_{t-1}\right), \tag{19}$$

for c. Using equations (4), (15) and (17), we can accomplish this task as follows,

$$\begin{aligned}
\tau &= \sum_{k=0}^{\kappa-1} \mathbb{E}\left(\underbrace{R_{t+k} - E_{t+k} - \Delta_{t+k}}_{\text{Equation (4)}} \mid \mathcal{F}_{t-1}\right), \qquad (20)\\
&= \sum_{k=0}^{\kappa-1} \mathbb{E}\left(\underbrace{\left(1+\beta_R^T \tilde{X}_{t+k}^q\right) R_{t-1} + \tilde{\epsilon}_t}_{\text{Equation (15)}} \middle| \mathcal{F}_{t-1}\right) - \mathbb{E}\left(\underbrace{\frac{c}{4}E_{t-1} + \left(1+\beta_E^T \tilde{X}_{t+k}^q\right) E_{t-1} + \tilde{\nu}_t}_{\text{Equation (17)}} \middle| \mathcal{F}_{t-1}\right) - \mathbb{E}\left(\Delta_{t+k} \mid \mathcal{F}_{t-1}\right),\\
&= \sum_{k=0}^{\kappa-1} \left(1+\beta_R^T \mathbb{E}\left(\tilde{X}_{t+k}^q \mid \mathcal{F}_{t-1}\right)\right) R_{t-1} - \frac{c}{4}E_{t-1} - \left(1+\beta_E^T \mathbb{E}\left(\tilde{X}_{t+k}^q \mid \mathcal{F}_{t-1}\right)\right) E_{t-1} - \mathbb{E}\left(\Delta_{t+k} \mid \mathcal{F}_{t-1}\right),\\
&= -cE_{t-1} + \sum_{k=0}^{\kappa-1} \left((R_{t-1} - E_{t-1}) + \left(\beta_R^T R_{t-1} - \beta_E^T E_{t-1}\right) \mathbb{E}\left(\tilde{X}_{t+k}^q \mid \mathcal{F}_{t-1}\right) - \mathbb{E}\left(\Delta_{t+k} \mid \mathcal{F}_{t-1}\right)\right),
\end{aligned}$$

where finally,

$$c = \frac{1}{E_{t-1}}\left(-\tau + \sum_{k=0}^{\kappa-1}\left[(R_{t-1} - E_{t-1}) + \left(\beta_R^T R_{t-1} - \beta_E^T E_{t-1}\right)^T \mathbb{E}\left(\tilde{X}_{t+k}^q \mid \mathcal{F}_{t-1}\right) - \mathbb{E}\left(\Delta_{t+k} \mid \mathcal{F}_{t-1}\right)\right]\right). \tag{21}$$

We can now interpret c as the fixed amount of spending to meet the fiscal target of τ over the next κ quarters. This value is determined using information as of time $t-1$. We can think, therefore, of c as the principal fiscal-policy parameter in our model.

Once the choice of c is determined, it remains fixed over the next four quarters (i.e., fiscal year). Each quarter, uncertainty is revealed about the evolution of the macroeconomy. In period t, for example, we use equations (15) and (17) to determine, for a fixed choice of c, the actual revenues and expenditures. We also determine the actual debt charges from the model. The amalgamation of these values determines the funding requirement for that quarter. This is repeated for each quarter in the fiscal year.

6.2 Some illustrative analysis

As is clear from the previous discussion, an important source of budgetary uncertainty stems from the necessity of forecasting the primary balance. If the fiscal manager, for example, knew the primary balance (i.e., the revenue and expenditure outcomes) in advance, then the only uncertainty regarding the financial requirement would be the debt-service charges. Of course, the fiscal manager must also forecast debt-service charges, which is another potential source of uncertainty. It is, in fact, quite interesting to look at the variation in the government's funding requirement. Recall from equation (4) that the government's funding requirement is,

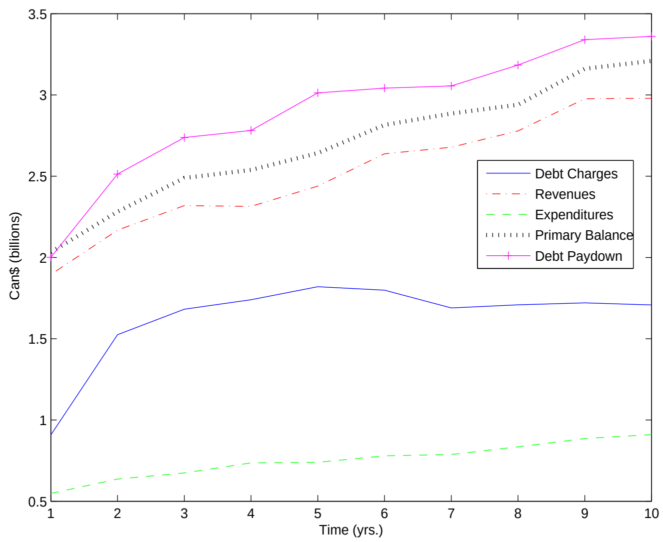
$$F_t = \underbrace{R_t - E_t}_{\substack{\text{Primary}\\ \text{balance}}} - \underbrace{\Delta_t}_{\substack{\text{Debt}\\ \text{costs}}}. \tag{22}$$

The variance, or the average dispersion, of the funding requirement, therefore, is given by,

$$\text{var}(F_t) = \text{var}(R_t - E_t) + \text{var}(\Delta_t) - 2\text{cov}(R_t - E_t, \Delta_t). \tag{23}$$

From the preceding equation, we can clearly observe that the dispersion of the budgetary outcomes depends on three factors: the dispersion of the primary balance, the dispersion of the debt-service charges and the interaction (i.e., covariance) of these two variables.[55] If the covariance between the primary balance and the debt-service charges is positive, it will dampen the volatility of the budgetary balance. Two additional points should be mentioned. First, the financing strategy ultimately influences the debt-charge volatility and the covariance between the primary balance and debt-service charges. The dispersion of the primary balance is independent of the government's financing strategy. Second, while a positive covariance is useful and can reduce budgetary volatility, the overall volatility of the debt-service charges remains important. Extremely volatile debt charges cannot generally be offset, even by very strong positive covariance.

Figure 19: **Fiscal-Position Forecast Errors**: This figure outlines the forecast errors for revenues, expenditures, debt charges, the primary balance and the debt-paydown target across 1,000 simulations. Note that, in all cases, the expected forecast errors are extremely close to zero.



To see how this works in the model, let us begin by examining the amount of uncertainty in the fiscal forecasts for the base parameterization of the model. Figure 19 outlines the root-mean-squared-error difference

[55] In reality, equation (23) applies to deviates of realized outcomes from expected outcomes, but the logic still applies, albeit with less notational complexity.

between actual-and-realized revenue, expenditure, primary-balance, debt-service charge and debt-paydown target forecasts across 1,000 simulations for an equally weighted financing strategy. While it is not shown in Figure 19, the average forecast errors for each of the fiscal quantities is essentially zero. This is important, because it implies, as we would expect, that the fiscal manager does not make systematic forecast errors.

Each line in Figure 19 basically denotes the average forecast error. Observe that revenues are more than three to four times more difficult to predict than expenditures. This is because revenue growth varies basically one-for-one with output and inflation. Inflation is relatively straightforward to forecast (given Canada's strong track record as an inflation-targeting nation), but output is very difficult to accurately forecast. This leads to noisy revenue forecasts. Expenditure, however, depends only weakly on output and inflation. We also observe that the primary-balance forecasting uncertainty is slightly greater than both the uncertainty of either the revenue or expenditures. This is due to the slightly negative correlation between revenues and expenditures. We further note that debt charges are about half as difficult to forecast as revenues, but twice as hard to forecast as expenditures. The level of difficulty in forecasting debt-service charges, however, will vary depending on the financing strategy. We should, for example, expect that financing strategies with large proportions of treasury bills, which are frequently refinanced, should be more volatile relative to portfolios dominated by longer-term debt.

A natural question is how these forecast errors actually translate into budgetary outputs. This will generally depend upon the choice of financing strategy. Figure 20 describes the optimal portfolio weights and cost-risk trade-off for the minimization of debt-service charges subject to a constraint on the conditional budgetary volatility. The minimum level of risk (Can\$2.725 billion) is achieved with a substantial allocation to Real Return Bonds (i.e., about 22 per cent), long bonds (i.e., 24 per cent and 30 per cent 10- and 30-year bonds, respectively), 5-year bonds (10 per cent) and treasury bills (i.e., 12 per cent 3-month treasury bills). Note, however, that moving to a conditional budgetary volatility level of Can\$3.0 billion results in a cost savings of about 60 basis points. Clearly then, there is a high cost to reducing budgetary volatility to its minimum level. Furthermore, the range of budgetary volatility levels is only Can\$1.0 billion, implying that a debt manager has a limited ability to affect conditional budgetary volatility. This is significantly smaller than the range of debt-charge volatility outcomes, but why is this the case?

To explore this further, we return to equation (23) (inserting $P_t = R_t - E_t$ to represent the primary balance):

$$\operatorname{var}(F_t) = \operatorname{var}(P_t) + \operatorname{var}(\Delta_t) - 2\operatorname{cov}(P_t, \Delta_t). \tag{24}$$

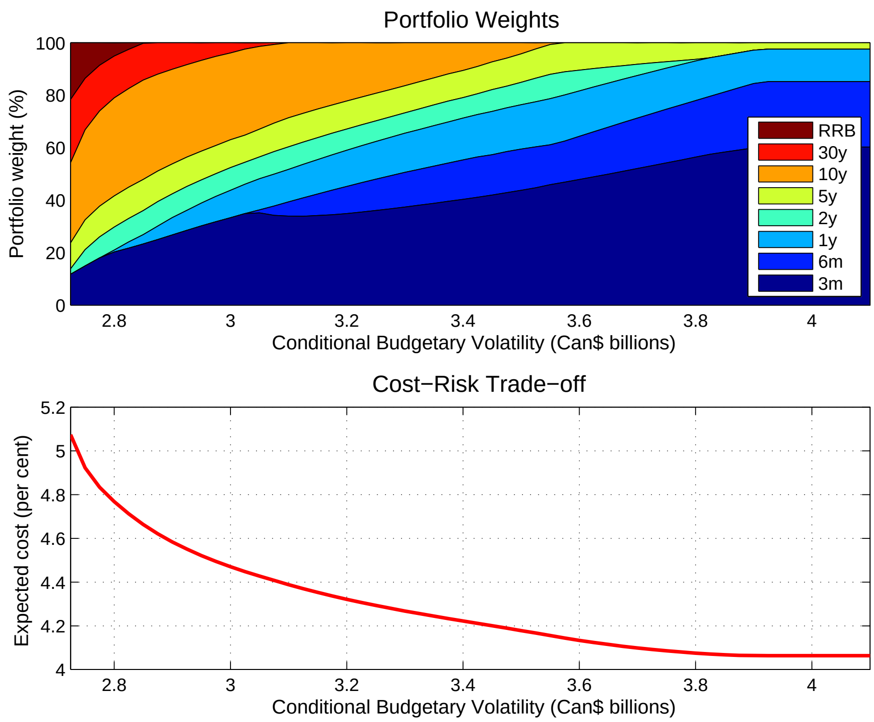
A few points need to be stressed. First, debt-management decisions do not influence the primary balance; thus, the contribution of the variance of the primary balance (i.e., $\operatorname{var}(P_t)$) to budgetary variance is beyond the control of debt-management policy. The choice of financing strategy, therefore, can influence only the debt-charge variance ($\operatorname{var}(\Delta_t)$) and, indirectly, the interaction between the primary balance and debt-service charges ($2\operatorname{cov}(P_t, \Delta_t)$). If there is no relationship between the primary balance and debt-service charges ($\operatorname{cov}(P_t, \Delta_t) = 0$), then only the debt-service-charge variance will matter. In this case, we have that

$$\operatorname{var}(B_t) = \operatorname{var}(P_t) + \operatorname{var}(\Delta_t), \tag{25}$$

and consequently, it will not matter whether risk is defined in a debt-charge or budgetary context, since they would be essentially equivalent.

The correlation between the primary balance and debt-service charges is typically positive ($\operatorname{cov}(P_t, \Delta_t) > 0$), which implies that budgetary uncertainty is lower than what one would expect to observe in the case of independence (i.e., equation (25)). In other words, positive correlation between the primary balance and debt-service charges acts to dampen budgetary volatility. This implies, of course, that negative correlation will have the opposite effect—it will act to augment budgetary volatility. The magnitude and sign of primary-balance and debt-service-charge correlation, therefore, will have important implications for the variance of budgetary outcomes.

Figure 20: **Conditional Budgetary Volatility**: The figure displays optimal portfolio weights and the cost-risk trade-off for the minimization of debt-service charges subject to a constraint on the conditional budgetary volatility.



The correlation between the primary balance and the debt-service charges is determined empirically through the statistical model of interest rates and the macroeconomy. We do have substantial information regarding the primary-balance, debt-service-charge and budgetary variance. We can, therefore, use this information and the structure of equation (24) to identify the role that this covariance plays across a variety of different possible portfolios. Specifically, rearranging equation (24) yields

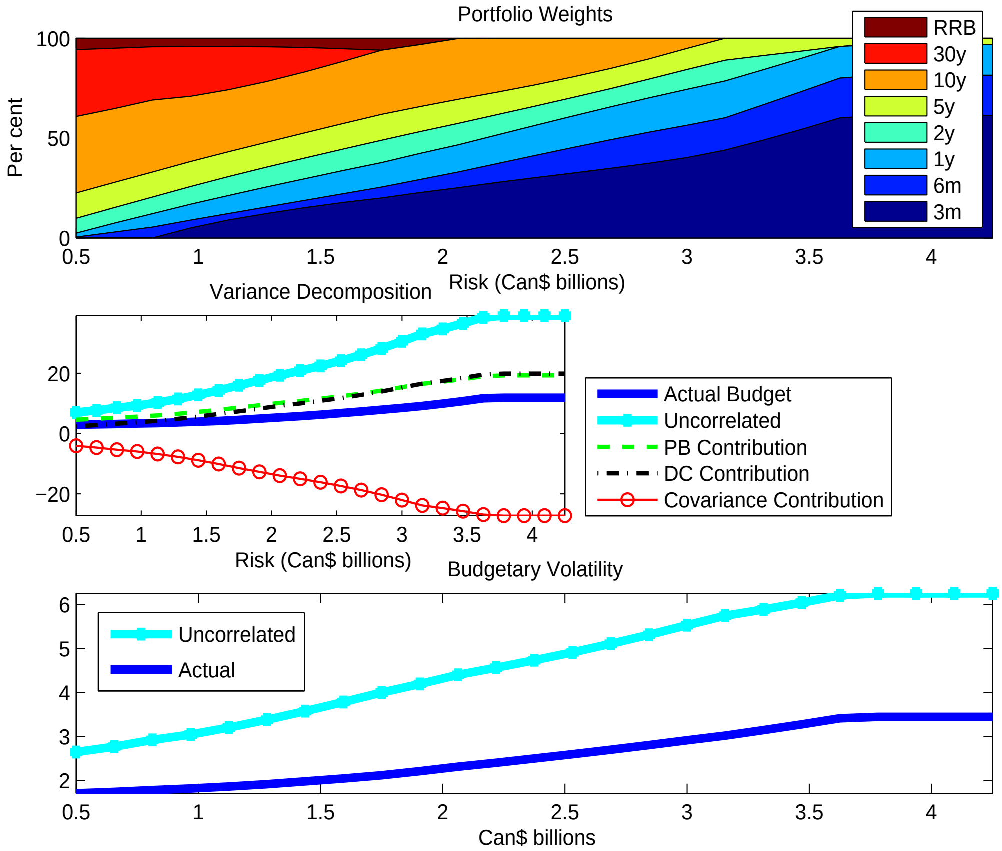
$$2\text{cov}(P_t, \Delta_t) = \underbrace{\text{var}(P_t) + \text{var}(\Delta_t)}_{\text{Equation (25)}} - \text{var}(B_t), \tag{26}$$

which describes the difference between budgetary variance and the sum of primary-balance and debt-service-charge variance.

If, for a given financing strategy, the primary balance and debt-service charges are uncorrelated, then the value in equation (26) should be zero. If the primary balance and debt charges are correlated, which is generally the case, this quantity will be non-zero. To better understand the implications of this correlation, let us perform a simple exercise. We will compute this quantity in equation (26) across a range of optimal portfolios determined by minimizing expected debt-service charges subject to a varying constraint on conditional debt-service-charge volatility. Figure 21 provides a summary of this exercise and thus includes three separate graphics related to the implications of primary-balance and debt-service-charge correlation.

The top graphic in Figure 21 outlines the optimal portfolio weights associated with a debt-service-charge definition of risk. Observe that the lowest levels of risk include substantial portfolio allocations to longer-term bonds, while higher levels of risk include predominately treasury bill issuance. The middle graphic provides a detailed decomposition of the budgetary variance associated with each of these optimal portfolios. This decomposition includes each of the terms in equation (26). In particular, it demonstrates the total budgetary variance (the dark blue line), the contribution of the primary-balance variance to budgetary variance (the dashed

Figure 21: **Budgetary-Risk Analysis**: This figure illustrates the optimal portfolio weights associated with a debt-service-charge definition of risk, a decomposition that includes each of the variance terms in equation (24) for each of these portfolios, and the standard deviation of actual and uncorrelated budgetary outcomes for each optimal portfolio.



green line), the contribution of the debt-service-charge variance to the budgetary variance (the dashed-dotted black line) and the contribution of the covariance term in equation (26) to the budgetary variance (the red line).

A few observations can be made immediately. First, the magnitude of the total budgetary variance increases gradually as the risk of the portfolio in debt-charge terms is increased. Second, the budgetary variance appears to be generally less than either the primary-balance or the debt-service-charge component. The reason is that the contribution of the covariance term is significant and negative, which acts to offset the combined impact of primary-balance and debt-service-charge variance. Moreover, as the debt-service-charge risk of the portfolio increases, the importance of the covariance contribution grows.[56] Indeed, Figure 21 also demonstrates what the budgetary variance would be in the event that the primary balance and debt-service charges were uncorrelated (i.e., equation (25)). It is consequently quite clear that budgetary variance would be substantially higher in the event that no relationship between the primary balance and debt-service charges existed. This is particularly the case for higher-risk portfolios.

The final graph in Figure 21 describes the budgetary *volatility*—that is, the standard deviation of actual and uncorrelated budgetary outcomes—across different levels of risk. This is useful because the units of budgetary

[56]Recall that if there are two random variables X and Y, the covariance is defined as,

$$\mathrm{cov}(X,Y) = corr(X,Y)\cdot\sigma(X)\cdot\sigma(Y), \tag{27}$$

where $\sigma(\cdot)$ is the standard deviation. Thus, it is possible for the covariance term to increase in size even if the correlation is constant as the standard deviation of the individual components increases.

variance are difficult to interpret, whereas the units of budgetary volatility are Can$ billions. We observe that for the lowest level of risk, the actual budgetary volatility is slightly less than Can$2 billion. If we assume no correlation contribution, this rises to approximately Can$2.5 billion. The impact of the no-correlation assumption, however, rises as we increase the risk of the portfolio. Specifically, for the highest amount of risk, actual budgetary volatility is about Can$3 billion, but it doubles to around Can$6 billion in the event of zero correlation between the primary balance and debt-service charges.

The above analysis demonstrated that the relatively narrower range of optimal conditional budgetary volatility portfolios arises from the covariance effect between the primary balance and debt charges partially offsetting increasing levels of debt-charge risk.

6.3 Conclusion

The government's fiscal position has been carefully modelled as outlined in Robbins, Torgunrud, and Matier (2007). This challenge involves forecasting macroeconomic variables (such as output, inflation and monetary policy) to arrive at a forecast of government revenues and expenditures. It also involves forecasting financial variables (such as interest rates) to assist in forecasting government debt-service charges. Using these forecasts, adjustments are made—admittedly in a stylized manner in our model—to the fixed component of expenditure to achieve, in expectation, the fiscal target. The actual realizations of these macroeconomic and financial variables, as well as the properties of the current financing strategy, will determine how close each realization is to the fiscal objective.

The two principal characteristics of a financing strategy that contribute to budgetary outcomes are the volatility of the debt-service charges and the correlation of these debt charges with the primary balance. Interestingly, these two components combine empirically to give a narrower range of optimal conditional budgetary volatility portfolios than optimal conditional debt-charge volatility portfolios.

7 Sensitivity to the Stochastic Model

This chapter looks at the sensitivity of the model results to alternative specifications of the random future evolution of interest rates and the macroeconomy. Interest rates and inflation determine the *cost* of nominal and inflation-linked debt issuance at any given point in time. Output, inflation, monetary policy and debt-service charges drive the government's funding requirement, which has implications for the *amount* of debt the government must borrow in any given period. Given these price and quantity implications, it is clear that the results of the debt-strategy model depend importantly on random interest rate and macroeconomic dynamics. This dependence creates risk. This type of risk—risk of misspecifying the stochastic model—is termed *model risk*. Since model risk has the potential to lead to suboptimal policy advice, it requires special attention.[57]

Special attention not only involves being aware of this risk, but also taking steps to mitigate it. To date, our principal mitigation approach involves the use of a broad range of different statistical models for joint term-structure and macroeconomic dynamics. Currently, there are *five* alternative statistical approaches that have been implemented in the enhanced debt-strategy model. While this reduces the reliance on any single statistical approach, it also imposes an increased burden on the group performing the analysis. For this reason, we have generally focused on a single model. Given the very real existence of model risk, however, it is important to understand how different statistical models can impact the results. The objective of this analysis, therefore, is to provide some intuition around the sensitivity of the results to the underlying stochastic framework.

This chapter is organized into two parts. In the first part, we provide a very high-level overview of the alternative statistical models. In the second part, we examine the sensitivity of the debt-strategy model outputs to the different stochastic frameworks. This occurs in two steps. First, we compare the results of the simulation of a single equally weighted portfolio with different statistical approaches. This permits us to look, in detail, at a number of key features of the debt-charge and budgetary distributions. Second, we examine how the optimal portfolio weights vary, for two relatively simple objective functions, as we alter the statistical model.

7.1 A brief overview of our stochastic models

Most of the popular models of the term structure of interest rates, despite the broad literature in this area, have a similar basic form.[58] Abstracting from the empirical technical details, there are, in fact, really only *three* key assumptions involved in this process. The first assumption is that the interest rate system is a function of some set of state variables. The idea is that the *state* of the interest rate system can be characterized by values of these variables, or factors, at each given point in time. These state variables may be latent (i.e., unobservable) or observable factors, such as macroeconomic variables. Generally, the models used in the debt-strategy model involve three finance-related state variables and three macroeconomic state variables—the macroeconomic variables are output, inflation and the monetary policy rate.

It might seem odd to describe something as complicated as the term structure of interest rates with only a few state variables. Academic and practitioner research has repeatedly demonstrated that, generally, the majority of the variance in interest rate movements is well described by three or four factors.[59] These results provide confidence that working with a low-dimensional state-variable system is reasonable. Consequently, it is rare to see a term-structure model with more than three latent and/or—in the new stream of joint macro-finance term-structure models—two or three macroeconomic state variables.

The second assumption involves the dynamics of the state-variable vector; in other words, how the state variable move together through time. There are a variety of possible assumptions. The most common is to use

[57]Different historical data periods have also been tested, with results being fairly insensitive to the time period used.

[58]A number of Bank of Canada working papers examine alternatives approaches for addressing this statistical and mathematical challenge. See, for example, Bolder (2001, 2006), and Bolder and Liu (2007) for a more detailed description and quantitative comparison of these models.

[59]See, for example, Litterman and Scheinkman (1991).

a vector autoregression (VAR) process to describe the evolution of the state-variable vector. The reason for its popularity is its stability and flexibility. This is not to say, however, that this is the only approach to describing state-variable dynamics. In particular, this is a rich area of research with a wide range of more complex mapping specifications. None of the current stochastic approaches currently implemented in the debt-strategy model considers these complications.

The third, and final, assumption relates to the mapping between the state variables and the term structure of interest rates. If you assume that the term structure is a function of some set of state variables, then you need a way to transform (or map) a given set of state variables into the term structure. This can, in principle, be quite *ad hoc*, since any function that maps the state vector into a collection of zero-coupon interest rates would do the job. An arbitrary mapping can, however, be somewhat dangerous, since it can lead to an interest rate system that permits arbitrage. Building a system upon a model that frequently allows for arbitrage opportunities is not desirable. Having said that, if a model has other positive properties, but permits a small probability of arbitrage opportunities, then it might *not* be a big problem.[60] In particular, if one is trying—as we are in the sovereign-debt management setting—to understand interest rate dynamics, then the permission of a small number of arbitrage opportunities may not be so problematic.

The majority of the extant models ensure that the mapping excludes arbitrage opportunities. The consequence is a number of restrictions on the nature of the state-variable dynamics and the corresponding form of the mapping. The most common model is the so-called affine model, where the yield curve is a linear—the fancy term is exponential-affine—function of the state variables.[61] There are, of course, other no-arbitrage mappings, but they find application in the pricing of contingent claims (i.e., options) and are not particularly relevant to the debt-strategy model.[62]

Another, more recent approach has suggested a mapping that is *not* motivated by no-arbitrage considerations.[63] In this setting, zero-coupon interest rates are described as a linear combination of the state variables where the coefficients are a collection of mathematical functions called Laguerre polynomials. Bolder (2006) extends this idea to consider alternative mappings including orthogonalized-exponential and Fourier-series function coefficients. These three approaches are termed Nelson-Siegel, exponential-spline and Fourier-series models, respectively. By virtue of the lack of no-arbitrage restrictions, these models operate under substantially fewer restrictions than the previously mentioned affine models. Moreover, there is a reasonable amount of empirical evidence suggesting that these models outperform the no-arbitrage models in their ability to generate out-of-sample forecasts and predict excess holding-period returns.

As previously indicated, we have implemented five models in the debt-strategy model. It is natural to ask how the three fundamental term-structure model assumptions apply to these models. On the first assumption, each of these models makes quite similar assumptions regarding the state variables. For the second assumption, each of the models assumes that the state-variable dynamics follow a VAR specification. The real difference between the models arises in the third assumption. The first three models—Nelson-Siegel, exponential-spline and Fourier-series approaches—use an empirical mapping between the state variables and the term structure. The fourth approach, called the OLS model, assumes a simple linear relationship between the state variables and the yield curve. None of these four models imposes no-arbitrage constraints. The fifth and final approach, termed the observed-affine model, has a complex mapping that ensures the absence of arbitrage.

[60] The thinking here is analogous to the negative interest rates that occur with positive probability in Gaussian affine term-structure models. Everyone can agree that negative nominal interest rates are a fairly unreasonable model output, but a small probabililty of negative interest rates has not stopped these models from widespread application.

[61] Dai and Singleton (2000) and Duffie, Filipovic, and Schachermayer (2003) are two excellent references for this very popular class of term-structure models. Leippold and Wu (2000) introduce the idea of an exponential-quadratic mapping between pure-discount bond prices and the state-variable vector leading to the quadratic term-structure approach. Flesaker and Hughston (1996) and Cairns (2004) consider a rather more complex integral expression for the price of a pure-discount bond as a function of the state variables; this has been termed the positive-interest rate model.

[62] One popular example is the well-known HJM-framework proposed by Heath, Jarrow, and Morton (1992).

[63] This work originated with Diebold and Li (2003).

To summarize, a term-structure model has three components: a collection of observable and/or unobservable state variables, a description of the dynamics of these state variables, and a mapping between these state variables and the term structure of interest rates. The mapping can either be theoretically motivated and constructed so as to avoid arbitrage opportunities, or constructed solely based upon empirical considerations.

7.2 Sensitivity analysis

The simplest way to understand the sensitivity of the debt-strategy model results is to merely run the model with different underlying stochastic approaches and compare the results. This is precisely what we plan to do. The first question that arises is "what financing strategy should we use?" The best answer is "all of them." This approach, however, presents rather severe computational and presentational challenges. For this reason, we perform the sensitivity analysis in two parts. First, we compare the results of alternative simulations with a *single* financing strategy consisting of an equal weight in each instrument. This permits us to focus on the differences in a highly detailed manner. In the second vein of analysis, we use the optimization module described in Chapter 5. Specifically, we examine how the optimal portfolio weights vary, for two relatively simple objective functions, as we alter the statistical model.

7.2.1 A brief statistical summary

Before we begin to examine the various aspects of the debt-charge and budgetary distributions, it makes sense to examine a few features of the different statistical models. As previously mentioned, we have five separate statistical models for the description of joint yield-curve and macroeconomic dynamics: the Nelson-Siegel (NS), exponential-spline (ES), Fourier-series (FS), ordinary least-squares (OLS) and observed-affine (OA) models. Figure 22 illustrates a number of important summary statistics arising from the simulation of these models. These statistics are derived from 10,000 simulations of each model, where each approach is estimated to exactly the same data period.

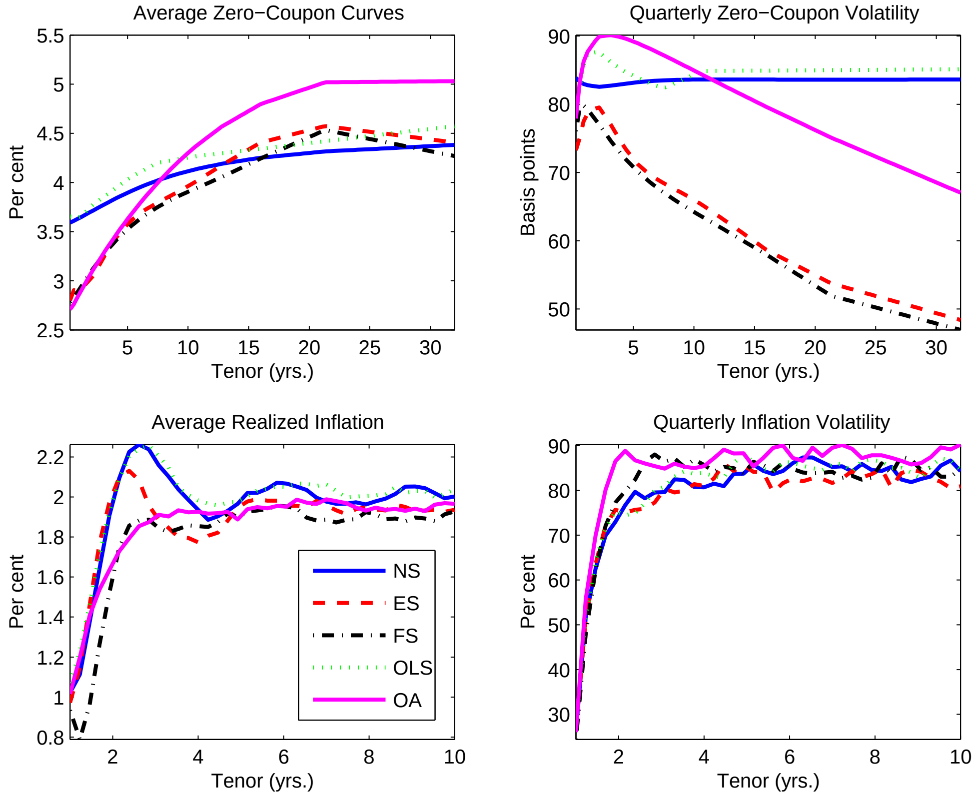
The top-left quadrant of Figure 22 illustrates the average shape, across the 10,000 simulations, of the zero-coupon term structure for each of our five models. Observe that the average exponential-spline, Fourier-series and observed-affine zero-coupon curve is approximately 50 basis points less at the short end of the curve relative to the Nelson-Siegel and OLS models. Each of the curves, with the exception of the observed-affine model, converges to an average long-term rate in-between 4 and 4.5 per cent. The observed affine model, however, generates the steepest curve with almost 200 basis points between 3-month and 30-year zero-coupon rates.

The average volatility of the zero-coupon curve, as evidenced in the top-right quadrant of Figure 22, also differs fairly substantially among the various models. The Nelson-Siegel and OLS models generate relatively flat quarterly zero-coupon curve volatility across all tenors of around 85 basis points. The exponential-spline and Fourier-series models, conversely, begin at approximately 80 basis points at the short end, but fall to approximately 50 basis points at the 30-year tenor. The observed-affine model volatility has a hump-shaped form that increases to 90 basis points at five years and then gradually decreases to about 70 basis points at the 30-year tenor.

The bottom two quadrants illustrate the mean and standard deviation of the realized quarterly inflation volatility in the model. In all models, the average realized inflation appears to vary tightly around the 2 per cent level. Moreover, the quarterly volatility of inflation is approximately 80 basis points. We can conclude, therefore, that each of the models appears to provide a similar description of inflation.

Figure 23 considers the correlation between key points on the zero-coupon curve: these include 3-month, 6-month, 1-year, 2-year, 5-year, 10-year and 30-year rates. These are contour plots of the average correlation matrices for these rates across each of our five models, as well as the actual correlation of the observed rates. Interpreting contour plots takes a bit of practice. Generally, the more pink the colour, the greater the correlation. Recall that the diagonal of a correlation matrix—in this case, a line from the top-left corner to the bottom-right

Figure 22: **Some Key Model Summary Statistics**: This figure illustrates a number of key summary statistics for our five alternative stochastic models. The principal focus is on interest rates and inflation, although these models also describe output. The upper two quadrants include the average shape of the term structure of zero coupon rates as well as the average volatility of the zero-coupon curve. The lower two quadrants demonstrate the average expected inflation and the average volatility of inflation.

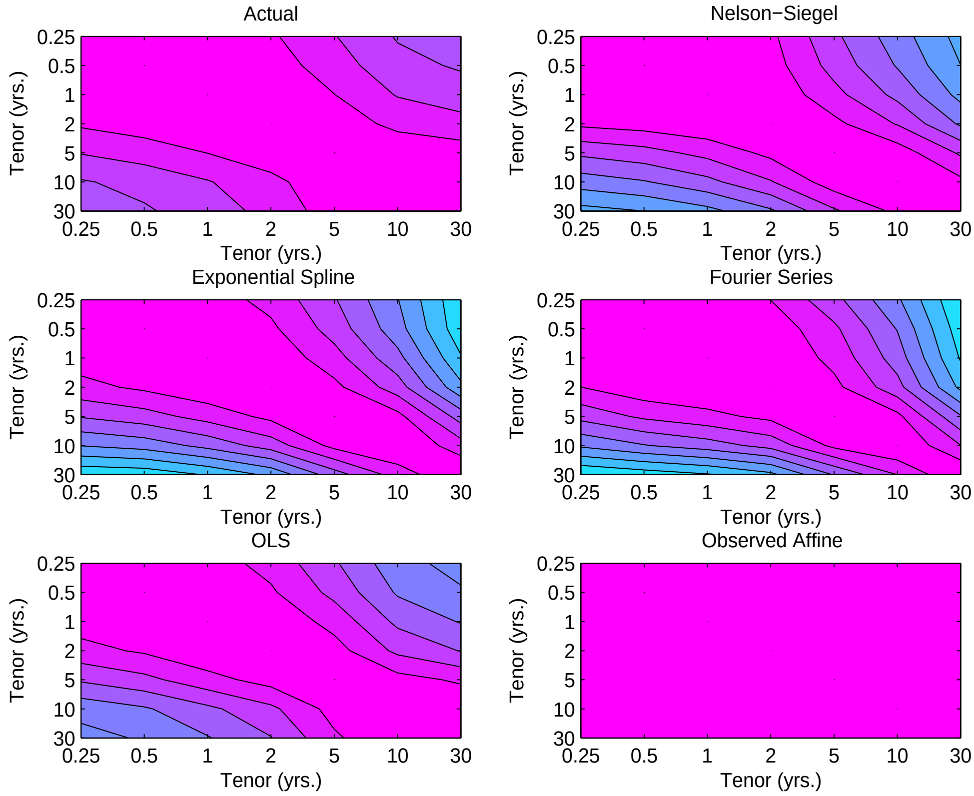


corner—is one. This represents the correlation of each rate with itself. We see, for example, that the diagonal and slightly off-diagonal points are completely light pink, representing a strong correlation with adjacent points. As we move to the corners, we observe the correlation with a given zero-coupon rate with more distant rates. The light blue, conversely, represents a correlation of less than about 30 per cent.

What can we conclude from Figure 23? First, we see from the actual data that there is extremely high correlation between all interest rates. All of the models, with the exception of the observed affine model, appear to underestimate the correlation between distant rates—this is evidenced by the greater amount of light blue in these graphs as compared to the top-left corner. The observed-affine model, interestingly, appears to suggest a correlation among all rates of greater than 90 per cent. The final point to note is that Nelson-Siegel, exponential-spline and Fourier-series models also tend to underestimate the correlation between 5-, 10- and 30-year zero-coupon rates. The actual data suggest that this correlation is about 0.99, whereas these three models generate a correlation of at least 0.10 less than this value.

In short, the models, estimated with the same data, generate different average yield-curve shapes, zero-coupon volatility and correlation structures. The macroeconomic data generated by each of these models, however, are quite similar. We next examine the impact these differences might have on the debt-strategy model results.

Figure 23: **Zero-Coupon Correlation Diagrams**: This figure includes contour plots of the correlation matrices of key points on the zero-coupon term structure for each of our five models. The more pink the colour, the greater the correlation. We see, for example, that the light pink represents a correlation of greater than 0.90. The light blue, conversely, represents a correlation of less than about 30 per cent. Recall that the diagonal of a correlation matrix—in this case a line from the top-left corner to the bottom-right corner—is one. This represents the correlation of each rate with itself. As we move to the corners we observe the correlation with a given zero-coupon rate with more distant rates. "Actual" uses the data period January 1994 to February 2007, the same as the data-generating period for all the models.



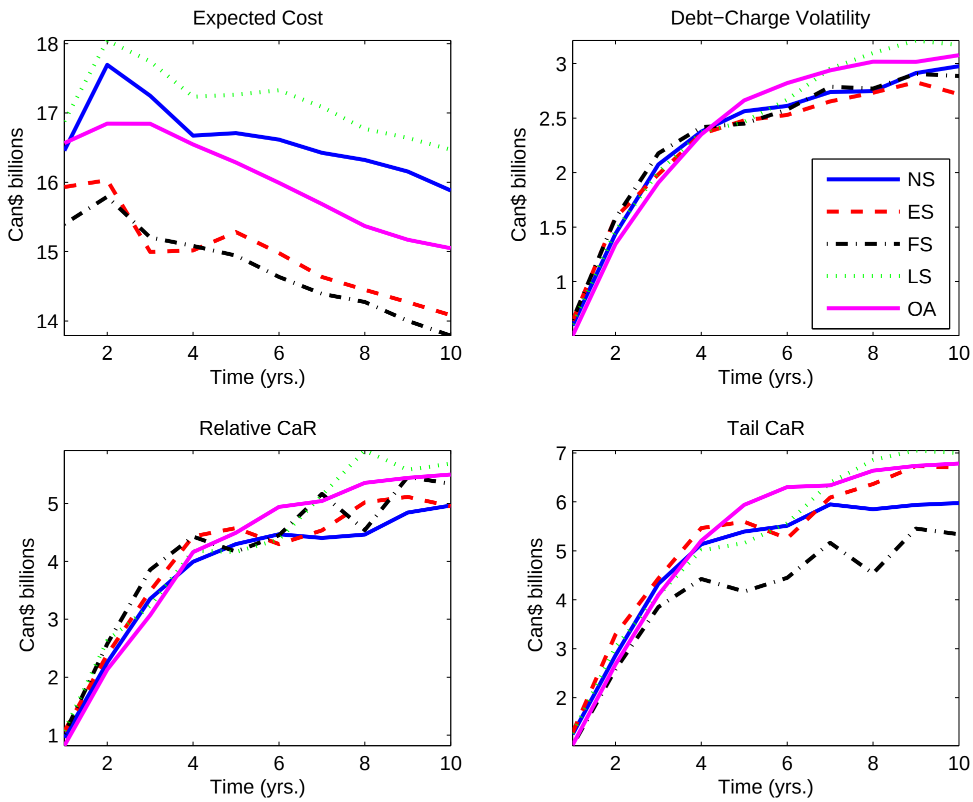
7.2.2 An equally weighted portfolio

As previously mentioned, the first component of our sensitivity analysis compares the results for alternative simulations of a financing strategy associated with a single equally weighted portfolio. Figure 24 outlines—for each of our five models run with 10,000 simulations—the associated expected debt-service charges, the debt-service-charge volatility, the relative CaR and the tail CaR. Each of these measures relates to the debt-service charges.

The top-left quadrant of Figure 24 provides the expected debt charges for each model across the 10-year simulation horizon. Observe that the expected debt-service charges for the Nelson-Siegel and OLS models dominate the other models across the entire time period. This is due to the fact that these models suggest a flatter average yield curve, which corresponds to more expensive short-term debt. The steeper curves generated by the exponential-spline and Fourier-series models lead to lower expected debt charges. The observed-affine model, with its more expensive long-term debt, lies between these two extremes; although, by virture of its higher long-term interest rates, the observed-affine average debt-service charges are closer to those generated by the Nelson-Siegel and OLS models.

The top-right quadrant of Figure 24 describes the volatility, or standard deviation, of the debt-service charges over each of the 10 years in the simulation. To understand these results, it is important to recall a key fact about

Figure 24: **Debt-Service-Charge Comparison**: This figure illustrates a number of debt-service-charge related risk and cost measures for our five alternative stochastic models. For an equally weighted portfolio—among 3-, 6- and 12-month treasury bills; 2-, 5-, 10- and 30-year nominal bonds; and 30-year inflation-linked bonds—this figure outlines the expected debt-service charges, the debt-service-charge volatility, the relative CaR and the tail CaR.



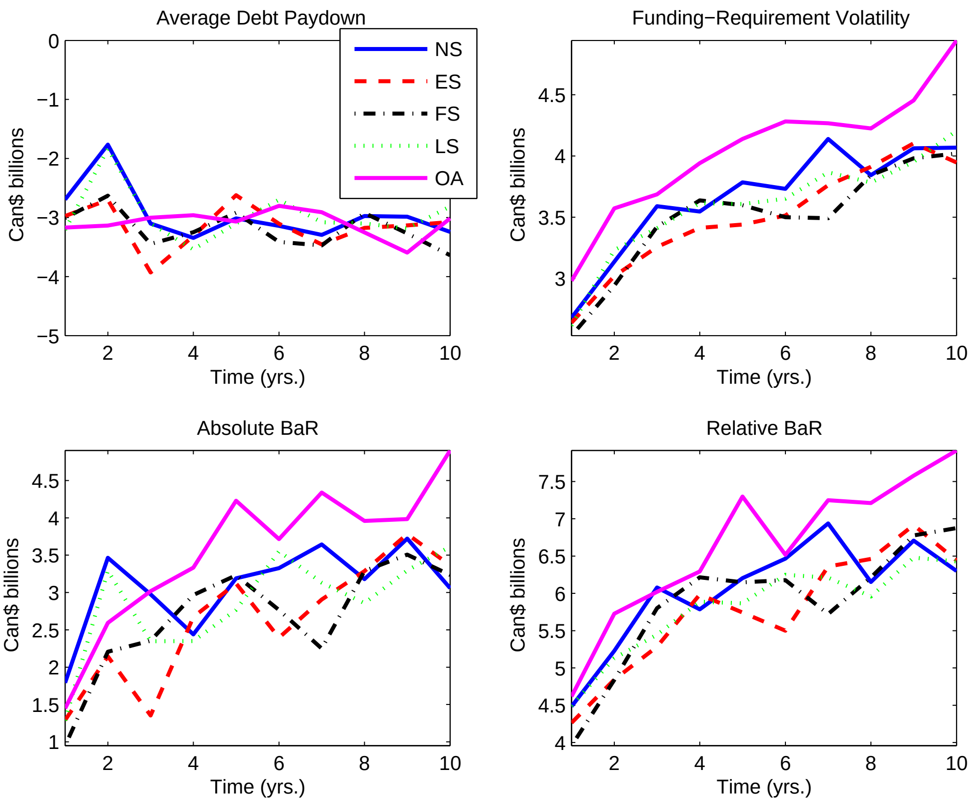
the issuance of a debt portfolio. In any given year, and for almost any given financing strategy, the issuance of short-term debt substantially outweighs the issuance of long-term debt. Take the simple example of a 100-billion portfolio consisting of equal parts 3-month treasury bills and 30-year bonds. The 50-billion stock of 3-month treasury bills is refinanced four times per year generating about 200 billion of annual issuance. Only about $\frac{1}{30}$th of the 50-billion stock of 30-year bonds is refinanced in each year, generating about 1.67 billion of annual issuance. Thus, the amount of treasury bill issuance in this admittedly extreme example is approximately 120 times more than the corresponding 30-year issuance. In our equally weighted portfolio example, however, the amount of short-term issuance is more than 20 times greater than that of long-term issuance.[64] The preponderance of short-term issuance in a given year implies that debt-service-charge volatility is primarily influenced by short-term interest rate volatility.

Returning to the top-right quadrant of Figure 24, therefore, we see that there is relatively little difference between the alternative models. This is evidenced by the fact that each of the models generates similar short-term interest rate volatility values. The observed-affine and OLS models do have a slight hump in short-term volatility, which is illustrated by a slightly higher debt-charge volatility for these models. These results carry through to the relative and tail CaR figures illustrated in the bottom two quadrants of Figure 24. In particular, we see that the observed-affine and OLS models are somewhat more risky, particularly in the latter years of the simulation horizon.

Figure 25 examines a number of summary measures of budgetary risk for our equally weighted portfolio across

[64]For the purpose of this calculation, we treat treasury bills and 2-year bonds as short-term debt.

Figure 25: **Budget and Funding-Requirement Comparison**: This figure illustrates a number of debt-service-charge related risk and cost measures for our five alternative stochastic models. For an equally weighted portfolio—among 3-, 6- and 12-month treasury bills; 2-, 5-, 10- and 30-year nominal bonds; and 30-year inflation-linked bonds—this figure outlines the average debt paydown, the funding-requirement volatility, the absolute BaR and the relative BaR.



the five models. The top-left quadrant shows, for illustration, the ability of the models to attain a debt paydown of Can\$3 billion per fiscal year. We observe that the five models are roughly equal in this regard. When we turn to the top-right quadrant, however, we do observe an interesting result in the funding-requirement volatility. In particular, the observed-affine model generates rather more funding-requirement volatility than the other four models. This result also shows up in both the relative and absolute BaR figures in the bottom two quadrants of Figure 25.

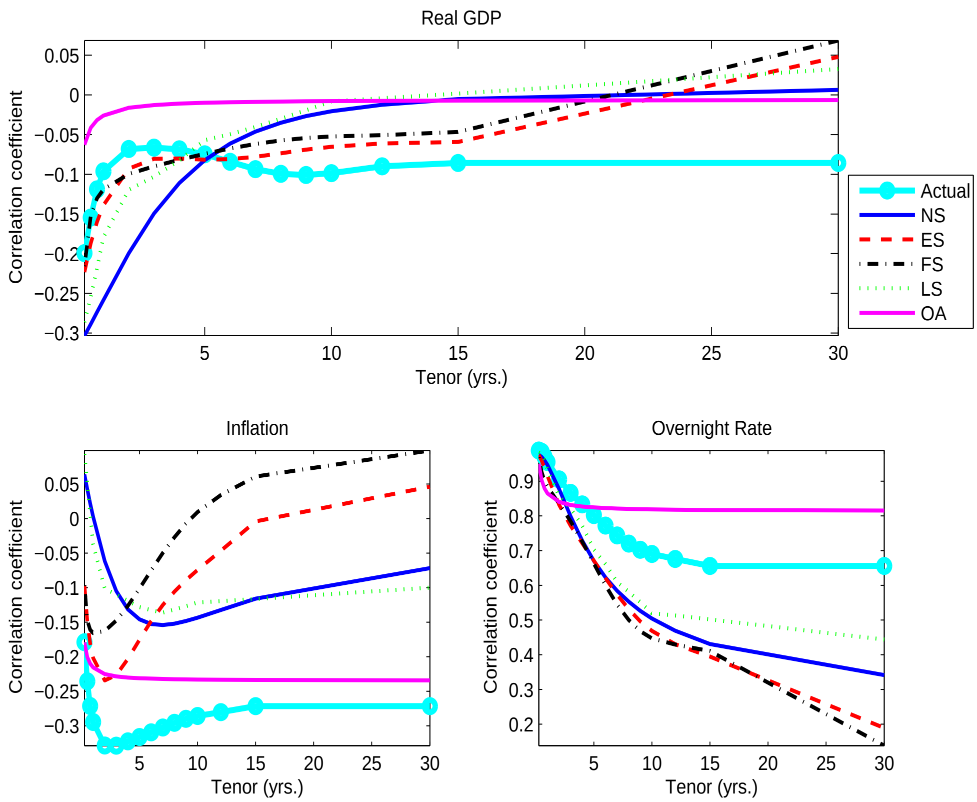
What is driving the larger budgetary volatility associated with the observed-affine model? Budgetary volatility is influenced by the correlation between debt charges and the macroeconomic variables that govern the primary balance; these are principally real GDP and inflation.[65] To permit this budgetary diversification, this correlation should be positive or, at least, close to zero. Figure 26 demonstrates the correlation structure suggested by the five different models relative to that suggested by the data. The first thing to notice is that the data suggest a modestly negative correlation between the term structure and both inflation and real GDP.[66] This would, therefore, appear to suggest a relatively limited scope for diversification.

We also note that the observed-affine model generates a flat correlation structure across all tenors. This is related to the correlation structure described in Figure 23. Specifically, the observed-affine model appears to overestimate the correlation between different points in the term structure—it essentially assumes that the entire

[65] See Chapter 6 for more detail. Note that errors in the forecasting of macroeconomic variables also contribute to budgetary volatility. There does not, however, appear to be much difference in the macroeconomic forecasting ability of the various models.

[66] This issue will be addressed in the next chapter.

Figure 26: **Term-Structure and Macroeconomic Correlation**: This figure illustrates the average contemporaneous correlation between the three principal macroeconomic variables—real GDP, inflation and the overnight rate—and a number of key points on the zero-coupon term structure for each of the five alternative models. We also include the actual contemporaneous correlation observed in the data.



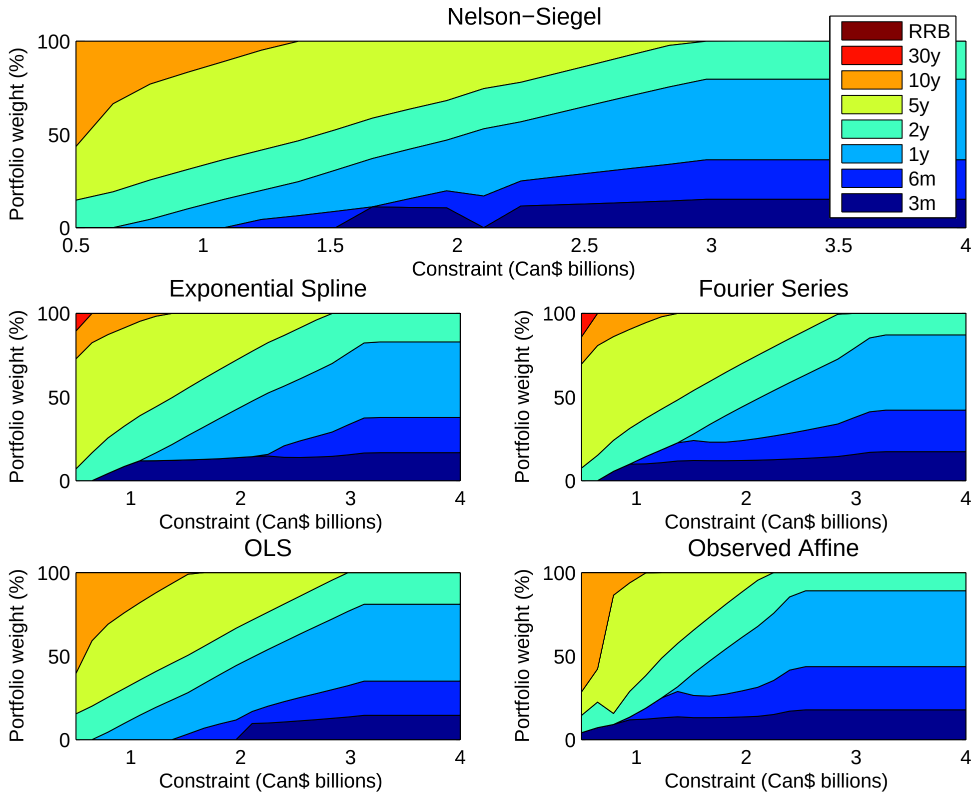
yield curve moves together. What appears to drive the greater budgetary volatility of the observed-affine model, therefore, is the flat and negative contemporaneous correlation with inflation. This negative correlation—which is greater than the other models—reduces the capacity of the debt charges to offset budgetary volatility. What is particularly interesting is that, in this one instance, the observed-affine model is closer to the true data than the other models.

7.2.3 Portfolio optimization

While the various models may generate significant differences in a number of key risk and cost measures, perhaps the most important question is how they differ in terms of the optimal portfolios that they recommend. To this end, we have used the optimization tool to examine how the portfolio weights vary for two relatively simple objective functions. As in previous chapters, we use 2,000 alternative financing strategies—with 10,000 random realizations of our stochastic model—to train our approximation algorithm. In this analysis, however, we repeat this sequence of computations for each of our five alternative stochastic models.

Our first objective function involves the minimization of the average percentage debt charges, over the 10-year simulation horizon, subject to a range of constraints on the conditional debt-service-charge volatility. The optimal portfolio weights, for conditional debt-charge volatility ranging from Can$500 million to 4 billion, are summarized in Figure 27. The first thing to note is that for a constraint beyond about Can$3 billion, the optimal portfolios are virtually identical for each stochastic model. We also observe that for highly constrained

Figure 27: **Some Optimal Portfolio Weights**: This figure illustrates the optimal portfolio weights—for each of the five stochastic models—when the optimizer seeks to minimize total discounted debt charges over the 10-year simulation horizon subject to a range of constraints on conditional debt-charge volatility.



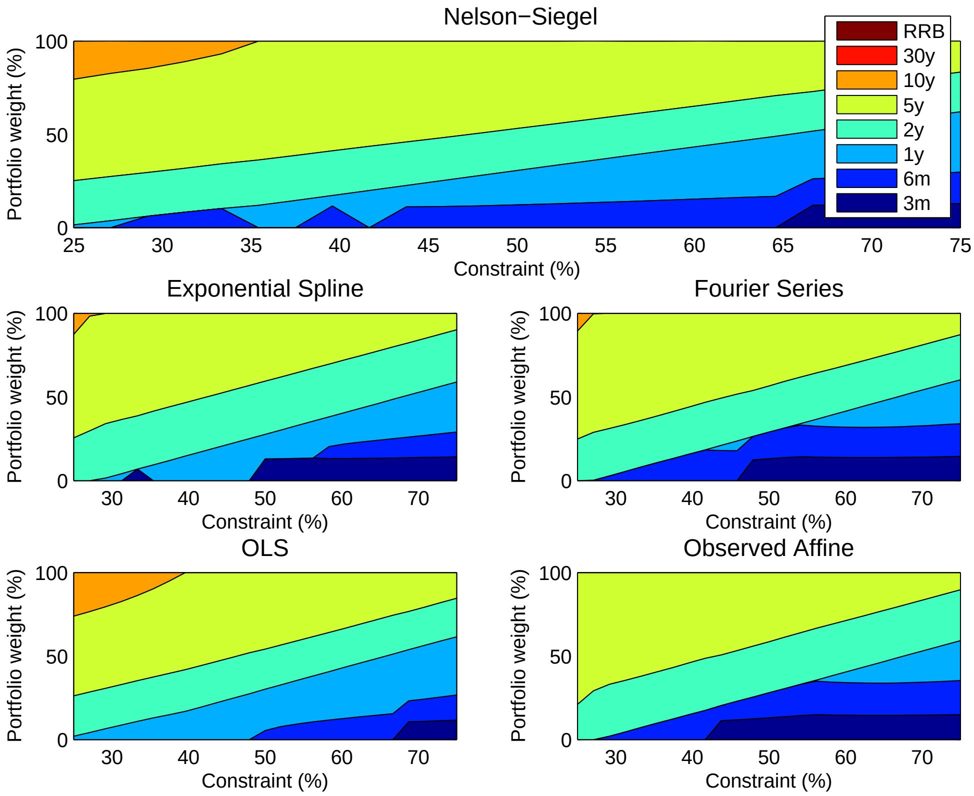
optimizations (i.e., around Can$500–600 million), the exponential-spline and Fourier series models, unlike the other approaches, recommend a small, although non-zero, allocation to 30-year nominal bonds. We suspect that this is due to the decreasing yield-curve volatility generated by these models and the relatively small cost differences between 10- and 30-year debt in these models. As well, the observed-affine model reduces the reliance on 10-year debt more quickly than the Nelson-Siegel and OLS models, given its relatively higher assessment of the cost. Finally, RRBs are not present for any of the models.

The second optimization, described in Figure 28, seeks again to minimize the average percentage debt charges, albeit with a restriction on the refixing share of debt. Here we observe that only the Nelson-Siegel and OLS models recommend a significant allocation to 10-year nominal bonds for lower refixing share of debt constraints. We suspect that this is due to the relatively flat yield curves generated by these models, which suggests that there are fewer cost savings to be gained by issuing short-term debt. For the same reason, we also observe that the exponential-spline, the Fourier-series and the observed-affine models all recommend 3- and 6-month treasury bill allocations more quickly as we increase the refixing share of debt. The steeper short-term yield curves forecast by these models make this more cost effective.

7.3 Conclusion

The important conclusion from this analysis is that different models of the stochastic environment, estimated with the same data, can yield rather different results. The differences show up in the average shape of the yield curve, the volatility of the yield curve, and correlation structure among and between the yield curve and the

Figure 28: **Some More Optimal Portfolio Weights**: This figure illustrates the optimal portfolio weights—for each of the five stochastic models—when the optimizer seeks to minimize total discounted debt charges over the 10-year simulation horizon subject to a range of constraints on the refixing share of debt.



macroeconomic variables. These differences manifest themselves in the form of differing forecasts of key risk and cost characteristics for a given financing strategy. In this analysis, we found that in some cases these differences were small, whereas in others they were important. This finding applies to both the examination of a single financing strategy and the optimization across the range of all admissible financing strategies. The bottom line, therefore, is that, in performing debt-strategy analysis, we cannot afford to ignore model risk.

8 The Macroeconomy

This chapter examines the interaction of macroeconomic variables and the term structure of interest rates, as well as the sensitivity of the model results to macroeconomic and financial shocks.

In the previous chapter, we examined how alternative models describing the macroeconomy and the term structure of interest rates can lead to different results. In this chapter, we focus on a single model—the Nelson-Siegel model—and walk through a number of detailed model diagnostics to better understand the interaction between the key components of the macroeconomy and the interest rate environment. Such analysis is important to verify that the interaction between key variables is both correct and consistent. As previously discussed, the interaction between the macroeconomy and interest rates will have important implications for the determination of the government's funding requirement and the relative attractiveness of inflation-linked debt. These diagnostics are also important to ensure that the sensitivity analysis can reasonably be performed in the context of this model. In particular, we want to examine how an initial shock to output, inflation or the shape of the yield curve might impact the portfolio allocations. Such analysis is useful only if the interactions between the key stochastic variables are correctly specified. Our diagnostic efforts, therefore, are designed to understand these interactions.

This chapter is organized into four sections. In the first section, we focus on model diagnostics. The principal diagnostic tool, given the reduced-form nature of our model, is the impulse-response function. This section is highly detailed to provide comfort that due diligence has been performed on the model, but is not required for analyzing the impact of macroeconomic changes on model results. In the second section, we turn to examine how an equally weighted portfolio reacts to shocks to key variables. The objective in that section is to understand how the shocks to the key stochastic variables translate into different risk and cost characteristics for a given financing strategy. The third section will provide optimization results for alternative long-term values in the model. Finally, the fourth section will make some concluding remarks.

8.1 The stochastic model

In an effort to mitigate model risk, as was described in Chapter 7, the debt-strategy model employs *five* alternative models to describe the future evolution of macroeconomic and interest rate variables. These models have rather different foundations and various implications for the dynamics of the stochastic environment. Fortunately, however, the interactions between macroeconomic and interest rate variables in each of the models are quite similar; this implies that examining the interactions within the context of the Nelson-Siegel model should give us a reasonably good idea of the interactions for all of the models. In particular, the joint dynamics of the macroeconomy and interest rates are represented in a VAR, which means that the current values of the stochastic variables depend on their past values. Current output, for example, depends on past output as well as past inflation, monetary policy and yield-curve variables. The advantage of VARs is that they provide a rich description of the dependence between a collection of variables. In other words, such a model permits each variable to have an influence on every other variable in the system.

The macroeconomic variables included in our joint model of the macroeconomy and the term structure of interest rates include the output gap, the overnight rate and the annual rate of change in the *core* consumer price index (CPI).[67] These factors represent the level of real economic activity relative to potential, the monetary policy instrument and the rate of price inflation, respectively. We use this set of variables since it is considered to be the minimum set of factors needed to capture basic macroeconomic dynamics. The growth in potential output is included as an exogenous variable to the system.[68] It is included to infer real GDP growth, which is instrumental in the estimation of the debt model's primary balance.[69] We also include total CPI exogenously,

[67]The output-gap series used in our analysis is the estimate produced by the Bank of Canada and is converted from quarterly to monthly frequency using a cubic spline.

[68]An exogenous variable influences the other variables in the system, but is *not* directly influenced by the other variables.

[69]More details on this are provided in Chapter 6.

since this quantity is required for the pricing of inflation-linked debt. The term-structure variables are the level, slope and curvature of the zero-coupon yield curve. An extensive literature shows that these three factors are sufficient to explain more than 95 per cent of term-structure variability. Armed with this set of variables, our stochastic model is able to generate key macroeconomic shocks to inflation, interest rates, aggregate demand, technology and supply.

One disadvantage of the VAR approach is that the relatively large number of reduced-form model parameters create a general difficulty with parameter interpretation. Interpreting the sign and statistical significance of the parameters for a VAR of even moderate dimension is a very difficult, and often impossible, task. Moreover, R^2 measures are not terribly helpful. For this reason, other tools are used to gauge the usefulness of a VAR model. The first tool involves the examination of the out-of-sample forecasting ability of a given VAR specification. Since the ability of these models to forecast future interest rate outcomes has been extensively addressed in Bolder and Liu (2007), we will not cover this diagnostic approach. Another tool for the examination of VARs is the impulse-response function. An impulse-response function traces the effect of a one standard-deviation shock to one of the error terms of the VAR on current and future values of the endogenous variables.[70] In other words, imagine that there was a one standard-deviation surprise in the current value of inflation: the impulse-response function would measure the corresponding impact, in the absence of any additional noise, on the other variables in the VAR.

In short, impulse-response functions are invaluable in understanding the structure of a VAR model, because they succinctly describe the model's dependence structure. In what follows, we will examine the key impulse-response functions for our standard VAR model. Before turning to the VAR impulse-response functions, however, it will be useful to outline the dynamics of a traditional small-scale structural macroeconomic model. This will provide a reference point for the analysis of the VAR dynamics.

Typical small macroeconomic models used for monetary policy analysis—often termed New-Keynesian models—generally include *three* key variables: the output gap, inflation and the monetary policy rate. In these models, current output depends positively on past/expected output and negatively on the real interest rate. Current inflation is determined by an expectations-augmented Phillips curve, whereas inflation is positively influenced by past/expected inflation and by the size of the current output gap. The monetary policy rate is determined using an estimated or calibrated Taylor rule, in which the short-term interest is set according to current/expected inflation and the output gap. In such a model, an aggregate demand shock would increase the output gap, and in turn raise inflation. This would cause the monetary authorities to increase the policy rate, which would eventually close the output gap and bring inflation back in line with their target.

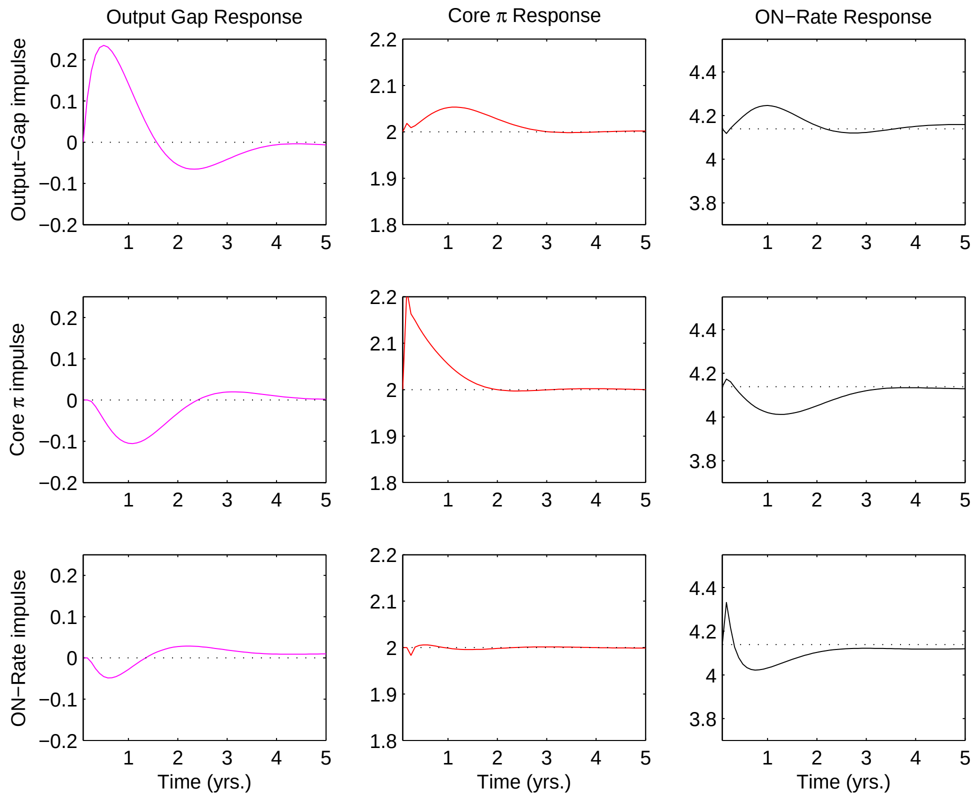
We can now examine the impulse-response functions to better understand our VAR dynamics. Figure 29 describes the impact (response) on the macroeconomic variables to a one standard-deviation shock (impulse) to the macroeconomic variables. The first row, for example, illustrates the impact on the output gap, the core inflation rate, and the ON-rate for a shock to the output gap. We observe that the overnight rate displays a positive response to a rise in the output gap, which is consistent with a typical monetary policy reaction function. In contrast, when examining the second row of Figure 29, we observe that the overnight rate falls with a positive shock to inflation. This is a seemingly counterintuitive finding.

However, the impulse-response functions indicate that a positive shock to inflation has the typical response of reducing output.[71] Hence, the negative response of the overnight rate to a positive inflation shock can be explained by the fact that output displays a greater degree of persistence over the estimation sample than inflation. That is, core inflation comes back to target more quickly than output. Indeed, the output gap displays

[70] The Cholesky decomposition is used to identify underlying structural shocks from the reduced-form shocks. Changing the ordering of the variables used for identification of the structural shocks in this approach will, generally, lead to different results. In this analysis, however, changing the ordering produced only minor differences in the impulse-response functions. The order presented in this analyis is level, curvature, slope, output gap, core inflation and the overnight rate.

[71] The negative response of the output gap to a positive shock to both inflation and the overnight rate is consistent with an aggregate demand shock in small macro models.

Figure 29: **Macro-Macro Impulse-Response Functions**: This figure illustrates the impulse-response functions of the macroeconomic variables for shocks to the macroeconomic variables. The first row demonstrates the response on the output gap, core inflation and the overnight rate to a shock to the output gap. The second row illustrates the response, for the same set of variables, to an inflation shock. The final row outlines the results for a shock to the overnight rate.



a high degree of persistence, taking about two years to return to its initial value following an aggregate demand shock.[72] Inflation, conversely, exhibits little persistence, falling back to its initial value following a price shock within 10 months. This observation suggests that the monetary authorities will react more to the fall in output than to the rise in inflation, because a stronger reaction is required to increase output than is required to decrease inflation.

A positive shock to the output gap raises inflation—which is consistent with the results of small monetary policy macroeconomic models—while shocks to the overnight rate and to the yield-curve factors have little impact. In fact, statistical tests indicate that inflation can be treated as an exogenous variable to the VAR system.[73] This is consistent with studies that have found that the inflation process, since the introduction of inflation targeting in the early 1990s, is well forecast by a simple constant.[74] This result stems from the fact that the Bank of Canada has been generally successful in keeping inflation expectations well anchored and that, on average, shocks to inflation do not permanently impact expectations.

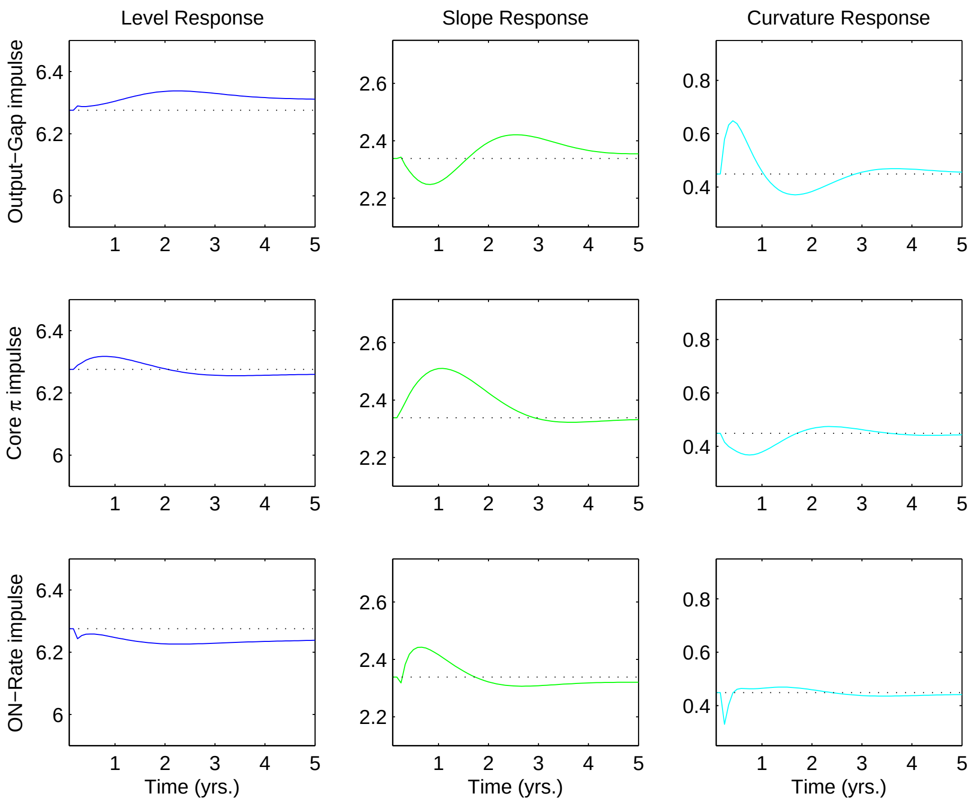
Figure 30 illustrates the response of the financial variables to a one standard-deviation shock to the macroeconomic variables. The first row outlines the impact on the level, slope and curvature of a shock to the output gap. As before, the second row of Figure 30 illustrates the response, for the same set of variables, to an infla-

[72]Incidentally, although it is not shown here, an exogenous shock to potential output growth leads to a decline in the output gap, consistent with a lagged adjustment of demand to supply shocks.

[73]We principally employ the so-called Wald exogeneity test. Loosely speaking, this test permits a multivariate examination of the statistical significance between the covariance terms in the endogenous variables of a linear system.

[74]See Demers (2003) for a more detailed discussion of this issue.

Figure 30: **Macro-Finance Impulse-Response Functions**: This figure illustrates the impulse-response functions of the financial variables for shocks to the macroeconomic variables. The first row demonstrates the response of the level, slope and curvature of the zero-coupon curve to a shock to the output gap. The second row illustrates the response, for the same set of variables, to an inflation shock. The final row outlines the results for a shock to the overnight rate.

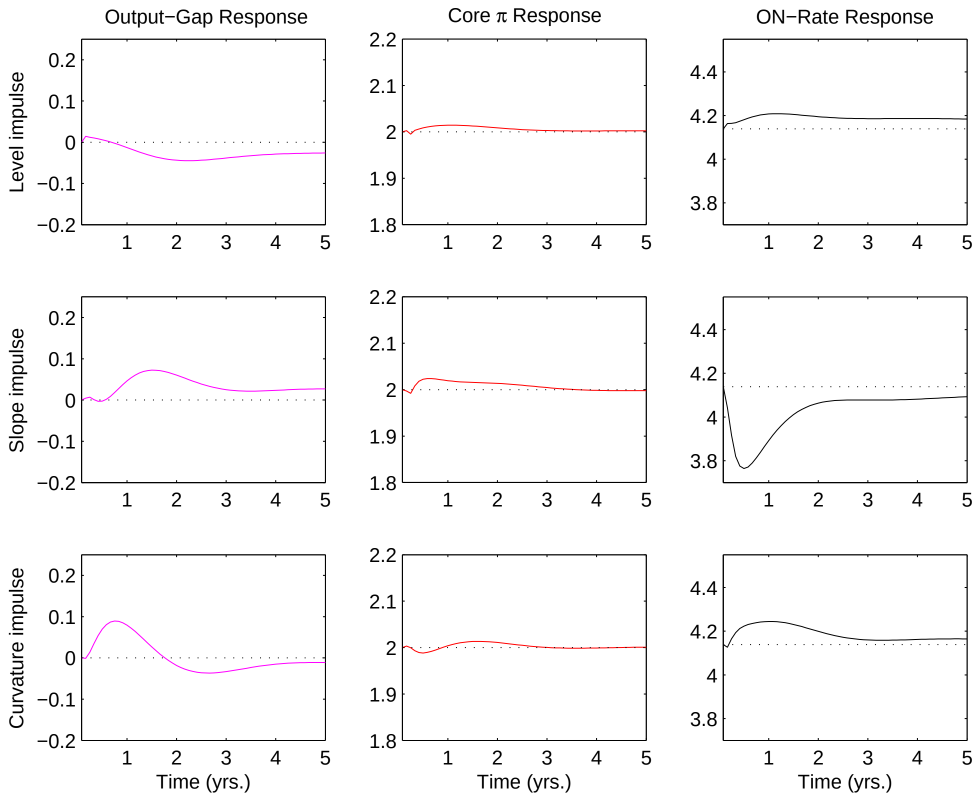


tion shock while the final row outlines the results for a shock to the overnight rate. The level factor is highly correlated with the nominal 10-year government bond yields as well as with measures of inflation. This suggests a relationship between the level of the yield curve and inflationary expectations. We observe in the first column of Figure 30, however, that shocks to core inflation, the output gap and the overnight rate appear to produce relatively modest responses in the level factor. Again, statistical tests indicate that the level factor could also be treated as an exogenous variable in the VAR system. This result is consistent with inflationary expectations being well anchored in Canada over the estimation period.

The slope factor exhibits a negative response to an increase in the overnight rate. What appears to be happening is that an increase in the output gap leads to an increase in the overnight rate (see Figure 29), which in turn leads to an increase in short-term interest rates and a subsequent decrease in the slope of the yield curve. Indeed, the overnight rate and the yield-curve slope are strongly negatively correlated. The reason is that with anchored inflationary expectations, we should expect to see increases (decreases) in the overnight rate associated with decreases (increases) in the slope of the yield curve. This is because the overnight rate and short-term interest rates are closely linked. Shocks to core inflation, therefore, generate decreases in the overnight rate, which trigger an increase in the slope of the yield curve.

Initially, the positive reaction of the slope to a positive shock to the overnight rate is a bit puzzling. Figure 29, however, demonstrates that a shock to the overnight rate persists only for a very short period and eventually leads to a decrease in the overnight rate—the consequence, therefore, is an increase in the yield-curve slope. We also observe that both the slope and curvature factors display a high degree of persistence, requiring more than

Figure 31: **Finance-Macro Impulse-Response Functions**: This figure illustrates the impulse-response functions of the macroeconomic variables for shocks to the financial variables. The first row demonstrates the response of the output gap, core inflation and the overnight rate to a shock to the level. The second row illustrates the response, for the same set of variables, to a slope shock. The final row outlines the results for a shock to curvature.



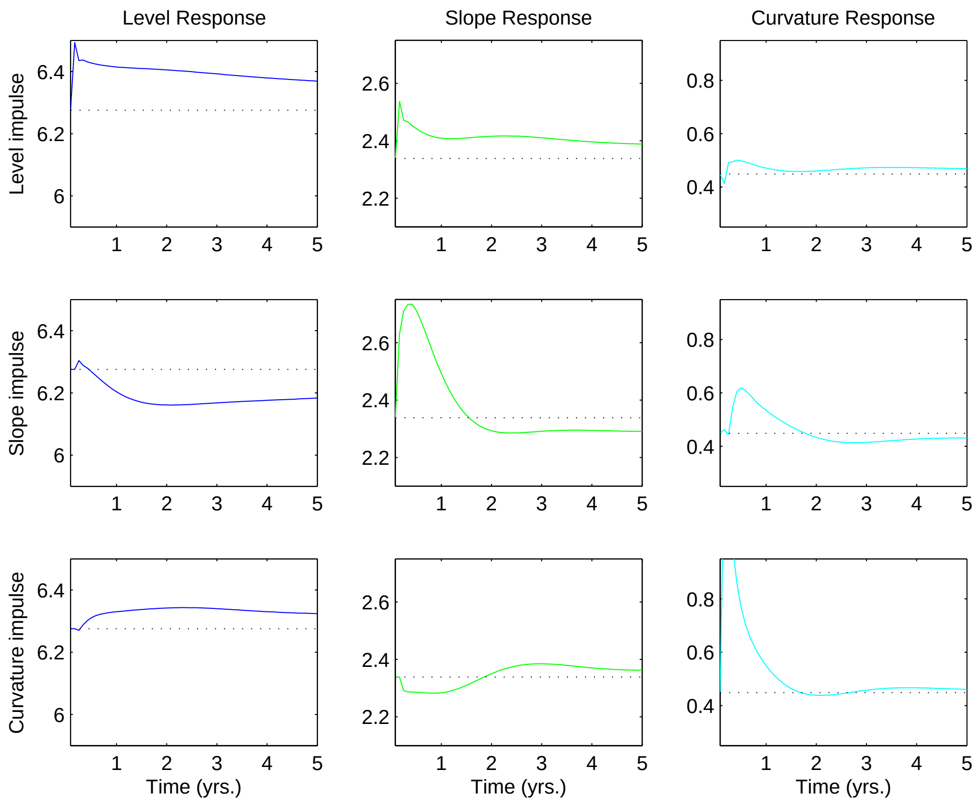
two years each to return to their initial values following an aggregate demand shock.

The yield-curve curvature factor is highly correlated with economic activity. Indeed, the contemporaneous correlation between real GDP and the curvature factor is slightly greater than 0.40. A shock to the output gap, therefore, leads to an increased curvature, while increases in core inflation, which dampen growth, negatively impact the curvature of the yield curve.

Figure 31 demonstrates the response of the macroeconomic variables to a one standard-deviation shock to the financial variables. Each row outlines the impact on the output gap, core inflation and the overnight rate of a one standard-deviation shock to the yield-curve level, slope and curvature, respectively. The level factor appears to have little impact on any of the macroeconomic variables. As we can see in Figure 32, a shock to the level factor is persistent and reasonably large, but it does not have a large impact on the macroeconomy. A positive shock to the slope variable, however, generates, as expected, a decrease in the overnight rate, which in turn leads to an increase in the output gap. Moreover, a positive shock to curvature raises the output gap. This is consistent with typical forward-looking macroeconomic models, where expectations of greater future economic activity and inflation lead to higher contemporaneous output and inflation.

Level, slope and curvature factors obtained from traditional statistical analysis are, by construction, independent. When these factors are extracted from the Nelson-Siegel model, however, we observe a strong contemporaneous correlation between the level and slope factors. In addition, the level and curvature factors demonstrate a modest positive contemporaneous correlation. Figure 32 shows the impulse-response functions of the financial variables for shocks to the financial variables. Again, each row demonstrates the response of the level, slope and

Figure 32: **Finance-Finance Impulse-Response Functions**: This figure illustrates the impulse-response functions of the financial variables for shocks to the financial variables. The first row demonstrates the response of the level, slope and curvature of the yield curve to a shock to the level. The second row illustrates the response, for the same set of variables, to a slope shock. The final row outlines the results for a shock to curvature.



curvature of the yield curve to a shock to the level, slope and curvature factors, respectively.

An increase in the level of the yield curve appears to generate a persistent increase in the slope of the yield curve, although it has little impact on the yield-curve curvature. These results seem to be driven by the previously mentioned correlation structure. Interestingly, a shock to the slope factor generates a lagged decrease in the level of the yield curve as well as a slight increase in the curvature. The slope returns back to its long-term mean within about 18 months and then actually falls below this value. Presumably, this is what is driving the fall in the level. Finally, a curvature shock generates a modest, but persistent, increase in the level of the yield curve and a slight decrease in the yield-curve slope.

The basic impulse-response functions, then, are generally consistent with the results of traditional small-scale structural macroeconomic models. The one exception is the response of the overnight rate to an inflationary shock, although this can be explained by the lack of persistence of inflation and the forward-looking nature of Canadian monetary policy. Nevertheless, we also estimated a structural VAR—in which a standard Taylor type rule was calibrated to explain the dynamics of the overnight rate—to address this issue. This model produced persistence in the inflation process, however, that was fundamentally inconsistent with the historical evidence. As a consequence of this distortion, we elected not to use the structural specification.

As a final point, we find that the macroeconomic variables have a significant influence on the term structure. Moreover, the term-structure factors also appear to have an important influence on the macroeconomic variables. While this is not entirely consistent with the findings in the literature, it does seem reasonable given the forward-looking nature of monetary policy in Canada and the expectational information contained in the term-structure factors.[75]

8.2 Impact of shocks on model results

We next focus on how shocks to our six principal stochastic variables impact the model results. This is, in fact, a natural extension of the analysis in the previous section. We are essentially asking "what happens to the risk and cost characteristics associated with a given financing strategy when the starting point of one of the stochastic variables is shocked?" At most points in this paper we begin the simulation of the stochastic system using the final datapoints in our historical time series.[76] In this analysis, however, we start from the long-term mean to provide a clean view of the interaction of the variables without mean reversion effects clouding the results. Using the long-term mean starting point, therefore, we again apply the idea of an impulse-response function with a one standard-deviation shock to each of the stochastic variables. We then run the simulation algorithm using these alternative starting points.

Let us consider an example. A shock to the output gap, using the results from the previous section, should, on average, lead to a slight increase in core inflation, the overnight rate and the yield-curve level, with a simultaneous decrease in the slope of the yield curve.[77] What impact, however, does this output-gap shock have on the risk and cost characteristics of a given financing strategy? Figure 33 addresses this question in part by demonstrating the impact on the zero-coupon curve of a one standard-deviation shock to the simulation's starting point for each of the key stochastic variables.

The top-left quadrant of Figure 33 describes the average distance between the unshocked curves across 10,000 simulations. The thick dashed blue line from the origin represents the results for an unshocked simulation. Observe that a shock to the level and curvature factors leads to the highest zero-coupon curves. While the level shock is self-explanatory, the positive and persistent increase of the level factor in response to a curvature shock appears to explain the increase in the expected zero-coupon curve. An output-gap shock also leads to a general increase in interest rates. This is consistent with the impulse-response analysis, insofar as the associated increase in the overnight rate and the yield-curve level should generate slightly higher zero-coupon curves.

A positive shock to the yield-curve slope appears to arise more through an decrease in short-term interest rates than through an increase in long-term rates. Indeed, longer-term rates appear to fall in reaction to a positive shock to the yield-curve slope. The consequence, therefore, is generally lower interest rates in response to a shock to the slope. A positive inflation shock appears to slightly decrease interest rates; this is consistent with the previous analysis, which indicated a negative output-gap and overnight-rate response to an inflationary shock. Finally, the overnight shock leads to a decrease in interest rates, although it does not appear to deviate substantially from the base scenario. This can be explained by Figure 29, where an initial shock to the overnight rate generates a decrease in output and a subsequent decrease in the overnight rate. The total result, therefore, is lower interest rates.

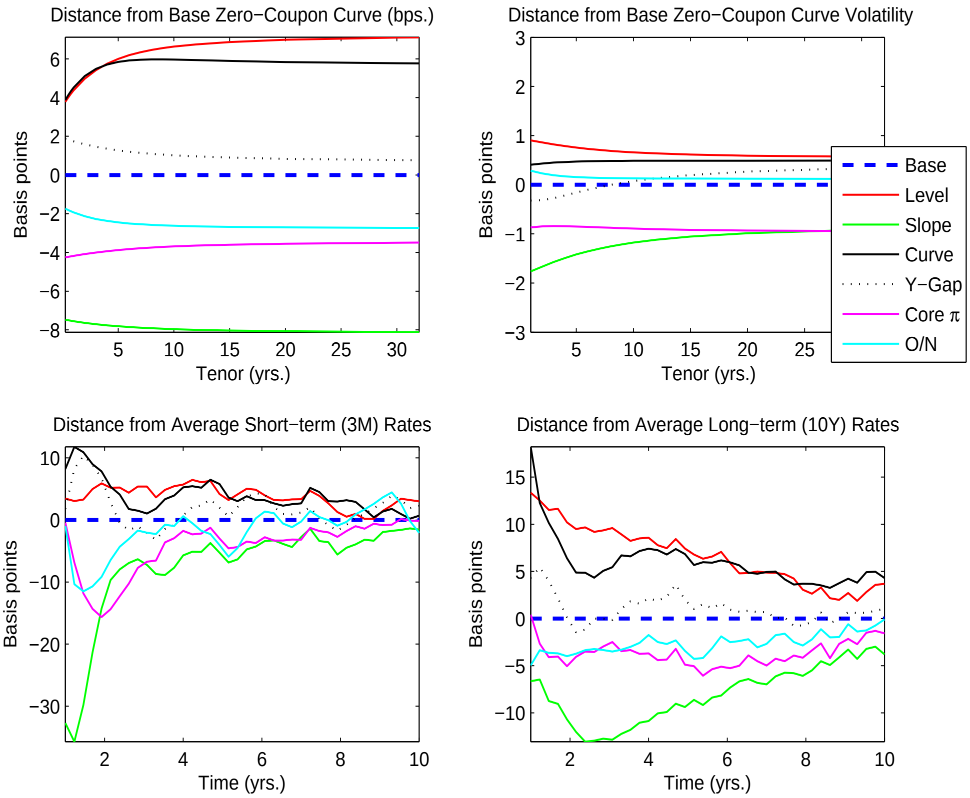
We also observe from Figure 33 that the standard deviation of the zero-coupon curve (see the top-right quadrant) does *not* substantially differ for the various shocks to our stochastic variables. None of the shocks leads to a change of more than five basis points from the base, unshocked scenario—considering that the overall zero-coupon volatility ranges from 80 to 120 basis points, this is not substantial. It is also not surprising, since only the starting points differ between the simulations. That is, the dynamics of the state variables do not change in response to a shock to the starting values. As such, there should be no change in the variability of zero-coupon outcomes. Moreover, the bottom two quadrants of Figure 33 indicate that the 3-month and 10-year

[75] The academic evidence is somewhat mixed. Some studies indicate a rather weak influence of macroeconomic variables in the term structure, while other studies demonstrate a stronger relationship.

[76] The data used in this analysis begin in November 1993 and terminate in February 2007. The starting point for the simulation of the stochastic system, therefore, would typically be February 2007.

[77] We say "on average" because the simulation, unlike the impulse-response analysis in the previous section, also includes the usual amount of noise.

Figure 33: **Impact of One Standard-Deviation Shocks on Term-Structure Outcomes**: This figure illustrates the impact of a one standard-deviation shock to the starting point of the simulation for each of the key stochastic variables: level, slope, curvature, the output gap, core inflation and the overnight rate. For each shock, the results are presented as the distance of the average zero-coupon curve, the zero-coupon curve volatility, the average evolution of the 3-month rate and the average evolution of the 10-year rate from the base, unshocked scenario. The thick dashed blue line from the origin, therefore, represents an unshocked simulation.

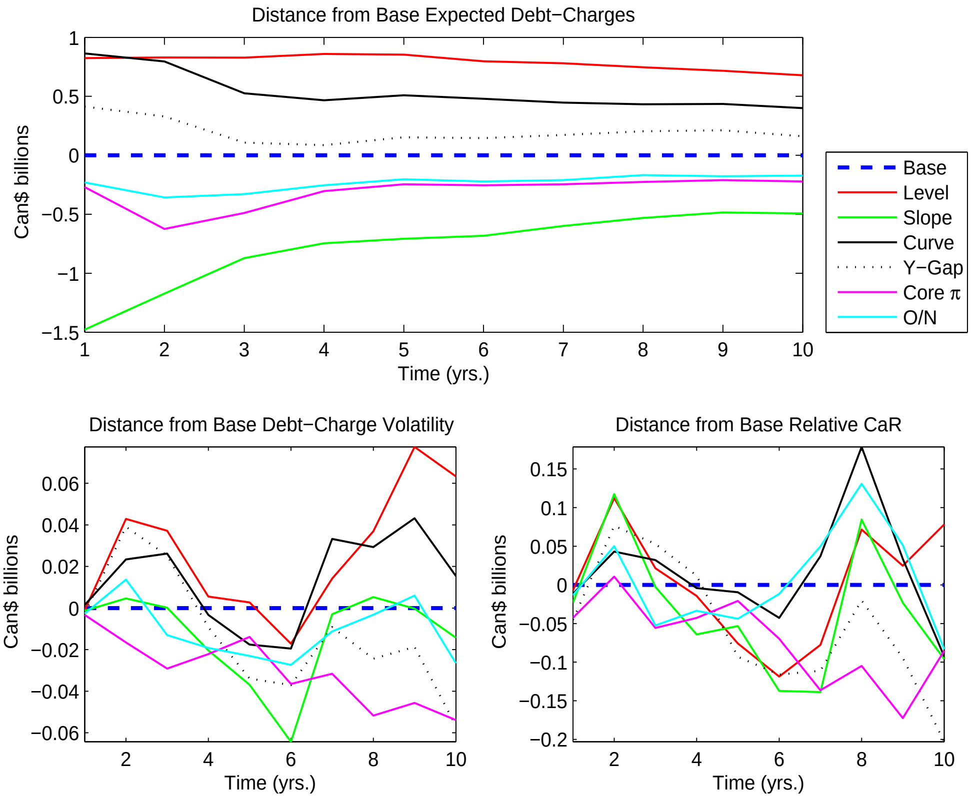


zero-coupon rates essentially converge from their alternative starting points to their long-term mean within three to four years.

Figure 34 shows how the zero-coupon curve results in Figure 33 translate into model results. In particular, Figure 34 illustrates the impact of each shock on the expected debt charges, debt-charge volatility and relative cost-at-risk (CaR) for an equally weighted portfolio of 3-, 6- and 12-month treasury bills; 2-, 5-, 10- and 30-year nominal bonds; and 30-year inflation-linked bonds. Again, the results are presented as the distance from the base, unshocked scenario. Those shocks that generate higher interest rates translate, unsurprisingly, into higher debt charges. Thus, shocks to the level of the yield curve, the curvature of the yield curve and the output gap generate higher expected debt-service charges. Conversely, shocks to the slope of the yield curve, core inflation and the overnight rate generate lower expected debt-service charges. In all cases, the magnitude of the differences from the unshocked scenario dissipate as we move through time and further away from the starting-value shock.

The small differences in zero-coupon curve volatility evident in Figure 33 are consistent with the minimal differences in debt-charge volatility and relative CaR for the different shocks. The differences in Figure 33 appear primarily because of simulation noise. Again, the shocks influence the starting point of the simulations, not the dynamics of the stochastic system. The consequence is that the level of debt-service charges is impacted, whereas their dispersion remains the same. In other words, shocks to the starting point of the stochastic system have cost implications, but not risk implications, for a given financing strategy. One could, of course, construct a more

Figure 34: **Impact of One Standard-Deviation Shocks on Equally Weighted Portfolio**: This figure illustrates the impact of a one standard-deviation (i.e., 2.3 standard-deviation) shock to the starting point of the simulation for each of the key stochastic variables on the expected debt charges, debt-charge volatility and relative cost-at-risk (CaR) of an equally weighted portfolio.



complex model, whereby the volatility of the stochastic system varies with the level of the individual stochastic variables.[78] This, however, cannot be accommodated in the current model.

8.3 Testing alternate long-term values

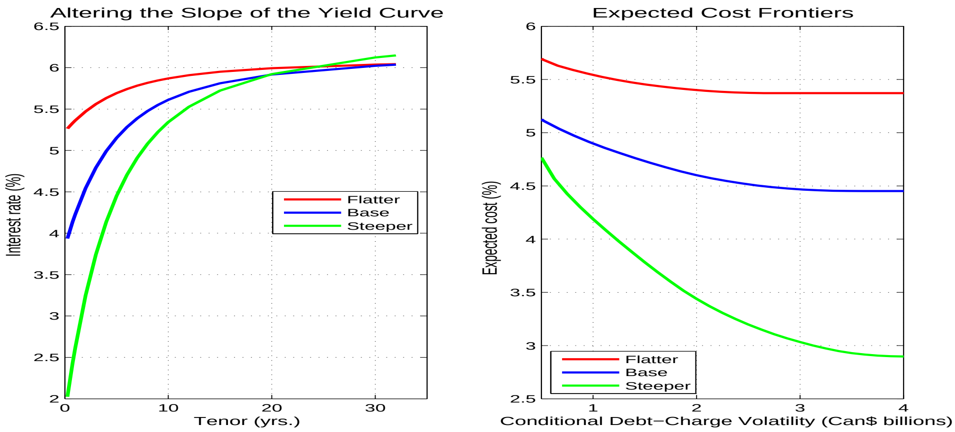
The above analysis demonstrated that shocks to the financial and macroeconomic variables did not cause changes to key risk measures. Furthermore, even if the risk measures were affected, it would have to be in an asymmetric (with respect to financing instruments) manner to be of great concern. Therefore, from the perspective of selecting an optimal portfolio, the impact of the shocks is not "portfolio-moving." In this section we analyze what happens to optimal portfolios when key components of the system are changed. These will be the slope of the yield curve and the yield-volatility structure.[79]

First, we will examine the results, using conditional debt-charge volatility as the risk constraint, when changing the yield-curve slope. When generating scenarios for the model, one can specify long-term values for financial or macroeconomic variables, or allow the model to determine the long-term values. The curves are the average yield curves for each of the three long-term value specifications, with the long-term means as the starting point. In this case, the financial variables' long-term values were changed to give the red and green lines in Figure 35.

[78] If increasing interest rates were, for example, more variable, then a shock to output might actually increase the risk of a given financing strategy.

[79] Analysis in this section uses a more recent dataset, and therefore base results will not match with results from Chapter 5.

Figure 35: **Different Yield-Curve Slopes**: The graph on the left shows the three different yield curves (Flatter, Base, Steeper) that were tested with the model. The expected cost-risk trade-off from the optimizations using conditional debt-charge volatility are shown in the graph on the right.



The blue line uses the model-determined long-term values. Long-term yields have remained relatively close in the three curves, while short-term rates are very different.

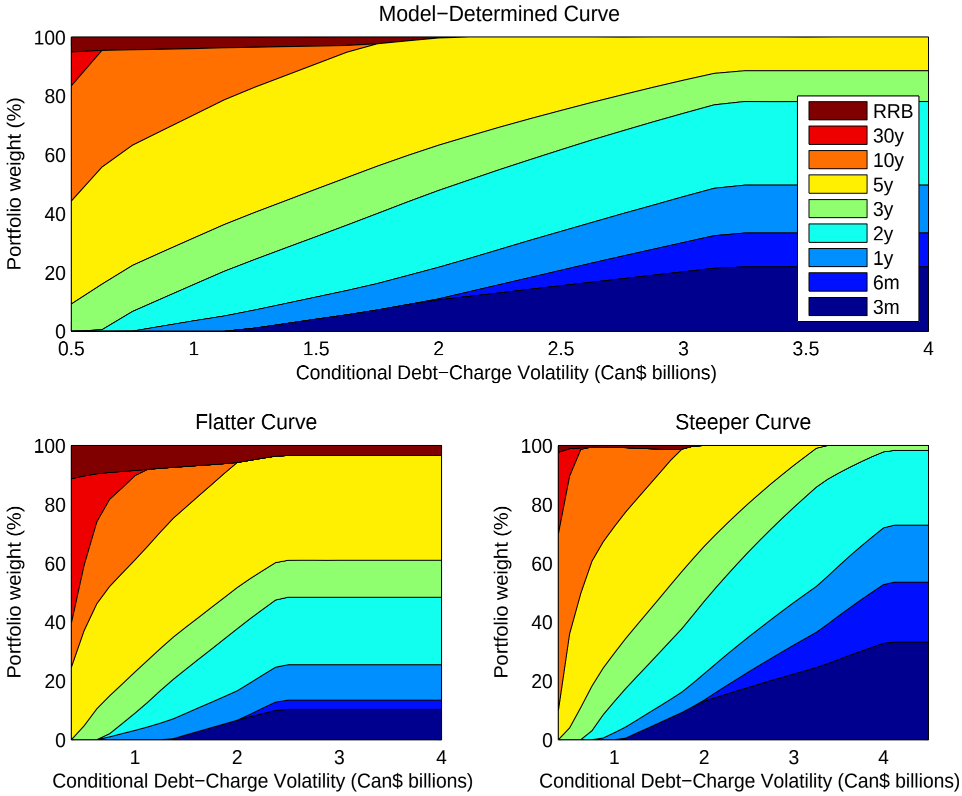
One would expect the different curves to impact the expected cost frontiers and the portfolio weights. The second pane of Figure 35 shows the expected cost frontiers, and it is immediately obvious how much the yield-curve slope impacts these frontiers. The slope of the expected cost frontiers is directly related to the slope of the yield curve, and one can see that the range of the frontiers is also affected. The frontiers level off at their respective minimum cost portfolios, with these differing between the curves because the cost savings from having more short-term debt in the portfolio (at a higher volatility) are different across the curves. Specifically, the lower cost savings in the flatter curve lead to less short-term debt in the minimum cost portfolio, and a lower conditional debt-charge volatility of that portfolio. This can be seen in Figure 36. Conversely, and not surprisingly, long-term debt plays a larger role the flatter the yield curve, with this extending to the 5-year sector as well.

Table 5: **Financing Strategies**: This table outlines the optimal financing strategies for the flatter, base and steeper yield-curve scenarios at a conditional debt-charge volatility of Can$2 billion.

Debt Type	**Tenor**	**Financing Strategies**		
		Flatter	**Base**	**Steeper**
Nominal	3-month	7%	11%	13%
	6-month	0%	0%	0%
	1-year	10%	11%	9%
	2-year	21%	26%	25%
	3-year	14%	15%	19%
	5-year	42%	36%	34%
	10- year	0%	0%	0%
	30-year	0%	0%	0%
Real	30-year	6%	0%	0%

Another relevant avenue for analysis is comparing the optimal portfolios at a given risk level. Table 5 shows the optimal portfolio for a conditional debt-charge volatility level of two, for all three yield curves. Although not identical, they have a significant amount in common. Specifically, they each have a large allocation to 5-year

Figure 36: **Portfolio Weights for Different Yield Curves**: This figure illustrates the portfolio weights for three different yield curves. The range of outcomes is lower the flatter the yield curve. As well, the flatter the yield curve, the lower the amount of short-term debt in the minimum cost portfolio, and in general.

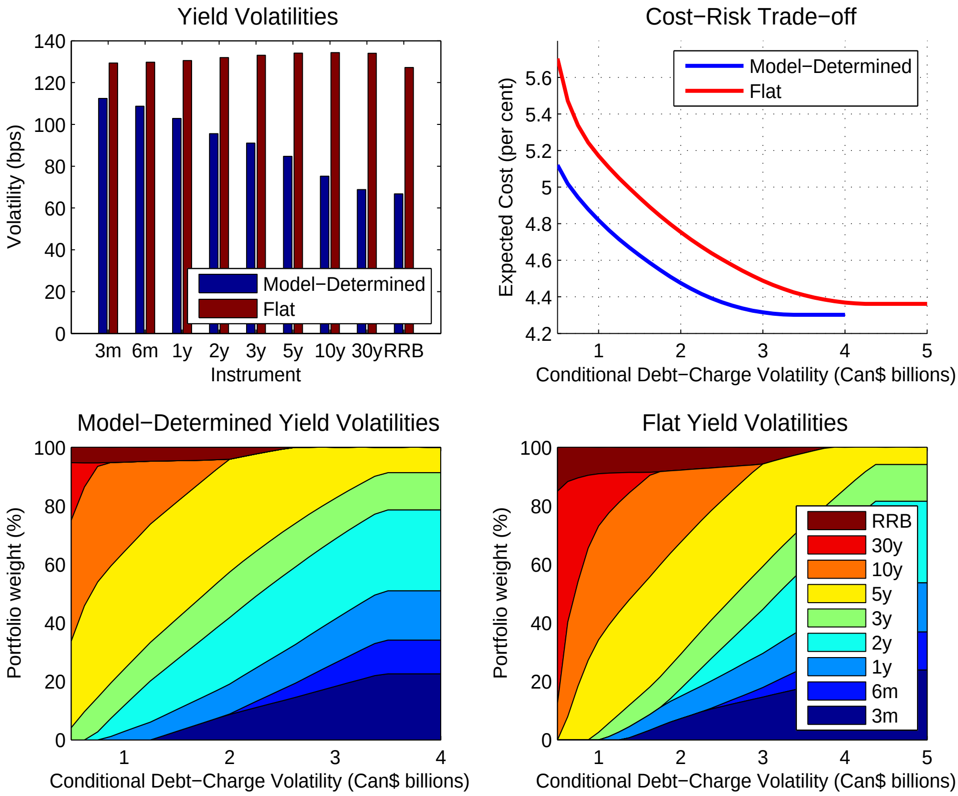


bonds; smaller, but significant allocations to 2- and 3-year bonds; about a 20 per cent allocation to treasury bills; with a small to non-existent allocation in RRBs as the only long-term debt. This is significant because if a particular optimal portfolio is targeted from a given scenario, changes in the average yield curve will not greatly impact its efficiency or risk level. This result is to be expected, since yield volatilities remained the same across the different average yield-curve scenarios.

A natural next step in the analysis is to look at the effects of changing the volatility structure of yields. Figure 37 has the relevant graphs for the change in volatility structure. The volatility structure was flattened and increased.[80] The effect of this on the expected cost-risk trade-off was higher cost at a given risk level. This is not surprising, since to reach a given conditional debt-charge volatility level, one must have more longer-term debt. One can see graphically that the lowest cost portfolios are not significantly different, nor do they have significantly different expected costs. This was anticipated because the average yield curve did not change. They do, however, have different conditional debt-charge volatilities. Furthermore, the bottom panes of Figure 37 show that, unlike when changing the slope of the yield curve, changing the yield volatilities *will* change the optimal portfolio at a given risk level.

[80] Although the change will affect the cost volatility for a given financing instrument, the effect will not be one-to-one. Cost volatility is significantly affected by the frequency of rollover, as well as yield volatility.

Figure 37: **Altering Yield Volatilities**: The top-left graph shows the model-determined and flatter yield volatilities. The top-right graph shows the expected cost differences between the two yield-volatility scenarios. The two bottom graphs show the financing strategies associated with the expected cost curves in the top-right graph.



8.4 Conclusion

This chapter examined the nature of the interaction between the individual macroeconomic and financial variables in our stochastic system. Such diagnostic analysis is critical for understanding, and ensuring the consistency of, our stochastic system. Our principal conclusion is that the impulse-response functions from our reduced-form VAR model, which describes the dependence structure of our stochastic system, are generally consistent with the results of traditional small-scale structural macroeconomic models. Moreover, shocks to the starting values of the macroeconomic and financial variables potentially impact the cost of a given financing strategy, but not its risk.

In addition, we examined how optimal portfolios and expected cost frontiers changed when key aspects of the system are changed. We found that, although significantly different yield curves will give different ranges of cost-volatility outcomes, the portfolio at a given risk level is similar across the three average curves tested. As well, we found that altering yield volatility will impact the optimal portfolios at a given risk level.

9 The Issuance-Penalty Function

This final chapter considers the interaction between the amount of debt issuance and its associated cost. In principle, a government's financing decisions have an impact on the state of the macroeconomy and interest rates. A government's decision, for example, to finance itself entirely with very short-term debt would have an impact on the financing costs of non-government entities. In particular, a surfeit of short-term, risk-free debt could *crowd out* other borrowers; the consequence might be higher borrowing costs for these entities and correspondingly lower economic activity. In addition to this *crowding-out* effect at the short end, one might expect to observe difficulty in pricing and hedging longer-term issuance given the dearth of government supply—again, this could influence the level of interest rates and negatively impact economic activity. While this logic is quite reasonable, the stochastic model does not currently describe the interaction between the financing strategy and the macroeconomy. Why not? We do *not* endogeneously add the financing strategy into the macroeconomy, quite simply because we do *not* have the data to estimate anything meaningful.

For the effect of the issuance amount on yield, we have two choices. We can (i) ignore this effect or, (ii) use a relatively arbitrary approach to capture it. With the enhanced debt-strategy model, we opted for the second approach, which involves including an issuance-penalty function to capture this effect. Implicitly, therefore, we have decided that it is preferrable to attempt to describe this effect—and to have a rough approximation included in the model—than to ignore it. This chapter describes the logic behind this issuance-penalty function and examines its influence on model results.[81] The chapter is organized into two parts. The first part describes the basic logic and structure of the penalty function and the second part examines how optimal portfolio weights vary for different specifications of the penalty function.

9.1 The penalty function

The logic behind the issuance-penalty function is quite simple. If the government issues an excessively small or large amount of a given debt instrument in a given sector, it will possibly be at a higher yield than if it issued a more "reasonable" amount. Reasonable in this context implies that supply approximately equals demand and that the issuance profile contributes to a liquid, benchmark issue. Imagine, for example, that the Canadian government decided to issue only Can$100 million of 30-year nominal bonds in a given quarter, compared to a typical issue amount of more than Can$1 billion. One might expect that the government would have to pay a liquidity premium to issue such a small amount. It is more likely, however, that this would create substantial excess demand from pensions and insurance companies, who use these long-tenor bonds to match their long-term liabilities. The consequence of such a small issuance amount, therefore, may actually be slightly better issuance costs for the government—the difference between an increase or a decrease in cost as we decrease supply will depend upon the demand schedule of investors and available substitutes. Conversely, imagine that the government decided to issue Can$40 billion of 2-year bonds in a given quarter.[82] Again, one can easily imagine that it would be difficult for the market to absorb such a large amount of supply. We assume that such extreme issuance can be achieved, but that it has a cost, and that the government pays this cost in the form of higher coupons. The penalty function determines the magnitude of this increase to the coupon.

We can probably all agree, in principle, that small issuance amounts lead to either the payment of a liquidity premium or a possible decrease in funding costs, and that large issuance amounts lead to the payment of a placement or absorption premium. What is more difficult to agree on, however, is the magnitude of these effects and when they begin and end. Unfortunately, these effects are generally impossible to estimate with statistical techniques. The reason is simple: we have no data. It is hard to know how much it would cost to issue Can$100 millon of 30-year nominal bonds or Can$40 billion of 2-year bonds in a given quarter, since the government has

[81] The penalty functions provided in this chapter are not those currently used in the model.

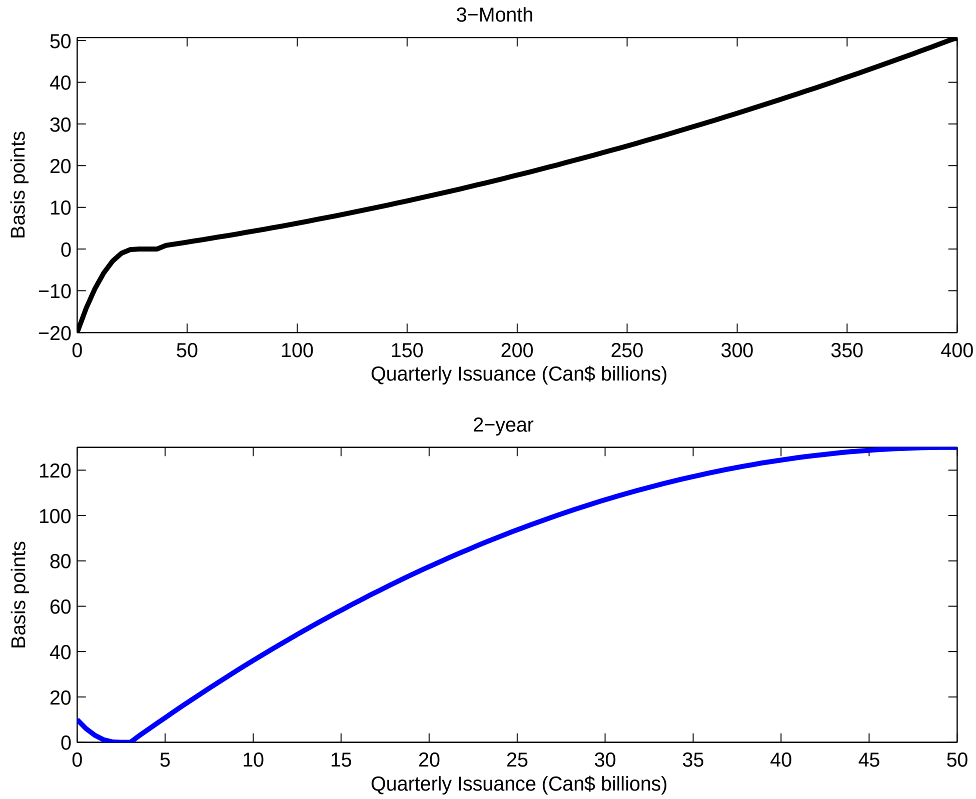
[82] It is not simply a case of absorbing a one quarter issuance amount. Given that the model uses fixed financing strategies, and assuming a constant debt stock, this amounts to an *every*-quarter issuance amount.

never engaged in anything close to such issuance behaviour in the past.

As a consequence, we find ourselves in uncharted territory. The inability to estimate the issuance-penalty function implies that we must determine its mathematical form in a subjective fashion. While we have attempted to do this in a reasonable and defensible manner, it should be stressed that there is an unavoidably arbitrary element to this effort. As such, users and consumers of the debt-strategy model should be clearly aware of the logic behind the construction of the issuance-penalty function.

The graphs in the following figures are for demonstration only. They generally show the form of the issuance-penalty functions, but do not represent those currently in use. One can see a flat portion at zero for both the curves. This represents the non-penalized zone; issuance in this area can be done at prevailing market yields. Issuance above the upper bound of this zone, or below the lower bound of this zone, will have a change in yield (and coupon) based on the instrument and issuance level. The formulations below, and those not shown, are used in subsequent sections to evaluate the impact of the issuance-penalty functions on the model results. In practice, the issuance-penalty functions are updated regularly, so that they are reflective (to the best of our ability) of the current environment.

Figure 38: **Sample Penalty**: This figure illustrates a possible penalty, in basis points, associated with the 3-month treasury bill and 2-year bond. The lower bound on issuance for each is, of course, zero. The upper bound, conversely, represents the maximum amount of quarterly issuance possible, assuming that a Can$400 billion portfolio was funded with only that specific tenor.



The second part of the issuance-penalty function relates to its mathematical form. As we have already established, we add a penalty to the par interest rate of each financing instrument when it falls below the lower bound or above the upper bound. Logically, the size of the penalty should increase the further we deviate from the unpenalized boundary (i.e., the region where issuance does *not* impact market prices). A small reduction or increase in issuance beyond the non-penalized range should *not* generate dramatic increases in cost. We opted, therefore, for a functional form that increases non-linearly as the distance from the boundary increases. As such, the penalty is quite small for small movements outside the boundary, but increases exponentially for substantial deviations. Essentially, the penalty function is a piecewise polynomial whereby issuance costs below the lower bound, and above the upper bound, increase parabolically.[83]

The choices of the penalty-function parameters—that decide on the magnitude of the incremental cost to the government for excessively large or small issuance—were initially made to provide reasonably conservative values. Figure 38 graphically demonstrates the magnitude of a sample issuance-penalty function for the 3-month treasury bill and the 2-year bond. The lower bound on issuance for each bill is, of course, zero. The upper bound, conversely, represents the maximum amount of quarterly issuance possible, assuming that a Can$400 billion portfolio was funded with only that specific tenor. Note that the maximum penalty, for a situation where the entire portfolio is financed with only one tenor, leads to a 50 and 130 basis point penalty for the 3-month bill and 2-year bond, respectively. When issuance falls below the lower issuance bound, however, the actual funding rate is assumed to decrease for the 3-month treasury bill. The logic behind this choice is the strong institutional demand for risk-free short-term investments that can be classified as cash. We assume that there is a modest increase in cost for the issuance of 2-year nominal bonds below the lower bound.

In the next section, we use the issuance-penalty functions in a portfolio-optimization exercise. It bears repeating that all the issuance-penalty functions presented in this chapter are for demonstration only. They are intended to show the general functional form and characteristics of the issuance-penalty functions, not to show their current iteration.

9.2 Optimal portfolio weights

The issuance-penalty functions should presumably have implications for the relative cost characteristics of different financing strategies. Figure 38, for example, indicates that issuing entirely 3-month treasury bills will increase the average issuance costs by approximately 50 basis points. This should lead to significantly larger debt-service charges relative to a more balanced portfolio. This section attempts to understand the nature of these implications. If we attempt to minimize debt-service charges, subject to some form of risk constraint, we would expect the optimizer to avoid a financing strategy composed entirely of 3-month treasury bills by virtue of its higher cost. Thus, we can see that the issuance-penalty function acts as something of a constraint against extreme portfolio allocations.

To better understand how the issuance-penalty function impacts portfolio allocations, we perform a simple experiment. Specifically, we examine three alternative issuance-penalty specifications: the base penalty, a zero-issuance penalty and an extreme penalty function. The extreme issuance penalty essentially involves a 5-fold increase in the coefficients on each of the piecewise-polynomial issuance-penalty functions. For each of these different issuance-penalty function choices, we evaluate 2,000 financing strategies with 10,000 randomly generated macroeconomic and interest rate outcomes. The stochastic scenarios are generated using the Nelson-Siegel model—with the specification described in Chapter 8—estimated with data from January 1994 to August 2007.

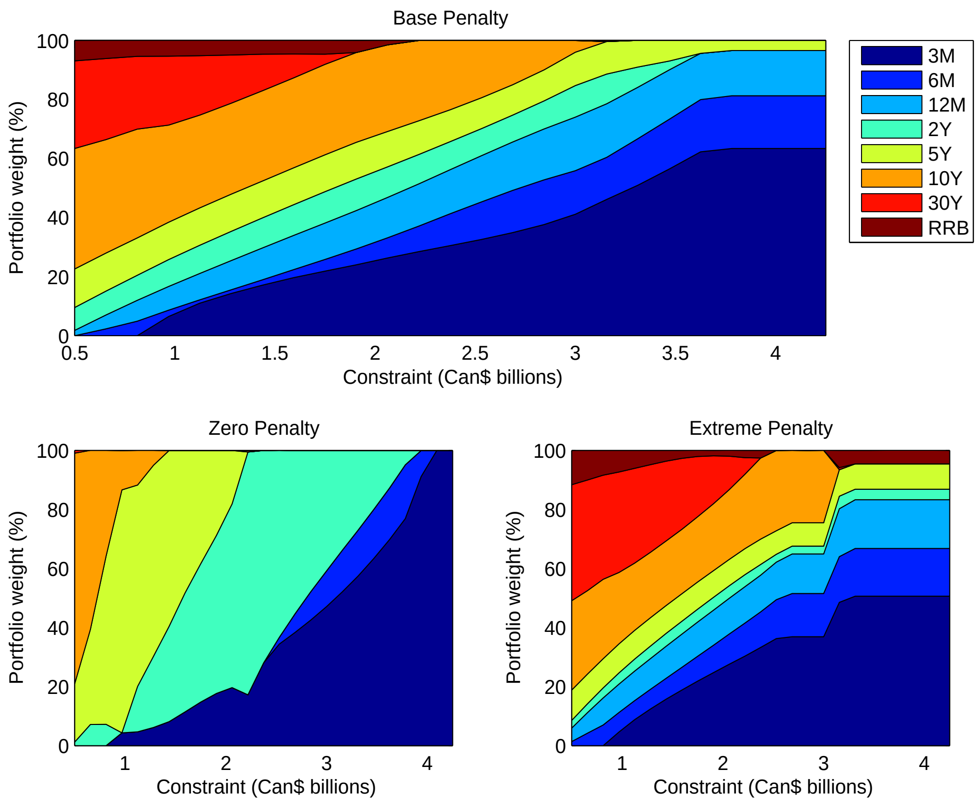
Using these three separate formulations, we use the optimization tool to find the optimal portfolio weights

[83] Mathematically, the penalty function $p_i(x)$ has the following form,

$$p_i(x) = \begin{cases} \alpha_i(a_i - x)^2 : x \in [0, a_i) \\ 0 : x \in [a_i, b_i] \\ \beta_i(x - b_i)^2 : x \in [0, \infty] \end{cases}, \tag{28}$$

where x denotes the issue of the ith financing instrument in a given quarter, a_i is the lower bound, b_i is the upper bound, and α_i, β_i are coefficients to be determined for $i = 1, ..., N$.

Figure 39: **Portfolio Allocations for Alternative Penalty Functions**: This figure illustrates the optimal-portfolio weights for three alternative penalty functions associated with the minimization of debt-service charges subject to a constraint on the conditional debt-charge volatility. The base penalty function, a zero-penalty function and a larger penalty function are illustrated.



to the following familiar optimization problem:

$$\min_{\substack{\text{Portfolio}\\ \text{weights}}} \textbf{Expected Average Percentage Debt-Service Charges}, \tag{29}$$

subject to:

$$\textbf{Conditional Cost Volatility} \leq \textbf{Can\$}\delta \textbf{ billion}.$$

where the conditional debt-charge volatility constraint (i.e., δ) ranges from Can$500 million to Can$3 billion. We choose this particular objective function to be consistent with analysis performed in previous chapters.

Figure 39 illustrates the optimal portfolio weights across the range of conditional debt-charge volatility constraints. The top graphic illustrates the optimal portfolio weights for the *base* issuance-penalty function. The optimal portfolio, for a conditional debt-service-charge volatility of Can$500 million, is composed principally of nominal bonds with a small allocation to inflation-linked bonds. Roughly 30 per cent of the portfolio is allocated to 30-year nominal bonds, 40 per cent to 10-year bonds, 15 per cent to 5-year bonds and the remaining 15 per cent divided more or less equally among the 2- and 30-year nominal bonds. As the risk constraint is relaxed, the allocation to longer-tenor bonds falls. Correspondingly, the allocation to treasury bills increases. Indeed, for a conditional debt-service-charge constraint of Can$4 billion, the portfolio consists of approximately 90 per cent treasury bills with a small allocation in the 5-year sector.

In the lower-left quadrant of Figure 39, we consider the optimal portfolio weights—again, for the optimization problem in equation (29)—in the complete absence of an issuance-penalty function. At a constraint level of

Can$500 million, the portfolio remains composed of longer-term bonds although the composition is somewhat different. In particular, we observe that approximately 78 per cent of the portfolio is allocated to 10-year bonds, with 20 per cent in 5-year bonds, and the remainder split between 2-year nominal and 30-year inflation-linked instruments. What is behind the decreased allocation to the 30-year sector? 10-year bonds are relatively less expensive, but similar in terms of risk, to 30-year nominal bonds. Elimination of the issuance penalty, therefore, leads to an extreme allocation to 10-year debt, whereas its presence leads to the replacement of 10-year issuance with 30-year bonds. Clearly, we can see that the penalty function can, and does, have an important impact on the model results. Misspecification of the issuance-penalty function can lead to a general misspecification of the results. Caution, therefore, is required.

Again, considering the lower-left quadrant of Figure 39, we observe a similar movement from long- to short-term issuance as the conditional debt-service-charge volatility constraint is relaxed. In the absence of an issuance-penalty constraint, however, we observe that a Can$4 billion risk constraint generates an optimal portfolio that consists entirely of 3-month treasury bills. This compares to an allocation of approximately 60 per cent for the base issuance-penalty function. The absence of an issuance-penalty function leads, therefore, to *corner* solutions. Intermediate portfolios, with a constraint ranging from Can$1-3 billion, are dominated by 2-year issuance. This compares with the base penalty function, where the allocation to the 2-year bond is relatively constant at approximately one-tenth of the portfolio until the constraint attains Can$3.5 billion, whereafter it falls to zero. Again, the 2-year allocation is sensitive to the imposition of the issuance-penalty function.

The bottom-right graphic in Figure 39 illustrates the results when the base issuance-penalty function is adjusted to be 5 times larger; we term this the *extreme* issuance-penalty function. At lower levels of risk, there is relatively little difference between the base and extreme issuance-penalty optimal portfolio allocations. Intermediate levels of risk look very similar to the base issuance-penalty function as well, likely because these issuance amounts occur over the unpenalized issuance ranges. Finally, at the highest risk levels, there is a small allocation to inflation-linked bonds in addition to the usual assortment of treasury bills and short-term nominal bonds. Again, this is due to the fact that the extreme issuance-penalty function makes the issuance of a small amount of inflation-linked debt very inexpensive.

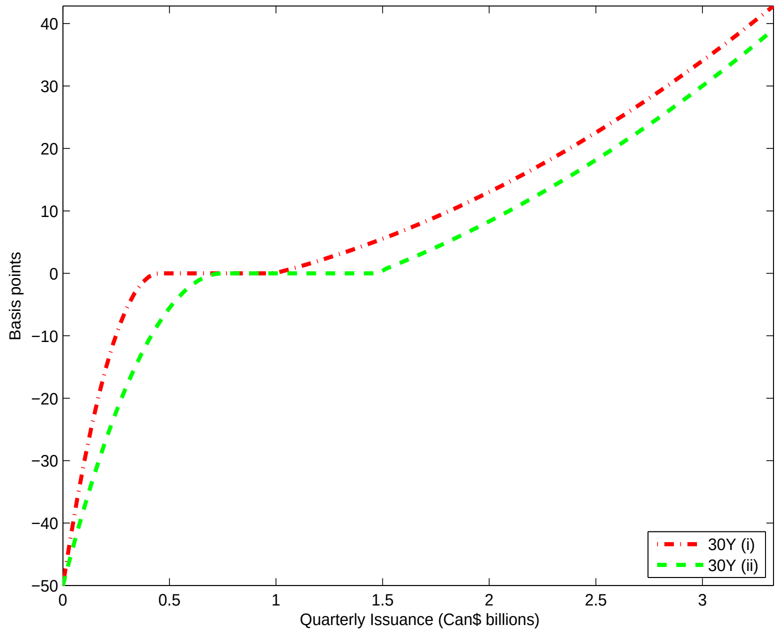
The results appear to suggest that the optimizer makes some trade-offs as the issuance-penalty function becomes more extreme, but that it is relatively limited. It is nonetheless clear that, as the issuance-penalty is relaxed, the optimizer makes different decisions. In this manner, the optimizer is quite sensitive. In general, it acts to minimize the expected debt-service charges subject to a risk constraint. It will, quite effectively, find the best solution to this problem. A difference of a few basis points in expected cost can nevertheless make the difference between a substantial allocation to a given financing instrument and a zero portfolio weight.

9.3 A brief sensitivity analysis

In this section, we extend the analysis to consider another situation where the issuance-penalty function can potentially influence results. Our objective is to demonstrate that not only the coefficients associated with the issuance-penalty functions are important, but that the upper and lower bounds demarcating the unpenalized region also matter. In Figure 40, we illustrate the issuance-penalty function for two alternative choices for the 30-year nominal bond issuance-penalty function. The two 30-year issuance penalties differ only in the specification of the upper and lower bound. The first choice, which we term 30Y (i), exhibits the original choice of a lower bound of Can$450 million of quarterly issuance while the second choice, denoted 30Y (ii), has a Can$750 million lower bound. The upper bound in 30Y (i) is Can$1 billion per quarter, whereas in 30Y (ii) it is increased to Can$1.5 billion per quarter.

The consequence of the boundary change to the 30-year nominal bond penalty function is shown at the right side of Figure 41. In the case of 30Y (i), a small allocation to inflation-linked bonds of approximately 5 per cent is observed for lower levels of risk. Increasing the lower bound, as in 30Y (ii), leads to an elimination of inflation-linked issuance across all levels of risk. Figure 40 demonstrates the reasoning behind this result.

Figure 40: **Alternative Long-Bond Penalties**: This figure includes two alternative 30-year issuance-penalty functions. The only difference is that the lower and upper bounds for the base 30-year issuance-penalty function (i) are shifted upwards in (ii).



Moving the upper bound on 30-year issuance to Can$750 million per quarter generates a lower cost associated with 30-year nominal debt.

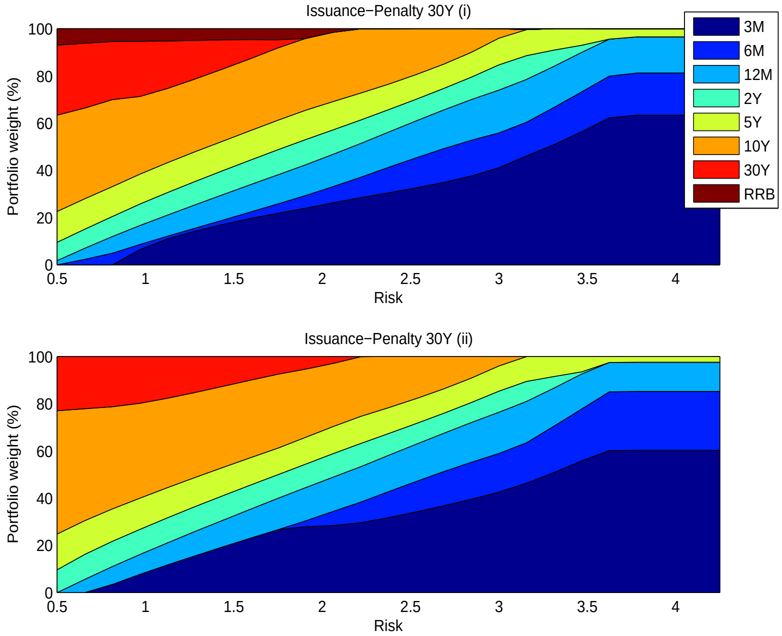
Certainly, not all of the bounds specified in the issuance-penalty function have the potential to impact the results. The previous example, however, does stress that the necessarily subjective choices made when specifying the issuance-penalty functions can potentially influence model results. Consequently, these choices must be made in a careful and transparent manner.

9.4 Magnitude of the issuance-penalty functions

This final element of our analysis attempts to describe the actual cost implications associated with the imposition of the issuance-penalty functions. This is difficult to understand when examining a large range of financing strategies, because there are too many moving parts. Instead, we select four alternative financing strategies: the minimum debt-service-charge risk portfolio, the minimum budgetary-risk portfolio, the maximum risk portfolio and the portfolio as of 31 March 2006. These four portfolios are, with the exception of the last portfolio, fairly extreme portfolios, with substantial issuance in one or more sectors combined with minimal issuance in other instruments. This makes them ideal for cost comparison, because the issuance-penalty functions are more likely to be invoked.

Figure 42 illustrates the average debt-service charges for our four alternative portfolios under the base, zero and extreme issuance-penalty specifications analyzed in the previous section. There are, at least, three points to draw from this figure. First, the difference between the base and zero issuance-penalty functions is quite modest—on the order of five to seven basis points—for the fairly extreme minimum debt-charge risk and maximum risk portfolios. This suggests that even relatively small changes in cost can lead to the rather important differences in portfolio allocations observed in Figure 39. Second, and not surprisingly, the extreme penalty-function specification generates rather extreme differences—on the order of about 50 basis points. Finally, the

Figure 41: **Portfolio Allocations for Alternative 30-Year Nominal Issuance Penalty**: This figure illustrates the optimal-portfolio weights associated with the two alternative 30-year nominal issuance-penalty functions described in Figure 40.



31 March 2006 portfolio, which involves issuance that almost entirely falls in the non-penalized ranges, exhibits very small cost differences between the three issuance-penalty specifications.

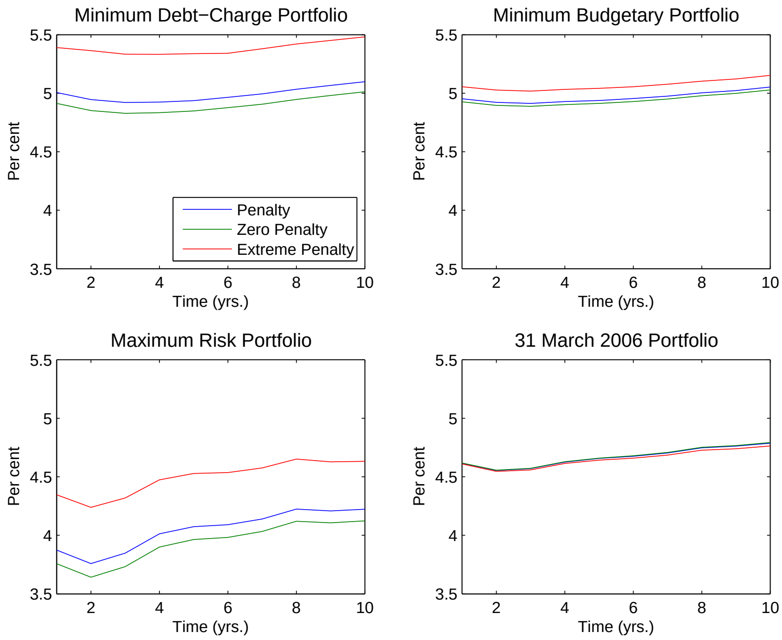
9.5 Conclusion

When constructing a model, one is occasionally faced with parameters that cannot be objectively determined. In the context of the debt-strategy model, we are faced with determining the cost of issuing a given financing instrument as a function of the amount issued. If we restricted our attention to relatively limited issuance ranges, this effect could probably be safely ignored. Strategic analysis of the government's debt strategy, however, involves consideration of a wide range of possible financing strategies. As a consequence, this interaction cannot be ignored. Moreover, as we have seen in the previous analysis, the link between issuance size and cost has an important impact on the model results.

We suggest the use of an issuance-penalty function that penalizes excessively large (or small) amounts of issuance in a given financing instrument; this function can also be used to describe excess demand for a specific financing instrument. Such an approach has the benefit of directly describing the interaction between issuance amount and price. The issuance-penalty function has the significant disadvantage, however, of being subjectively determined. It is subjective, because we simply do not have data describing this interaction for extreme issuance levels. This chapter demonstrates, in an illustrative manner, the impact of a given set of choices on the optimal portfolio weights for a straightforward objective function. Our goal in this analysis is not to specify the correct penalty functions, but to provide the general functional forms and graphical representations of these functions, and to test what happens when they are altered.

Determination of the appropriate parameters for the issuance-penalty function is difficult, important and unavoidable. Going forward, the following guidelines should continue to be followed with respect to the issuance-penalty function. The parameters should be determined by consensus, there should be a general understanding

Figure 42: <u>**Average Debt-Service Charges for Various Issuance-Penalty Specifications**</u>: This figure illustrates the average debt-service charges over a 10-year simulation horizon for four alternative portfolios under three different issuance-penalty function specifications. The four portfolios include the minimum debt-service-charge risk portfolio, the minimum budgetary-risk portfolio, the maximum risk portfolio and the portfolio as of 31 March 2006. The issuance-penalty function specifications include the base, zero and extreme forms analyzed in the previous section.



of their relative impact on model results (hence this chapter), they should be subject to extensive sensitivity analysis, and they should be revisited on a frequent basis. These are essential to efficiently using the issuance-penalty functions to improve the quality of our debt-strategy analysis.

10 Conclusion

A model is, after all, only a simplified mathematical representation of a complex reality. Understanding this fact is critical to the sensible use of any model. A key part of this process, which has been the objective of this paper, is to understand where the simplification has the potential to influence the model results in an important way. The preceding chapters have analyzed the main components of the Canadian debt-strategy model and the key risks inherent in them. Examining the specific components of the model and their risks is essential to using and understanding the results and providing useful advice from the model.

Chapter 2 examined the key assumptions underlying the model including the relevance of the macroeconomic term-structure models. Decision makers must be both aware and comfortable with these key assumptions. Next, in Chapter 3, a broad set of cost and risk measures calculated in the model, which form the basis for comparison among alternative financing strategies, was presented and evaluated. Budgetary volatility was presented in that chapter as a theoretically attractive measure. We should note, however, that this list of cost and risk measures is not exhaustive. Following introduction of the cost and risk measures, the intricacies of incorporating inflation-linked debt were illustrated in Chapter 4. Included in the discussion of inflation-linked instruments was a comparison of otherwise identical portfolios with different levels of 30-year inflation-linked and 30-year nominal debt. How we use the debt-strategy model to optimize over various criteria was discussed in Chapter 5. The discussion centred around the variety of possible objectives and constraints that form the heart of any optimization. A policy-maker's role is perhaps most apparent here given that there is not, as yet, full consensus among practitioners and academics on the exact set of objectives and constraints to be used in determining an optimal debt strategy. How fiscal policy is incorporated into the model was addressed in Chapter 6. The highlight of this relatively technical chapter was the introduction of a fiscal rule approximating how the government might react to surprise fiscal outcomes. We also examined debt management's impact on budgetary volatility, finding that the covariance between debt charges and the primary balance appears to be the cause of the modest range of conditional budgetary-volatility outcomes. Model risk was thoroughly analyzed in Chapter 7. We demonstrateed how *five* alternative term-structure models with macroeconomic variables can be used in the debt-strategy model to mitigate model risk, and compared the results associated with each model along numerous dimensions. Chapter 8 had two important sections. The first section used impulse-response functions to provide an overview of the linkages among and between macroeconomic variables and term-structure factors. The second section of Chapter 8 examined the impact of shocking variables and changing long-term values on model results. Finally, Chapter 9 described the issuance-penalty functions and performed some sensitivity analysis, including a comparison of results between three fundamental cases: a base-, a non-penalized and an extreme-penalty case.

We conclude this paper with a note of caution. As we hope the preceding pages have made evident, interpreting the model results requires a thorough understanding of its underlying parts and their inherent risks. It also requires a thorough understanding of the debt-management problem and an intuitive feel for the important aspects of the government's debt strategy. The debt-strategy model is an important part of the decision-making process for determining the government's debt strategy. It is *not*, however, the only part. The debt-strategy model remains *one* tool that is available to debt managers, and is to be used in conjunction with (not as a substitute for) the judgment of an experienced debt manager. Simply put, we require models because we cannot fully trust our intuition. We require intuition, however, because we cannot fully trust our models. It is the controlled interplay between these two elements that contributes to good debt-strategy policy.

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