Competing Portfolio Constraints

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May 20, 2009

Keywords: Portfolio Constraints, International Finance, Correlation, Volatility
JEL classification: G11, G12, G15, G18

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Abstract

Over the last decades, different countries have had quite varied experiences with liberalization, which have not been fully explained. Models of segmentation have often focused on the effects of a particular type of portfolio constraint in isolation, contrasting it with perfect markets. This paper shows that accounting for the presence of multiple international constraints could help to explain countries' varying liberalization experiences. Partial liberalization will make stock markets' volatility deviate even further from the perfect market benchmark than leaving multiple constraints in place. The types of constraints, and how tightly they bind, determine whether liberalization will increase or decrease stock market volatility. Similarly, depending on how tightly constraints bind, correlation across stock markets may increase or decrease following liberalization. By extension, time variation in market correlation increases following liberalization. Likewise, liberalization can change the correlation between expected excess returns and volatility from negative to positive. The policy question of whether to reimpose restrictions on international capital flow or rather to push for further liberalization is still being debated. This paper discusses in general terms how different types of constraints combine to exacerbate or mitigate each other's effects on volatility and correlation.

1 Introduction

Financial market liberalization has had different effects in different countries. On average, liberalization seems to have a positive but relatively small effect on a market's correlation with world markets. However, results vary significantly across countries. Notably, a comparative study by Edwards et al. (2003) finds that Latin American and Asian financial markets reacted in systematically different ways to their liberalization through the early 1990's. Volatility and the amplitude of stock market cycles decreased in Latin America, but increased in Asia. Latin American markets' correlation with the US market increased, whereas that of Asian markets initially did not see strong changes, and has even decreased in recent years. Post-liberalization, Latin American market dynamics seem to be more similar to those of the US stock market then before, while Asian markets' dynamics have become less simlar to the US market since liberalization.

I propose a model accounting for the different effects of liberalization across markets. While empirical studies have allowed for varying degrees of liberalization across the world in any time period studied, much of the theory models a country's liberalization assuming other countries to be fully liberalized. This implies comparing the market impact of a particular constraint to a benchmark perfect market equilibrium. Instead, I model a country's liberalization assuming other constraints on investors' portfolio choices are kept in place. I consider two countries, each with one stock market, and populated with a representative investor. Critically, both investors face portfolio constraints, and hold different beliefs about the countries' economic fundamentals. How tightly a constraint binds depends on the dispersion of these beliefs. Our main results explain existing empirical findings on liberalization effects.

First, liberalization can either reduce or increase the correlation of stock market returns, depending on the constraints' tightness — i.e. on belief dispersion. In particular, liberalization can increase the volatility of countries' stock market correlation over time. Second, eliminating a constraint impacts the volatility of stock markets differently depending on whether further constraints remain in place, or full liberalization is achieved. Liberalizing markets only partially will make stock markets' volatility deviate even further from the perfect market benchmark than under multiple constraints. Whether this means a volatility increase or decrease depends on the type of constraint removed. Third, these effects of liberalization on stock volatility also imply a change in the relationship between expected excess returns and volatility. When both constraints bind, higher belief dispersion leads to expected excess returns being higher, while volatility is lower. After eliminating a constraint, both volatility as well as returns are higher when beliefs are dispersed.

I consider a continuous-time pure-exchange economy with two countries, *home* and *foreign*, each with one good and one representative investor. Each good is produced by a Lucas tree, with country-specific supply shocks. Investors consume both goods but have a preference for their respective domestic good. Each country has one stock (market) that is a claim on domestic output, and a zero-net supply, locally riskless bond. We make two main assumptions.

First, both investors face portfolio constraints: The *home* investor faces a leverage constraint, i.e. the total amount he can invest in *home* and *foreign* stocks is limited. The *foreign* investor's holding in the *home* stock is limited. The assumption that both investors are constrained contrasts with much of the literature, and has distinct and new implications as we discuss below. The fact that the two constraints are not symmetric in nature contributes to the results.I compare instances of both partial and full liberalization. Where 'partial' means the elimination of one constraint — either that imposed on the *home* investor or that on the *foreign* investor. Full liberalization means the elimination of both constraints.

Second, the investors hold different beliefs about the countries' fundamental economic growth rates, and thus about the stock returns. How tightly a constraint binds, and indeed whether it binds at all, depends on the dispersion of these beliefs. For example, an investor's constraint on his position in a stock binds more tightly when this investor is very optimistic about this stock.[1]

In this context, stock returns depend on both output and terms-of-trade effects. Assume for instance that the *home* investor's portfolio overweighs the *home* stock relative to the *foreign* investor's (and therefore the market) portfolio.[2] All else equal, when the *home* stock has a positive return due to higher than expected *home* output, the *home* investor will experience a wealth increase relative to the *foreign* investor. He will increase his current consumption, predominantly of the *home* good, for which he has a preference. The price of the *home* good will increase due to higher aggregate demand, again raising the price of the *home* stock. In other words, the initial positive stock return due to the output effect is amplified by the terms of trade effect. This positive feedback effect tends to increase the volatility of the *home* stock market and decreases the correlation between the two stock market returns.

[1]Differences in investment behavior can stem from differences in beliefs, in information, or in attitudes to risk. (Other papers studying the effect of differences in beliefs are, e.g. Harrison and Kreps (1978) and Basak (2005). The literature on differences in risk attitude, e.g. Kogan and Uppal (2003), focuses on a different source of heterogeneity in agents.) We do not aim to distinguish between these explanations, but rather use differences in beliefs as a tractable modeling tool.

[2]Note that in this model, the investor's preference for their domestic good does not itself imply a "home bias" in their portfolio choice.

Instead, a negative feedback effect arises when the *home* investor's portfolio overweighs the *foreign* stock. Following a higher than expected *foreign* output, the *home* investor's relative wealth increases as does his demand for the *home* good. The terms of trade effect decreases the *foreign* good's price and therefore that of the *foreign* stock, thus dampening the initial positive return due to the output effect. Hence this negative feedback effect tends to decrease the volatility of the *foreign* stock, as well as the markets' correlation.

We have three main results.

First, the effect of liberalization on the correlation between the stock market returns depends on belief dispersion, and therefore how tight a constraint is. Liberalization increases correlation if belief dispersion is low, and decreases correlation when belief dispersion is high.

The intuition is as follows. Note first that the feedback effects are stronger when the *home* and *foreign* investors' portfolios are very different. Indeed, very different portfolios mean that an output shock will have a large effect on the investors' relative wealth, and therefore a large feedback effect. Next note that how different these two portfolios are depends on the dispersion in beliefs and on the constraints. Therefore the effect of liberalization on correlation depends on whether the freely chosen portfolios post-liberalization are more or less similar than before.

Suppose that the *foreign* investor is more optimistic than the *home* investor about the *home* stock market. Pre-liberalization, his constraint will prevent him from tilting his portfolio strongly towards the *home* stock and thus will limit the divergence in the two investors' portfolios. As a result, the negative feedback effect is kept small by the constraint. Liberalization will then allow the *foreign* investor to purchase more of the *home* stock from the (less optimistic) *home* investor. The holdings diverge, implying a larger negative feedback effect. In this case, liberalization leads to a decrease in the correlation of stock market returns.

Suppose now that both investors' beliefs about the *home* stock market are very similar. Pre-liberalization, the *foreign* investor's constraint will force the him to put less weight on the *home* stock than the *home* investor himself does, making the *foreign* investor look significantly more pessimistic. Liberalization will allow the *foreign* investor to form a portfolio that reflects his true beliefs. Since these are in line with those of the *home* investor, the magnitude of the difference in portfolio holdings is lower than pre-liberalization. In that case, liberalization leads to a smaller feedback effect and thus increases correlation.

This finding also implies that liberalization would make stock markets' correlation vary more strongly over time, as belief dispersion varies. Through the described effects, portfolio constraints would mitigate time-variation in correlations.

The second result is that the effect on volatility of eliminating a constraint depends on whether liberalization is partial or full, and which constraint is removed.

The intuition on volatility follows similar arguments to those for the correlation effect. Dispersion in beliefs will, through investors' different portfolio choices and the resulting feedback effect, affect the sensitivity of stock prices to shocks. When liberalization leads to investors' portfolios diverging and becoming less similar, volatility tends to increase. How tightly the constraints bind therefore plays a critical role. Eliminating the *home* investor's leverage constraint increases volatility, while eliminating the *foreign* investor's constraint on *home* stock holding decrease volatility. Either of these two effects can dominate in the case of full liberalization — i.e. when eliminating both constraints simultaneously. If the *foreign* investor's constraint binds more tightly, its effect will dominate and volatility decreases. If the *home* investor's constraint binds more tightly, volatility increases.

Pre-liberalization, with both constraints in place, suppose the *foreign* investor reaches his constraint not necessarily because of innate optimism, but rather because the *home* investor's leverage constraint binds very tightly and he cannot take more of the *home* stock in his portfolio. The *foreign* investor faces an upper limit on his stock holdings through his own constraint, while facing a lower limit indirectly through the *home* investor's constraint. Jointly, the constraints put relatively severe restrictions on portfolio divergence. Once the restrictions are lifted, portfolios will diverge and volatility will rise — the effect of the more tightly binding leverage constraint dominates.

Now suppose instead of lifting both constraints, only the constraint faced by the *foreign* investor is lifted, allowing him to freely adjust his investment into the *home* stock. Therefore, the *home* investor himself now has the chance to shift his portfolio towards the *foreign* stock (though still leverage constrained, he prefers a different composition), by trading with the *foreign* agent. These new portfolios are more similar to one another, lowering volatility.

These effects are driven by how the constrained investor(s) individually decides to 'undo' his constraint, albeit imperfectly. For example, in order to achieve a portfolio that is as close as possible to his desired risk exposure, the *foreign* investor may choose to allocate those funds that he cannot put into the *home* stock, to either the *foreign* stock or bonds, depending on his own set of beliefs. But this 'imperfect' replication of his actual desired portfolio drives a wedge between investors' choices: with this adjustment to his portfolio, it will seem to reflect different levels of belief dispersion than that of the unconstrained investor. I show that this can be expressed as reflecting belief dispersion regarding another, extraneous source of risk, one which would not be priced in a fully liberalized market.

The third result is that the relationship between excess stock returns and volatility is systematically different whether both constraints are imposed or partial liberalization has been achieved. As with volatility, high dispersion in beliefs — as reflected in more diverging portfolios — also implies higher expected excess returns. The description of the first two results showed that binding constraints drive a wedge between the 'true' dispersion of beliefs and that reflected in (forced) portfolio choice and thus in stock prices: if constraints make portfolios more similar than they would be in a liberalized market, the belief dispersion reflected in asset prices is lower than the true belief dispersion. This shift between true and reflected dispersion affects volatility more strongly than expected excess returns. Pre-liberalization, changes in expected returns and volatility will be negatively correlated: as investors' true beliefs converge, volatility will rise and expected excess returns will drop. After liberalization, a convergence of beliefs will result in positive correlation: both volatility and expected excess returns will drop.

My paper relates to the empirical literature along two main dimensions: first, it provides an explanation for the differences in the effects of liberalization that have been found in cross-country studies. Second, it provides additional testable implications that have not been studied in an international context of liberalization.

Empirical studies across countries, treating each country's liberalization event as one datapoint, have generally found that 'on average', liberalization has a small effect on cross-country stock market correlations and volatilities (e.g.Bekaert and Harvey (2000), Bailey and Lim (1992)). Papers that have looked at individual countries' liberalization experiences in more detail find that effects can be significant and vary across countries. Edison and Warnock (2003) confirm the results of Bekaert and Harvey (2000) in aggregate, but show that in some countries, liberalization has lead to decreases in correlation. Carrieri et al. (2007) do not find consistent patterns of correlation changes. DeSantis and Imrohoroglu (1997) and Miles (2002) show that liberalization increases stock return volatility in some countries, while lowering it in others. In the cross-section, Bae et al. (2004) show that volatility is higher among assets that are highly accessible, compared to assets whose ownership is restricted. Kim and Singal (2000) link capital inflows to increases in volatility, and Levine and Zervos (1998) link higher volatility with stronger integration of markets. [3] As

[3]Harvey (1995), while not looking at liberalization events in particular shows that some countries have stable levels of world correlation, while for other countries levels have increased.

mentioned earlier, Edwards et al. (2003) contrast the liberalization experience of Latin American with that of Asia, and they find that the behavior of Latin American markets has become more similar to that of US markets. They find the opposite is true of Asian markets since liberalization. Bekaert et al. (2002) show that even though financial integration has vastly improved, significant constraints on international investment remain.[4]

In this paper I provide a model that can explain these disparate findings by linking the effects observed upon liberalization to two factors: the tightness of the binding constraints, and the presence of other constraints for the assets' investor base.

In addition, the model provides additional testable implications that have so far not been widely studied in an international context. The literature on the relationship between stocks' expected excess returns and volatility has produced mixed results, which the model in this paper can reconcile. (See, e.g. Whitelaw (2000) and Wu (2001).) While these empirical results are generally based on US stock data, expanding the studies to a larger set of assets that allows for conditioning information on binding investment constraints could shed more light on the relationship between returns and volatility.

Some recent empirical papers have contributed to looking jointly at the interaction of constriants and differences in beliefs in an international context. The Chinese regulation on Class A and Class B shares was a natural experiment that allowed looking at both aspects. The price premia have been attributed to local investors' information advantage or to differences in risk exposure of local and *foreign* investors, as discussed by e.g. Fernald and Rogers (2002). Bailey et al. (1999) and Bailey and Jagtiani (1994) suggest that the positive A-B share premia in China — in comparison to generally negative premia elsewhere — are due to the fact that unlike investors from other countries studied, Chinese investors face stringent restrictions on investing abroad, thus pushing up local prices for lack of investment alternatives.[5]

My paper also contributes to the theoretical asset pricing literature on constrained markets along one dimension in particular. To date, the literature on portfolio constraints' impact on equilibrium market outcomes has generally assumed that only some investors are constrained, while the marginal (and thus price-setting) investor is free to provide any amount of liquidity necessary to clear the market (most often characterized by a single asset). While these assumptions have

[4] The survey by Stulz (1995) as well as Karolyi and Stulz (2002) provide a good overview of the empirical findings on international integration and asset pricing. Bekaert et al. (2003) provide a synthesis of empirical methods to allow a differentiated study of the different liberalization paths countries have followed.

[5] Bailey (1995), and Mei et al. (2005) study this regulation's effects on Chinese stock prices.

provided interesting findings and tractability, the models have had little to say on how constraints (and their removal) affect volatility and correlation.[6]

Pavlova and Rigobon (2008) introduce an economy with multiple assets, but retain a universally advantaged price-setting international investor. Their model shows that a single constraint on a large investor will result in high correlations among small (developing) stock markets and low correlation between the large market and the cluster of developing markets.Our analysis emphasizes that this effect crucially relies on all investors holding the same beliefs. [7]

Detemple and Murthy (1997) introduce techniques to deal with multiple constraints in equilibrium. Due to requirements of market clearing, the example of a single-asset economy in that paper collapses to the benchmark solution, where stock dynamics are equivalent to those of a perfect market. We extend their approach and find a tractable solution to a two-asset setting, where deviations from the benchmark model can be studied.

The paper proceeds as follows. Section 2 develops the model. Section 3 derives the equilibrium. Section 4 discusses implications for stock market dynamics. Section 5 concludes. Proofs are provided in the appendix.

2 Model

2.1 Exchange Economy

Two countries, *home* and *foreign*, are each represented by one investor $i = H, F$, respectively. Each country produces one good, $j = h, f$, which both investors have access to. There are no transportation costs or other trade frictions, goods markets are assumed to be perfectly integrated. The endowment economy setup imposes exogenously the production or dividend processes

$$dY_t^j = \mu_t^{Yj} Y_t^j dt + \sigma_t^{Yj} Y_t^j dW_t^j \qquad j = h, f \tag{1}$$

where parameters μ_t^{Yj} and σ_t^{Yj} are adapted processes and each country's production is driven by its own domestic source of risk, dW_t^h and dW_t^f.[8]

[6] One exception is Basak and Croitoru (2000), where all agents are constrained and the existence of a derivative asset in a spanned market ensures tractability.

[7] Other models of segmentation include Errunza and Losq (1989), He and Modest (1995), Heaton and Lucas (1996), Zapatero (1998), Caballero and Krishnamurthy (2001) (who focus on crises), and Bhamra (2007). Soumare and Wang (2006) also study constraints in a multiple-good economy, but again focuses on individual constraints.

[8] This is purely for the sake of simplicity and without loss of generality. As a special case, the production processes can be assumed to follow geometric Brownian motions.

Each good j is sold at its time-t price p_t^j. Equilibrium in a real economy, depends on the relative price of goods, captured by $\bar{p}_t = p_t^f / p_t^h$. For generality, I use a reference consumption basket as a numeraire. Weights on the *home* and *foreign* goods are β and $1-\beta$, respectively.[9]

The dividend streams Y_t^j provide consumption for the two agents, while the traded stocks are claims to the production output and, together with country bonds, make up the international financial markets.

The stocks' behavior follows[10]

$$\begin{aligned} dS_t^h &= \mu_t^{S^h} S_t^h dt + \sigma_t^{S^h} S_t^h d\overrightarrow{W}_t, && (2) \\ dS_t^f &= \mu_t^{S^f} S_t^f dt + \sigma_t^{S^f} S_t^f d\overrightarrow{W}_t, && (3) \end{aligned}$$

where $d\overrightarrow{W}_t = \left(dW_t^h, dW_t^f, dW_t^* \right)^\top$ is a three-dimensional vector of mutually uncorrelated Brownian Motions. The return and variance parameters of these asset prices will be identified in equilibrium.

Each country issues a locally riskless bond in zero net supply, both of which are available to both investors.

$$\begin{aligned} dB_t^h &= r_t^h B_t^h dt \qquad \text{in terms of good } Y_t^h, && (4) \\ dB_t^f &= r_t^f B_t^f dt \qquad \text{in terms of good } Y_t^f. && (5) \end{aligned}$$

Note that these bonds are riskless in their respectively domestic good. In terms of the numeraire consumption basket, a portfolio of these bonds, $B_t = \beta B_t^h + (1-\beta) B_t^f$, represents the truly riskless investment opportunity.

The *home* (H) and the *foreign* (F) investor both consume both goods, their consumption is described by the set (C_{it}^h, C_{it}^f), $i = H, F$. Agent i maximizes $E\left[\int_0^T u_i \left(C_{it}^h, C_{it}^f \right) dt \right]$, subject to the budget constraint

$$dX_t^i = X_t^i \left[\sum_{j=h}^{f} \pi_{it}^{S_j} (dS_t^j + p_t^j Y_t^j)/S_t^j + \sum_{j=h}^{f} \pi_{it}^{B_j} dB_t^j / B_t^j \right] - \sum_{j=h}^{f} p^j C_{it}^j dt \qquad (6)$$

where agent i's wealth X_t^i must satisfy $X_t^i \geq 0$, for $i = H, F$.

[9]While consumption choices will of course be time-varying in equilibrium, leaving β as a constant takes reference to the fact that consumption bundles used to determine indices like the CPI, are rarely updated and thus very stable compared to relative prices and true consumption.

[10]t-subscripts will be dropped occasionally for parsimony of notation.

Investors' utility functions are assumed to be identical and additive over goods, with a preference for their respectively domestic good: when $\alpha^i > 0.5$, the agent has a home bias in consumption:

$$u_H\left(C^h_{Ht}, C^f_{Ht}\right) = \alpha^H_t \log C^h_{Ht} + (1-\alpha^H_t)\log C^f_{Ht}, \quad (7)$$

$$u_F\left(C^h_{Ft}, C^f_{Ft}\right) = (1-\alpha^F)\log C^h_{Ft} + \alpha^F \log C^f_{Ft}. \quad (8)$$

As well as supply shocks (through production process dY^j_t), the economic system also experiences demand shocks. These come through changes in α^H_t, which is assumed to follow a martingale, uncorrelated with production shocks:

$$d\alpha^H_t = \sigma^{\alpha*}_t dW^*_t. \quad (9)$$

For tractability, α^F is assumed to be constant. Rather than interpreting this as only one country being fickle in their relative demands of the goods, it may be more intuitive to think about α^H_t representing a relative shift in demands. The importance of such demand shifts in a multiple-good economy was established by Dornbusch et al. (1977), whose modeling approach I follow here.This could be interpreted as an ad-hoc way to take into accountexogenous events that change demand. For example, heating oil prices tend to rise as soon as the seasonal forecast is made, well before cold weather actually requires heating. Likewise, hurricane warnings cause large spikes in the demand for timber. Import negotiations between countries can also trigger changes in demand, like the so-called Banana Wars between the U.S. and the EU. None of these events have (had) any impact on the production of oil, timber or bananas. [11]

Throughout the paper, both α^H_t and α^F are assumed to be greater than 0.5, implying a home bias in consumption. The empirical pervasiveness of a home bias in consumption across the world has been linked to reasons of familiarity, transportation costs, and non-tradability of certain goods and services.[12]

[11] In order to study the effects of such demand shocks on the supply, a model of a production economy would be required, which is an interesting question for future research.

[12] It can also be seen as a tractable substitute for imperfections in the goods market. As Dumas and Uppal (2001) have shown, imperfections in goods markets have no great impact on the benefits of financial integration, so this does not seem to be a critical shortcoming.

2.2 Information Structure

Investors are assumed to have dispersed beliefs about the true parameters of the fundamental economic processes: they disagree about the expected growth rate of the two dividend processes, while both are able to perfectly observe output as well as demand. The model setup easily allows for incorporating an explicit learning process of investors about these parameters. For parsimony however, I abstract from studying learning explicity and assume belief dispersion to be exogenous, which nevertheless allows us to study the effect in terms of comparative statics. How learning and thus changes in belief dispersion may potentially interact with constraints is left for future research.

The filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$ defines the uncertainty. $\{\mathcal{F}_t^W\}$ is the augmented filtration generated by Brownian Motion W, which does not represent the agents' true information set. Both agents have only the incomplete information filtration $\{\mathcal{F}_t^{Y_{i,j}}\}$ available, generated by processes Y_t^h and Y_t^f. Through quadratic variation, both can draw exact inferences about the diffusion terms of fundamental processes, but they have incomplete knowledge about the drift parameters $\mu_t^{Y^h}$ and $\mu_t^{Y^f}$ of the goods' production, i.e. the growth rate.

Investors are fully rational, so observational equivalence must hold:

$$dY_t^h = \mu_t^{Y^h} Y_t^h dt + \sigma_t^{Y^h} Y_t^h dW_{h,t} = m_{Y^h}^{(i)} Y_t^h dt + \sigma_t^{Y^h} Y_t^h dW_{h,t}^{(i)} \tag{10}$$

The right side of eq. (10) characterizes agent i's belief, where $dW_h^{(i)}$ represents the innovation process perceived by agent i and $m_{Y^h}^{(i)}$ is his belief regarding the mean output rate of good Y_t^h. Since α_t^H follows a martingale, there cannot be (rational) disagreement about the demand process.

The dispersion in investors' beliefs can be captured by the difference in perceived innovation processes of investors H and F.

$$dW_{h,t}^{(F)} = dW_{h,t}^{(H)} - \Delta m_t^{Y^h} dt \quad ; \quad dW_{f,t}^{(F)} = dW_{f,t}^{(H)} - \Delta m_t^{Y^f} dt \quad ; \quad dW_{*,t}^{(F)} = dW_{*,t}^{(H)} \tag{11}$$

where $\Delta m_t^{Y^j} = \frac{m_{Y^j}^{(F)} - m_{Y^j}^{(H)}}{\sigma_t^{Y^j}}$ for $j = h, f$.

Both agents have access to the same public information, they simply choose to interpret it differently. As there is no asymmetry of information, neither agent will try to infer information from the actions of the other. The foundations for this assumption have been discussed at length for the general case in Morris (1994) and similar setups can be found, e.g. in Basak (2000) who includes extraneous risk, Yan (2005), and Gallmeyer and Hollifield (2008). For the financial market equilibrium, observational equivalence will again require that agents will agree on stock prices and total realized (observed) returns, even while potentially disagreeing on future *expected* returns.

3 Equilibrium

3.1 Optimal Consumption

The economy has a competitive equilibrium whose endogenous goods prices and consumption rates can be established in a representative-agent setup with the following aggregate utility function.[13]

$$U(C_H, C_F) = u_H\left(C^h_{Ht}, C^f_{Ht}\right) + \lambda_t u_F\left(C^h_{Ft}, C^f_{Ft}\right), \tag{12}$$

where $u_H(\cdot)$ and $u_F(\cdot)$ are as stated in eqs. (7) and (8).

The progressively measurable state variable λ_t is central in characterizing the equilibrium and the effects of constraints and dispersion of beliefs on the financial market. It represents the relative weight of agent F in the economy.[14] This "weight" can be interpreted as the importance a fictitious solcial planner would give to agent F, or alternatively the impact he has on the equilibrium. As will be described in more detail, λ_t is determined by agents' relative state price densities.

In the simplest benchmark equilibrium where both agents have the same beliefs and no constraints are in place, they will choose to hold identical portfolios. In this case, the state variable λ_t will be constant, and reflect only their initial wealth as given by their endowment. In the model of this paper, λ_t will be stochastic due to two features: Since investors' beliefs regarding the countries' growth rates differ, their assessment of the two stocks as investment opportunities also differ. Thus the *home* and *foreign* investors choose to hold different portfolios, and stock returns are distributed asymmetrically between the investors. The investor who chooses to put a higher proportion of his wealth into a particular stock (because he anticipates a higher growth rate) will see his wealth rise disproportionately when this stock has positive returns. Analogously, he will also lose relatively more money when the stock does poorly. A portfolio constraint likewise affects investors' portfolio choice.

In equilibrium, consumption market clearing leads to the following individual consumption

[13] The fact that agents are initially endowed with their home stock market raises the issue of market completeness, but as Cass and Pavlova (2004) have shown, Pareto optimality still holds in this setting.

[14] The relative weight of agent H is normalized to one without loss of generality.

choices of goods h and f.

$$\begin{array}{lll} & \underline{\textit{agent H}} & \underline{\textit{agent F}} \\ \textit{good h:} & C^h_{Ht} = \frac{\alpha^H_t Y^h_t}{\alpha^H_t + (1-\alpha^F)\lambda_t} & C^h_{Ft} = \frac{\lambda_t (1-\alpha^F) Y^h_t}{\alpha^H_t + (1-\alpha^F)\lambda_t} \\ \textit{good f:} & C^f_{Ht} = \frac{(1-\alpha^H_t) Y^f_t}{1-\alpha^H_t + \alpha^F \lambda_t} & C^f_{Ft} = \frac{\lambda_t \alpha^F Y^f_t}{1-\alpha^H_t + \alpha^F \lambda_t} \end{array}$$

Accordingly, optimal wealth for agents H and F respectively, is

$$\begin{aligned} X^H_t &= C^h_{Ht} \cdot \frac{p^h_t}{\alpha^H_t}(T-t) = C^f_{Ht} \cdot \frac{p^f_t}{1-\alpha^H_t}(T-t), \\ X^F_t &= C^h_{Ft} \cdot \frac{p^h_t}{1-\alpha^F}(T-t) = C^f_{Ft} \cdot \frac{p^f_t}{\alpha^F}(T-t). \end{aligned}$$

These consumption choices show that the marginal propensity to consume is stochastic only through the effects of demand and price shifts, not through changes in saving motive due to incomplete information. Likewise, as is typical for agents with logarithmic utility functions, wealth and consumption choice do not reflect any desire to hedge against potential future binding of constraints. [15]

3.2 Asset Valuation

Pinning down equilibrium stock prices in this setting is complicated by the fact that both investors face constraints on their portfolio decisions. The price-setting investor is not always the same across the possible equilibria, since both constraints may bind simultaneously.

Methodologically, I follow the approach of Detemple and Murthy (1997) and I attain closed-form solutions for consumption decisions and stock prices.

In general, the methodology introduced by Cvitanic and Karatzas (1992) allows a variety of constraints to be studied in an endowment economy.[16] In this paper I look at two particular constraints that are not symmetric in nature. Investor H, located in the *home* country, is subject to a leverage constraint. He can hold both long and short positions in the two country bonds, but his net position in the riskless asset cannot be negative.[17] In contrast, the *foreign* investor F is

[15]See appendix for details and proof.

[16]The technical literature on how to incorporate constraints also include e.g. He and Pearson (1991), Karatzas et al. (1991), and Liptser and Shiryaev (1977).

[17]This is the simplest form of analyzing borrowing restrictions, where no borrowing is allowed. Conceptually, a less strict leverage constraint should lead to similar implications for wealth transfers but would impede tractability.

limited in the amount he invests into the stock abroad, *home* stock S_t^h.

While both of these types of constraints seem to be reasonable assumptions individually — we can observe similar limitations imposed on e.g. mutual funds, who are restricted from levering up or overinvesting into a certain asset classes — the combination of two different constraints may seem ad-hoc. It is indeed a caveat that cannot be circumvented in this paper, for reasons of market clearing. In a model with only two assets and two investors, it must be ensured that constraints allow markets to clear at all times — someone must always be able to hold the asset. If the studied constraints are of a type that only allow one to bind at any point in time, it would be sufficient to use the established methodologies looking at single constraints, as has been discussed in the literature.[18]

Looking at "asymmetric" constraints such as these two has an additional benefit. The constraint imposed on investor F affects only the home stock S_t^h directly. The leverage constraint imposed on investor H affects both stocks, S_t^h and S_t^f. This allows us to study how the presence of two 'competing' constraints, that both directly affect holdings of the same stock, S_t^h, will affect its dynamics. When both constraints bind, the market in S_t^h is essentially frozen market - even new information may not shift the market sufficiently to induce either investor to trade.

I make no particular assumption on who imposes these constraints — i.e. whether a country is responsible for imposing restrictions on their own citizens or on the foreign investors. Examples of both types, domestically imposed as well as foreign-government imposed, still exist today. Chile and Indonesia are examples of countries that have imposed strict rules on foreigners' investment and trading in their financial markets. Domestically imposed constraints largely come from institutional characteristics and moral hazard or agency problems in the money management industry, like the examples on mutual funds mentioned earlier. In-depth welfare analysis would require taking a stand on which constraint a government is able to impose (or eliminate). Any government has jurisdiction only over their own laws, and likewise is concerned with the welfare of only one group of investors.

Regardless of the origin of these constraints, they can technically be expressed as

$$\iota_H^\top \pi_{H,t} \leqslant 1, \qquad \iota_F^\top \pi_{F,t} \leqslant \varphi, \tag{13}$$

where $\iota_H = [1, 1, 0]^\top$ and $\iota_F = [1, 0, 0]^\top$ and $\pi_{j,t}$ is the vector of agent j's portfolio choice.

[18]Technically, expanding the model to n countries is not a problem, but the increase in the number of parameters is substantial.

One of the constraints most commonly studied both theoretically and with respect to their empirical effects are short-selling restrictions. These are implicitly included here, through the joint constraints: either agent is prevented from soaking up too much excess supply, thus putting an upper bound on how much the other agent can short. Likewise, constraints like margin requirements can be thought of as an upper and lower bound to the portfolio. While here I only consider one-sided constraints to keep the model simple, the indirect lower limit on an investor's portfolio coming from the other investor's constraint can roughly approximate such a margin requirement.

The constraints the two investors face are reflected in their state price densities, adjusted to take into account the unattainability of certain payout states:

$$d\xi_t^i = -\left(r_t + \delta(\upsilon_t^i)\right)\xi_t^i dt - {\kappa_t^i}^\top \xi_t^i d\overrightarrow{W}_t^{(i)}, \tag{14}$$

where the adjusted market price of risk of agent $i = H, F$ is

$$\kappa_t^i = \sigma_{S,t}^{-1}\left(m_{S,t}^{(i)} + \upsilon_t^i \iota_i - r_t\right) = \kappa_{0,t}^i + \sigma_{S,t}^{-1}\upsilon_t^i\iota_i, \tag{15}$$

and υ_t^i is the adjustment term due to the constraint imposed on agent i. This adjustment term brings a (fictitious) change in agent i's expected returns, such that he is happy to invest within his allowed boundaries. Both types of constraints discussed here limit (albeit in different ways) investors' long portfolio positions, thus their υ_t^is will adjust their beliefs downward, such that they do not want to invest more than allowed. These fictitious adjustments drive a wedge between the beliefs "expressed" through portfolio choice and their truly held beliefs, which they are prevented from fully acting upon.

The *home* and *foreign* investors will choose their portfolios in line with their beliefs, conditional on their budget constraint.[19]

$$\begin{aligned} \overrightarrow{\pi}_{Ht} &= (\sigma_{S,t}^{-1})^\top\left(\kappa_{o,t}^H + \sigma_{S,t}^{-1}\upsilon_t^H\iota_H\right), && (16)\\ \overrightarrow{\pi}_{Ft} &= (\sigma_{S,t}^{-1})^\top\left(\kappa_{o,t}^F + \sigma_{S,t}^{-1}\upsilon_t^F\iota_F\right), && (17) \end{aligned}$$

where $\kappa_{o,t}^H$ is the three-dimensional vector of market prices of risk, under the beliefs held by agent H. $\kappa_{o,t}^F$ is defined accordingly for the *foreign* agent. Both agents hold the mean-variance portfolio

[19] The fact that log-utility investors do not hedge anticipated changes in their investment opportunity set like the potential future binding of constraints or the changes in belief dispersion helps in retaining tractability of this model.

in line with their own beliefs about growth rates, adjusted for their constrained investment set, as detailed in proposition 1. If an agent i's desired portfolio according to his beliefs is attainable and his constraint is not binding, this adjustment term $\upsilon_t^i \iota_i$ will be zero. Since both agents' constraints impose limitations on their long positions, the vectors $\upsilon_t^i \iota_i$ will always be non-positive.

Observational equivalence requires that both agents must agree on observed stock prices, therefore assets can be priced from either agent's perspective.[20] Since both agents face potential constraints, the adjusted state price density and market price of risk from eqs. (14) and (15) must be taken into consideration when appropriately discounting stock prices to arrive at the equilibrium valuation.

$$S_t^j = \frac{1}{\xi_t^H} E_t \left[\int_t^T \xi_s^H p_s^j Y_s^j ds \right] + \frac{1}{\xi_t^H} E_t \left[\int_t^T \left(\upsilon_t^H + \delta(\upsilon_t^H) \right) S_s^j \xi_s^H ds \right] \qquad j = h, f. \tag{18}$$

The first term is the usual value imparted by expected future dividends. The last two terms inside the second integral have been interpreted by Detemple and Murthy (1997) as speculative and collateral premia.

The interpretation of $E_t \int_t^T \delta(\upsilon_t^H) S_s^j \xi_s^H ds$ as a "collateral" value stems from the characteristic of the support function $\delta(\upsilon_t^H)$, it effectively acts as an additional endowment to the agent in the fictitious market.

A speculative premium arises in markets with constrained participants as investors speculate on the future necessity of other traders to buy (or sell) the specific asset for non-fundamental reasons, namely portfolio restrictions.

In their paper, Detemple and Murthy show the closed-form solution for a single-asset economy. As in this model, market clearing requirements limit the types of constraints that can bind in equilibrium. In the single-asset case, the constraints neutralize each other, and effects disappear. Asset prices and dynamics are the same as in a benchmark unconstrained economy.

Extending their analysis to multiple assets facilitates looking at transfers of wealth between countries and how comovement of assets is affected within the different possible equilibria. The appendix details the proof of gaining explicit values for the stock prices via market clearing.

[20] The valuation and thus the remainder of the paper are done using agent H's perception of risk. This is without loss of generality, as results that are affected by this, such as returns, can easily be translated into agent F's or the "benchmark" perception by applying the relationships known from section 2.2.

Proposition 1. *In equilibrium, stocks are priced by*

$$S_t^h = \frac{1}{\beta + (1-\beta)\,\bar{p}_t} Y_t^h (T-t) \tag{19}$$

$$S_t^f = \frac{\bar{p}_t}{\beta + (1-\beta)\,\bar{p}_t} Y_t^f (T-t) \quad \forall t \tag{20}$$

where

$$\bar{p}_t = \frac{(1-\alpha_t^H) + \alpha^F \lambda_t}{\alpha_t^H + (1-\alpha^F)\lambda_t} \frac{Y_t^h}{Y_t^f}. \tag{21}$$

There are four possible cases of the equilibrium — neither agent is constrained, only the home agent's constraint is binding, only the foreign agent's constraint is binding, and both agents' constraints are binding. The state variable λ_t thus follows a case-dependent process:

case U (both agents unconstrained)

$$d\lambda_t^{\mathtt{U}} = \lambda_t \overrightarrow{\Delta m_Y}^\top \overrightarrow{dW}_t^{(H)},$$

when $\Delta m_{Yh} < \big(\varphi\,(1+\lambda_t) - \alpha_t^H - (1-\alpha^F)\lambda_t\big)\,\sigma_{Yh}$ *&* $\Delta m_{Yf} > -\frac{\sigma_{Yf}}{\sigma_{Yh}} \Delta m_{Yh}$

case F (agent F limited in holding S_t^h)

$$d\lambda_t^{\mathtt{F}} = \Big[\Delta m_Y + \sigma_{S,t}^{-1} \upsilon_t^F \iota_F\Big]^\top \lambda_t \overrightarrow{dW}_t^{(H)},$$

when $\Delta m_{Yh} > \big(\varphi\,(1+\lambda_t) - \alpha_t^H - (1-\alpha^F)\lambda_t\big)\,\sigma_{Yh}$ *&*
$\Delta m_{Yf} > \frac{-\Delta m_{Yh}\left[\lambda_t(1-\varphi)(1-\alpha_t^H-\alpha^F)+\alpha_t^H\left(1-\alpha_t^H+\alpha^F\lambda_t\right)\right]^2\sigma_{Yh}\sigma_{Yf}-\left(\varphi(1+\lambda_t)-\alpha_t^H-(1-\alpha^F)\lambda_t\right)\sigma_{Yf}\sigma_{\alpha*}^2}{\left[\lambda_t(1-\varphi)(1-\alpha_t^H-\alpha^F)+\alpha_t^H\left(1-\alpha_t^H+\alpha^F\lambda_t\right)\right]^2\sigma_{Yh}^2+\sigma_{\alpha*}^2}$

case H (agent H leverage constrained)

$$d\lambda_t^{\mathtt{H}} = \Big[\Delta m_Y - \sigma_{S,t}^{-1} \upsilon_t^H \iota_H\Big]^\top \lambda_t \overrightarrow{dW}_t^{(H)}$$

when $\Delta m_{Yh} < \big(\varphi\,(1+\lambda_t) - \alpha_t^H - (1-\alpha^F)\lambda_t\big)\,\sigma_{Yh}$ *&*
$-\big(\varphi\,(1+\lambda_t) - \alpha_t^H - (1-\alpha^F)\lambda_t\big)\,\frac{\sigma_{Yh}^2+\sigma_{Yf}^2}{\sigma_{Yf}} + \Delta m_{Yh}\frac{\sigma_{Yh}}{\sigma_{Yf}} < \Delta m_{Yf} < -\frac{\sigma_{Yf}}{\sigma_{Yh}}\Delta m_{Yh}$

case FH (both agents constrained)

$$d\lambda_t^{\mathtt{FH}} = \Big[\Delta m_Y + \sigma_{S,t}^{-1}\left(\upsilon_t^F \iota_F - \upsilon_t^H \iota_H\right)\Big]^\top \lambda_t \overrightarrow{dW}_t^{(H)}$$

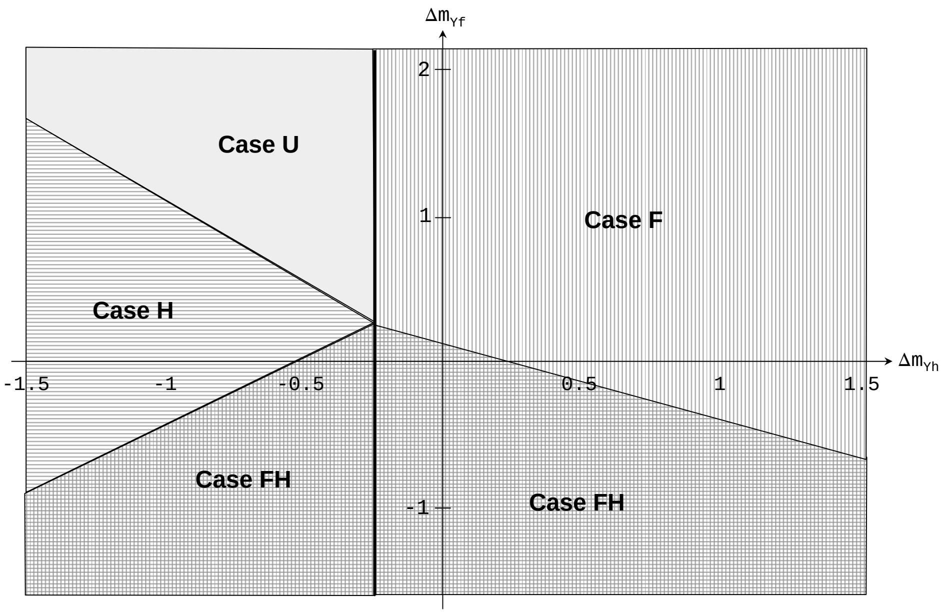


Figure 1: **Equilibria with two constraints:** The constraints imposed on investors F and H can each bind individually (case F, case H), jointly (case FH), or not at all (case U). Which of the four possible equilibria holds, depends on the belief dispersion regarding fundamental economic growth rates, Δm_{Y_h} and Δm_{Y_f}.

when

$\Delta m_{Yh} < \left(\varphi\left(1+\lambda_t\right) - \alpha_t^H - (1-\alpha^F)\lambda_t\right)\sigma_{Yh}$ *&*

$\Delta m_{Yf} < -\left(\varphi\left(1+\lambda_t\right) - \alpha_t^H - (1-\alpha^F)\lambda_t\right)\frac{\sigma_{Yh}^2+\sigma_{Yf}^2}{\sigma_{Yf}} + \Delta m_{Yh}\frac{\sigma_{Yh}}{\sigma_{Yf}}$

or

$\Delta m_{Yh} > \left(\varphi\left(1+\lambda_t\right) - \alpha_t^H - (1-\alpha^F)\lambda_t\right)\sigma_{Yh}$ *&*

$\Delta m_{Yf} < \frac{-\Delta m_{Yh}\left[\lambda_t(1-\varphi)(1-\alpha_t^H-\alpha^F)+\alpha_t^H\left(1-\alpha_t^H+\alpha^F\lambda_t\right)\right]^2\sigma_{Yh}\sigma_{Yf}-\left(\varphi(1+\lambda_t)-\alpha_t^H-(1-\alpha^F)\lambda_t\right)\sigma_{Yf}\sigma_{\alpha*}^2}{\left[\lambda_t(1-\varphi)(1-\alpha_t^H-\alpha^F)+\alpha_t^H\left(1-\alpha_t^H+\alpha^F\lambda_t\right)\right]^2\sigma_{Yh}^2+\sigma_{\alpha*}^2}.$

In equilibrium, υ_t^H *and* υ_t^F *are*

$$\upsilon_t^H = \min\left(\frac{1-\iota_H^\top(\sigma_S^{-1})^\top\kappa_{ot}^H}{\iota_H^\top(\sigma_S^{-1})^\top\sigma_S^{-1}\iota_H}, 0\right); \quad \upsilon_t^F = \min\left(\frac{\varphi-\iota_F^\top(\sigma_S^{-1})^\top\kappa_{ot}^F}{\iota_F^\top(\sigma_S^{-1})^\top\sigma_S^{-1}\iota_F}, 0\right). \tag{22}$$

For reasons of space, the appendix gives the technical details and shows the adjustments υ_t^iin terms of fundamentals, thereby closing the model.

Figure (1) illustrates graphically the conditions under which the four possible equilibria from Prop. 1 hold. As belief dispersion changes, the economy may jump between equilibria. Belief dispersion determines portfolio choice of the two investors, and thus also which constraint may bind. Together, figs. (1) and (2) illustrate the difference between this model and the one that

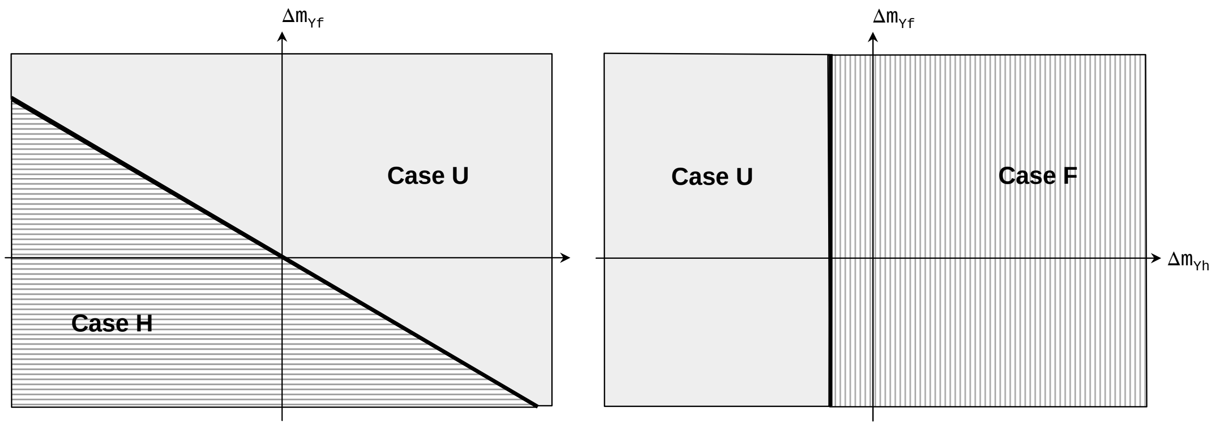


Figure 2: **Economies with a single constraint:** The left graph shows the possible equilibria for an economy where only investor H faces the leverage constraint. This constraint will bind when H is relatively optimistic, i.e. when Δm_{Y_h} and Δm_{Y_f} are negative. The right graph shows possible equilibria when only investor F faces his constraint on holdings of S_t^h. This constraint will bind when Δm_{Y_h} is relatively large.

has been discussed in more depth in the literature - where only one agent (either F or H) is constrained.

Compare the bottom left quadrant of fig. (1) to the same quadrant in the right graph of fig. (2).When both constraints are imposed, investor F's constraint will also bind in parameter regions where, if his constraint were imposed in isolation, it would not: when he is relatively pessimistic about the stock he is restricted in: $\Delta m_{Y_h} < 0$. This illustrates the importance of considering constraints' equilibrium interaction when looking at policy implications. In order to predict the effects of eliminating (or, perhaps, imposing) a particular constraint, the remaining imperfections of the market must be considered.

The figures demonstrate that comparing conditional stock market dynamics pre- and post-liberalization is only useful within certain ranges of belief dispersion.Eliminating a constraint will only have an effect on capital flows across countries (and thus first and second moments of stock dynamics) if it is binding at the time. For example, assume that currently, Δm_{Y_f} (the dispersion of beliefs about the *foreign* country's growth rate) is small in magnitude, but Δm_{Y_h} is strongly negative (implying that H is much more optimistic about *home* country's growth rate than F is). Fig. (1) shows that only H's leverage constraint will bind (case H), therefore a decision to remove the constraint on F's holdings of S_t^h would not have any effect on portfolios or stock markets.[21]

[21]Due to the log utility of investors, they do not hedge changes in the investment opportunity set, i.e. 'potentially' binding constraints. For other utility functions, lifting constraints could have an effect on portfolios even at times when they are not currently binding.

Therefore, when looking at the effect of partial or full liberalization on stock price dynamics, it is critical that this comparison be made within the relevant range of belief dispersion.

4 Equilibrium Effects on Stock Prices and Dynamics

Despite individual goods' production processes being completely uncorrelated, the stock prices will not be. In equilibrium they are driven by the consumption and investment choices of both agents through the goods' relative price dynamics.[22]

The relative price of the goods, $\bar{p}_t = p^f_t / p^h_t$, is the axis around which shocks propagate through the system. Perfect integration of goods markets implies that $\bar{p}_t$ is determined by the relation of marginal utilities with respect to the two goods, $\bar{p}_t = u^i_{C^F}(\cdot) / u^i_{C^H}(\cdot)$, which has the form described in eq. (21):

$$\bar{p}_t = \frac{(1-\alpha^H_t) + \alpha^F \lambda_t}{\alpha^H_t + (1-\alpha^F)\lambda_t} \frac{Y^h_t}{Y^f_t}.$$

$\bar{p}_t$ captures both supply and demand effects, the latter being driven by consumption preferences and the distribution of wealth.[23] A positive supply shock to *home* country's good Y^h_t will lower its price relative to p^f_t, as it is now more abundant — $\bar{p}_t$ will rise. Analogously, a positive supply shock to good Y^f_t will lower $\bar{p}_t$.

From the demand side — higher demand for a good will raise that good's price in equilibrium. As α^H_t and $(1-\alpha^F)$ represent the relative preferences for good Y^h_t, high levels of these will lead to high demand for Y^h_t and lower demand for Y^f_t, thus lowering relative price $\bar{p}_t$.

The dynamics of relative goods prices $\bar{p}_t$ follow

$$\begin{aligned} d\bar{p}_t &= (\cdot)dt + \frac{1-\alpha^H_t + \alpha^F\lambda_t}{\alpha^H_t + (1-\alpha^F)\lambda_t} \frac{1}{Y^f_t} dY^h_t - \frac{1-\alpha^H_t + \alpha^F\lambda_t}{\alpha^H_t + (1-\alpha^F)\lambda_t} \frac{Y^h_t}{(Y^f_t)^2} dY^f_t - \\ &- \frac{\lambda_t + 1}{(\alpha^H_t + (1-\alpha^F)\lambda_t)^2} \frac{Y^h_t}{Y^f_t} d\alpha^H_t + \frac{2\alpha^H_t - 1}{(\alpha^H_t + (1-\alpha^F)\lambda_t)^2} \frac{Y^h_t}{Y^f_t} d\lambda_t. \end{aligned}$$

The described demand effect links, through λ_t, the goods markets to the financial markets. Due to the log-utility of both agents, the ratio of their state prices λ_t is uniquely captured by

[22] Despite there being a sufficient number of four assets — two bonds and two stocks — to span the three sources of risk, completeness does not follow necessarily, but can be shown to hold in this setting.

[23] In contrast to the model of Dumas and Solnik (1995), where consumers in different countries face different prices for the same good, here all consumers face the same price as there are no frictions in the goods market. Therefore there is no "market price of exchange rate risk" in this model.

their relative wealth, $wealth^F/wealth^H$, which is determined by their investments, or rather, the differences in their investments. If investors hold different portfolios, their levels of wealth will be differently affected by stock returns. Both belief dispersion (differences in portfolio holdings by choice) as well as constraints (differences in portfolio holdings by force) will thus affect relative wealth over time, thereby affecting consumption choice, which ultimately feeds back into stock prices.

An example of this wealth effect is the development of copper prices. Copper is a commodity heavily used in industrial economies. The rising relative wealth of these countries have lead to rising copper prices, as demand rose alongside wealth. These rising prices meant a windfall for economies with an important copper industry, like Chile.

4.1 Stock Returns

Investors differences in beliefs regarding the two countries' economic growth rates will transmit into different beliefs regarding stock returns.[24] Equations (23) and (24) describe the expected stock returns on both the *home* and the *foreign* stock from the viewpoint of the *home* investor H. The stock returns expected under the *foreign* investor's beliefs follow directly from the dispersion of beliefs as defined in eq. (11).

$$\mu^H_{S_h} = m^H_{Y_h} - \frac{(1-\beta)\bar{p}_t}{\beta+(1-\beta)\bar{p}_t}\mu_{\bar{p}} + \frac{(1-\beta)^2\bar{p}_t^2}{(\beta+(1-\beta)\bar{p}_t)^2}\sigma^2_{\bar{p}} - \frac{(1-\beta)\bar{p}_t}{\beta+(1-\beta)\bar{p}_t}\sigma_{(Y_h,\bar{p})}, \tag{23}$$

$$\mu^H_{S_f} = m^H_{Y_f} + \frac{\beta}{\beta+(1-\beta)\bar{p}_t}\mu_{\bar{p}} - \frac{\beta(1-\beta)\bar{p}_t}{(\beta+(1-\beta)\bar{p}_t)^2}\sigma^2_{\bar{p}} + \frac{\beta}{\beta+(1-\beta)\bar{p}_t}\sigma_{(Y_f,\bar{p})}. \tag{24}$$

where $\mu_{\bar{p}}$ and $\sigma_{\bar{p}}$ stand for the drift and volatility of equilibrium goods prices $\bar{p}_t$ as determined above. Both expected stock returns will increase when (expected) dividend growth rises. Since $\bar{p}_t$ is the price of good Y^f_t relative to Y^h_t, an expected rise in this price ratio ($\mu_{\bar{p}} > 0$) will be detrimental for S^h_t, as it implies a downward trend on the relative value of future output. Conversely, it will benefit the expected return of S^f_t.

Analogously, the variance of the relative price $\bar{p}_t$ also has an opposite effect on the two stocks' expected returns: expected returns of a stock will be depressed when it's own price relative to

[24] This has been established in other models of heterogeneous agents. By observational equivalence, differences in beliefs about the mean growth rate of a stochastic process imply disagreement about the realizations of the random shocks. This disagreement will thus be reflected in stock dynamics as well.

the other good's price varies a lot.[25] Lastly, the expected return also depends on the covariance between output and price. For both countries' stock markets, if output of the local good is high when the price of this local good is high, this will increase the expected return, as it exacerbates the risk of the stock.

For this covariance between output and relative price, two factors play a role, as $\bar{p}_t$ is directly affected by both the supply and demand side.

$$\sigma_{(Y_h,\bar{p})} = \text{cov}(d\bar{p}_t, dY_t^h) = \bar{p}_t Y_t^h \sigma_{Y_h} + \frac{\lambda_t(\alpha_t^H + \alpha^F - 1)}{(\alpha_t^H + (1-\alpha^F)\lambda_t)(1-\alpha_t^H + \alpha^F\lambda_t)} \bar{p}_t Y_t^h \sigma_{Y_h} \Delta\kappa_i,$$
$$\sigma_{(Y_f,\bar{p})} = \text{cov}(d\bar{p}_t, dY_t^f) = -\bar{p}_t Y_t^f \sigma_{Y_f} + \frac{\lambda_t(\alpha_t^H + \alpha^F - 1)}{(\alpha_t^H + (1-\alpha^F)\lambda_t)(1-\alpha_t^H + \alpha^F\lambda_t)} \bar{p}_t Y_t^f \sigma_{Y_f} \Delta\kappa_{ii}.$$

The first term reflects the supply effect. Higher production of the *home* good leads to higher relative price $\bar{p}_t = p_t^f/p_t^h$.[26] The second effect is determined through the demand side — how changes in relative wealth affect aggregate demand, and thus prices. This effect will be discussed in more detail in the next section.

4.2 Stock Market Dynamics

From equilibrium stock prices as determined in Proposition 1 stock price dynamics can be determined as follows:

$$dS_t^h = (\cdot)dY_t^h + (\cdot)dY_t^f + (\cdot)d\alpha_t^H - (\cdot)d\lambda_t^{(\text{case})}, \quad (25)$$
$$dS_t^f = (\cdot)dY_t^h + (\cdot)dY_t^f - (\cdot)d\alpha_t^H + (\cdot)d\lambda_t^{(\text{case})}, \quad \text{all } (\cdot) > 0 \quad (26)$$

where the dynamics of λ_t will depend on which equilibrium investors find themselves in, i.e. which constraints are binding. In a direct supply or 'dividend' effect, a positive production shock to, e.g., good Y_t^h leads to an appreciation of the stock price S_t^h. But indirectly, via the 'terms-

[25]Recall that the volatility of the price of good Y_t^h relative to that of Y_t^f is $-\sigma_{barp}$, which allows symmetric interpretation of the price variance effect.

[26]Empirical studies more often link stock returns to inflation variables rather than price variation in goods directly, but Kaul and Seyhun (1990) show that the explanatory effect of inflation on stock returns is actually a proxy for price variability.

of-trade' effect, this increase in supply will simultaneously tend to push down the good's price — putting downwards pressure on S_t^h and pushing up S_t^f. The 'dividend' effect of a production shock will however always dominate the negative 'terms-of-trade' effect on its own stock. This shock to Y_t^h is uncorrelated to production shocks to the *foreign* good, and thus affects S_t^f only through the terms-of-trade effect, which pushes up S_t^f.[27]

An increase in demand α_t^H for the *home* good Y_t^h will increase its relative price, thus increasing the value of output, pushing up stock price S_t^h. Conversely it will put downward pressure on the stock price of the *foreign* country's good, as demand for Y_t^f now shrinks.

Constraints will affect stock price dynamics through relative wealth dynamics $d\lambda_t$, arising from the divergence in portfolio choice — which is partly desired, and potentially forced by the constraints.

The relative wealth distribution λ_t affects stock price dynamics via the relative goods price: $\partial S_t^h/\partial\lambda_t = \partial S_t^h/\partial\bar{p}_t \cdot \partial\bar{p}_t/\partial\lambda_t$. The first of these two terms, the 'terms-of-trade' effect discussed above, is negative for S_t^h. For stock S_t^f, $\partial S_t^f/\partial\bar{p}_t$ is positive. $\partial\bar{p}_t/\partial\lambda_t$ is positive: λ_t increases when investor F's relative wealth increases. As both agents have a home bias in consumption, a shift of wealth towards agent F will lead to a relative increase in demand for good f, raising $\bar{p}_t$, thus benefiting stock S_t^f. Conversely, a wealth transfer to agent H (lower λ_t) will boost returns of S_t^h, as H will channel more of his money into his own local good.

These 'feedback effects' play a critical role for asset prices. Belief dispersion and (binding) constraints determine the direction of wealth transfers through portfolio choice. The investor that holds a larger fraction of his wealth in a particular stock will effectively receive a wealth transfer from the other agent when this stock experiences dividend or capital gains. This demand-side price effect determines whether the feedback effect is "positive" or "negative". If the investor that overweighs stock S_t^i (relative to the market portfolio) also has a preference for consuming the good produced in country i, then the wealth gain resulting from positive returns to S_t^i will again benefit this same stock through the demand-side price effect: the feedback effect is positive. If on the other hand this investor prefers consuming good j, its relative price will rise, and the feedback effect will be negative.

For example, if agent H exhibits, due to his beliefs or perhaps a binding constraint, a home bias in his portfolio, a positive return to his domestic stock S_t^h will make him relatively wealthier, thus allowing him to consume more. He will consume more of his own good Y_t^h, thus pushing

[27]The analogous intuition about dividend and terms-of-trade effects applies to both stocks.

up the good's price, again benefiting the local stock S_t^h. This is a positive feedback effect. If, on the other hand, he is very optimistic regarding the foreign investment opportunity, he will have a high exposure to S_t^f in his portfolio. Returns to *this* stock will now give rise to a wealth transfer towards agent H. But again, he will channel a large fraction of his new wealth into consumption of the *home* good and the resulting price increase of Y_t^h will harm the price of stock S_t^f — a negative feedback effect.

The volatility vectors $\sigma_t^{S^h}$ and $\sigma_t^{S^f}$ from eqs. (2) and (3) of the stock prices are three-dimensional in equilibrium, with both stocks exhibiting sensitivity with respect to all three sources of risk — production risk of the *home* good (dW_t^h), production risk of the *foreign* good (dW_t^f), and demand risk (dW_t^*):[28]

$$\sigma_t^{S^h} = \begin{pmatrix} -\frac{\bar{p}(1-\beta)}{\beta+(1-\beta)\bar{p}} \frac{\lambda(\alpha^H+\alpha^F-1)}{(1-\alpha^H+\lambda\alpha^F)(\alpha^H+\lambda(1-\alpha^F))} \Delta\kappa_i + \sigma_{Y_h} \frac{\beta}{\beta+(1-\beta)\bar{p}} \\ -\frac{\bar{p}(1-\beta)}{\beta+(1-\beta)\bar{p}} \frac{\lambda(\alpha^H+\alpha^F-1)}{(1-\alpha^H+\lambda\alpha^F)(\alpha^H+\lambda(1-\alpha^F))} \Delta\kappa_{ii} + \sigma_{Y_f} \frac{(1-\beta)\bar{p}}{\beta+(1-\beta)\bar{p}} \\ -\frac{\bar{p}(1-\beta)}{\beta+(1-\beta)\bar{p}} \frac{\lambda(\alpha^H+\alpha^F-1)}{(1-\alpha^H+\lambda\alpha^F)(\alpha^H+\lambda(1-\alpha^F))} \Delta\kappa_{iii} + \sigma_\alpha \frac{(1-\beta)\bar{p}}{\beta+(1-\beta)\bar{p}} \frac{1+\lambda}{(1-\alpha^H+\lambda\alpha^F)(\alpha^H+\lambda(1-\alpha^F))} \end{pmatrix} \tag{27}$$

$$\sigma_t^{S^f} = \begin{pmatrix} \frac{\beta}{\beta+(1-\beta)\bar{p}} \frac{\lambda(\alpha^H+\alpha^F-1)}{(1-\alpha^H+\lambda\alpha^F)(\alpha^H+\lambda(1-\alpha^F))} \Delta\kappa_i + \sigma_{Y_h} \frac{\beta}{\beta+(1-\beta)\bar{p}} \\ \frac{\beta}{\beta+(1-\beta)\bar{p}} \frac{\lambda(\alpha^H+\alpha^F-1)}{(1-\alpha^H+\lambda\alpha^F)(\alpha^H+\lambda(1-\alpha^F))} \Delta\kappa_{ii} + \sigma_{Y_f} \frac{(1-\beta)\bar{p}}{\beta+(1-\beta)\bar{p}} \\ \frac{\beta}{\beta+(1-\beta)\bar{p}} \frac{\lambda(\alpha^H+\alpha^F-1)}{(1-\alpha^H+\lambda\alpha^F)(\alpha^H+\lambda(1-\alpha^F))} \Delta\kappa_{iii} - \sigma_\alpha \frac{\beta}{\beta+(1-\beta)\bar{p}} \frac{1+\lambda}{(1-\alpha^H+\lambda\alpha^F)(\alpha^H+\lambda(1-\alpha^F))} \end{pmatrix} \tag{28}$$

where $\Delta\kappa_i$, $\Delta\kappa_{ii}$ and $\Delta\kappa_{iii}$ are the three elements of vector $\Delta\kappa$, the dispersion in investors' perception of the market prices of risk.

Empirically, the data show very clearly that by far the largest share of consumption falls to domestic goods and services. However, it is nevertheless valuable to keep in mind that the results of stock dynamics hinge on this characteristic of the real economy. Some goods, like services or perishable goods, will be consumed predominantly domestically. It is intuitive that firms who sell their goods internationally are more susceptible to 'non-domestic' shocks. In a world of many goods and many countries, this would indeed suggest that this is not so much a function of whether or not a firm has markets all over the world, but rather how evenly they are distributed across the world. The stock price of a firm who exports their goods to many countries with a moderate taste for this good would be much less reactive to changes in any single country's wealth and consumption, acting

[28]Time-t subscripts have been dropped in eqs. (27) and (28) for parsimony of notation.

as a natural hedge. Quantitatively, a consumption home bias in a two-country model may seem to exaggerate the impact of non-domestic consumption on the price level of export goods. This concern is alleviated by extending this model to n countries, while the qualitative effect remains desirable.

The model clearly distinguishes between a home bias in consumption and a home bias in portfolio choice, the latter of which is explicitly not assumed. In the nested benchmark equilibrium — assuming beliefs about economic growth rates are identical and capital markets are perfect — investors can be shown to both hold the world market portfolio, investing into each stock according to its weight in the world index.In the benchmark case, relative wealth λ is a constant, determined entirely by the initial endowment.[29]

I allow investors to differ along two dimensions: they hold different beliefs about the countries' economic growth rates, and both face a portfolio constraint. These constraints drive a wedge between the true dispersion of beliefs and the beliefs that are reflected in the investors' portfolios. Although the two constraints are not identical, both impose a limit on the long position held by the respective investor.[30] For investor F, the constraint binds if he would choose to hold more of S_t^h than φ, his exogenously imposed limit. This means his portfolio will reflect a less optimistic belief about the *home* country's growth rate than he actually has: he is prevented from 'betting' as much as he would like on positive returns to S_t^h. Depending on the true level of belief dispersion, his (constrained) portfolio may be more or less similar to that of investor H compared to the portfolio he would have chosen in absence of his constraint.

The vector of dispersion in investors' perceptions of the market prices of risk $\Delta\kappa$ are determined by belief dispersion Δm_{Y_h} and Δm_{Y_f} about fundamentals. When belief dispersion is such that neither constraint binds (case U), any time-variation in λ_t arises from the investors choosing to hold different portfolios, each in line with their respective perceptions of the market prices of risk. Agents agree about the distribution of demand shocks dW_t^*, and will evenly share this risk.In the three equilibria where either or both of the constraints bind — cases F, H and FH — the binding constraints will change how belief dispersion is reflected in portfolios, and thereby in equilibrium market prices of risk: $\Delta\kappa = \Delta m_Y + \sigma_{S,t}^{-1}\left(\upsilon_t^F \iota_F - \upsilon_t^H \iota_H\right)$, where $\upsilon_t^i \leq 0$ for $i = F, H$. When only investor F's constraint binds, $\upsilon_t^H = 0$, which implies $\Delta\kappa < \Delta m_Y$: F's smaller-than-desired

[29]The model of Uppal (1993) shows that directly incorporating imperfections in the goods markets would create a home bias in portfolios even in a log-utility setting. In this model I abstract from such imperfections to clearly show the effects of financial constraints.

[30]In equilibrium, the other investor's constraint implies an endogenous limit on short positions.

portfolio holdings in S_t^h will make him look less optimistic. Note that this does not imply an unambiguous effect on the magnitude $|\Delta\kappa|$. That can increase or decrease, depending on the level of Δm_Y. A binding leverage constraint on H tends to increase $\Delta\kappa$. Which of these effects dominates when both constraints bind is ambiguous. The next section discusses the resulting implications for correlation between the two stock markets and volatilities.

4.2.1 Stock Return Correlation between S_t^h and S_t^f

Consider case F, in which only the *foreign* investor is constrained. Focusing first on the case of a single binding constraint demonstrates the contribution of belief dispersion in a constrained environment, separately from the issue of multiple constraints.

Stock return correlation is determined through the feedback effects via the relative goods price, as described above. Consider the benchmark model where both investors hold the market portfolio. Production shocks and the supply-side effect on the terms of trade imply that stocks are perfectly correlated.[31] Now suppose, for example, that investor H is much more optimistic about his domestic growth rate than investor F (belief dispersion is high). His overweighing of his domestic asset in his portfolio will lead to a strong positive feedback effect from the demand side of the terms-of-trade effect. A positive return to the *home* stock will push up S_t^h further, and put downwards pressure on S_t^f: correlation will be lower than in the benchmark case where both investors hold the market portfolio. The more dissimilar the portfolio choices of the two investors are, the lower is the correlation between the two stock markets. When investor F's constraint binds, he cannot hold as much of stock S_t^h as he would like to.Liberalization in this case means the exogenous decision to eliminate this constraint. Immediately after liberalization, F will trade with H and purchase more of S_t^h. Whether this trade will increase or decrease correlation depends on whether this trade makes the two investors' portfolios more or less similar to one another.

Assume first that belief dispersion about *home* country's growth rate is very small, indeed one can look at the special case of homogeneous beliefs. In the right graph of fig. (2), this would imply investors are close to the origin on the horizontal axis. H and F would like to hold the same proportion of their wealth in S_t^h, but the upper bound on F's investment, φ, does not allow this. F's portfolio now reflects a more pessimistic view than he actually has. $\Delta\kappa_i$ is negative, even though $\Delta m_{Y_h} = 0$: the magnitude of belief dispersion is artificially increased by the constraint,

[31] The only factor bounding correlation away from one is the demand shock $d\alpha_t^H$, which has a positive impact on S_t^h and a negative impact on S_t^f. This effect is constant at any level of belief dispersion, and does not alter the intuition about the effect of belief dispersion on correlation.

resulting in a positive feedback effect in S_t^h. Liberalizing markets by lifting F's constraint will thus eliminate this feedback effect, increasing correlation between the two markets. Fig. (3) provides a numerical illustration.

Now assume instead that F is much more optimistic about growth rates in *home*: $m_{Y_h}^F - m_{Y_h}^H$ is strongly positive. If he were unconstrained, F would overweight the *home* stock heavily. H would accordingly underweight the stock in his own portfolio, thereby inducing strong feedback effects. But the constraint induces F's portfolio to reflect a more pessimistic view.It forces portfolios to be more similar, mitigating the negative feedback effect. Upon liberalization, investors' trades will make portfolios diverge, and correlation across markets will decrease. A numerical illustration of this can be seen in fig. (4).

The empirical studies mentioned earlier find that on average, liberalization events seem lead to an increase in correlation, albeit weakly. However, this effect is not consistent across countries. My model provides an explanation for this disparity. The results indicate that whether liberalization leads to higher or lower correlations with the world market depends on how tightly a constraint binds, and thus the amount of capital flows across countries following the liberalization. Linking the effect on correlation to how tightly a constraint binds may be able to explain why, on average, we see a (small) positive effect on correlation. The political pressure to lift constraints on capital inflow (here into *home* country) is presumably larger when these constraints bind tightly, i.e. when foreign investors are very interested to hold these assets. Lifting them at this point in time would lead to large immediate capital inflows and higher correlation with world markets. When foreign investors are skeptical of investment opportunities abroad and their constraint binds only midly, there is likely to be less political pressure. This would suggest a selection bias in the liberalization events we see, tilting the results toward observing correlation increases on average.

In this paper I model only two countries, though the concept can be extended to a world with n countries. When multiple countries with different investment opportunities exist, this correlation effect would depend on the relative optimism or pessimism regarding the entire international investment opportunity set. Accordingly, seeing waves of coinciding liberalizations could likewise mitigate an expected increase in correlation post-liberalization.

In interpreting the magnitudes of correlations and volatilities in figures (3) and (4), it must be stressed that these are conditional moments. To put it into perspective, when neither investor is constrained under these fundamental parameters, the correlation and volatilities of S_t^h and S_t^f are

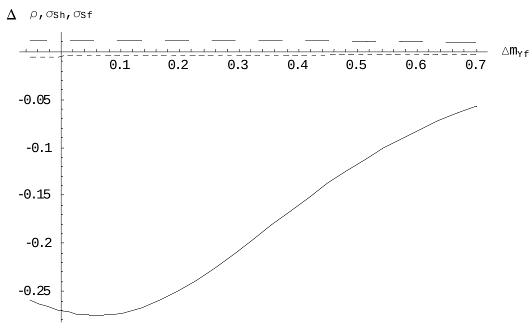


Figure 3: **Difference in Correlation between case F and case U:** When agent F is not very optimistic (low Δm_{Yh}, here 0.2, implying $m^F_{Y_h} - m^H_{Y_h}$=0.024) and nevertheless bound by his constraint on S^h_t, then correlation between stocks returns (solid line) is lower before liberalization. Removing a single constraint has only small effects on stock volatility of S^h_t (dashed line) and S^f_t (dotted line).

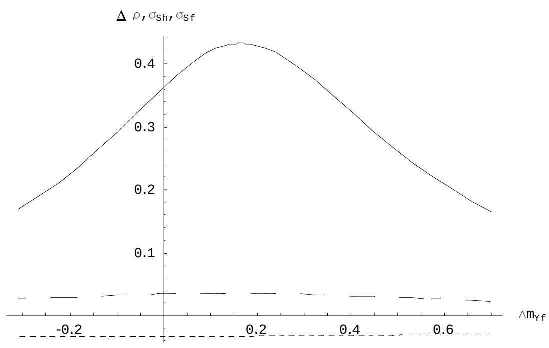


Figure 4: **Difference in Correlation between case F and case U:** When agent F is very optimistic (high Δm_{Y_h}) and his constraint on S^h_t binds severely, correlation between stock returns (solid line) is higher before liberalization.

at reasonable levels — volatilities of $10-15\%$ and positive correlations.[32]

Another way of interpreting these results is along the time series dimension. As discussed above, a constraint of the type that investor F faces has a 'mitigating' effect on correlation. At levels of belief dispersion where the unconstrained market displays high correlation, the constraint decreases correlation. At levels of belief dispersion that imply a low correlation in the unconstrained market, the constraint increases correlation. I do not from modeling time variation in belief dispersion explicitly in this paper, for reasons of parsimony. But it is intuitive to think that beliefs may change over time, as investors try to learn about fundamental growth rates by observing output.[33] The results show that in the benchmark 'perfect' economy without constraints, variation in beliefs over time would likewise make correlation between stock markets time-varying. A constraint of the type imposed on the *foreign* investor would mitigate this time-variation, indicating that liberalization would tend to increase the volatility of a market's correlation with the world market.

Alongside correlations, the effects on the composition of volatility implied by this model are consistent with empirical findings. A constraint on foreigners owning a country's stock will induce a stronger positive feedback (or lessen a negative feedback). As a result, this "restricted" stock will be dominated by local risk and less responsive to foreign shocks. This dominance of domestic risk in the stock market is consistent with Harvey (1995), who finds that emerging market returns are more likely than developed countries to be influenced by local information.[34]

4.2.2 Effects of Liberalization on Stock Volatilities, and their Relation to Excess Returns

In contrast to mean growth rates, investors can determine total stock volatility through quadratic variation of stock dynamics. Dispersion in beliefs affects equilibrium stock volatility via λ_t, as portfolios (and thus relative wealth) are determined by market prices of risk, $\Delta\kappa$ (see eqs. (27) and (28)). The variance of stock returns is a parabolic function of belief dispersion: a stock's variance is at its minimum when belief dispersion is zero, and as dispersion becomes very positive or very negative, variance increases. This relationship also holds in the presence of constraints, but is shifted. Depending on how tightly they bind, constraints will lead asset prices to reflect a

[32]Figs. (3) and (4) are based on the following fundamental economic parameters: $\alpha^H = \alpha^F = 0.7$, $\beta = 0.4$, $\varphi = 0.2$, $\lambda_t = 2$, $\sigma_{Yh} = 0.12$, $\sigma_{Yf} = 0.05$, $Y_t^h = 10$, $Y_t^f = 45$, $\sigma_{\alpha*} = 0.02$, $\Delta m_{Yh} = 1$, implying $m^F_{Yh} - m^H_{Yh}$=0.12.

[33]Incorporating a process of learning for the two investors is technically trivial, but increases the parameter space. I have therefore relegated this to separate work.

[34]Harvey (1991) more generally finds that countries have a time-varying exposure to world covariance risk. While my model supports this finding, it is surely not a unique link.

level of belief dispersion that is larger or smaller than the true level of dispersion. Therefore, a constraint will shift the parabola along the axis of belief dispersion.

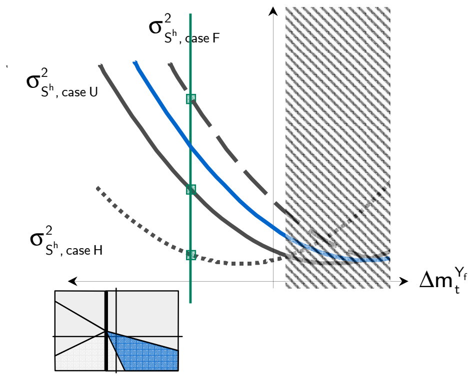
The previous section of the paper provided the intution for the effect of liberalization on stock markets' correlation using the simpler case where only one constraint is imposed. The intuition that tightness of constraints determines whether correlation increases or decreases in response to liberalization remains the same for an economy where both investors face constraints. However, when both investors are bound by their constraints, either of the constraints can dominate the effect on 'reflected' versus 'true' dispersion in beliefs. This competition for dominance is also salient in the effect of constraints on stock volatility, which we discuss below. In the interest of brevity, I focus on the volatility of the *home* stock S_t^h. In the context of the two particular constraints I study in this paper, this stock is the more interesting to analyze. When both constraints bind simultaneously, S_t^h is directly affected by both investors' constraints, whereas only H's leverage constraint has a direct effect on the *foreign* stock S_t^f.

For example, if only the constraint on the *foreign* investor binds, the *home* investor has the flexibility to provide the necessary liquidity for market clearing. If, however, both constraints bind (case FH), investor H is not able to provide this extra liquidity, as his own investment constraint limits his portfolio choices as well. In order to achieve market clearing, a shift has to occur in holdings of all assets in the market, and along with them the compensation for risk, the effect is no longer isolated to S_t^h. The general intuition provided below on how constraints affect volatility is analogous for the volatility of stock S_t^f, which is therefore not discussed in detail.

The effect of removing a constraint from the economy will vary, depending on whether removing this constraint moves the economy to full liberalization or the other constraint remains. A parameter range of $\Delta m_{Y_h} > 0$ and $\Delta m_{Y_f} < 0$ serves best to compare all four possible equilibria. In figs. (1) and (2) demonstrating the conditions for binding constraints, this is within the bottom right quadrant. In an economy where both investors face constraints, these will both bind. In an economy where only investor F faces the constraint on investment abroad, this constraint will bind. And if only investor H faces a leverage constraint, this will likewise bind.

This parameter range thus provides a setting where the effects of partial and full liberalization can be compared directly. A caveat of the model should be mentioned here: the level of belief dispersion required for the binding of constraints and quantitatively reasonable effects on stock market dynamics may seem excessive.[35] This is an artifact of the two-country model. Extending

[35]Recall that Δm_{Y_i} for $i = h, f$ is normalized by fundamental volatility, meaning a value of 1 translates into



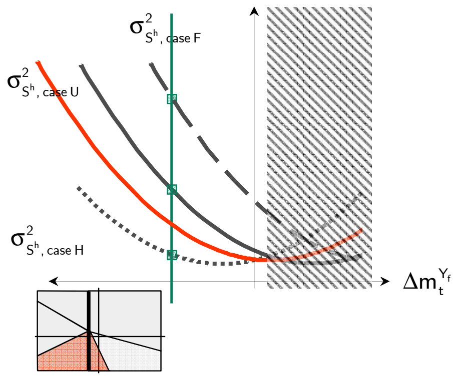


Figure 5: **The Dominant Constraint:** The binding constraint on investor F will push volatility higher than in the unconstrained case, while H's binding constraint will push volatility down. When both constraints bind, the one that binds more tightly will dominate the effect. For $\Delta m_t^{Y_h}$ quite large and positive (left graph), F is exceedingly optimistic regarding investment opportunities in *home*, thus his constraint binds more tightly. When $\Delta m_t^{Y_h}$ small or potentially negative (right graph), F is only constrained due to the liquidity he needs to provide in response to H's constraint binding — he is not himself optimistic, H's constraint binds more tightly and thus dominates the effect on volatility.

the model to n countries with active stock markets, a much smaller level of belief dispersion would lead to constraints binding.While this is a limitation of the model in terms of immediate calibration, it is not critical to the assessment of the implications more generally.

For levels of belief dispersion within this bottom right quadrant, the inequalities below show how the levels of S_t^h's volatily vary when moving from one equilibrium to another by lifting one or both constraints. There are two possible cases.

Magnitude of Δm_{Y_f} is small:

$$\sigma_{\texttt{caseF}}^{S_t^h} > \sigma_{\texttt{caseFH}}^{S_t^h} > \sigma_{\texttt{caseU}}^{S_t^h} > \sigma_{\texttt{caseH}}^{S_t^h} \tag{29}$$

Magnitude of Δm_{Y_f} is large (and negative):

$$\sigma_{\texttt{caseF}}^{S_t^h} > \sigma_{\texttt{caseU}}^{S_t^h} > \sigma_{\texttt{caseFH}}^{S_t^h} > \sigma_{\texttt{caseH}}^{S_t^h} \tag{30}$$

This ranking demonstrates that removing the constraint on F will decrease volatility (case F $\rightarrow$ case U), whereas removing the constraint on H will increase volatility (case H $\rightarrow$ case U). Note that when both constraints bind, volatility can be higher or lower than in the unconstrained market (case U), depending on which of the two constraints dominates. Eq. (29) shows the case where disagreement is small about *foreign* growth rates, but larger about *home* growth rates. This indicates that the constraint of investor F is binding more tightly — his optimism means he would prefer to invest much more into S_t^h than he is permitted to. H's leverage constraint also binds, but not severly so. In the case displayed in eq. (30) the opposite is true. H is very optimistic regarding growth rates of both countries. He is more severely constrained than investor F, so the volatility effect of removing H's leverage constraint dominates. In what follows, I provide the intuition for different steps of liberalization. First I focusing on the case of removing a single constraint on F from the market, leaving the market fully liberalized (case $F \rightarrow$ case U).

When F's constraint binds, he invests exactly proportion φ into S_t^h. The money that he would like to, but cannot, invest into this stock must be reallocated to other assets, S_t^f and bonds. The dispersion of beliefs regarding the *foreign* country will determine whether he considers S_t^f or bonds the better substitute for his desired asset S_t^h.

Changes in optimal portfolios in response to the constraint introduce a new priced risk factor

expected growth rates differing by about 10%.

that does not exist in the unconstrained equilibrium: belief dispersion about demand risk.

$$\Delta\kappa_i^{\texttt{caseF}} < \Delta\kappa_i^{\texttt{caseU}}; \qquad \Delta\kappa_{ii}^{\texttt{caseF}} = \Delta\kappa_{ii}^{\texttt{caseU}}; \qquad \Delta\kappa_{iii}^{\texttt{caseF}} < \Delta\kappa_{iii}^{\texttt{caseU}}.$$

Fundamentally, investors agree on demand risk, so in a perfect market they share demand risk equally, $\Delta\kappa_{iii}^{\texttt{caseU}} = 0$. The binding constraint seems to make investors disagree about demand risk.

To explain this, note that the belief dispersion reflected about growth rates in the *foreign* country the true belief dispersion: $\Delta\kappa_{ii}^{\texttt{caseF}} = \Delta\kappa_{ii}^{\texttt{caseU}} = \Delta m_{Yf}$. Both investors are free to trade in this country's stock, so portfolio holdings must be consistent with true levels belief dispersion. But the constraint prevents the same from happening for the *home* stock — F invests less than his beliefs relative to those of H would suggest. This 'underinverstment' in S_t^h is not consistent with the levels of belief dispersion (about *home* and *foreign*) reflected in holdings of S_t^f. This can be interpreted as F's portfolio reflecting a further source of disagreement. Increases in α_t^H benefit S_t^h via the demand effect on goods price $\bar{p}_t$. Underweighting S_t^h in his portfolio seems to imply that F is expecting negative trend in α_t^H: $\Delta\kappa_{iii}^{\texttt{caseF}} < 0$.[36]

$$\text{investor F} \quad : \qquad \pi_{S_h}^{\texttt{caseF}} < \pi_{S_h}^{\texttt{caseU}}; \qquad \pi_{S_f}^{\texttt{caseF}} > \pi_{S_f}^{\texttt{caseU}} \text{ for } \Delta m_{Yf} \text{ small.} \tag{31}$$

$$\text{investor H} \quad : \qquad \pi_{S_h}^{\texttt{caseF}} > \pi_{S_h}^{\texttt{caseU}}; \qquad \pi_{S_f}^{\texttt{caseF}} < \pi_{S_f}^{\texttt{caseU}} \text{ for } \Delta m_{Yf} \text{ small.} \tag{32}$$

From eqs. (31) and (32) we can see that upon liberalization, F will always increase his holdings of S_t^h. The effect on S_t^f is not unambiguous. If he is relatively pessimistic about growth rates in the *foreign* country (Δm_{Yf} small), he will pull money out of S_t^f upon liberalization and allocate it to S_t^h.[37] If Δm_{Yf} is large (positive), F is optimistic enough to increase his investment after liberalization. Even though he was free to invest as much as desired into S_t^f even before liberalization, his constraint on holdings of S_t^h would have implied a strongly tilted portfolio. Due to diversification considerations, he did not want to overweight the *foreign* stock when the constraint was binding.

For the effect on stock volatility however, not only directions of changes in portfolio holdings and reflected belief dispersions are important, but also magnitudes. As with the intuition for correlations, the more similar portfolios are, ie. the smaller the magnitude of $\Delta\kappa$'s, the lower

[36] Ghysels and Juergens (2001) show empirically that belief dispersion is a priced risk factor. The model here would imply that stocks subject to constraints (i.e. with a liimited investor set) would reflect additional, extraneous, risk factors based on belief dispersion.

[37] This is the case for the bottom right quadrant discussed above: Δm_{Yf} is small (i.e. negative).

volatility. In this case, where only the constraint on investor F is considered, the effects of liberalization on $\Delta\kappa$ compensate such that volatility of S_t^h always decreases upon liberalization. If, for example, F is much more optimistic regarding the *home* stock than H, lifting the constraint will cause him to heavily overweight this stock, the constraint mitigates disparity in portfolios: $|\Delta\kappa_i^{\texttt{caseF}}| < |\Delta\kappa_i^{\texttt{caseU}}|$. Accordingly, liberalization would seem to increase volatility. At the same time however, the 'fictitious' disagreement about demand risk will also be stronger, $\Delta\kappa_{iii}^{\texttt{caseF}}$ is more strongly negative, its magnitude increases: Investments into S_t^h and S_t^f are increasingly inconsistent in the belief dispersion they reflect as the constraint binds more tightly. Analytically, $\sigma_{\texttt{caseF}}^{S_t^h} > \sigma_{\texttt{caseU}}^{S_t^h}$ can be shown to hold for any parameter range where F's constraint is binding.

Even in conjunction with another binding constraint on investor H, removing the constraint on F will tend to decrease volatility. But as mentioned above, whether it is the dominant effect in a jointly-constrained environment depends on how tightly it binds. Fig. (6) is a numerical illustration of the change in volatility when investor F's constraint is removed, for the case where the leverage constraint remains in place (partial liberalization, case $FH \rightarrow$ case H) as well as for the case where F's is the last constraint to be removed (case $F \rightarrow$ case U).[38] Fig. (7) shows the same results for removing the leverage constraint on H, in which case partial liberalization means going from case FH to case F. If H is the last remaining constraint, removing it implies going from case H to case U.[39]

The results on volatility for the remaining cases are discussed below, the intuition follows the same logic as above.

case $H \rightarrow U$: The results are unambiguous for the entire parameter range indicated in the left graph of fig. (2).

$$\Delta\kappa_i^{\texttt{caseH}} > \Delta\kappa_i^{\texttt{caseU}}; \qquad \Delta\kappa_{ii}^{\texttt{caseH}} > \Delta\kappa_{ii}^{\texttt{caseU}}; \qquad \Delta\kappa_{iii}^{\texttt{caseH}} = \Delta\kappa_{iii}^{\texttt{caseU}}$$

$$\begin{array}{lcll} \text{investor F} & : & \pi_{S_h}^{\texttt{caseH}} > \pi_{S_h}^{\texttt{caseU}}; & \pi_{S_f}^{\texttt{caseH}} > \pi_{S_f}^{\texttt{caseU}} \\ \text{investor H} & : & \pi_{S_h}^{\texttt{caseH}} < \pi_{S_h}^{\texttt{caseU}}; & \pi_{S_f}^{\texttt{caseH}} < \pi_{S_f}^{\texttt{caseU}} \end{array}$$

[38] The parameter range of Δm_{Y^h} and Δm_{Y^f} is again such that all of these constraints could bind in the appropriate economy, i.e. the bottom right quadrant. Note that the situation of eq. (30) holds — H is the dominant constraint in this numerical example.

[39] Anlytical proofs of when volatility (variance) increases or decreases upon partial or full liberalization are based on a linear approximation (first-order Taylor expansion). While the model is solved entirely in closed form, the inherent non-linearity of stock return variance did not allow for the determination of signs without some form of approximation. Details of the proof can be requested from the author.

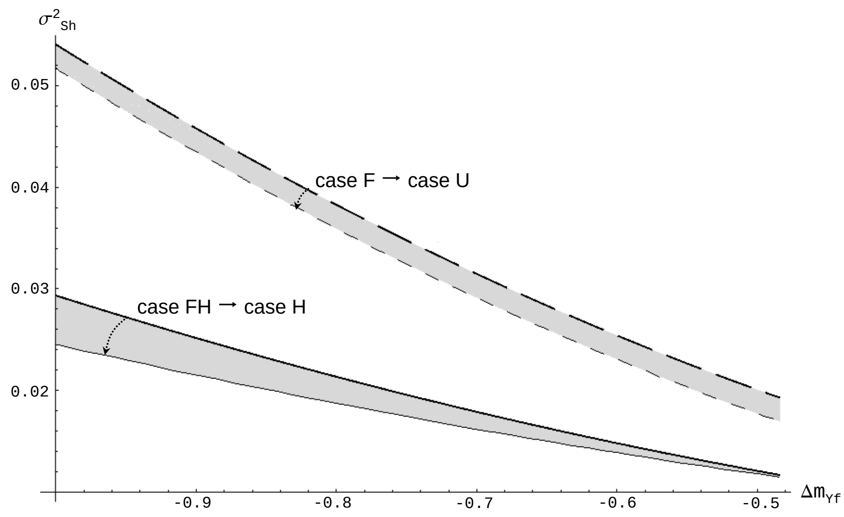


Figure 6: **Variance changes on removing constraint on F:** Partial liberalization (case $FH \rightarrow$ case H) decreases volatility of S_t^h. When F's constraint is the last one to remain, its elimination leads to fully liberalized markets, and volatility also decreases, albeit at a higher absolute level.

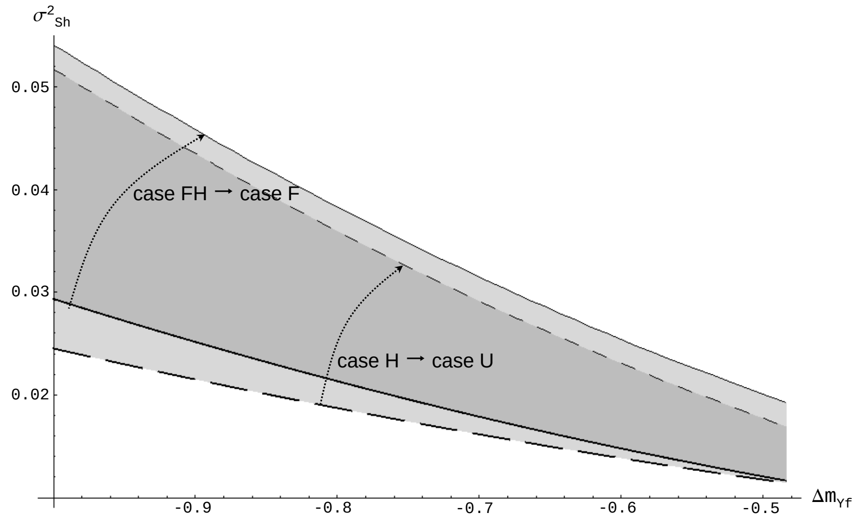


Figure 7: **Variance changes on removing constraint on H:** Both incidences of liberalization, case $FH \rightarrow$ case F and ase $H \rightarrow$ case U increase volatility of S_t^h.

Despite H's leverage constraint binding, he is free to adjust the proportions of his portfolio as he wishes. So while he has less of both stocks than he would want to hold, the proportions are consistent with the levels of belief dispersion about both countries. Therefore in the case of a constraint that limits *total* stock holding rather than that of individual stocks, the constraint will not imply an additional priced risk factor of belief dispersion: $\Delta\kappa_{iii}^{\texttt{caseH}} = \Delta\kappa_{iii}^{\texttt{caseU}}$. In general, H's constraint will bind when he is optimistic enough (jointly across countries' growth rates) to make him want to lever up his overall portfolio. Under these conditions, liberalization will make the investors' portfolios less similar, and volatility will increase: $\sigma_{\texttt{caseH}}^{S_t^h} < \sigma_{\texttt{caseU}}^{S_t^h}$.

case $FH \to U$**:** Full liberalization (lifting both constraints simultaneously) can occur in two cases, as displayed in fig. (1): the bottom right quadrant and the bottom left quadrant. The differences between the two are emphasised by the different volatility effects displayed in eqs. (29) and (30).

For the left quadrant of fig. (1):[40] $\Delta m_{Y^h} < 0$ and $\Delta m_{Y^f} < 0$

$$\Delta\kappa_{i}^{\texttt{caseFH}} > \Delta\kappa_{i}^{\texttt{caseU}}; \qquad \Delta\kappa_{ii}^{\texttt{caseFH}} > \Delta\kappa_{ii}^{\texttt{caseU}}; \qquad \Delta\kappa_{iii}^{\texttt{caseFH}} < \Delta\kappa_{iii}^{\texttt{caseU}}$$

$$\text{investor F} \quad : \qquad \pi_{S_h}^{\texttt{caseFH}} > \pi_{S_h}^{\texttt{caseU}}; \qquad \pi_{S_f}^{\texttt{caseFH}} > \pi_{S_f}^{\texttt{caseU}} \tag{33}$$

$$\text{investor H} \quad : \qquad \pi_{S_h}^{\texttt{caseFH}} < \pi_{S_h}^{\texttt{caseU}}; \qquad \pi_{S_f}^{\texttt{caseFH}} < \pi_{S_f}^{\texttt{caseU}} \tag{34}$$

In this situation, investor H is overall the more optimistic investor regarding both countries' growth rates. F's constraint is also binding, but, critically, not because he is optimistic regarding growth rates in the *home* country. The dispersion of beliefs indicates that left to his own devices, he would not reach his limit on investment abroad. But because H is so optimistic about both countries, his leverage constraint is binding, forcing F to soak up more of the *home* stock's supply, and finally reaching his limit — his constraint binds. This explains why, after full liberalization, investor F is left holding less of the *home* stock than before.[41] Under these conditions, the constraint on H bind more tightly, and dominates the aggregate effect on volatility. After liberalization, volatility will increase (see eq. (30): $\sigma_{\texttt{caseFH}}^{S_t^h} < \sigma_{\texttt{caseU}}^{S_t^h}$.

[40]Technically, this case extends partly into the right quadrant, i.e. this intuition also holds for small levels of $\Delta m_{Y^h} > 0$. But for purposes of exposition, the separation into two quadrants is more clear.

[41]In the case $F \to U$ discussed earlier, he was constrained due to his own beliefs regarding growth rates in *home* country, and liberalization accordingly led to an increase in his holdings of the restricted stock.

For the right quadrant of fig. (1):[42] $\Delta m_{Y^h} >> 0$ and $\Delta m_{Y^f} < 0$

$$\Delta\kappa_i^{\texttt{caseFH}} > \Delta\kappa_i^{\texttt{caseU}}; \qquad \Delta\kappa_{ii}^{\texttt{caseFH}} > \Delta\kappa_{ii}^{\texttt{caseU}}; \qquad \Delta\kappa_{iii}^{\texttt{caseFH}} < \Delta\kappa_{iii}^{\texttt{caseU}}$$

$$\text{investor F} \quad : \qquad \pi_{S_h}^{\texttt{caseFH}} < \pi_{S_h}^{\texttt{caseU}}; \qquad \pi_{S_f}^{\texttt{caseFH}} > \pi_{S_f}^{\texttt{caseU}} \tag{35}$$

$$\text{investor H} \quad : \qquad \pi_{S_h}^{\texttt{caseFH}} > \pi_{S_h}^{\texttt{caseU}}; \qquad \pi_{S_f}^{\texttt{caseFH}} < \pi_{S_f}^{\texttt{caseU}} \tag{36}$$

When Δm_{Y^h} is sufficiently positive, investor F's constraint on holding *home* stock will bind. In constrast to the case in the left quadrant, it is no longer the case that the constraint binds purely due to H's constraint, through market clearing. For this subset of the parameter range, F's constraint will bind more tightly and will thus dominate the volatility effect in liberalization: $\sigma_{\texttt{caseFH}}^{S_t^h} > \sigma_{\texttt{caseU}}^{S_t^h}$. This is the case when Δm_{Y^h} is relative large and positive (F is more optimistic than H about fundamentals in the *home* country), and dispersion in beliefs Δm_{Y^f} is not very strong, meaning H's constraint binds less tightly.

case $FH \to H$ In an economy where both constraints are initially imposed, lifting the constraint on F still leaves H's leverage constraint to bind in the bottom left and the bottom right quadrant.

$$\Delta\kappa_i^{\texttt{caseFH}} < \Delta\kappa_i^{\texttt{caseH}}; \qquad \Delta\kappa_{ii}^{\texttt{caseFH}} \lesseqgtr \Delta\kappa_{ii}^{\texttt{caseH}};^{43} \qquad \Delta\kappa_{iii}^{\texttt{caseFH}} < \Delta\kappa_{iii}^{\texttt{caseH}}$$

$$\text{investor F} \quad : \qquad \pi_{S_h}^{\texttt{caseFH}} < \pi_{S_h}^{\texttt{caseH}}; \qquad \pi_{S_f}^{\texttt{caseFH}} > \pi_{S_f}^{\texttt{caseH}} \tag{37}$$

$$\text{investor H} \quad : \qquad \pi_{S_h}^{\texttt{caseFH}} > \pi_{S_h}^{\texttt{caseH}}; \qquad \pi_{S_f}^{\texttt{caseFH}} < \pi_{S_f}^{\texttt{caseH}} \tag{38}$$

Eliminating the constraint on investor F again makes portfolios of the two investors more similar, and volatility decreases after liberalization: $\sigma_{\texttt{caseFH}}^{S_t^h} > \sigma_{\texttt{caseH}}^{S_t^h}$.

case $FH \to F$ Lifting the leverage constraint on H, achieving partial liberalization will only be relevant in the bottom right quadrant of fig. (1): $\Delta m_{Y^h} > 0$ and $\Delta m_{Y^f} < 0$

$$\Delta\kappa_i^{\texttt{caseFH}} > \Delta\kappa_i^{\texttt{caseF}}; \qquad \Delta\kappa_{ii}^{\texttt{caseFH}} > \Delta\kappa_{ii}^{\texttt{caseF}}; \qquad \Delta\kappa_{iii}^{\texttt{caseFH}} < \Delta\kappa_{iii}^{\texttt{caseF}}$$

[42] Technically this case only holds for relatively extreme values of belief dispersion. Dispersion about the *foreign* country must not be too strong relative to that about the *home* country.

[43] In the bottom left quadrant of fig. (1) the sign is >. In the bottom right quadrant it is < for reasons analogous to those explained earlier. This will not change the aggregate effect on volatility.

$$\text{investor F} \quad : \qquad \pi_{S_h}^{\texttt{caseFH}} = \pi_{S_h}^{\texttt{caseF}}; \qquad \pi_{S_f}^{\texttt{caseFH}} > \pi_{S_f}^{\texttt{caseF}} \tag{39}$$

$$\text{investor H} \quad : \qquad \pi_{S_h}^{\texttt{caseFH}} = \pi_{S_h}^{\texttt{caseF}}; \qquad \pi_{S_f}^{\texttt{caseFH}} < \pi_{S_f}^{\texttt{caseF}} \tag{40}$$

Note that in this case, portfolio holdings in S_t^h do not change when the leverage constraint is eliminated. F is still constrained — he is not willing to sell and is not able to buy. But eliminating the leverage constraint allows investor H to purchase even more of stock S_t^f, portfolios diverge. Thus volatility increases after this partial liberalization step: $\sigma_{\texttt{caseFH}}^{S_t^h} < \sigma_{\texttt{caseF}}^{S_t^h}$.

Among all these cases, the results regarding jointly binding constraints are perhaps initially most counterintuitive. For all but the most extreme parameter values, the model indicates that volatility is lower when both investors are bound by their constraints than in an unconstrained market. Conceputally, our idea of how markets work would suggest that when new information arrives, it will be incorporated into equilibrium prices by trading, quantities and prices will adjust. Lack of available quantity is generally linked to low levels of liquidity, implying that prices will be more sensitive to any changes. Since portfolio constraints fix the quantities of stocks held, it would be intuitive to assume new information or shocks would in that case be incorporated via prices — i.e. high volatility. But the model implies the opposite — lower volatility in a constrained economy than in an unconstrained one with identical fundamentals.

But this line of argument limits the scope to one asset market only. In general, as in the model of this paper, investors aim to circumvent the constrained market by moving their trades into other assets — the stock market abroad (if possible) and the countries' respective bond markets. By doing so, they can partially 'undo' the constraints they face. This finding is particularly interesting in light of empirical studies that find bond and stock markets react differently to international exposure. While the previously mentioned series of papers by Harvey and Bekaert have shown that stock markets in less liberalized countries are strongly driven by local shocks and news, evidence on bond markets seem to indicate the opposite — bonds of countries that restrict access to their stock market tend to be more responsive to world shocks than to local shocks.

The model's implications regarding volatility in constrained markets are consistent with Miles (2002) who shows that investors' beliefs regarding their own local investment opportunities have a large role in explaining capital flows into Emerging Markets. Bae et al. (2004) find that highly investible stocks, i.e. stocks that are more accessible to a wide range of investors worldwide, tend to have higher volatility. The authors explain this with investible stocks having a higher exposure to world risk, which would however indicate that the regional stock markets in question otherwise have a below average volatility compared to the world. Since they study mainly young stock markets and Emerging Markets, this cannot necessarily be taken as given. Kim and Singal (2000) and Henry (2000) show that high levels of foreign ownership tend to increase stocks' volatilities.

My model suggests a different explanation for these findings. Depending on the type of constraint, we should expect to see different effects on stock volatility post-liberalization. If there are multiple constraints binding simultaneously, the dominant one — the one that binds more tightly — will determine the aggregate effect on volatility. Therfore, when studying effects of past liberalization experiences, conditioning on information about remaining constrainedness of the investor set may help to explain the empirical results. In terms of policy implications, refining the empirical results regarding past liberalizations could help the discussion for further changes to international portfolio restrictions.

The above implications about the effects of liberalization on stock volatility can also be linked back to expected returns as described in section 4.1 to provide further unique testable implications. The results demonstrate that accounting for constraints can reconcile the mixed findings in the literature. Assets that have a more limited investor base due to binding constraints tend to exhibit different volatility—return relationship than unconstrained assets.

First, consider again the case where only the constraint on investor F is in place and binding. The model predicts that both volatility and expected excess returns of the *home* stock rise together as the magnitude of differences in beliefs increases. This is diplayed in a numerical example in the left graph of fig. (8). The right graph of the same figure shows the relationship between expected excess returns and volatility of stock S_t^h when both constraints bind, under the same economic parameters.[44]

In contrast, when both investors are bound by their respective constraints, the *home* stock that is affected by both investors' constraints reacts asymmetrically to dispersion in beliefs. When beliefs are very dispersed, expected excess returns on this stock is high while volatility is low. As the *foreign* investor regains confidence in his local stock market and beliefs therein converge, volatility of the *home* stock S_t^h increases while expected excess returns decrease.

As the *foreign* investors' optimism regarding his own country's investment opportunity rises, the *home* stock S_t^h becomes indirectly less attractive. While investor H must still be incentivized to hold the extra supply of S_t^h as both constraints still bind, the burden of his constraint is shifted, thus requiring lower returns on S_t^h. Note that even as belief dispersion changes, portfolio holdings

[44]Belief dispersion must again be such that both scenarios can occur, the bottom right quadrant as above.

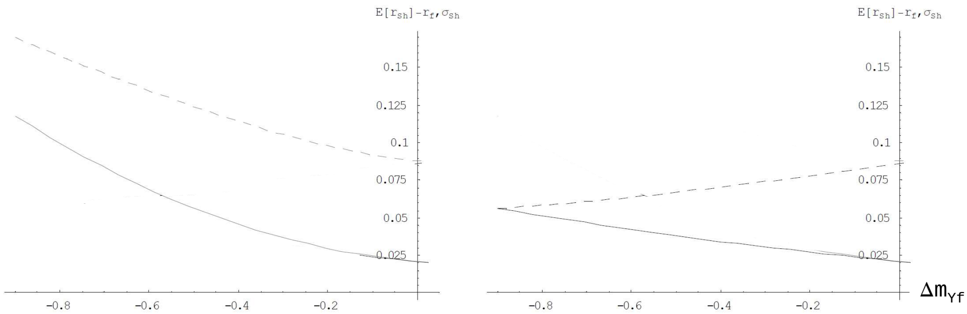


Figure 8: The left graph shows the relationship between excess returns (solid) and volatility (dashed) of stock S_t^h in the case where only agent F's constraint binds. The right graph shows excess returns (solid) and volatility (dashed) of S_t^h when both constraints on F and H are binding.

in the stock do not change — constraints are still binding. The two agents use the local bond markets to even out their changes in beliefs.[45]

The empirical evidence on the correlation between volatility and excess returns is mixed, as can be seen in the studies of Turner et al. (1989), Glosten et al. (1993), Wu (2001), and Whitelaw (1994).[46] The link between risk and return implied by the model suggests that using information about the accessibility of stocks in the cross section may help to explain some of these disparate findings.

5 Conclusion

In this paper I construct a model that allows me to study how portfolio constraints interact in equilibrium. To date, the theoretical literature on portfolio constraints has studied various types of constraints in isolation. I show that a constraint's impact on stock returns and volatility will be different, depending on whether it is binding in isolation or in jointly with other constraints. If, for example, one type of constraint tends to increase stock volatility while the other constraint tends to decrease it, the aggregate effect will depend on which constraint binds more tightly, or 'dominates' in the market. Within the policy debate about international financial market liberalization, this is an important fact to consider. These results are able to explain the differences in several countries' liberalization experiences, and thus provide some insight into the debate about further liberalization efforts.

The model provides testable implications. Whether liberalization increases or decreases correlation of the market with the world, and increases or decreases market volatility, is linked to the tightness of binding constraints. While it is not trivial to obtain information on the severity of constraints, the relationship between belief dispersion and the tightness of a constraint may give a first proxy on this for empirical testing.

Another interesting avenue for future research is a more detailed analysis of the relationship between stocks' excess expected returns and volatility, which has been studied mostly domestically on the empirical side. The model suggests that a limitation on foreign ownership of an asset leads to a positive correlation between volatility and expected returns when this constraint binds in

[45]Note that the dashed lines in fig. (8) display the volatility pattern of S_t^h as discussed above. It is lower in case FH, where both constraints bind than in the partially liberalized case where only F's constraint remains and binds.

[46]An obvious caveat is that these studies are based on US data. To the extent that constraints also exist for certain stocks within any one stock market, the results remain indicative. Separate studies using international markets would have to be conducted to confirm these results.

isolation. If it is imposed jointly with other constraints, the correlation will be negative. This indicates that in the 'risk-return' relationship, the concept of what is the relevant measure of risk for an asset depends on how restricted that stock's ownership is.

Despite the fact that I study two particular examples of constraints, the results of the model are generalizable. The impact a constraint has on stock market dynamics depends on its effect on portfolio similarity. If a constraint makes the investors' portfolios have more similar levels of risk exposure than they would have in absence of the constraint, then liberalization will make volatility increase. In this sense, depending on its impact on portfolio choice across world investors, contstraints can have a mitigating or an exacerbating effect on volatility and correlation.

6 Appendix

6.1 Optimal Consumption

$$U(C_H, C_F) = u_H\left(C^h_{Ht}, C^f_{Ht}\right) + \lambda_t u_F\left(C^h_{Ft}, C^f_{Ft}\right)$$

where

$$u_H\left(C^h_{Ht}, C^f_{Ht}\right) = \alpha^H_t \log C^h_{Ht} + (1-\alpha^H_t)\log C^f_{Ht},$$
$$u_F\left(C^h_{Ft}, C^f_{Ft}\right) = (1-\alpha^F)\log C^h_{Ft} + \alpha^F \log C^f_{Ft}.$$

FOC: $u^i_{C^j}(\cdot) = \frac{\partial u_i(C^i_{it}, C^j_{it})}{\partial C^j_{it}} = y_i p^j_t \xi^i_t$ for goods $j = h, f$ and agents $i = H, F$.

	agent H:	*agent F:*
good h:	$\frac{\alpha^H_t}{C^h_{Ht}} = y_H p^h_t \xi^H_t$	$\frac{1-\alpha^F}{C^h_{Ft}} = y_F p^h_t \xi^F_t$
good f:	$\frac{1-\alpha^H_t}{C^f_{Ht}} = y_H p^f_t \xi^H_t$	$\frac{\alpha^F}{C^f_{Ft}} = y_F p^f_t \xi^F_t$

Market clearing requires, $\sum_i C^j_i = Y^j$, giving equilibrium consumption.

6.2 Optimal Wealth

Current wealth is an appropriately discounted value of all future consumption levels, here described for agent H. Agent F follows directly.

$$X^H_t = \frac{1}{\xi^H_t} E\left[\int_t^T \left(\xi^H_s p^h_s C^h_{Hs} + \xi^H_s p^f_s C^f_{Hs}\right) ds\right]$$

From FOC above, $\frac{\alpha^H_t}{y_H} = C^h_{Ht} p^h_t \xi^H_t \, and \frac{1-\alpha^H_t}{y_H} = C^f_{Ht} p^f_t \xi^H_t$ holds, therefore:

$$X^H_t = \frac{1}{\xi^H_t} E\left[\int_t^T \left(\frac{\alpha^H_s}{y_H} + \frac{1-\alpha^H_s}{y_H}\right) ds\right] = \frac{1}{y_H \xi^H_t}(T-t).$$

Comparing X^i_t and C^j_{it}: $X^H_t = C^h_{Ht} \cdot \frac{p^h_t}{\alpha^H_t}(T-t)$. Analogously for agent F: $X^F_t = \frac{1}{y_F \xi^F_t}(T-t)$.

6.3 Relative Prices

The relative price of the two goods is $\bar{p}_t = \frac{p^f_t}{p^h_t} = \frac{u^i_{C^f}(\cdot)}{u^i_{C^h}(\cdot)}$. The basket of goods $\beta p^h_t + (1-\beta)\, p^f_t = 1$ defines the numeraire. β can take any value between 0 and 1, representing the weight of the *home* good in the basket.

Using the equilibrium marginal utilities from market clearing restrictions above gives:

$$\bar{p}_t = \frac{p_t^f}{p_t^h} = \frac{u_{C^f}^H(\cdot)}{u_{C^h}^H(\cdot)} = \frac{y_H p_t^f \xi_t^H}{y_H p_t^h \xi_t^H} = \frac{(1-\alpha_t^H) + \alpha^F \lambda_t}{\alpha_t^H + (1-\alpha^F)\lambda_t} \frac{Y_t^h}{Y_t^f}.$$

6.4 Auxiliary Market

6.4.1 Portfolio Choice in Constrained Markets

I assume that portfolio positions $\pi_{i,t}^j$ of investor $i = H, F$ in assets $j = S_t^h, S_t^f, B_t^h, B_t^f$ are constrained to lie in a closed, convex, non-empty set K that contains the origin. This is the methodology developed in Cvitanic and Karatzas (1992) and several other papers of these authors and others, e.g. Karatzas et al. (1991).

The martingale analysis of incomplete markets requires the construction of a fictitious market that hypothetically augments the market parameters of the original constrained market. Under these augmented market parameters, the constrained investor will optimally choose a portfolio permissible within the constraints. This is then the optimal portfolio also under the original, constrained market.[47]

The set of admissible trading strategies is defined by the set K, the support function is $\delta(\upsilon_t^i) \equiv \delta(\upsilon_t^i|K) \equiv \sup\left(-\pi_{i,t}^\top \upsilon_t^i : \pi_{i,t} \in K\right)$ and the barrier cone of the set $-K$ is defined as $\bar{K} \equiv \left\{\upsilon_t^i \in \mathbb{R}^2 | \delta(\upsilon_t^i) < \infty\right\}$. υ_t^i is a square-integrable, progressively measurable process taking values in $\bar{K}$ to ensure boundedness.

Both investors' respective state price densities adjust to reflect these augmented market perceptions due to the constraints:

$$d\xi_t^i = -\left(r_t + \delta(\upsilon_t^i)\right)\xi_t^i dt - \kappa_t^{i\top}\xi_t^i d\overrightarrow{W}_t^{(i)}, \tag{41}$$

where the adjusted market price of risk is $\kappa_t^i = (\sigma_{S,t}^{-1})\left(m_{S,t}^{(i)} + \upsilon_t^i - r_t\mathbf{1}\right) = \kappa_{0,t}^i + \sigma_{S,t}^{-1}\upsilon_t^i$.

In the auxiliary market, the asset prices follow:

$$\begin{aligned} dB_t &= \left(r_t + \delta(\upsilon_t^i)\right) I_B dt, \\ dS_t &= I_S\left(m_{S,t}^{(i)} + \upsilon_t^i + \delta(\upsilon_t^i)\right)dt + I_S \sigma_{S,t} d\overrightarrow{W}_t^{(i)}, \end{aligned}$$

[47] This setting is a straightforward application of that in Cvitanic and Karatzas (1992), and it can be easily shown that their convex duality approach for convex constraint sets holds here.

where I_S (I_B) is a diagonal matrix whose entries are the stock prices (bond prices) and $dW_t^{(i)}$ is the 3×1 vector of investor i's innovation processes. For the proof of convex duality, see Cvitanic and Karatzas (1992).

Investor H's leverage constraint $\iota_H^\top \pi_t^H \leqslant 1$ paired with his optimal trading strategy $\pi_t^H = (\sigma_S^{-1})^\top \left[\kappa_{ot}^H + \sigma_S^{-1}\upsilon_t^H\right]$ gives

$$\upsilon_t^H = \min\left(\frac{1-\iota_H^\top(\sigma_S^{-1})^\top\kappa_{ot}^H}{\iota_H^\top(\sigma_S^{-1})^\top\sigma_S^{-1}\iota_H}, 0\right), \qquad \delta(\upsilon_t^H) = -\upsilon_t^H = \max\left(-\frac{1-\iota_H^\top(\sigma_S^{-1})^\top\kappa_{ot}^H}{\iota_H^\top(\sigma_S^{-1})^\top\sigma_S^{-1}\iota_H}, 0\right). \tag{42}$$

Note that a leverage constraint does not impact the two assets individually but rather only the joint holding, so investor H's adjustments υ_t^H are the same for both assets.

For agent F's constraint, the adjustments in the auxiliary market are:

$$\upsilon_t^F = \min\left(\frac{\varphi-\iota_F^\top(\sigma_S^{-1})^\top\kappa_{ot}^F}{\iota_F^\top(\sigma_S^{-1})^\top\sigma_S^{-1}\iota_F}, 0\right), \qquad \delta(\upsilon_t^F) = -\varphi\upsilon_t^F = \max\left(-\varphi\frac{\varphi-\iota_F^\top(\sigma_S^{-1})^\top\kappa_{ot}^F}{\iota_F^\top(\sigma_S^{-1})^\top\sigma_S^{-1}\iota_F}, 0\right). \tag{43}$$

6.5 State Price Density

Agent H consumes a fraction $\frac{\alpha_t^H}{\alpha_t^H+(1-\alpha^F)\lambda_t}$ of good Y_t^h and a fraction $\frac{1-\alpha_t^H}{1-\alpha_t^H+\alpha^F\lambda_t}$ of good Y_t^f. This and equilibrium relative prices $\bar{p}_t$ gives

$$\xi_t^H = \beta\frac{\alpha_t^H+(1-\alpha^F)\lambda_t}{y_H Y_t^h} + (1-\beta)\,\frac{1-\alpha_t^H+\alpha^F\lambda_t}{y_H Y_t^f}. \tag{44}$$

Accordingly, agent F consumes a fraction $\frac{\lambda_t(1-\alpha^F)}{\alpha_t^H+(1-\alpha^F)\lambda_t}$ of good Y_t^h and a fraction $\frac{\lambda_t\alpha^F}{1-\alpha_t^H+\alpha^F\lambda_t}$ of good Y_t^f:

$$\xi_t^F = \beta\frac{\alpha_t^H+(1-\alpha^F)\lambda_t}{\lambda_t y_F Y_t^h} + (1-\beta)\,\frac{1-\alpha_t^H+\alpha^F\lambda_t}{\lambda_t y_F Y_t^f}. \tag{45}$$

6.6 Asset Valuation

Proof of Proposition 1:

Under constraints, the properly discounted gains process (using the adjusted state price density as described above) must be a martingale.[48]

[48]The proof of the stock price valuation closely follows that of Detemple and Murthy (1997).

Valuing the stock under either investor i's information measure, there must be an $\mathcal{F}_t^i$ measurable process z_t^i such that $S_t^j \equiv E_t\left[\int_t^T \xi_s^i/\xi_t^i p_s^j Y_s^n ds\right] + E_t\left[\int_t^T z_s^{i,j}/\xi_t^i ds\right]$.

Expanding this, $\xi_t^i S_t^j + \int_0^t \xi_s^i p_s^j Y_s^j ds + \int_0^t z_s^i ds = E_t\left[\int_0^T \xi_s^i p_s^j Y_s^j ds\right] + E_t\left[\int_0^T z_s^{i,j} ds\right]$ is a martingale for all $t \in [0, T]$.

Accordingly, discounted cum-dividend stock returns using the adjusted state price density will have an expected value of $E_t \int_t^T \left[d\xi_s^i S_s^j + \xi_s^i p_s^j Y_s^j ds\right] = -E_t \int_t^T z_s^{i,j} ds$.

Using the adjusted state price density in eq. (41) and the general notation for stock dynamics $dS_t^j = m_{S^j}^{(i)} S_t^j dt + \sigma_{S^j}^{(1\times 3)} S_t^j d\overrightarrow{W}_t^{(i)}$ gives $z_s^{i,j} = \left(\delta(\upsilon_t^i) + \upsilon_{(j),t}^i\right) S_t^j \xi_t^i$. $\upsilon_{(j),t}^i$, the j-th element of the (3×1) vector υ_t^i, is the speculative premium, and $\delta(\upsilon_t^i)$ the collateral premium.

Market clearing in asset markets requires

$$S_t^h + S_t^f = X_t^H + X_t^F = p_t^h Y_t^h (T-t) + p_t^f Y_t^f (T-t). \tag{46}$$

Each asset $j = h, f$ is valued as

$$S_t^j = \frac{1}{\xi_t^H} E_t\left[\int_t^T \xi_s^H p_s^j Y_s^j ds\right] + \frac{1}{\xi_t^H} E_t\left[\int_t^T \left(\upsilon_t^H + \delta(\upsilon_t^H)\right) S_s^j \xi_s^H ds\right] \qquad j = h, f.$$

Using $\frac{1}{p_t^h \xi_t^H} = \frac{Y_t^h y_H}{\alpha_t^H + (1-\alpha^F)\lambda_t}$ and goods market clearing, as well as $\lambda_t = \frac{y_H \xi_t^H}{y_F \xi_t^F}$ in the pricing function of S_t^h:

$$\begin{aligned} S_t^h = p_t^h Y_t^h (T-t) &+ \frac{p_t^h Y_t^h}{\alpha_t^H + (1-\alpha^F)\lambda_t}(1-\alpha^F)\left[E_t \int_t^T \lambda_s ds - \lambda_t (T-t)\right] \\ &+ \frac{y_H p_t^h Y_t^h}{\alpha_t^H + (1-\alpha^F)\lambda_t} E_t\left[\int_t^T \left(\upsilon_t^H + \delta(\upsilon_t^H)\right) S_s^h \xi_s^H ds\right] \end{aligned} \tag{47}$$

$$\begin{aligned} S_t^f = p_t^f Y_t^f (T-t) &+ \frac{p_t^f Y_t^f}{1 - \alpha_t^1 + \alpha^2 \lambda_t}\alpha^F \left[E_t \int_t^T \lambda_s ds - \lambda_t (T-t)\right] \\ &+ \frac{y_H p_t^f Y_t^f}{1 - \alpha_t^H + \alpha^F \lambda_t} E_t\left[\int_t^T \left(\upsilon_t^H + \delta(\upsilon_t^H)\right) S_s^f \xi_s^H ds\right] \end{aligned} \tag{48}$$

Under the given constraints $E_t\left[\int_t^T \left(\upsilon_t^H + \delta(\upsilon_t^H)\right) S_s^f \xi_s^H ds\right] \leqslant 0$ can be shown to hold. $d\lambda_t$ is a supermartingale under all possible equilibria. All terms in eqs. (48) and (49) except $p_t^j Y_t^j (T-t)$ are non-positive, thus for the equilibrium pinned down by eq. 46, they must all be zero in equilibrium.

Therefore,

$$
\begin{aligned}
S_t^h &= p_t^h Y_t^h (T-t), \\
S_t^f &= p_t^f Y_t^f (T-t).
\end{aligned}
\tag{49}
$$

where p_t^h and p_t^f can be rewritten in terms of $\bar{p}_t$.

In equilibrium, the adjustments to perceived investment opportunities (eqs. (42), (43)) are

case F:

$$
\upsilon_t^F = \frac{\left[\Delta m_{Y_h}\sigma_{Y_h} - \left(\varphi(1+\lambda_t) - \alpha_t^H - (1-\alpha^F)\lambda_t\right)\sigma_{Y_h}^2\right]\sigma_\alpha^2}{\left[\lambda_t(1-\varphi)\left(\alpha_t^H + \alpha^F - 1\right) - \left(1 - \alpha_t^H + \alpha^F\lambda_t\right)\alpha_t^H\right]^2 \sigma_{Y_h}^2 + \sigma_\alpha^2}; \qquad \upsilon_t^H = 0.
$$

case H:

$$
\upsilon_t^F = 0; \qquad \upsilon_t^H = \frac{\sigma_{Y_h}\sigma_{Y_f}\left(\Delta m_{Y_h}\sigma_{Y_f} + \Delta m_{Y_f}\sigma_{Y_h}\right)}{\sigma_{Y_h}^2 + \sigma_{Y_f}^2}.
$$

case FH:

$$
\upsilon_t^F = \frac{\left[\Delta m_{Y_f}\sigma_{Y_f} - \Delta m_{Y_h}\sigma_{Y_h} + \left(\varphi(1+\lambda_t) - \alpha_t^H - (1-\alpha^F)\lambda_t\right)(\sigma_{Y_h}^2 + \sigma_{Y_f}^2)\right]\sigma_\alpha^2}{\left[\lambda_t(1-\varphi)\left(\alpha_t^H + \alpha^F - 1\right) - \left(1 - \alpha_t^H + \alpha^F\lambda_t\right)\alpha_t^H\right]^2 (\sigma_{Y_h}^2 + \sigma_{Y_f}^2) + \sigma_\alpha^2};
$$

$$
\begin{aligned}
\upsilon_t^H = & \frac{\left[\lambda_t(1-\varphi)\left(\alpha_t^H + \alpha^F - 1\right) - \left(1 - \alpha_t^H + \alpha^F\lambda_t\right)\alpha_t^H\right]^2 \sigma_{Y_h}\sigma_{Y_f}\left(\Delta m_{Y_h}\sigma_{Y_f} + \Delta m_{Y_f}\sigma_{Y_h}\right)}{\left[\lambda_t(1-\varphi)\left(\alpha_t^H + \alpha^F - 1\right) - \left(1 - \alpha_t^H + \alpha^F\lambda_t\right)\alpha_t^H\right]^2 (\sigma_{Y_h}^2 + \sigma_{Y_f}^2) + \sigma_\alpha^2} \\
& + \frac{\Delta m_{Y_f}\sigma_{Y_f}\sigma_\alpha^2 + \left(\varphi(1+\lambda_t) - \alpha_t^H - (1-\alpha^F)\lambda_t\right)\sigma_{Y_f}^2\sigma_{\alpha *}^2}{\left[\lambda_t(1-\varphi)\left(\alpha_t^H + \alpha^F - 1\right) - \left(1 - \alpha_t^H + \alpha^F\lambda_t\right)\alpha_t^H\right]^2 (\sigma_{Y_h}^2 + \sigma_{Y_f}^2) + \sigma_\alpha^2}.
\end{aligned}
$$

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