A Model of Indivisible Commodity Money with Minting and Melting

Angela Redish
University of British Columbia
email: angela.redish@ubc.ca

Warren E. Weber*
Federal Reserve Bank of Minneapolis
University of Minnesota
e-mail: wew@minneapolisfed.org

Very preliminary, comments invited

November 4, 2008

Abstract

Keywords: commodity money, random matching, Gresham's Law, optimal denominations
JEL: E40,N23

*We thank Miguel Molico, François Velde, Randy Wright, and participants at seminars at the Federal Reserve Bank of Boston, the University of British Columbia, for helpful comments. The views expressed in this paper are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

1 Introduction

Monetary economists frequently imagine commodity money systems to be smoothly operating regimes where the fixed supply (or costs to produce) of the commodity provide a nominal anchor for the economy. However, while the commodity could anchor the value of the unit of account, commodity money systems had more difficulty in performing the medium of exchange function of money. Jevons famously pointed out that to provide a medium of exchange a commodity must be durable, portable and divisible, and for much of the last millennium gold and silver were adopted as monetary commodities because they had such qualities. Identifiability (for example, of the purity of the metal in a coin) and uniformity (permitting payments to be made by tale rather than by weight) were further desirable characteristics of money. The desire for these attributes promoted the use of coined metals and the monopolization of the right to mint coins. In turn, the monopoly production of coins gave the monetary authority (typically the Crown) two instruments of monetary policy: the rate of seignorage and the size of coins - the denomination structure.

Monetary historians have documented the difficulties created by these monetary systems and argued that denominational structures are an important contributor to those difficulties. [**footnote mayhew quote re need to take into account?**] In medieval and early modern England, contemporaries frequently complained that there was a scarcity of small denomination coin. Similar complaints were common in the colonial US and Canada.

In a previous paper we built a random matching monetary model with two indivisible coins and illustrated that a small change shortage could exist in the sense that adding small coins to the economy with large coins was welfare improving. However, in that paper we assumed a fixed quantity of coins of each type and compared the characteristics of equilibria with different stocks of, and sizes of, coins. The commodity value of the coins was imposed by assuming that each coin paid a dividend (essentially was a Lucas tree).

In this paper we endogenize the quantity of money. We assume that there is a fixed stock of each monetary metal and allow agents to choose to mint coins (at a cost) or to melt the coins into jewelry. The value of the coins derives from their potential to be used to make jewelry. Thus the model combines a more fundamental way of generating valued commodity money, and an endogenous quantity of money (though not quantity of monetary metal) providing a much richer framework to discuss the shortage of small coins.

We use the model to analyze the implications of alternative monetary policy choices. In particular, we examine the impact of the denomination structure on welfare, and find that there is **an** denomination structure. We also show that the optimal coin size may not (likely will not) maximize seignorage revenue. Finally, we show that as the trading opportunities rise the optimal size of the silver coins shrinks.

The paper proceeds as follows. In the next section we present the model. Section 3 discusses the results. The final section concludes.

2 The Model

2.1 Environment

The model has discrete time and an infinite number of periods. There is a nonstorable, perfectly divisible good and two metals (durable commodities) — silver and gold — in the economy. There are m_s ounces of silver and m_g ounces of gold in existence.

The way in which we model commodity money follows that used by **?**. We assume that these metals can be held in either of two forms, which we will refer to as *coins* and *jewelry*. Metal held in the form of jewelry yields utility directly to the holder, whereas metal held in the form of coins yields no utility to the holder. However, because of difficulties in determining the weight and fineness of jewelry, only coins can be traded.

Because of the limitations of the technology, metals can be divided, but only imperfectly. This means that coins — because they are made from metal — must be indivisible, and the role of the monetary authority in this environment is to choose how many ounces of metal to put into a coin of that metal. We let b_s be the ounces of silver that it puts in a silver coin and b_g be the number of ounces of gold that it puts in a gold coin. As was the case with coins throughout most of the time during which commodity monies were used, these gold and silver coins do not have denominations. They are simply amounts of the two metals that have been turned into coins with some type of standardized markings that allow one type of coin to be easily differentiated from a different type of coin.

Let s and g be an agent's holdings of silver and gold coins, respectively, and j^s and j^g be an agent's holdings of silver and gold jewelry, respectively. The agent's portfolio of metal holdings is

$$y = \{(s, g, j^s, j^g) : s, g, j^s, j^g \in \mathbb{N}\}.$$

There is a $[0, 1]$ continuum of infinitely lived agents in the model that maximize expected discounted lifetime utility. The discount factor is β. Let q denote the quantity of the good. The agent's period preferences are

$$u(q) - q + \mu(b_s j^s + b_g j^g)$$

with $u(0) = 0$, $u' > 0$, $u'' < 0$, and $u'(0) = \infty$. The disutility of good production is assumed to be linear without loss of generality. Further we assume that gold is intrinsically more valuable to account for that fact that this was the case historically. Specifically, we assume

$$\mu(x', x) > \mu(x, x') \quad \forall x' < x.$$

Each period in the model is divided into two subperiods. In the first subperiod agents can trade in bilateral matches. At the beginning of this subperiod, an agent has a probability $\frac{1}{2}$ of being a consumer but not a producer and the same probability of being a producer but not a consumer. This assumption rules out double coincidence matches, and therefore gives rise to the essentiality of a medium of exchange.

After agents' types (consumer or producer) are revealed, a fraction $\theta \in (0, 1]$ of agents are matched bilaterally. In a match, the portfolios of both agents are known. However, past trading histories are private information and agents are anonymous. These assumptions rule out gift-giving equilibria and the use of credit. Thus, trading can only occur through the use of media of exchange, which is the role that the gold and silver coins can play.

In the second subperiod, agents can alter their mix of coins and jewelry by minting or melting. That is, in the second subperiod agents can change the form in which they hold the stock of each metal, but they cannot change the total quantity of each metal in the economy.

2.2 Consumer choices

We assume that in a single coincidence pairwise meeting, the potential consumer gets to make a take-it-or-leave-it (TIOLI) offer to the potential seller. This offer will be the triple (q, p_s, p_g), where $q \in \Re_+$ is the quantity of production demanded, $p_s \in \mathbb{Z}$ is the quantity of silver coins offered, and $p_g \in \mathbb{Z}$ is the quantity of gold coins offered. Offers with $p_s < 0$ or $p_g < 0$ can be thought of as the seller being asked to make change.

Let $v(y) : \mathbb{N} \bigotimes \mathbb{N} \rightarrow \Re_+$ be the expected value of an agent's portfolio at the beginning of the second subperiod. The set of feasible TIOLI offers is a combination of special good output and coins that is a feasible coin offer and satisfies the condition that the seller be no worse off than not trading. Denoting this set by $\Gamma(y, \tilde{y})$,

$$\begin{aligned} \Gamma(y, \tilde{y}) \quad &= \quad \{(q, p_s, p_g) : q \in \Re_+, -\tilde{s} \leq p_s \leq s, -\tilde{g} \leq p_g \leq g, \\ & \quad\quad -q + v(\tilde{s} + p_s, \tilde{g} + p_g, \tilde{j}^s, \tilde{j}^g) \geq v(\tilde{y})\}, \end{aligned}$$

where $\tilde{y}$ denotes the seller's portfolio. In the second subperiod, the agent can mint or melt. Define $z^s \in \mathbb{Z}$ to be the quantity of silver coins minted (> 0) or melted (< 0) and $z^g \in \mathbb{Z}$ to be the quantity of gold coins minted (> 0) or melted (< 0). Agents cannot mint more coins that they have jewelry and cannot melt more jewelry than they have. Denoting this set by $\Omega(y)$

$$\Omega(y) = \{(z^s, z^g) : s \geq z^s \leq j^s, g \geq z^g \leq j^g\}$$

Agents can melt coins without incurring a cost. However, if agents mint, they incur a seigniorage cost $S(j_t^s, z_t^s, j_t^g, z_t^g)$ in terms of foregone utility from holding jewelry. Specifically, we assume

$$S(j_t^s, z_t^s, j_t^g, z_t^g) = \max[\mu(b_s j_t^s, b_g j_t^g) - \mu(b_s j_t^s - b_s \sigma_s z_t^s, b_g j_t^g - \sigma_g b_g z_t^g), 0]$$

where σ_j seigniorage rate on coin j. The term $\mu(b_s j_t^s, b_g j_t^g)$ is the utility from holding jewelry before doing any minting. The term $\mu(b_s j_t^s - b_s \sigma_s z_t^s, b_g j_t^g - \sigma_g b_g z_t^g)$ is the utility from holding that amount of jewelry less the amount of jewelry given up to pay the seigniorage tax on minting coins from jewelry. The max with 0 is because seigniorage is only paid on minting. As will become clear later, we do not assume that any units of metal are lost by an agent when minting. Instead, we assume that the agent has to pay a utility cost which depends on the agent's initial holdings of jewelry, the seigniorage rate, and how many coins are minted.

2.3 Equilibrium

The three components needed are the value functions (Bellman equations), the asset transition equations, and the market clearing conditions. We proceed to describe each in turn.

Value functions

Let $w_t(y_t)$ and $v_t(y_t)$ be the expected values of holding y_t at the beginning of the first and second subperiods of period t, respectively. Then the Bellman equation at the beginning of the second subperiod is

$$v_t(y_t) = \max_{(z_t^s, z_t^g) \in \Omega(y)} \{\beta w_{t+1}(s_t + z_t^s, g_t + z_t^g, j_t^s - z_t^s, j_t^g - z_t^g) - S(j_t^s, z_t^s, j_t^g, z_t^g)\}$$

Note that we require that when coins are minted, it must be from jewelry of the same metal. This was a restriction imposed by all real world mints under commodity money regimes. Note also that when coins are melted, agents receive jewelry of the same metal. Imposing this seem obvious. Further, note that no metal is lost in minting or melting as mentioned in the discussion of seigniorage above.

The Bellman equation at the beginning of the first subperiod is

$$\begin{aligned} w_t(y_t) &= \frac{\theta}{2} \sum_{\tilde{y}_t} \pi_t(\tilde{y}_t) \max_{(q_t, p_t^s, p_t^g) \in \Gamma(y, \tilde{y})} [u(q_t) + v_t(s_t - p_t^s, g_t - p_t^g, j_t^s, j_t^g)] \\ &\quad + (1 - \frac{\theta}{2}) v_t(y_t) + \mu(b_s j_t^s, b_g j_t^g) \end{aligned}$$

where $\pi_t(y_t)$ is the fraction of agents with y_t at the beginning of the first subperiod. The first term on the right-hand side is the expected payoff from being a buyer in a single coincidence meeting, which occurs with probability $\frac{\theta}{2}$. The second term is the expected payoff either from being the seller in a single coincidence meeting or from not have a meeting. The final term is the utility from holding silver and gold jewelry.

Asset holdings

Define $\lambda^b(k, k'; y_t, \tilde{y}_t)$ to be the probability that a buyer with y_t meeting a seller with $\tilde{y}_t$ leaves with $s = k, g = k'$. That is,

$$\lambda^b(k, k'; y, \tilde{y}) = \begin{cases} 1 & \text{if } k = s - p_s(y, \tilde{y}) \text{ and } k' = g - p_g(y, \tilde{y}) \\ 0 & \text{otherwise.} \end{cases}$$

and define $\lambda^s(k, k'; y_t, \tilde{y}_t)$ similarly for the seller.

Then the post-trade (pre-mint/melt) asset distribution is

$$\begin{aligned} \tilde{\pi}_t(k, k', j^s, j^g) &= \frac{\theta}{2} \sum_{y_t, \tilde{y}_t} \pi_t(y_t) \pi_t(\tilde{y}_t) [\lambda^b(k, k'; y_t, \tilde{y}_t) + \lambda^s(k, k'; y_t, \tilde{y}_t)] \\ &\quad + (1 - \theta) \pi_t(k, k', j^s, j^g) \end{aligned}$$

The first term on the right-hand side is the fraction of single coincidence meetings in which the buyer leaves with k silver coins and k' gold coins. The second term is the fraction of such meetings in which the seller leaves with k silver coins and k' gold coins. The final term is the probability that no meeting occurs, in which case no coins change hands.

Next define $\delta(k, k', h, h')$ to be the probability an agent with y_t after trade has $s = k$, $g = k'$, $j^s = h$, and $j^g = h'$ after minting or melting. Then the post-mint/melt (pre-trade next period) asset distribution is

$$\pi_{t+1}(k, k', h, h') = \sum_{y_t} \tilde{\pi}_t(y_t)\delta(k, k', h, h')$$

Of course, asset holdings must also satisfy $\sum_y \pi(y) = \sum_y \tilde{\pi}(y) = 1$.

Market clearing

The market clearing conditions are that the stocks of gold and silver metal must be held. That is,

$$\sum_y b_s(s + j^s)\pi(y) = m_s$$

$$\sum_y b_g(g + j^g)\pi(y) = m_g.$$

Definition 1 ***Steady state equilibrium:*** *A steady state equilibrium is value functions w, v; asset holdings π and $\tilde{\pi}$; and quantities p^s, p^g, z^s, z^g, q that satisfies the value functions, the asset transition equations, and market clearing.*

3 Results

We are unable to prove the existence of steady state equilibria or to obtain analytic results for our model. Therefore, we rely instead on computed equilibria for numerical examples to obtain our results. Specifically, we assume $u(q) = \sqrt{(q)}, \mu(b_s j_s, b_g j_g) = 0.02(b_s j_s)^{1/2} + 0.2(b_g j_g)^{3/4}, \beta = 0.9, \theta = \frac{2}{3}, m_g = 0.1$, and $m_s = 0.05$. For most of our examples we will assume a gold coin size of $b_g = 0.1$, which implies that there is a total of 1 unit of coins plus jewelry. Because we are interested in small change, we will consider various values for b_s. Although we have not attempted to calibrate the parameters in the numerical analysis, the relative weights of silver and gold in the jewelry utility function are chosen to approximate the relative market ratio of 1:10 for these metals during the early medieval period. Further, we use seigniorage rates of $\sigma_s = 0.04$ and $\sigma_g = 0.02$ which were approximately the average seigniorage rates on silver and gold during this period.[1] Throughout the analysis, our welfare criterion is ex ante welfare, computed as $\bar{w} = \sum_y \pi(y)w(y)$.

The size of the silver coin affects the distribution of coin and jewlry holdings of the agents in the model. The distribution of coin holdings in sample steady states for coin sizes of $b_s = .0083$ and $b_s = .0167$ are shown in Figure 1. For the smaller coin size, $b_s = .0083$,

[1]Even though the theoretical model assumes no upper bound on coin holdings, we impose upper bounds $S = 20$ on silver coin holdings and $G = 2$ on gold coin holdings.

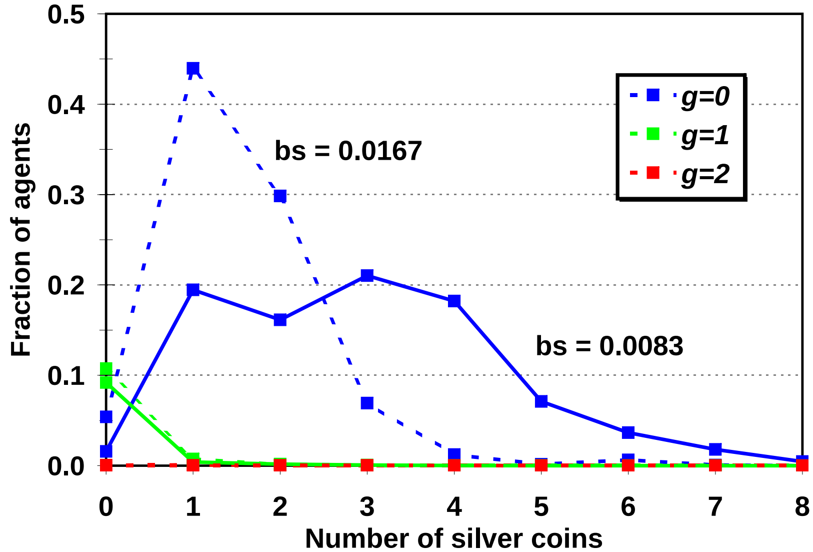

Figure 1: Distribution of pre-trade coin holdings for various b_s

approximately 1.2% of agents hold zero coins of either type and 6.1 % of agents hold no jewelry. The most widely held portfolio is three silver coins, one gold coin, and one unit of jewelry of each kind. It is held by 7.7 % of the agents. For the larger coin size, more agents (5.4 %) hold no coins and more agents (14.8 %) hold no jewelry. The most widely held portfolio also has fewer coins and less jewelry. It consists of one silver coin and one unit of gold jewelry only. It is held by 5.1 % of agents.

The agents in the economy both trade coins for goods and mint and melt coins. This is shown in Figure 2, which is constructed assuming a silver coins size of $b_s = 0.0083$. The solid lines in the Figure are the pre-trade distribution of coin holdings and the dashed lines are the post-trade distribution of coin holdings. If there were no trade and no minting or melting, then these two lines would be the same. However, clearly they are not.

The intuition for the changes in the distribution of coin holdings due to trade is the following. When agents buy, they run down their holdings of coins, particularly silver coins. For this reason, there are many more agents with no silver coin holdings and fewer agents holding relatively few silver coins after trade than before trade. However, because coins do not disappear during trade, coins have to be held by some agent. This is the reason that the post-trade line lies above the pre-trade line for higher silver coin holdings. Then after trade, agents rebalance their coin portfolios by minting or melting.

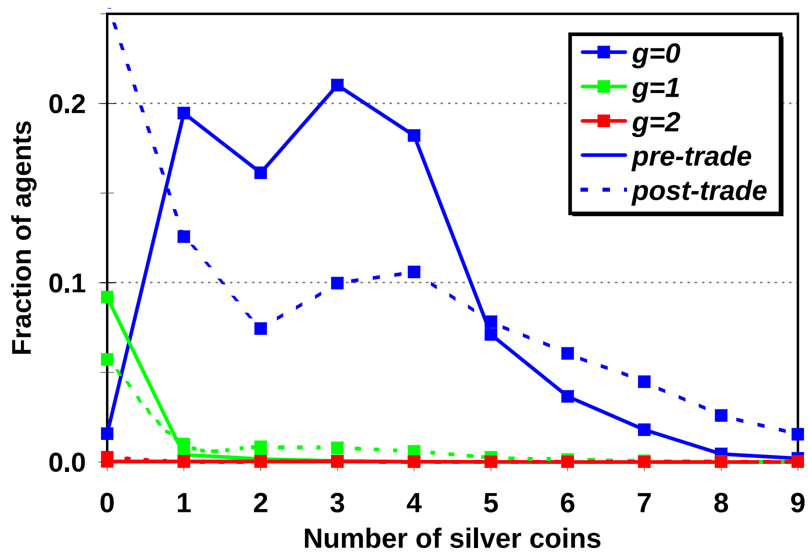

Figure 2: Distribution of pre- and post-trade coin holdings, $b_s = 0.0083$

3.1 Optimal coin size

We now examine whether there is an optimal (in terms of ex ante welfare) ratio between the sizes of the coins and whether changes in the trading opportunities available to agents in an economy (think: development of organized markets) affect the optimal size of silver coins.

Our model's results concerning whether there is an optimal ratio between large and small coins are given in Figure 3. There we plot as a function of the size of a silver coin the ex ante welfare of the agents in our example economy.

Figure 3 shows that decreasing the size of the silver coin can improve welfare up to some point, but welfare can also decrease if silver coins become too small. Some insight into why this occurs is provided in Figures 4 and 5. Figure 4 shows the contribution to welfare from the ability to use silver coins in trade which is equal to

$$\hat{w} = \frac{\theta}{2(1-\beta)} \sum_{y} \sum_{\tilde{y}} \{u[(q(y,\tilde{y}] - q(y,\tilde{y})\},$$

where $q(y,\tilde{y})$ is the quantity traded in $y,\tilde{y}$ matches. This curve shows the same general shape as that total ex ante welfare, although it peaks at a large size for the silver coin. The shape of this curve is explained by how $q(y,\tilde{y})$ varies with silver coin size and how that affects $u[q(y,\tilde{y}] - q(y,\tilde{y})$. As silver coins become larger, buyers demand more output from the seller in order to trade. Up to some point, this larger $q(y,\tilde{y})$ increases $u(q(y,\tilde{y}) - q(y,\tilde{y})]$, which is the surplus in the trade. However, after some point, surplus decreases with $q(y,\tilde{y})]$, and welfare from transactions declines.

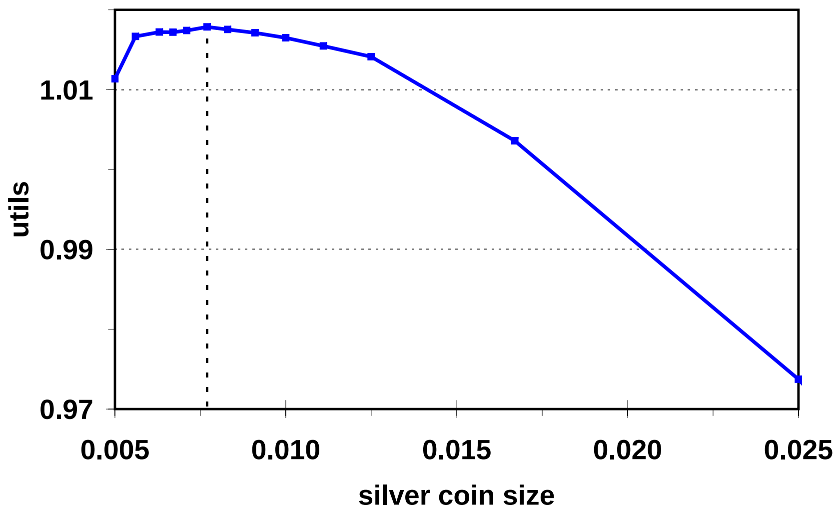

Figure 3: Ex ante welfare for various sizes of small coins

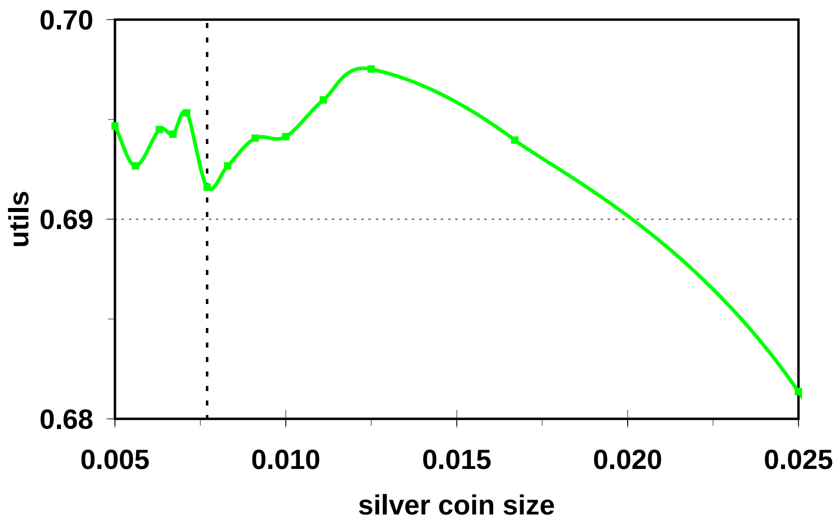

Figure 4: Ex ante welfare from transactions for various sizes of small coins

However, agents also obtain utility from their jewelry holdings, and this is shown in 5. The figure shows that ex ante welfare from silver jewelry holdings is roughly constant for small coins sizes and then declines. For gold jewelry, ex ante welfare at first increases and then declines, and this decline in utility from gold jewelry more than offsets the fact the ex ante welfare tends to increase for slightly larger silver coins.

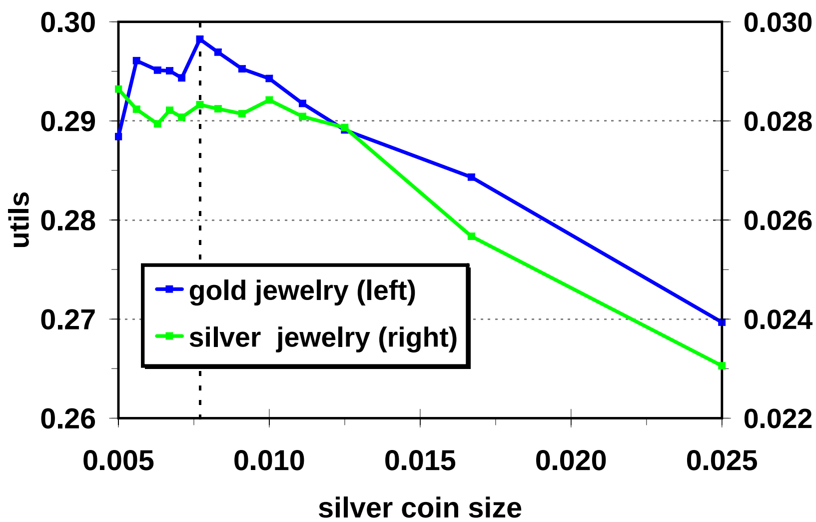

Figure 5: Ex ante welfare from gold and silver jewelry for various sizes of small coins

This reason these effects occur is shown in Figure 6. This figure shows the fraction of matches in which silver coins are used in trade alone, are given as change, or are given with gold coins. The fraction of matches in which only gold coins are used in trade is also shown. [2]

When silver coins are very small, gold coins are not very necessary for buyers to precisely adjust the TIOLI offer made to the seller. In fact, almost 90 % of all trade occurs solely with silver coins. Because gold coins are not required for many trades, agents hold their gold in the form of utility-yielding jewelry. However, as silver coins are made large, gold coins, because they are not integer multiples of silver coins in utility, have a larger role to play in enabling buyers to calibrate their offers. This is shown in the figure by the increasing fraction of trades (the sum of all trades except those with silver only) involving gold. So, less gold is devoted to jewelry and the welfare from gold jewelry falls as shown in Figure 5.

The decline in the welfare from silver jewelry occurs because after some point, as silver coins become larger, more silver gets devoted to coins and less to jewelry. This is explained by the fact that even when silver coins are relatively large, the vast majority of trades still

[2] The fractions do not add to one because there are matches in which no trade occurs because either the buyer has no coins or cannot make an offer that makes him better off and that the seller will accept.

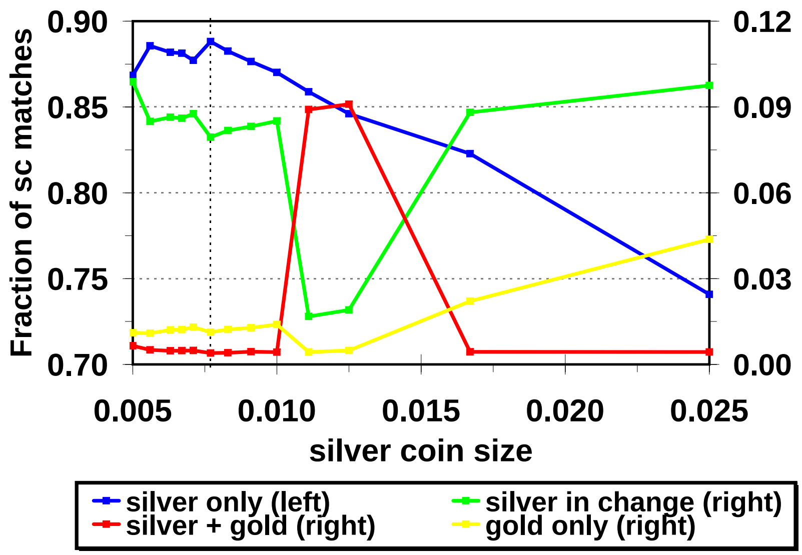

Figure 6: Fractions of matches by type of trade by size of silver coin

involve silver coins, and as silver coins become large, this requires more silver to be devoted to coinage.[3]

3.2 Welfare losses relative to planner's allocation

Suppose that a benevolent planner could implement the ex ante welfare maximizing allocation through a gift-giving allocation that did not require the use of coins to facilitate trade. This allocation would have the quantity q^* exchanged in each match, where $u'(q^*) = 1$, have agents holding the entire stocks of gold and silver as jewelry, and have no seigniorage collected by the sovereign. The question we examine is the extent to which the welfare losses in our model economy are due to the fact that q^* is not exchanged in each match, the extent to which they are due to metals being allocated to non-utility yielding use as coins, and the extent to which they are due to the cost of seigniorage.

To do this we computed the welfare loss due to

- q^* not being exchanged in each match:

$$\frac{\theta}{2(1-\beta)}\{[u(q^*) - q^*] - \sum_{y,\tilde{y}} \pi(y)\pi(\tilde{y})\{u[q(y,\tilde{y})] - q(y,\tilde{y})\}\}$$

[3]Note also the change in how silver and gold coins are used for payments **as not are** the size of the silver coin changes. Usually, silver coins are demanded as change by the buyer. However, for a few silver coin sizes this pattern changes and both types of coins are offered by the buyer. Then as silver coins become even larger, the offers go back to the more usual pattern of silver coins being used to make change.

- metal not held as jewelry:

$$\frac{1}{1-\beta}\{\mu(ms, mg) - \sum_y \pi(y)\mu[b_s j^s(y), b_g j^g(y)]\}$$

- seigniorage:

$$\frac{1}{1-\beta}\sum_y \pi(y) S[j^s(y), z^s(y), j^g(y), z^g(y)]$$

  where $j^s(y)$ is silver jewelry held by agent with y, $j^g(y)$ is gold jewelry held by agent with y, $z^s(y)$ is silver minted by agent with y, and $z^g(y)$ is gold minted by agent with y.

The results are shown in Figure 7. We find that by far the largest losses are due to either more or less than the optimal amount of output being produced and consumed in each match. The next largest loss is due to the use of gold for coinage rather than jewelry. The loss due to having silver as coinage is relative small, which is expected due to its low weight in the jewelry utility function. Finally, the welfare loss due to seigniorage is so small that it does not show in the figure even though it is plotted.

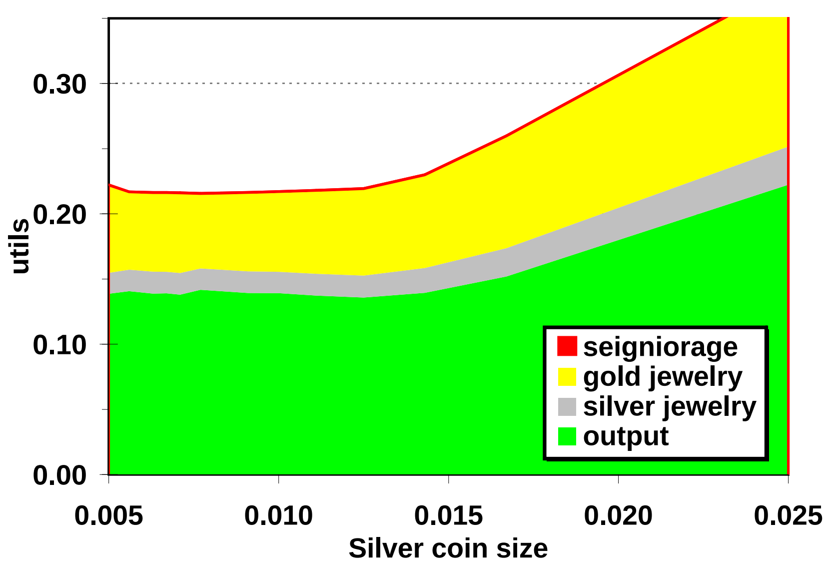

Figure 7: Welfare loss by source by size of silver coin

3.3 Sovereign seigniorage

To this point we have only considered how the size of the silver coin affects the welfare of agents in the economy. We have not considered how the sovereign, the agent who runs the mint and collects the seigniorage revenue, is affected.

There are two possible ways we could do this. One would be to take the sovereign's revenue to be the utility loss in terms of jewelry experienced by the other agents when they pay the seigniorage tax to mint coins. The second is to assume that the seigniorage tax paid by agents is actually gold and silver and sovereign's obtain utility from this jewelry according to the same utility function as that of the other agents. That is, the utility of the sovereign is

$$\mu(\sigma_s b_s Z^s, \sigma_g b_g Z^g)$$

where $Z^s(Z^g)$ is total quantity of silver (gold) minted. We choose this second way.

The utility of the sovereign from seigniorage is shown in Figure 8. It shows that for small silver coin sizes, the sovereign's utility is lower the larger the coin. However, after some point this changes and the sovereign's utility is higher the larger the silver coin. Thus, there is a tension between the sovereign and the other agents in the economy. The sovereign would like to have large silver coins to maximize the utility from seigniorage. Ordinary agents would like small silver coins to allow them to maximize the welfare from trading.[4]

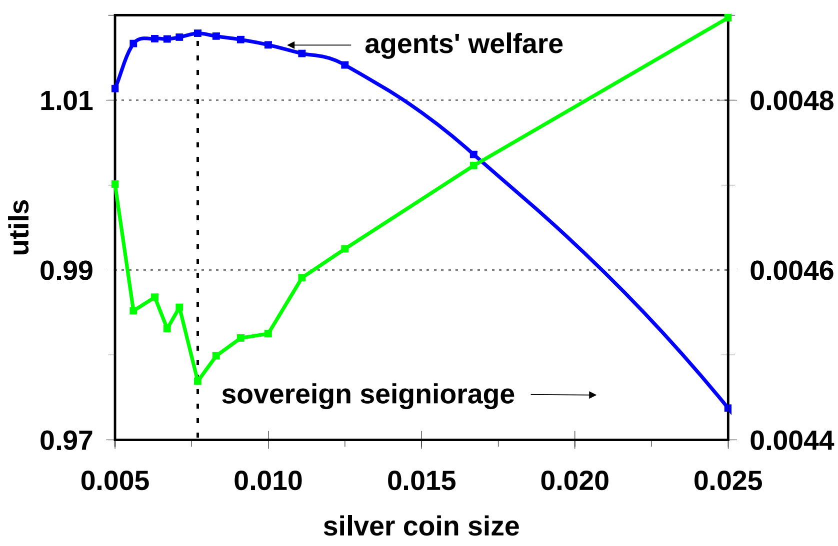

Figure 8: Sovereign utility from seigniorage by size of silver coin

[4]In the figure the size of silver coin that maximizes the ex ante welfare for ordinary agents minimizes the utility from seigniorage for the sovereign. This is a special case for this numerical example and is not true in general.

3.4 Trading opportunities

In the model, θ is the probability of a match, so changing θ can be interpreted as changing the trading opportunities available to agents in the economy. We now examine how changes in trading opportunities available to agents in an economy affect the optimal size of silver coins. To do so, we compute ex ante welfare for $\theta = \{\frac{1}{5}$ and compare it to the welfare computed above for various sizes of silver coins, holding the size of the gold coin fixed.

The results are shown in Figure 9. When $\theta = \frac{1}{5}$, that is, when an agent has a one-in-five chance of having a single coincidence match in any period, the ex ante welfare maximizing size of the silver coin is $b_s = 0.0063$. When $\theta = \frac{2}{3}$, the case studied above, the optimal coin size is larger, $b_s = 0.0077$. Thus, our model indicates that increasing the size of silver coins would be an optimal response to an increase in a country's trading opportunities.

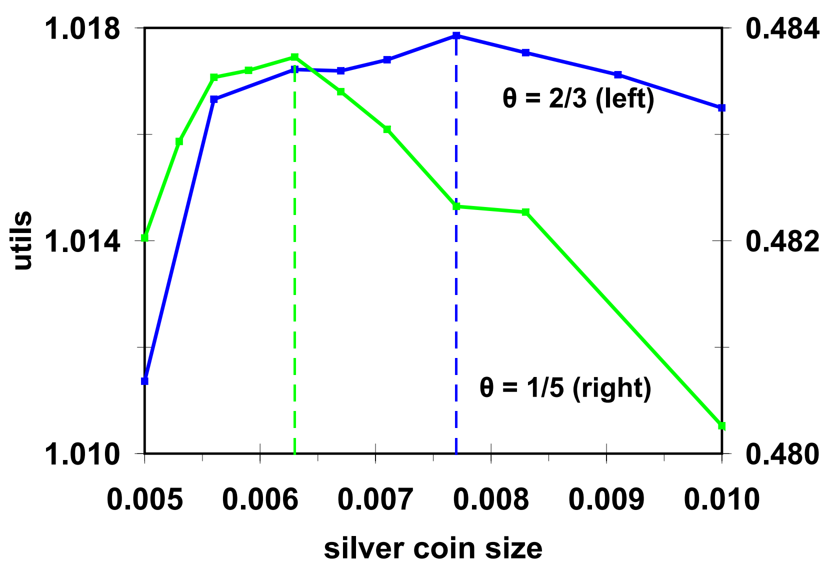

Figure 9: Ex ante welfare as a function of silver coin size for various θ

Intuitively, the smaller the silver coin, the more finely a potential buyer is able to calibrate an offer of wealth (in the form of coins) for goods to a potential seller. The finer the wealth offer that a buyer can make, the less likely it is that the buyer will have to take a smaller quantity of goods from the seller in exchange for a given wealth transfer or have to give up additional wealth to get the desired quantity. This is more important the less likely it is that a buyer will meet a potential seller.

Bibliography