An Option Pricing Approach to Stress-testing the Canadian Mortgage Portfolio.

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Abstract

In this work, we propose an option-pricing approach to stress-testing the Canadian mortgage portfolio. This approach combines results from a theoretical two-factor contingent-claims pricing model with microdata on the Canadian mortgage market. It uses the assumption that rational homeowners default on valuable property only when it is in their financial interest to do so. The decision to default is then analyzed as an intertemporal optimization problem in a stochastic economic environment. In order to illustrate its usefulness, we apply this approach to a base case reflecting the very favourable environment that prevailed over the 2001-2006 period, and to stress situations where housing prices are falling. Compared with actual default rates, our estimated measure of overall defaults appear reasonable and in the general range of historical experience.

Key words*:* Mortgage default, option pricing, fixed rate mortgages, stress test.

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1. Introduction

Since the early 2000s, there has been a sustained and rapid growth in the Canadian housing and residential mortgage markets. This growth, likely driven by historically low interest rates, strong labour market and by a more intense competition in the housing finance market[1], has led to a high reliance of the loan portfolio of Canadian commercial banks on residential mortgage loans.

Despite the high exposure of Canadian chartered banks to the housing and mortgage markets, most of the risk of default on mortgages rests with the mortgage insurers rather than with the commercial banks. This is because, in Canada, mortgages with a down payment less than 20 per cent must be insured. These insured mortgages represent around half of the total residential mortgage balances outstanding at chartered banks. On the remaining uninsured mortgages, banks take low risks since these loans are backed by relatively high collateral.

There is currently no sign of a deterioration in credit quality in Canada, with current default rates on residential mortgages being at a near-historical low. Nevertheless, the recent episode of strong growth in housing credit and prices could be the background for a potentially riskier housing loan environment assuming a trend reversal of the growth in housisng credit and price. This is supported by the empirically negative relationship between nominal housing price growth and default rates, and between residential mortgage credit growth and default rates (see Figure 3 in Appendix A).[2]

It is therefore important to improve our ability to assess the risks to the financial system associated with the housing and the mortgage markets. In this work, we outline the development of an option-pricing based approach to implementing a stress test of the housing mortgage portfolio of Canadian banks. We focus exclusively on fixed-rate mortgage loans which account for about 75 per cent of total mortgage loans outstanding in Canada.

This approach diverges from previous empirical work on mortgage default in at least two respects. First, it does not rely on econometric modeling to identify factors (both systematic and idiosyncratic) that determine the probability of default on actual mortgages[3]. It

[1] The increased competition among housing credit providers in Canada (which are mainly commercial banks) has resulted in an increasing range of financial products enabling Canadian consumers to gain easier access to mortgage credit. For more details on the main developments and trends in the Canadian mortgage market, see Traclet [35].

[2] This fact is well documented in the literature. See among others Brio, Furfine and Lowe[5].

[3] For example, Coleman et al.[10] presents a case study for developing stress tests of housing loan portfolios for the Austrian case. Their study uses logistic regressions relating real house price growth and mortgage loan characteristics (such as the loan-to-value (LTV) ratio at origination, the age of the mortgage loan and its size) to the probability of default and to the loss given default.

rather uses a structural two-factor mortgage pricing model in which rational mortgage-holders choose when to default in response to changes in, specifically, house prices and interest rates. To estimate the overall risk of default on Canadian mortgages, theoretical results from this model are combined with microdata on the Canadian mortgage market. To our best knowledge, we are the first to develop such a methodology within the framework of stress-testing mortgage portfolios. Second, our approach is confined to analyze only financially motivated defaults rather than observed defaults which, among other reasons, may be caused by income constraints such as those caused by job loss.

In order to illustrate how this approach could be applied to assessing potential risks to the Canadian financial system, we calculate an overall default rate for several scenarios: a base case, reflecting the average economic situation in Canada over the 2001-2006 period, and stress scenarios in which housing prices are falling. Our measure of the overall risk of default is estimated by applying the probabilities of default on mortgage loans with different LTV ratios at origination to the empirical LTV ratios distribution obtained from the Canadian Financial Monitor (CFM), a survey conducted by Ipsos-Reid Canada. Our estimated overall default rate will then be compared with actual default rates. This comparison is intended only to provide a rough test of whether our estimates are in the general range of historical experience.

Our simulations suggest that the impact of decreasing housing prices on the overall risk of default is not linear. The risk of default on high LTV ratio loans increases more, for a given shock, than the risk of default on low LTV ratio loans. Compared with actual default rates, our estimated default rates appear reasonable and in the general range of historical experience.

The paper proceeds as follows. In the second section, we present the model used to estimate the probability of default of the representative mortgagor. Section three lists the main caveats that apply to this work. Section four is dedicated to present and analyze the main results and conclusions from the simulation process. The paper concludes with some comments on possible extensions of the current study.

2. The option-pricing model

There is a growing body of literature on mortgage default risk and how it relates to house prices and interest rates. One strand of this literature is motivated by option theory and follows the seminal work on pricing contingent claims in capital markets by Black and Scholes [3] and Merton [30]. It maintains that, under conditions of limited liability, negligible transactions costs and no exogenous reasons for residential mobility, the decision to default can be viewed as a financial one.[4]

In this work, we follow this literature in analyzing the homeowner's decision to default. We consider that mortgage borrowers in a perfectly competitive market can increase their wealth by defaulting when the market value of the mortgage equals or exceeds the market value of the collateral, which depends on the price of the house. We use a standard two-factor theoretical contingent-claims pricing model. The two factors are the housing price and the short term interest rate. These factors are assumed to be stochastic. Subsection 2.1 contains a brief description of the assumed continuous time processes for how these variables evolve over time.

This model, which was initially developed to evaluate fixed-rate mortgage contracts[5], generates all the information we need to compute the probability of default on any fixed-rate mortgage contract. This computation requires a two-step default analysis. In the first step, we analyze the decision to default of a representative mortgagor and determine where defaults occur in the state space defined by housing prices and interest rates. The borrower's problem is resolved by 'discretizing' the evolution of the processes of these variables. This is done using the bivariate binomial approximation technique outlined in Hilliard, Kau and Dlawson [21]. This technique and the possible decisions of the mortgagor, and the solving of its problem are outlined in subsections 2.2 and 2.3 respectively. In the second step, we use the forward recursion technique developed in Capozza, Kazarian and Thomson [8] to determine the probability of reaching such default regions. This technique is briefly described in Appendix D.[6]

2.1 Description of the factors' stochastic processes

House prices (H) are assumed to follow the standard Geometric Brownian Motion, with α_H representing the instantaneous total expected return and σ_H the proportionate volatility. The return from owning a house (α_H) consists of price appreciation ($\frac{dH}{H}$) and a service flow, s, from using the house over time. The relevant stochastic process is

$$\frac{dH}{H} = (\alpha_H - s)dt + \sigma_H dz_H \tag{1}$$

[4] For a more detailed discussion, see Deng, Quigley and Van Order [14] and Kau, Keenan, Muller, and Epperson [28].

[5] As was pointed out in Chatterjee, et al. [9], the two-factor model is efficient in predicting market mortgage values.

[6] A summary of methodology and results in this paper will be published in the December 2007 Bank of Canada Financial System Review.

where dz is a Wiener process.

This process is originally adapted from the stock process in the tradition of Black and Scholes [3] and Merton [30]. It is a continuous-time Markov process where the house price depends only upon most recent history. It has an absorbing barrier at zero, meaning that if H ever becomes zero, it remains zero thereafter and can never be negative.

Interest rates are the discounting rates and are assumed to follow the Cox, Ingersoll and Ross [11] process:

$$dr = \gamma(\theta - r)dt + \sigma_r \sqrt{r} dz_r \tag{2}$$

This is a mean-reverting process, with θ representing the long-term value for the interest rate, γ the speed of adjustment, and $\sigma_r \sqrt{r}$ the volatility of the interest rate. This process assumes that the interest rate reverts toward its long-term value θ at rate γ, but that it is constantly disturbed by stochastic events, as represented by the Wiener process dz_r. This process ensures that if initially $\gamma \geq 0$ and $r \geq 0$, then subsequent negative interest rates are precluded.

We suppose that the correlation between dz_H and dz_r is ρ (i.e. $dz_H.dz_r = \rho dt$). We might expect ρ to be negative since decreased interest rates are likely to stimulate the demand for durable assets. This was the case in Canada during the last housing boom.

As is standard convention in the existing literature, we use the risk-neutral valuation principal. For this, the expected rate of total return of the house (α_H) needs to be adjusted for market risk such that the risk-free rate of interest (r) can be substituted for the expected total return of owning the house[7]. The substitution yields the following risk-neutral process for house price:

$$\frac{d\widehat{H}}{\widehat{H}} = (r - s)dt + \sigma_H dz_H \tag{3}$$

We refer to Ingersoll [24] for a further discussion of the above procedure and merely note that the entire risk-adjustment argument follows not from some restriction on risk preferences but from arbitrage arguments based on the assumption that there are perfectly competitive markets continuously open in the housing asset[8].

[7] Note that in the course of transforming the house price process to its risk-adjusted form, all reference disappears to α_H. In this sense, the values of the mortgage and the default option embedded on the mortgage are independent of α_H.

[8] As showed in details in Appendix B, for consistency, the risk-neutralized house price process described by equation (3) is used when modeling the default decision; however, the original process is used when modeling the probability of default.

2.2 The borrower's problem

Consider a 5-year term mortgage contract, with a 25-year amortization period. In our model, for every possible outcome for housing prices and interest rates over the length of the contract, the borrower faces three exclusive options: making the required monthly payment, defaulting, or prepaying the mortgage.

The opportunity to default is treated as a put option since it enables the borrower to sell his property to the mortgagee at a price equal to the loan's outstanding balance. The modeling of defaults on mortgages as put options held by the mortgagor was developed by Foster and Van Order [18, 19] and extended by Epperson, et al. [16]. Such treatment has become standard in the mortgage termination literature. Simply stated, the opportunity to default has value if the value of the home falls below the expected value of remaining payments (see among others Crawford and Rosenblatt [12] and Ambose and Capone [1]).

The mortgagor also has the opportunity to prepay his mortgage loan.[9] Prepayment can be viewed as refinancing. We treat the opportunity to prepay a mortgage as a call option, in that it allows the borrower to buy all future obligations remaining under the mortgage at a price equal to the loan's outstanding balance (Maris and Yang, [31]). Prepayment has value if interest rates fall below one's fixed mortgage rate to such an extent that the expected present value of remaining payments becomes higher than the unpaid mortgage balance.

Note that closed mortgages can not generally be paid off before maturity without paying a penalty. Prepayment penalties in Canada are frequently calculated as three months interest applied to the outstanding balance. This is what we will use as prepayment penalty in our simulations.

These options are embedded in the sense that they give the mortgagor, not only the opportunity to default or prepay now, but also the opportunity to postpone the default or the prepayment by at least one period to see if postponement will provide additional value. Note that these options compete against each other. For example, when an individual decides to exercise the default option, she is making the decision to forego both current and future exercise of the prepayment option.[10].

Note that a necessary condition for exercising these options is that they be 'in the money' but that is not sufficient. For example, a borrower whose house price declines below the

[9] As suggested in Deng, Quigley and Van Order [14] and in Deng and Gabriel [13], one cannot calculate accurately the economic value of the default option without considering simultaneously the financial incentive for prepayment.

[10] Kau et al. [28] and Kau and Keenan [25] have outlined the theoretical relationships among default and prepayment options, and Schwartz and Torous [34] have demonstrated their practical importance.

mortgage balance (negative equity) may not find optimal to default at the current period because, by defaulting, the borrower would forfeit the options to default and to prepay later.

As mentioned above, the analysis of the mortgagor's decisions requires a transformation of the continuous time processes of r and $\widehat{H}$ to time discrete ones. This is done using the bivariate binomial technique which was initially developed by Nelson and Ramaswamy [32] and then extended by Hilliard, Schwartz and Tucker [22] to the case of two correlated stochastic variables. This technique is detailed in Appendix B.

Using this technique, we obtain a two-dimensional binomial lattice for r and $\widehat{H}$. As shown is Figure 1, from any node $\left(r_t, \widehat{H}_t\right)$ at time t, the lattice evolves to four nodes at the next time step, $\left(r^+_{t+1}, \widehat{H}^+_{t+1}\right)$, $\left(r^+_{t+1}, \widehat{H}^-_{t+1}\right)$, $\left(r^-_{t+1}, \widehat{H}^+_{t+1}\right)$ and $\left(r^-_{t+1}, \widehat{H}^-_{t+1}\right)$, where subscripts $+$ and $-$ represent respectively the up and down jumps in our variables. In a nutshell, at every period, the borrower solves a dynamic problem where not only today's options are considered but also the potential options during the rest of the contract. This is done on the basis of the current values of the transformed housing price and the interest rate $\left(r_t, \widehat{H}_t\right)$, their ex-ante possible values in the next period and their respective probabilities.

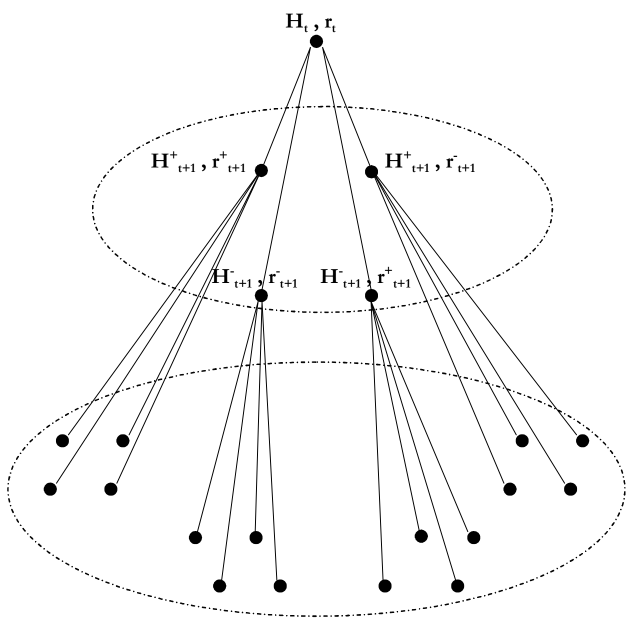

Figure 1: Two-period representation of the bivariate binomial tree

2.3 Solving of the mortgagor's problem

In what follows, we will focus on the borrower's decisions during the first five years of the contract. Before we value the borrower's options, it is useful to introduce the used notation.

- L is the amount initially loaned to the borrower and c_1 is the fixed yearly interest rate of the first 5-year mortgage contract. The corresponding monthly mortgage payment is given by the standard annuity formulae:

$$M_{c_1} = \frac{(\frac{c_1}{12})(1+\frac{c_1}{12})^{(25*12)}}{(1+\frac{c_1}{12})^{(25*12)}-1} * L$$

- UMB_t is the unpaid mortgage balance at time t. At any date before the maturity of the 5-year contract ($0 \leq t < 60$), UMB_t can defined as follows:

$$UMB_t = \frac{(1+\frac{c_1}{12})^{(25*12)} - (1+\frac{c_1}{12})^{t}}{(1+\frac{c_1}{12})^{(25*12)}-1} * L$$

- $PVRP_{t,r}$ is the present value of remaining payments. At any time t before the maturity of the contract ($0 \leq t < 60$), $PVRP_{t,r}$ is given by:

$$PVRP_{t,r} = q_t \left(\frac{PVRP^u_{t+1}}{1+\frac{r^u_{t+1}}{12}} \right) + (1-q_t) \left(\frac{PVRP^d_{t+1}}{1+\frac{r^d_{t+1}}{12}} \right)$$

  where q_t is the probability of an up-jump in r process at time t (see Appendix B for the exact formula of q_t) and $PVRP^u_{t+1}(PVRP^d_{t+1})$ is the present value of the remaining payment at time $(t+1)$ if the up (down) state for the interest rate occurs at that time, i.e. if $r_{t+1} = r^u_{t+1}$ (if $r_{t+1} = r^d_{t+1}$).

At any point in time, the borrower maximizes his wealth by choosing one among three exclusive actions: default, prepayment and continuing the contract. In our context, the wealth of the borrower is given by his position which equals the value of the house, $\widehat{H}$, less the value of the mortgage ($V_{t,r,\widehat{H}}$). The value of the mortgage equals the present value of the remaining payments (a liability for the borrower), $PVRP_{t,r}$, minus the value of the default option, $D_{t,r,\widehat{H}}$, minus the value of the prepayment option, $P_{t,r,\widehat{H}}$ (these two options are assets on the borrower's balance sheet). At any node in the bivariate binomial tree during the life of the mortgage contract,

$$V_{t,r,\widehat{H}} = PVRP_{t,r} - D_{t,r,\widehat{H}} - P_{t,r,\widehat{H}} \tag{4}$$

Since the house price is exogenous and is not affected by the borrower's actions, then, at any point of time, maximizing his position is equivalent to minimizing the value of the mortgage. Thus, the borrower determines which action among default, prepayment and continuation provides the lowest value of the mortgage. Before the value of the mortgage is determined, the values of the options are calculated for each of the three actions.

Option values

At a particular node, there are three possible scenarios: default, prepayment and continuation.

- The borrower may be tempted to *default on the loan* if the expected present value of the remaining payments is higher than the current house price. Otherwise, immediate default is worthless. Then, the value of an immediate default is :

  $$D_{t,r,\widehat{H}} = \max\left\{PVRP_{t,r} - \widehat{H}, 0\right\}$$

  If default occurs, the borrower loses the option to prepay, so that

  $$\begin{cases} D_{t,r,\widehat{H}} = PVRP_{t,r} - \widehat{H} \\ P_{t,r,\widehat{H}} = 0 \end{cases}$$

- The borrower may decide to *prepay the loan* if the present value of the remaining payments is higher than the unpaid mortgage balance at time t, UMB_t. Otherwise, immediate prepayment has no value. Then, the value of an immediate prepayment is:

  $$P_{t,r,\widehat{H}} = \max\{PVRP_{t,r} - UMB_t, 0\}$$

  If prepayment occurs, the borrower loses the option to default, so that

  $$\begin{cases} D_{t,r,\widehat{H}} = 0 \\ P_{t,r,\widehat{H}} = PVRP_{t,r} - UMB_t \end{cases}$$

- The borrower may choose to *continue the loan* at least one period and makes the current scheduled mortgage payment. If the mortgage continues, the borrower retains both the option to default and the option to prepay. In this case, $D_{t,r,\widehat{H}}$ and $P_{t,r,\widehat{H}}$ are the present value of future default options and the present value of future prepayment options, respectively, given that the house price and interest rate change in the next period according to the processes given by (2) and (3), so that

$$\begin{cases} D^c_{t,r,\widehat{H}} = & \delta_t \Big[p_t.q_t.V_{t+1,r^u_{t+1},\widehat{H}^u_{t+1}} + (1-p_t).q_t.V_{t+1,r^u_{t+1},\widehat{H}^d_{t+1}} + \\ & p_t.(1-q_t).D_{t+1,r^d_{t+1},\widehat{H}^u_{t+1}} + (1-p_t).(1-q_t).D_{t+1,r^d_{t+1},\widehat{H}^d_{t+1}} \Big] \\ \\ P^c_{t,r,\widehat{H}} = & \delta_t \Big[p_t.q_t.P_{t+1,r^u_{t+1},\widehat{H}^u_{t+1}} + (1-p_t).q_t.P_{t+1,r^u_{t+1},\widehat{H}^d_{t+1}} + \\ & p_t.(1-q_t).P_{t+1,r^d_{t+1},\widehat{H}^u_{t+1}} + (1-p_t).(1-q_t).P_{t+1,r^d_{t+1},\widehat{H}^d_{t+1}} \Big] \end{cases}$$

where:

- p_t (q_t) is the probability of an up-jump in $\widehat{H}$ (r) process at time t.
- r^u_{t+1} (r^d_{t+1}) is the interest rate at time $(t+1)$ if the up (down) state occurs.
- $\widehat{H}^u_{t+1}$ ($\widehat{H}^d_{t+1}$) is the house price at time $(t+1)$ if the up (down) state occurs.
- δ_t is the one-period discount factor for the current spot interest rate[11].

Possible values of the mortgage before maturity

Using equation (4), we obtain different values of the mortgage for the three scenarios. Indeed,

- if default occurs at time t, then the borrower transfers the house back to the lender (at the current price $\widehat{H}_t$). Then, the value of the mortgage is simply $\widehat{H_t}$. Indeed,

$$\begin{aligned} V_{t,r,\widehat{H}} &= PVRP_{t,r} - D_{t,r,\widehat{H}} - P_{t,r,\widehat{H}} \\ &= PVRP_{t,r} - (PVRP_{t,r} - \widehat{H}_t) - 0 \\ &= \widehat{H}_t \end{aligned}$$

- if prepayment occurs, then the borrower pays the unpaid mortgage balance and terminate the loan. In this case, the mortgage value is UMB_t. Indeed,

$$\begin{aligned} V_{t,r,\widehat{H}} &= PVRP_{t,r} - D_{t,r,\widehat{H}} - P_{t,r,\widehat{H}} \\ &= PVRP_{t,r} - 0 - (PVRP_{t,r} - UMB_t) \\ &= UMB_t \end{aligned}$$

[11] Let r_t be the annualized interest rate at date t. Then, we can write:

$$\delta_t = \frac{1}{1+\frac{r_t}{12}}$$

- if the borrower decides to continue the loan, the value of the mortgage is

$$V_{t,r,\widehat{H}} = PVRP_{t,r} - D^c_{t,r,\widehat{H}} - P^c_{t,r,\widehat{H}}$$

Using these results, the borrower assesses whether it is less costly to default, to prepay or to make the scheduled mortgage payment. This decision can be written as[12]:

$$V_{t,r,\widehat{H}} = \min\left[\widehat{H}_t, UMB_t, PVRP_{t,r} - D^c_{t,r,\widehat{H}} - P^c_{t,r,\widehat{H}}\right] \tag{5}$$

Terminal condition

To complete the borrower's problem, we have to specify the appropriate boundary condition at maturity (at time $t = 12 * 5 = 60$). At this time, the borrower has the choice between three actions: continuation, default and prepayment.

The choice to continue the mortgage contract means that the mortgagor holds the house but also that he has an obligation to make the final payment M_{c_1} and to renegotiate a new mortgage contract for the next five years. In this case, the value of the mortgage loan at a given node is:

$$V_{t=60,r,\widehat{H}} = M_{c_1} + \overline{V}_{t=60}$$

where $\overline{V}_{t=60}$ is the expected value (at time $t = 60$) of the next new 5-years mortgage loan starting at time $t = 61$. It depends on the mortgage contract rates of the four remaining five-year mortgage loans that the mortgagor would sign during the next twenty years. This is because, at the maturity date of each mortgage contract, the value to continue equals the last monthly payment plus the expected value, at that time, of subsequent mortgage loan. Also note that these contract rates should reflect the risk of mortgage default considering the evolution of house prices and interest rates during the next twenty years. Then, the exact way to value the option to continue at time $t = 60$ is to use the right contract rates when subsequent mortgage loan contracts will be renewed. To keep things simple, we assume in what follows that all future mortgage contract rates are equal to c_1. Under this assumption, we can value all subsequent mortgage contracts beginning with the last one and working recursively (see Appendix C for details of the valuation procedure).

Prepayment could be of value at the maturity of the loan if, at time $t = 60$, the present value of remaining payments is higher than the unpaid mortgage balance (i.e. $PVRP_{t=60,r} > UMB_{t=60}$), where:

[12] Note that if there are transactions costs to prepay (TC_P), equation (5) becomes:

$$V_{t,r,\widehat{H}} = \min\left[\widehat{H}_t, UMB_t + TC_P, PVRP_{t,r} - D^c_{t,r,\widehat{H}} - P^c_{t,r,\widehat{H}}\right]$$

$$\begin{cases} PVRP_{t=60,r} = q_t \left(\frac{PVRP^u_{61}}{1+\frac{r^u_{t+1}}{12}} \right) + (1-q_t) \left(\frac{PVRP^d_{61}}{1+\frac{r^d_{t+1}}{12}} \right) \\ UMB_{t=60} = \frac{(1+\frac{c_1}{12})^{(25*12)} - (1+\frac{c_1}{12})^{(5*12)}}{(1+\frac{c_1}{12})^{(25*12)}-1} * L \end{cases}$$

In that case,

$$V_{t=60,r,\widehat{H}} = UMB_{t=60}$$

Default could be of value in the case where the house price has fallen to less than the present value of remaining payment $PVRP_{t=60,r}$ (plus the last monthly payment). In that case, $P^D(\widehat{H}, r, t) = \widehat{H}_{60}$. This implies the following boundary equation at maturity[13] (at time $t = 60$):

$$V_{60,r,\widehat{H}} = \min\left(\widehat{H}_{60}, UMB_{t=60}, M_{c_1} + \overline{V}_{t=60}\right) \tag{6}$$

The solution for the optimal decision sequence and values is obtained by backward induction starting with the boundary condition at maturity and working to the present. Then, as detailed in Appendix D, we compute the *real* conditional probabilities of reaching these default regions using the forward recursion technique by Capozza, Kazarian and Thomson [8].

3. Caveats

Several caveats apply to our approach:

- As noted earlier, only voluntary defaults are considered in this work. It doesn't capture involuntary defaults caused by income constraints.
- Limited liability is assumed. This assumption may lead to an exaggerated measure of the risk of default on uninsured mortgages because, in Canada, uninsured borrowers remain liable for the unpaid balance of the mortgage loan over the value of the house. The extent of this exaggeration could be reduced by imposing a cost term to the unpaid balance of the mortgage over the current value of the house (at the time of default).
- Costs associated with the loss of reputation of a defaulting borrower are left out. As suggested in Kan, Keenan and Kim [27], these costs can be significant. The decision to

[13] Note that if there are transactions costs to prepay (TC_P), equation (6) becomes:

$$V_{60,r,\widehat{H}} = \min\left(\widehat{H}_{60}, UMB_{t=60} + TC_P, M_{c_1} + \overline{V}_{t=60}\right)$$

default can make it more difficult for the individual to obtain credit in the future. This creates an upward bias in our estimated probability of default. These costs could be incorporated into the default decision by adding a cost term to the outstanding balance at the time of default.

- As mentioned above, prepayment can be viewed as refinancing. Although refinancing, like prepayment, implies termination of the current mortgage contract, it also implies the origination of a new mortgage loan on which the borrower may default. This is not modeled in this work because of its complexity. Consequently, the probability of default that we compute at a given time is specific to the original mortgage contract. This leads to a downward bias in our estimated probability of default since refinanced mortgages are assumed not to default.

4. Simulations

The objective of these simulations is to illustrate how this model could be used in order to stress-testing the Canadian mortgage portfolio under a scenario of decreasing house prices.[14] We measure the overall default rate using a two-step default analysis. First, the probabilities of default for different LTV ratios are estimated using an option-pricing model as described in the above section. The overall default rate is then estimated by applying these probabilities to the empirical LTV distribution, which we construct from the CFM database.

4.1 Parameters of the simulations

We consider a representative homeowner who has taken out a five-year mortgage contract with a twenty five-year amortization period. To illustrate how the model works, we calibrated the parameters of our model such that they reflect as closely as possible the economic situation in Canada over the 2001Q1 - 2006Q1 period. This is what we call our base case. In fact, we used the average values, over the period, of the five-year discounted mortgage rate, the rate of nominal appreciation in housing prices, and the one-month treasury bill interest rate. The latter is used for both the original interest rate (r_0) and the steady state interest rate (θ) to which it reverts over the given five-year period of interest. Note that, to better reflect the current interest rate environment, we also simulated the model using 4.5 per cent as the value of r_0 and θ. We also assume that some transactions costs are charged in the case of a prepayment.

[14] The same method could be used to examine the potential impact of a change in interest rates.

Values of the parameters related to the stochastic behaviour of housing prices and the interest rate are chosen as follows. The standard deviation of stochastic disturbances to the change in house prices (σ_H) has been estimated over the 2001-2006 period at 4 per cent per year. The standard deviation of stochastic disturbances to interest rates (σ_r) and the reversion parameter (γ), which measures the speed of return to the mean interest rate , are set equal to 10 per cent and 25 per cent per year respectively. These values are within the range of those reported in previous works by McManus and Watt [29] and Bolder [4]. All parameters describing our base case are given in Table 1.

Table 1: Base-case parameters for numerical modeling

Parameters	**Base Case**
Mortgage term (years)	5
Amortization period (years)	25
Mortgage contract rate at origination (percentage)	$c_1 = 5.70$
Expected rate of house price appreciation (percentage per year)	$(\alpha - s) = 6.50$
Original one month interest rate (percentage per year)	$r_0 = 3.00$
Steady state 1 month interest rate (percentage per year)	$\theta = 3.00$
Reversion parameter (percentage per year)	$\gamma = 25$
Standard deviation of r (percentage per year)	$\sigma_r = 10$
Standard deviation of H (percentage per year)	$\sigma_H = 4$
Correlation coefficient	$\rho = -10$
Transaction cost of prepayment (dollars)	1% of loan balance

After valuing the probability of default for different LTV ratios at origination in the base case, we repeat the exercise assuming other scenarios of housing prices' evolution. In the first of three further scenarios considered in this illustration, the moderate case, we assume that housing prices are expected to increase moderately at the annual rate of 2.5 per cent. The second scenario is the extreme case in which nominal housing prices decline at an annual rate of 2 per cent (the rate of decline observed over the period of 1990Q1-1995Q1) for five years. In the third scenario, the very extreme case, nominal housing prices decrease at an annual rate of -5 per cent. This value reflects a real decrease in housing prices equivalent to that observed in the early 1980s. All other parameters are equal to those in the base case. Note that the parameter s, which measures the service flow from using the house over time, is assumed to be constant for all scenarios.

4.2 Results

Table 2 and Figure 2 display the cumulative conditional probabilities within one, two, three, four years and until the expiration of the mortgage loan. This is done for different LTV ratios at origination. Our results suggest that the probability of default rises slowly at the very beginning since it is unlikely that the house decreases in value in such a short amount of time. Within the first two years, this effect disappears and the likelihood of default is higher. Indeed, most of the acceleration in default rates comes before amortization lowers the LTV ratio significantly.

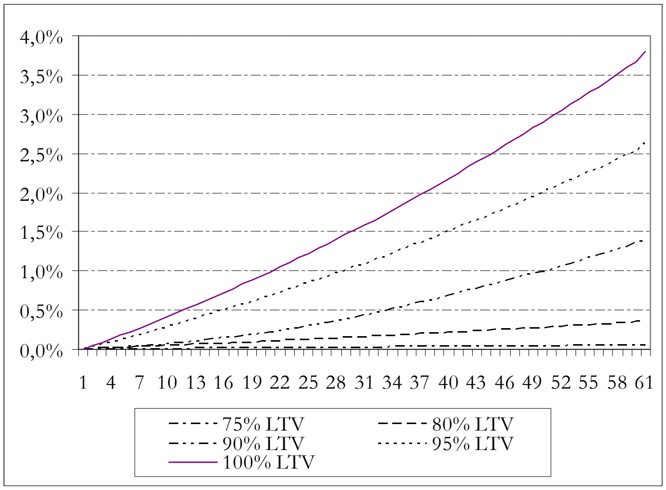

Figure 2: Cumulative probabilities of default over time for the base case.

In addition, as expected, the higher the LTV ratio the higher the conditional probability of default over time.[15] For example, over the entire life of the mortgage, a loan with a 75 per cent LTV has a 0.05 per cent chance of reaching a point where it is optimal to default, compared to 3.8 per cent for a 100% LTV loan.

Table 2: Probabilities of Default from Origination in the Base Case

Default period in years	**LTV Ratios** 40%	75%	80%	90%	95%	100%
1	0.00%	0.01%	0.06%	0.10%	0.39%	0.57%
2	0.00%	0.02%	0.12%	0.29%	0.85%	1.23%
3	0.00%	0.03%	0.19%	0.60%	1.36%	1.97%
4	0.00%	0.04%	0.27%	0.96%	1.95%	2.82%
5	0.00%	0.05%	0.36%	1.39%	2.62%	3.80%

[15] The insurance premium paid by a mortgagor, whose down payment is less than 20 per cent, increases with the LTV ratio. This is consistent with our results showing that probabilities of default increase with LTV ratios (at origination).

The results of our simulations for other scenarios are summarized in Table 3. The first six columns provide the cumulative probabilities of default over the five years of the loan for mortgages with different LTV ratios. Our results suggest that, for a given LTV ratio at origination, the probability of default is higher the more extreme is the scenario. For example, for a 100 per cent LTV ratio, it is 6.98 per cent in the moderate scenario, and increases to 12.10 per cent and 16.22 per cent in the extreme and the very extreme scenarios respectively.

Table 3: Risk of Default for different scenarios

LTV Ratios						**Overall**
40%	75%	80%	90%	95%	100%	**Default**
Base Case						
0.00	0.05	0.36	1.39	2.62	3.80	0.31
Moderate Case						
0.00	0.19	1.08	2.51	5.10	6.98	0.63
Extreme Case						
0.00	0.77	2.89	5.53	9.11	12.10	1.35
Very Extreme Case						
0.00	2.01	5.96	8.13	12.47	16.22	2.25

Our results also suggest that the impact of decreasing housing prices on the overall risk of default is not linear. Indeed, the risk of default on high LTV ratio loans increases by a greater magnitude than the low LTV ratio ones. For example, the probability of default on a 75 per cent mortgage increases from 0.05 per cent in the base case to 2.01 per cent in the very extreme case, while it increases from 3.80 per cent to 16.22 for a 100 per cent LTV mortgage. This may be explained by the fact that for a given fall in housing prices, it is more likely for a mortgagor with higher LTV ratio to be in a position of negative equity and therefore more prone to default.

For a given LTV ratio, the cumulative probabilities of default over the five years of contracts can be interpreted as the proportion of default in the pool of current mortgages which share the same LTV ratio, and were signed five years earlier. The overall default rate is a weighted average calculated by multiplying these cumulative probabilities by the weights given by the empirical distribution of LTV ratios. For simplicity, we used the 2006 distribution in our examples (see Table 4).

Table 4: Empirical Distribution of LTV ratios in 2006

LTV Ratios	Frequency
- Less than 75%	79.45%
- Between 75% and 80%	5.34%
- Between 80% and 90%	8.81%
- Between 90% and 95%	1.53%
- Between 95% and 100%	0.00%
- 100% and more	4.87%

In what follows, we will compare our estimated overall default rate with actual default rates. For several reasons, the simulated default rates will differ from the observed rates. For example, we consider only fixed-rate mortgages in our model, while actual default rates reflect defaults on both fixed-rate and variable-rate mortgages. Defaults on variable-rate mortgages may be more sensitive to changes in interest rates than defaults on fixed-rate mortgages. In addition, our simulated rate of default measures cumulative defaults for a given vintage while the actual rate of default reflects defaults that were observed for all vintages in a given year. For these measures to be perfectly equivalent, we have implicitly assumed that, over a given year, all mortgage contracts are spread equally across the LTV ratios. We were not be able to validate this hypothesis due to data limitations. This is why this comparison is intended only to provide a rough test of whether our estimates are in the general range of historical experience.

Our estimated rates of default appear reasonable and broadly within the range of historically observed default rates. The overall rate of default estimated for the base case (0.31 per cent) is slightly higher than the observed rate of default in 2006 (0.23 per cent). Also, our results suggest that the rate of default would reach 1.35 per cent following a persistent decrease in housing prices similar to the one observed over the 1990Q1-1995Q1 period. This rate is higher than the peak observed in Canada in 1992Q1 (0.62 per cent[16]). Among other reasons, this can be explained, as mentioned in the caveats, by the assumption of limited liability which may lead to an exaggerated measure of the risk of default, particularly under scenarios where defaults are more likely to happen (i.e. decreasing housing prices). The rate of default is much higher in the very extreme scenario still (2.25 per cent).

These rates do not reflect actual losses to banks and mortgage insurers because the loss given default on mortgages is considerably less than 100 per cent of the mortgage balance. Anecdotal evidence suggests that the loss given default on mortgages may be around 10

[16]The 0.62 per cent rate is measured as a percentage of the number of mortgage loans in arrears three months or more. Data on default rates as percentage of assets values are not available before 1997.

per cent. Also, these comparisons should be interpreted with caution given all the caveats mentioned above. Nevertheless, they suggest that the methodology applied here can be useful for stress testing the portfolio of Canadian mortgage loans.

4.3 Sensitivity analysis

We have performed several exercises to analyze the sensitivity of our results to key parameters related to the stochastic behaviour of housing prices and interest rates. In what follows, we will just outline the main conclusions of these exercises.[17]

First, we have repeated the same exercise for all scenarios using 4.5 per cent as the value of the original discounting rate and the rate to which it reverts over the coming five-year period. Our results did not change significantly suggesting that the effect of adverse changes in interest rate is less important than housing prices.

Second, we have simulated the probabilities of default for different values of house price volatility. As anticipated, we found that the probabilities of default are positively related to house price volatility. This is because, by definition, higher house price volatility implies larger regions of default.

Third, we have analyzed the sensitivity of the probabilities of default to the correlation factor between the processes of housing prices and interest rates (ρ). The effect of an increase in ρ is to raise the probability of default. According to Kau, Keenan and Kim [27], this is not surprising. Default is more likely to happen when intermediate falls in house prices are combined with intermediate falls in interest rates, as such combinations occur most readily when the housing prices and interest rates are positively correlated.

Finally, we have introduced a cost to default which we have set as a percentage of the house value (which represents in our model the entire wealth of the mortgagor). Obviously, the higher is this percentage, the lower are the probabilities of default. No defaults were observed beyond a certain threshold of the default cost (35 per cent of the house value at time of default).

5. Concluding remarks

This work applies a contingent-claims based approach to analyze the impact of changes in housing prices on the risk of default. This approach uses the assumption that rational homeowners default on valuable property only when it is in their financial interest to do so.

[17] The detailed results may be given by the author upon request.

We posed the decision to default as an intertemporal optimization problem in a stochastic economic environment.

Our simulations suggest that the impact of decreasing housing prices on the overall risk of default is not linear. The risk of default on high LTV ratio loans increases by a greater magnitude than the low LTV ratio ones. Compared with actual default rates, our estimated default rates appear reasonable and in the general range of historical experience.

This approach has limitations. In particular, it is technically very difficult to introduce other factors into this framework to take into account other important aspects of the default decision, such as the risk of income loss. This would require the introduction of a third stochastic variable which would make the solution of the model extremely complex. Also, we do not model explicitly the fact that, besides the options to default and to prepay, the mortgagor can choose to refinance his loan at a new mortgage market rate.

On the whole, however, this work appears helpful in gauging the risk of default on mortgage loans under different scenarios and assumptions regarding the evolution of the distribution of LTV ratios. The same method could be used to examine the potential impact of a change in interest rates.

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A Negative relationship between residential mortgage growth, real housing price growth and default rates.

Figure 3 shows clearly that there is negative correlation between annual residential mortgage growth and default rates, and between real house price growth and default rates. Despite it is well documented in the literature (see among others, Esho and Liaw [17] and Salas and Saurina (2002)), this empirical fact does not imply a clear relation of causality between the housing credit and price growth and default rates. Indeed, it is compatible with the fact that increasing interest payments, combined with a slowing housing market, result unavoidably in increased default rates. Also, it is compatible with the fact that as a higher percentage of poor credit risks are accepted during a period of rapid loan growth, resulting in increased credit losses should a shock occur.

Figure 3: Annual Growth of Housing Price and Residential Mortgage Credit versus the Rate of Defaults in Canada between 1986 et 2006.

B The bivariate binomial approximation technique

We use the bivariate binomial options pricing technique which was initially developed by Nelson and Ramaswamy [32], who demonstrate how a binomial model is used to approximate nearly all diffusions once the heteroskedasticity of each process is removed, and then extended by Hilliard, Schwartz and Tucker [22] to the case of two correlated stochastic state variables by implementing Hull and White's [23] procedure for removing the correlation between state variables.

In order to obtain constant volatility for processes, the so-called risk-adjusted house price ($\widehat{H}$) and interest rate (r) processes, which are described by (1) and (2), must undergo the following transformation:

$$\begin{cases} S = \ln(\widehat{H}) \\ R = 2\sqrt{r} \end{cases}$$

Next, to orthogonalize these constant-volatility processes, S and R are jointly transformed as following:

$$\begin{cases} X_1 = \sigma_r S + \sigma_H R \\ X_2 = \sigma_r S - \sigma_H R \end{cases}$$

Hilliard, Kau and Slawson [21] have showed that the drift terms of X_1 and X_2 are respectively given by

$$\begin{cases} \mu_1 = \sigma_r \left[\left(\frac{X_1 - X_2}{4\sigma_H} \right)^2 - s - \frac{\sigma_H^2}{2} \right] + \frac{\sigma_H^2}{X_1 - X_2} \left[4\gamma \left(\theta - \left(\frac{X_1 - X_2}{4\sigma_H} \right)^2 \right) - \sigma_r^2 \right] \\ \\ \mu_2 = \sigma_r \left[\left(\frac{X_1 - X_2}{4\sigma_H} \right)^2 - s - \frac{\sigma_H^2}{2} \right] - \frac{\sigma_H^2}{X_1 - X_2} \left[4\gamma \left(\theta - \left(\frac{X_1 - X_2}{4\sigma_H} \right)^2 \right) - \sigma_r^2 \right] \end{cases}$$

They also showed that the volatilities of X_1 and X_2 are respectively given by:

$$\begin{cases} \sigma_1 = \sigma_r \sigma_H \sqrt{2(1+\rho)} \\ \sigma_2 = \sigma_r \sigma_H \sqrt{2(1-\rho)} \end{cases}$$

To construct a recombining, two-dimensional binomial lattice for the variables X_1 and X_2, we divide the time-interval $[0, T]$ into N equal intervals of length Δt. From a node (X_{1t}, X_{2t}) at time t, the lattice evolves to four nodes, $\left(X^+_{1t+1}, X^+_{2t+1}\right)$, $\left(X^+_{1t+1}, X^-_{2t+1}\right)$, $\left(X^-_{1t+1}, X^+_{2t+1}\right)$ and $\left(X^-_{1t+1}, X^-_{2t+1}\right)$ at time $t+1$, where:

$$\begin{cases} X^+_{1t+1} = X_{1t} + (2k_1 + 1)\sigma_1\sqrt{\Delta t} \\ X^-_{1t+1} = X_{1t} + (2k_1 - 1)\sigma_1\sqrt{\Delta t} \\ X^+_{2t+1} = X_{2t} + (2k_2 + 1)\sigma_2\sqrt{\Delta t} \\ X^-_{2t+1} = X_{2t} + (2k_2 - 1)\sigma_2\sqrt{\Delta t} \end{cases}$$

where k_i ($i = 1, 2$) is the appropriate jump multiple of X_i. It has to be chosen such that both binomial means and variances match local diffusion means and variances, and that binomial probabilities are well defined. Indeed, as pointed Hilliard, Schwartz and Tucker [22], X_1 and X_2 are allowed to jump more than one node (i.e. $k_i = 0, \pm 1, \pm 2, .., i = 1, 2$) so that the following conditions hold[18]:

$$\begin{cases} (2k_1 - 1)\sigma_1\sqrt{\Delta t} \leq \mu_1 \Delta t \leq (2k_1 + 1)\sigma_1\sqrt{\Delta t} \\ (2k_2 - 1)\sigma_2\sqrt{\Delta t} \leq \mu_2 \Delta t \leq (2k_2 + 1)\sigma_2\sqrt{\Delta t} \end{cases}$$

[18] To make the lattice for each state variable recombine, the variable can only move an integral number of increments $\sigma_i\sqrt{\Delta t}$ ($i = 1, 2$). When the drift terms μ_1 and μ_2 are large in magnitude, for instance, at low interest rates when the speed of mean reversion is high, multiple jumps, that is, nonzero k_1 or k_2, occur.

The four nodes have associated risk-neutral probabilities $p_t q_t$, $p_t(1-q_t)$, $(1-p_t)q_t$ and $(1-p_t)(1-q_t)$, respectively. The probabilities, p_t, of an up-jump in X_1 process at time t, and q_t, of an up-jump in X_2 process at time t, are picked to ensure the right moments at the node (X_{1t}, X_{2t}):

$$\begin{cases} p_t = \frac{1}{2} + k_1 + \frac{\mu_1\sqrt{\Delta t}}{2\sigma_1} \\ q_t = \frac{1}{2} + k_1 + \frac{\mu_2\sqrt{\Delta t}}{2\sigma_2} \end{cases}$$

These jump probabilities are non-constant since μ_1 and μ_2 vary with time.

Using reverse transformation, we transform from X_1 and X_2 back to H and r at each node as follows:

$$\begin{cases} \widehat{H}_t = \exp\left(\frac{X_{1t}+X_{2t}}{2\sigma_r}\right) \\ r_t = \left(\frac{X_{1t}-X_{2t}}{4\sigma_H}\right) \end{cases}$$

C Valuation procedure of the remaining mortgage contracts

Assuming that the borrower will not terminate any of the first four mortgage contracts, he will obtain a loan whose value corresponds to the unpaid mortgage value after 20 years from the origination of the first loan contract, i.e.

$$L_{241} = UMB_{t=240} = \frac{(1+\frac{c_1}{12})^{(25*12)} - (1+\frac{c_1}{12})^{20*12}}{(1+\frac{c_1}{12})^{(25*12)} - 1} * L$$

For any time t before the maturity date of the last 5-year mortgage contract, the decision rule of the borrower is similar to the one described by equation (5) with

$$\begin{cases} UMB_t = \frac{(1+\frac{c_1}{12})^{(25*12)} - (1+\frac{c_1}{12})^{t}}{(1+\frac{c_1}{12})^{(25*12)} - 1} * L_{241} \\ PVRP_{t,r} = q_t \left(\frac{PVRP^u_{t+1}}{1+\frac{r^u_{t+1}}{12}}\right) + (1-q_t)\left(\frac{PVRP^d_{t+1}}{1+\frac{r^d_{t+1}}{12}}\right) \\ PVRP_{299} = M_{c_1} + q_{299}\left(\frac{M_{c_1}}{1+\frac{r^u_{300}}{12}}\right) + (1-q_{299})\left(\frac{M_{c_1}}{1+\frac{r^d_{300}}{12}}\right) \end{cases}$$

However, at the maturity date ($t = 300$), the borrower has only the choice between defaulting and making the final monthly payment (M_{c_1}). Of course, at that date, prepayment could not be of any value, since after the final payment, the loan is paid in full. In contrast, default could be of value in the extreme case where the house price has fallen to less than the final mortgage payment M_{c_1}. This implies the following terminal condition:

$$V_{300,r,\widehat{H}} = \min(\widehat{H}_{300}, M_{c_1})$$

The solution for the optimal decision sequence and values (of the mortgage and the options to default and to prepay) is obtained by backward induction starting with the terminal condition at time $t = 300$ and working to date $t = 240$. The model computes the value of the last mortgage loan, $\overline{V}_{t=240}$.

We use $\overline{V}_{t=240}$ to value the mortgage loan that would be signed by the borrower at date $t = 180$. For any time between the origination of this contract and its maturity date, the decision rule of the borrower is the similar to the one described by equation (5). At the maturity date, the borrower decides whether to default, to prepay or to continue by accepting a new mortgage loan according to the following equation:

$$V_{t=240,r,\widehat{H}} = \min\left[\widehat{H}_t, UMB_{t=240}, M_{c_1} + \overline{V}_{t=240}\right]$$

where:

$$UMB_{t=240} = \frac{(1+\frac{c_1}{12})^{(25*12)} - (1+\frac{c_1}{12})^{(20*12)}}{(1+\frac{c_1}{12})^{(25*12)} - 1} * L$$

By backward induction starting with the terminal condition at time $t = 240$ and working to date $t = 180$, we compute the value of this contract $\overline{V}_{t=180}$.

Using this value and the same valuation procedure described above, we valuate the mortgage contract that would be signed at time $t = 12\dot{0}$. We repeat the same exercise to calculate the value of the second mortgage contract $\overline{V}_{t=60}$.

D Numerical computation of the default's conditional probability

While hedging arguments provide that the default option and optimal stopping boundary are determined using the risk-neutral pricing process of the house price, valuing the probability default is done using the actual house process of equation (1). Indeed, to implement the computation of the default probabilities, the probability of making k step jump is computed using the same Hillard, Kau and Slawson's [21] derivation, but with the gross return to housing $(\alpha_H - s)$ used in place of the risk-free interest rate, that is,

$$\begin{cases} p_t = \frac{1}{2} + k_1 + \frac{\sigma_r\left[(\alpha_H - s) - \frac{\sigma_H^2}{2}\right] + \frac{\sigma_H^2}{X_1 - X_2}\left[4\gamma\left(\theta - (\alpha_H - s)^2\right) - \sigma_r^2\right]\sqrt{\Delta t}}{2\sigma_r\sigma_H\sqrt{2(1+\rho)}} \\ q_t = \frac{1}{2} + k_1 + \frac{\sigma_r\left[(\alpha_H - s) - \frac{\sigma_H^2}{2}\right] - \frac{\sigma_H^2}{X_1 - X_2}\left[4\gamma\left(\theta - (\alpha_H - s)^2\right) - \sigma_r^2\right]\sqrt{\Delta t}}{2\sigma_r\sigma_H\sqrt{2(1-\rho)}} \end{cases}$$

Operationally, the model is run and the optimal default and prepayment boundaries are stored. Then, starting with the probability of one from the initial node, the probability of reaching each interest rate and house price node in the next period is computed as following:

- If that node involves a prepayment, the probability of reaching that node is credited to the probability of a prepayment (for this specific node, the probability of default is set to zero), and then the probability of that node is set to zero.
- If that node involves a default, the probability of reaching that node is credited to the probability of a default (for this specific node, the probability of prepayment is set to zero), and then the probability of that node is set to zero.
- If that node involves continuation, the probability of reaching that node is credited to the probability of a continuation. However, the probability of that node is not set to zero.

The process is then continued for another stage. The correct conditional probabilities of default are computed because nodes where default or prepayment has occurred have their probability set to zero, so forward movements from these nodes are made with zero probabilities.

One this process is finished, the conditional probability of default is computed by summing the probabilities of all default nodes at that stage.