How Arbitrage-Free is the Nelson-Siegel Model?

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Abstract

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JEL classification codes: C14, C15, G12
Keywords *Nelson-Siegel model; No-arbitrage restrictions; affine term structure models; non-parametric test*

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Abstract

We test whether the Nelson and Siegel (1987) yield curve model is arbitrage-free in a statistical sense. Theoretically, the Nelson-Siegel model does not ensure the absence of arbitrage opportunities, as shown by Bjork and Christensen (1999). Still, central banks and public wealth managers rely heavily on it. Using a non-parametric resampling technique and zero-coupon yield curve data from the US market, we find that the no-arbitrage parameters are not statistically different from those obtained from the NS model, at a 95 percent confidence level. We therefore conclude that the Nelson and Siegel yield curve model is compatible with arbitrage-freeness on the US market. To corroborate this result, we show that the Nelson-Siegel model performs as well as its no-arbitrage counterpart in an out-of-sample forecasting experiment.

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Fixed-income wealth managers in public organizations, investment banks and central banks rely heavily on Nelson and Siegel (1987) type of models to fit and forecast yield curves. According to BIS (2005), the central banks of Belgium, Finland, France, Germany, Italy, Norway, Spain, and Switzerland, use these models to estimate zero-coupon yield curves. The European Central Bank (ECB) publishes daily Eurosystem-wide yield curves on the basis of the Soderlind and Svensson (1997) model, which is an extension of the Nelson-Siegel model.[1] In its foreign reserve management framework the ECB uses a regime-switching extension of the Nelson-Siegel model, see Bernadell, Coche and Nyholm (2005).

There are at least four reasons for the popularity of the Nelson-Siegel model. First, it is easy to estimate. In fact, if the so-called time-decay-parameter is fixed, then Nelson-Siegel curves are obtained by linear regression techniques. If this parameter is not fixed, one has to resort to non-linear regression techniques. In addition, the Nelson-Siegel model by virtue of its empirical nature, can easily be extended. Second, the model provides by construction yields for all maturities, i.e. also maturities that are not covered by the data sample. As such, it lends itself as an interpolation and extrapolation tool for the analyst who often is interested in yields at maturities that are not directly observable.[2] Third, estimated yield curve factors obtained from the Nelson and Siegel model have intuitive interpretations, as

[1]For Eurosystem-wide yield curves see http://www.ecb.int/stats/money/yc/html/index.en.html.

[2]This is relevant e.g. in a situation where fixed-income returns are calculated to take into account the roll-down/maturity shortening effect.

the level, the slope, and the curvature of the yield curve. This interpretation is akin to that obtained by a principal component analysis, see e.g. Litterman and Scheinkman (1991) and Diebold and Li (2006). Due to its intuitive appeal estimates and conclusions drawn on the basis of the model are easy to communicate. Fourth, empirically the Nelson-Siegel model fits data well and performs well in out-of-sample forecasting exercises, as shown by e.g. Diebold and Li (2006) and De Pooter, Ravazzolo and van Dijk (2007).

However, despite its empirical merits and wide-spread use in the finance community, two theoretical concerns can be raised against the Nelson-Siegel model. First, by conventional standards, it is not arbitrage-free, as shown by Bjork and Christensen (1999). Second, as demonstrated by Diebold, Ji and Li (2006b), it falls outside the class of affine yield curve models defined by Duffie and Kan (1996) and Dai and Singleton (2000).

In its original form, the Nelson and Siegel (1987) model does not prespecify any dynamic evolution for the underlying yield curve factors. Rather, it presents a parsimonious and intuitive description of the forward and spot yield curve at a given point in time. A dynamic version of the Nelson-Siegel model is proposed by Diebold and Li (2006), where a time-series model is suggested to account for the evolution of the level, slope and curvature factors over time. Data, as opposed to theory, guides the parametrisation of the introduced time series model. In this case the Nelson-Siegel yield-curve model can be set in state-space form and estimation can be carried out using the Kalman filter.

The modelling advance made in Diebold and Li (2006), allows us to better analyse the connection between the Nelson-Siegel and the classic multi-factor no-arbitrage yield curve models. When viewed through the lens of a state-space model, this difference becomes clear. Setting the Nelson-Siegel model in state space form amounts to defining the observation equation, which translates the yield curve factors into observed yields, as the original Nelson-Siegel model; the state equation is then represented by a time-series model for the yield curve factors, e.g. the Vector Autoregressive model of order 1, VAR(1), as suggested by Diebold and Li (2006). It is important to note that there is no relationship between the parameters in the observation equation (also called factor loadings) and the parameters of the state equation, when setting the Nelson-Siegel model in state space form. Contrary to this, in a classic dynamic no-arbitrage model, such a connection exists and is dictated by theory. In fact, the no-arbitrage constraints impose a certain connection between yield curve factor loadings (the parameters in the the observation equation) and the parameters describing the dynamic evolution of yield curve factors over time (the parameters included in the state equation). In addition, classic no-arbitrage models specify yield curve dynamics under the risk neutral measure, and as a consequence, a functional form for the market price of risk also has to be specified.

Hence, the Nelson-Siegel model predefines the functional form for the yield-curve factor-loadings with an aim to obtain model-derived yield curves that provide a good fit to data, as well as model-parameters/factors that are intuitively appealing. The no-arbitrage models derive yield-curve factor-loadings explicitly on the basis of the parameters that describe the time-series evolution of the yield curve factors and the market price of risk. The particular way these parameters enter into the yield-curve factor-loadings is determined by no-arbitrage principles.

The Nelson-Siegel yield-curve model operates at the level of yields, as they are observed, i.e. under the so-called empirical measure. In contrast, (affine) arbitrage-free yield curve models specify the dynamic evolution of yields under a risk-neutral measure and then map this dynamic evolution back to the physical measure via a functional form for the market price of risk. The advantage of the no-arbitrage approach is that it automatically ensures a certain consistency between the parameters that describe the dynamic evolution of the yield curve factors under the risk-neutral measure and the translation of yield curve factors into yields under the physical measure. An arbitrage-free setup will, by construction, ensure internal consistency as it cross-sectionally restricts, in an appropriate manner, the estimated parameters of the model. It is this consistency that guarantees arbitrage-freeness. Since a similar consistency is not hard-coded into the Nelson-Siegel model, this model is not necessarily arbitrage-free.[3]

It is an empirical fact that both modelling approaches produce model-derived yields that have good in-sample fits. Hence, for a given set of yield curve factors and yield curve factor dynamics, the factor loadings of the two models will be different only to the extent that the no-arbitrage constraints are binding. The main contribution of our paper lies in exploiting this idea to test the significance of the no-arbitrage constraints in relation to the Nelson-Siegel model. We treat the estimated Nelson-Siegel factors as "observables" in an affine arbitrage-free model and we apply the technique suggested by

[3] An illustrative example of this issue for a two-factor Nelson-Siegel model is presented by Diebold, Piazzesi and Rudebusch (2005).

Ang, Piazzesi and Wei (2006) to estimate the implied no-arbitrage loadings. Moreover, using zero-coupon yield curve data from the US market we apply a data resampling scheme, and for each resampled data set we estimate Nelson-Siegel yield curve factors (using a fixed Nelson-Siegel factor loading matrix). The obtained Nelson-Siegel yield curve factors are then used as exogenous factors in a no-arbitrage yield curve model, and following Ang et al. (2006) we obtain no-arbitrage factor loadings, which are consistent with the Nelson-Siegel factors. The applied resampling scheme allows us to build empirical distributions for the no-arbitrage factor-loadings, and these distributions facilitate statistical testing for the difference between the no-arbitrage loadings and the Nelson-Siegel loadings.

In a recent study Christensen, Diebold and Rudebusch (2007) also reconcile the Nelson and Siegel modelling setup with the absence of arbitrage by deriving a class of dynamic Nelson-Siegel models that fulfill the no-arbitrage constraints. They maintain the original Nelson-Siegel factor-loading structure and derive a correction term that, when added to the dynamic Nelson-Siegel model, ensures the fulfillments of the no-arbitrage constraints. The correction term is shown to mainly impact very long maturities, in particular maturities beyond the ten-year segment.

While being different in setup and analysis method, our paper confirms the findings of Christensen et al. (2007). In addition, we outline a general method for empirically testing for the fulfillment of the no-arbitrage constraints in yield curve models that are not necessarily arbitrage-free. Our results furthermore indicate that non-compliance with the no-arbitrage constraints is most likely to stem from "mis-specification" in the Nelson-Siegel factor loading structure pertaining to the third factor, i.e. the one often referred to as the curvature factor.

Our test is conducted on U.S. Treasury zero-coupon yield data covering the period from January 1970 to December 2000 and spanning 18 maturities from 1 month to 10 years. We rely on a non-parametric resampling procedure to generate multiple realizations of the original data. Our approach to regenerate yield curve samples can be seen as a simplified version of the yield-curve bootstrapping approach suggested by Rebonato, Mahal, Joshi, Bucholz and Nyholm (2005).

In summary, we (1) generate a realization from the original yield curve data using a circular block-bootstrapping technique; (2) estimate the Nelson-Siegel model on the regenerated yield curve sample; (3) use the obtained Nelson-Siegel yield curve factors as input for the essentially affine no-arbitrage model; (4) estimate the implied no-arbitrage yield curve factor loadings on the regenerated data sample. Steps (1) to (4) are repeated 1000 times in order to obtain bootstrapped distributions for the no-arbitrage parameters. These distributions are then used to test whether the implied no-arbitrage factor loadings are significantly different from the Nelson-Siegel loadings.

Our results show that the Nelson Siegel factor loadings are not statistically different from the implied no-arbitrage factor loadings at a 95 percent level of confidence. Moreover, in an out-of-sample forecasting experiment, we show that the performance of the Nelson-Siegel model is as good as the no-arbitrage counterpart. We therefore conclude that the Nelson and Siegel model is compatible with arbitrage-freeness for the US market at this level of confidence.

I Modeling framework

Term-structure factor models describe the relationship between observed yields, yield curve factors and loadings as given by

$$y_t = a + bX_t + \epsilon_t, \tag{1}$$

where y_t denotes a vector of yields observed at time t for N different maturities; y_t is then of dimension $(N \times 1)$. X_t denotes a $(K \times 1)$ vector of yield curve factors, where K counts the number of factors included in the model. The variable a is a $(N \times 1)$ vector of constants, b is of dimension $(N \times K)$ and contains the yield curve factor loadings. ϵ_t is a zero-mean $(N \times 1)$ vector of measurement errors.

The reason for the popularity of factor models in the area of yield curve modeling is the empirical observation that yields at different maturities generally are highly correlated - when the yield for one maturity changes, it is very likely that yields at other maturities also change. As a consequence, a parsimonious representation of the yield curve can be obtained by modeling fewer factors than observed maturities.

This empirical feature of yields was first exploited in the continuous-time one factor models, where, in terms of equation (1), $X_t = r_t$, r_t being the short rate, see e.g. Merton (1973), Vasicek (1977), Cox, Ingersoll and Ross (1985), Black, Derman and Toy (1990), and Black and Karasinski (1993).[4] A richer structure for the dynamic evolution of yield curves can be obtained by adding more yield curve factors to the model. Accordingly, X_t becomes

[4]The merit of these models mainly lies in the area of derivatives pricing.

a column-vector with a dimension equal to the number of included factors.[5] The multifactor representation of the yield curve is also supported empirically by principal component analysis, see e.g. Litterman and Scheinkman (1991).

Multifactor yield curve models can be specified in different ways: the yield curve factors can be observable or unobserved. In the latter case they have to be estimated alongside the other parameters of the model; the structure of the factor loadings can be specified in a way such that a particular interpretation is given to the unobserved yield curve factors, as e.g. Nelson and Siegel (1987) and Soderlind and Svensson (1997); or the factor loadings can be derived from no-arbitrage constraints, as in, among many others, Duffee (2002), Ang and Piazzesi (2003) and Ang, Bekaert and Wei (2007).

Yield curve models that are linear functions of the underlying factors can be written as special cases of equation (1).[6] In this context, the two models used in the current paper are presented below.

A The Nelson-Siegel model

The Nelson and Siegel (1987) model, as re-parameterized by Diebold and Li (2006), can be seen as a restricted version of equation (1) by imposing the following constraints:

$$a^{NS} = 0 \tag{2}$$

$$b^{NS} = \begin{bmatrix} 1 & \frac{1-\exp(-\lambda\tau)}{\lambda\tau} & \frac{1-\exp(-\lambda\tau)}{\lambda\tau} - \exp(-\lambda\tau) \end{bmatrix}, \tag{3}$$

[5] Yield curve factor models are categorized by Duffie and Kan (1996) and Dai and Singleton (2000).

[6] Excluded from this list are naturally the quadratic term structure models as proposed by Ahn, Dittmar and Gallant (2002).

where λ is the exponential decay rate of the loadings for different maturities, and τ is time to maturity. This particular loading structure implies that the first factor is responsible for parallel yield curve shifts, since the effect of this factor is identical for all maturities; the second factor represents minus the yield curve slope, because it has a maximal impact on short maturities and minimal effect on the longer maturity yields; and, the third factor can be interpreted as the curvature of the yield curve, because its loading has a hump in the middle part of the maturity spectrum, and little effect on both short and long maturities. In summary, the three factors have the interpretation of a yield curve level, slope and curvature.

[FIGURE 1 AROUND HERE]

A visual representation of the Nelson and Siegel (NS) factor loading structure is given in Figure 1. By imposing the restrictions (2) to (3) on equation (1) we obtain

$$y_t = b^{NS} X_t^{NS} + \epsilon_t^{NS}, \tag{4}$$

where $X_t^{NS} = [L_t \quad S_t \quad C_t]$ represents the Nelson-Siegel yield curve factors: Level, Slope and Curvature, at time t.

A structure can be imposed on the dynamic evolution of the yield curve factors as suggested by Diebold and Li (2006). In general this means that

$$X_t^{NS} = f(X_{t-1}^{NS}, ..., X_{t-j}^{NS}, Z) \tag{5}$$

where j counts the number of lags of X^{NS} to be included and Z is a vector of exogenous variables, that also can include lags.

Empirically the Nelson-Siegel model fits data well, as shown by Nelson and Siegel (1987), and performs relatively well in out-of-sample forecasting exercises, see among others, Diebold and Li (2006) and De Pooter et al. (2007). However, as mentioned in the introduction, from a theoretical viewpoint the Nelson-Siegel yield curve model is not necessarily arbitrage-free, see Bjork and Christensen (1999) and does not belong to the class of affine yield curve models, see Diebold et al. (2006b).

B Gaussian arbitrage-free models

The Gaussian discrete-time arbitrage-free affine term structure model can also be seen as a particular case of equation (1), where the factor loadings are cross-sectionally restricted to ensure the absence of arbitrage opportunities. This class of no-arbitrage (NA) models can be represented by

$$y_t = a^{NA} + b^{NA} X_t^{NA} + \epsilon_t^{NA}, \tag{6}$$

where the underlying factors are assumed to follow a Gaussian VAR(1) process

$$X_t^{NA} = \mu + \Phi X_{t-1}^{NA} + u_t,$$

with $u_t \sim N(0, \Sigma\Sigma')$ being a $(K \times 1)$ vector of errors, μ is a $(K \times 1)$ vector, and Φ is a $(K \times K)$ autoregressive matrix. The elements of a^{NA} and b^{NA} in equation (6) are defined by

$$a_\tau^{NA} = -\frac{A_\tau}{\tau}, \quad b_\tau^{NA} = -\frac{B_\tau}{\tau}, \tag{7}$$

where, as shown by e.g. Ang and Piazzesi (2003), A_τ and B_τ satisfy the following recursive formulas that preclude arbitrage opportunities

$$A_{\tau+1} = A_\tau + B'_\tau (\mu - \Sigma\, \lambda_0) + \frac{1}{2} B'_\tau \Sigma\Sigma' B_\tau - A_1, \tag{8}$$

$$B'_{\tau+1} = B'_\tau (\Phi - \Sigma\, \lambda_1) - B'_1, \tag{9}$$

with boundary conditions $A_0 = 0$ and $B_0 = 0$. The parameters λ_0 ($(K \times 1)$ vector) and λ_1 ($(K \times K)$ matrix) govern the time-varying market price of risk, specified as an affine function of the yield curve factors

$$\Lambda_t = \lambda_0 + \lambda_1 X_t^{NA}.$$

The coefficients $A_1 = -a_1^{NA}$ and $B_1 = -b_1^{NA}$ in equations (8) to (9) refer to the short rate equation

$$r_t = a_1^{NA} + b_1^{NA} X_t^{NA} + v_t,$$

where usually r_t is approximated by the one-month yield.

If the factors X_t^{NA} driving the dynamics of the yield curve are assumed to be unobservable, the estimation of affine term structure models requires a joint procedure to estimate the factors and the parameters of the model. This is a difficult task, given the non-linearity of the model and that the number of parameters grows with the number of included factors. As the factors are latent, identifying restrictions have to be imposed. Moreover, as mentioned by Ang and Piazzesi (2003), the likelihood function is flat in the market-price-of-risk parameters and this further complicates the numerical estimation process.

To overcome these difficulties Chen and Scott (1993) describes an estimation procedure that draws on the assumption that as many yields, as factors, are observed without measurement error. Hence, it allows for recovering the latent factors from the observed yields by inverting the yield curve equation. Unfortunately, the estimation results will depend on which yields are assumed to be measured without error and will vary according to the choice made. Alternatively, to reduce the degree of arbitrariness induced by the Chen and Scott (1993) procedure, observable factors can be used. For example, Ang et al. (2006) use the short rate, the spread and the quarterly GDP growth rate as yield curve factors. It is also possible to rely on pure statistical techniques in the determination of the yield curve factors, as e.g. De Pooter et al. (2007) who use extracted principal components as yield curve factors.

C Motivation

The affine no-arbitrage term structure models impose a structure on the loadings a^{NA} and b^{NA}, presented in equations (7) to (9), such that the resulting yield curves, in the maturity dimension, are compatible with the estimated time-series dynamics for the yield curve factors. This hard-coded internal consistency between the dynamic evolution of the yield curve factors, and hence the yields at different maturity segments of the curve, is what ensures the absence of arbitrage opportunities. A similar constraint is not integrated in the setup of the Nelson-Siegel model, see e.g. Bjork and Christensen

(1999).

However, in practice, when the Nelson-Siegel model is estimated, it is possible that the no-arbitrage constraints are approximately fulfilled, i.e. fulfilled in a statistical sense, while not being explicitly imposed on the model. It cannot be excluded that the functional form of the yield curve, as it is imposed by the Nelson and Siegel factor loading structure in equations (2) and (3), fulfils the no-arbitrage constraints most of the time.

As a preliminary check for the comparability of the Nelson-Siegel model and the no-arbitrage model, Figure 2 compares extracted, standardized yield curve factors i.e. $\widehat{X}_t^{NA}$ and $\widehat{X}_t^{NS}$ for US data from 1970 to 2000 (the data is presented in Section II). We estimate the Nelson-Siegel factors as in Diebold and Li (2006), and the no-arbitrage model as in Ang and Piazzesi (2003) using the Chen and Scott (1993) method, and assuming that yields at maturities $3, 24, 120$ months are observed without error.

[FIGURE 2 AROUND HERE]

Although the two models have different theoretical backgrounds and use different estimation procedures, the extracted factors are highly correlated. Indeed, the estimated correlation between the Nelson-Siegel level factor and the first latent factor from the no-arbitrage model is 0.95. The correlation between the slope and the second latent factor is 0.96 and between the curvature and the third latent factor is 0.65.[7]

On the basis of these results and in order to properly investigate whether the Nelson-Siegel model is compatible with arbitrage-freeness, we conduct a

[7] Correlations are reported in absolute value.

test for the equality of the Nelson-Siegel factor loadings to the implied no-arbitrage ones obtained from an arbitrage-free model. To ensure correspondence between the Nelson-Siegel model and its arbitrage-free counterpart, we use extracted Nelson-Siegel factors as exogenous factors in the no-arbitrage setup. The model that we estimate is the following

$$y_t = a^{NA} + b^{NA} \widehat{X}_t^{NS} + \epsilon_t^{NA}, \quad \epsilon_t^{NA} \sim (0, \Omega), \tag{10}$$

where $\widehat{X}_t^{NS}$ are the estimated Nelson-Siegel factors from equations (2) to (4), the observation errors ϵ_t^{NA} are not assumed to be normally distributed and a^{NA} and b^{NA} satisfy the no-arbitrage restrictions presented in equations (7) to (9). In order to impose these no-arbitrage restrictions we have to fit a VAR(1) on the estimated Nelson-Siegel factors

$$\hat{X}_t^{NS} = \mu + \Phi \hat{X}_{t-1}^{NS} + u_t, \tag{11}$$

with $u_t \sim N(0, \Sigma\Sigma')$, to specify the market price of risk as an affine function of the estimated Nelson-Siegel factors

$$\Lambda_t = \lambda_0 + \lambda_1 \hat{X}_t^{NS}, \tag{12}$$

and the short rate equation as

$$r_t = a_1^{NA} + b_1^{NA} \hat{X}_t^{NS} + v_t. \tag{13}$$

In this way, we estimate the no-arbitrage factor loading structure that emerges when the underlying yield curve factors are identical to the Nelson-Siegel yield curve factors. The test is then formulated in terms of the equality between the intercepts of the two models, a^{NS} and a^{NA}, and the relative loadings, b^{NA} and b^{NS}.

II Data

We use U.S. Treasury zero-coupon yield curve data covering the period from January 1970 to December 2000 constructed by Diebold and Li (2006), based on end-of-month CRSP government bond files.[8] The data is sampled at a monthly frequency providing a total of 372 observations for each of the maturities observed at the $(1, 3, 6, 9, 12, 15, 18, 21, 24, 30, 36, 48, 60, 72, 84, 96, 108, 120)$ month segments.

[FIGURE 3 AROUND HERE]

The data is presented in Figure 3. The surface plot illustrates how the yield curve evolves over time. Table 1 reports the means, standard deviations and autocorrelations across maturities to further illustrate the properties of the data.

[TABLE 1 AROUND HERE]

The estimated autocorrelation coefficients are significantly different from zero at a 95 percent level of confidence for lag one through twelve, across all ma-

[8] The data can be downloaded from http://www.ssc.upenn.edu/ fdiebold/papers/ paper49/FBFITTED.txt and Diebold and Li (2006, pp. 344-345) give a detailed description of the data treatment methodology applied.

turities.[9] Such high autocorrelations could suggest that the underlying yield series are integrated of order one. If this is the case, we would need to take first-differences to make the variables stationary before valid statistical inference could be drawn, or we would have to resort to co-integration analysis. However, economic theory tells us that nominal yield series cannot be integrated, since they have a lower bound support at zero and an upper bound support lower than infinity. Consequently, and in accordance with the yield-curve literature, we model yields in levels and thus disregard that their in-sample properties could indicate otherwise.[10]

III Estimation Procedure

To estimate the Nelson-Siegel factors $\widehat{X}_t^{NS}$ in equation (4), we follow Diebold and Li (2006) by fixing the decay parameter $\lambda = 0.0609$ in equation (3) and by using OLS.[11] We treat the obtained Nelson-Siegel factors as observable in the estimation of the no-arbitrage model presented in equations (7) to (13). To estimate the parameters of the arbitrage-free model we use the two-step procedure proposed by Ang et al. (2006). In the first step, we fit a VAR(1) for the Nelson-Siegel factors to estimate $\widehat{\mu}$, $\widehat{\Phi}$ and $\widehat{\Sigma}$ from equation (11). And, to estimate the parameters in the short rate equation (13), we project the short rate (one-month yield) on the Nelson-Siegel yield curve factors. In

[9] A similar degree of persistence in yield curve data is also noted by Diebold and Li (2006).

[10] It is often the case in yield-curve modeling that yields are in levels. See, among others, Nelson and Siegel (1987), Diebold and Li (2006), Diebold, Rudebusch and Aruoba (2006a), Diebold, Li and Yue (2007), Duffee (2006), Ang and Piazzesi (2003), Bansal and Zhou (2002), and Dai and Singleton (2000).

[11] This value of λ maximizes the loading on the curvature at 30 months maturity as shown by Diebold and Li (2006).

the second step, we minimize the sum of squared residuals between observed yields and fitted yields to estimate the market-price-of-risk parameters $\widehat{\lambda}_0$ and $\widehat{\lambda}_1$ of equation (12). Finally, we compute $\widehat{a}^{NA}$ and $\widehat{b}^{NA}$.

Our goal is to test whether the Nelson-Siegel model in equations (2) to (4) is statistically different from the no-arbitrage model in equations (7) to (13). Since the estimated factors, $\widehat{X}_t^{NS}$ are the same for both models we can formulate our hypotheses is the following way:

$$
\begin{aligned}
H_0^1 &: a_\tau^{NA} = a_\tau^{NS} \equiv 0, \\
H_0^2 &: b_\tau^{NA}(1) = b_\tau^{NS}(1), \\
H_0^3 &: b_\tau^{NA}(2) = b_\tau^{NS}(2), \\
H_0^4 &: b_\tau^{NA}(3) = b_\tau^{NS}(3),
\end{aligned}
$$

where $b_\tau^{NA}(k)$ denotes the loadings on the k-th factor in the no-arbitrage model at maturity τ, and $b_\tau^{NS}(k)$ denotes the corresponding variable from the Nelson-Siegel model.

We claim that the Nelson-Siegel model is compatible with arbitrage-freeness if H_0^1 to H_0^4 are not rejected at traditional levels of confidence. Notice that to test for H_0^1 to H_0^4 we only need to estimate a^{NA} and b^{NA}, since the Nelson-Siegel loading structure is fixed from the model. To account for the two-step estimation procedure of the no-arbitrage model and for the generated regressor problem, we construct confidence intervals around $\hat{a}^{NA}$ and $\hat{b}^{NA}$ using the resampling procedure described in the next section.

A Resampling procedure

Bootstrap methods come in many guises in the field of financial econometrics, see among others Davidson and MacKinnon (2006) for a general overview with an emphasis on hypothesis testing. There exist three generic forms of bootstrapping: parametric, semi-parametric and non-parametric. Parametric bootstrapping refers to a situation where the data generating process can be written down explicitly, and where the error-term distribution fulfils standard criteria (Niid). In this case new data samples are generated by drawing innovations from a normal distribution and feeding them through the true data generating process. In semi-parametric bootstrapping the distributional assumption on the error-term made under the parametric setting is relaxed. Rather it is assumed that errors are iid, but not necessarily normal, in which case new data samples can be generated by resampling from the empirical residuals, and feeding the resampled residuals through the data generating process. A variant of semi-parametric bootstrapping, called (circular) block-bootstrap, exists to cater for the event where the empirical residuals are serially autocorrelated. Here "blocks" of the empirical residuals are drawn and the blocks are concatenated to form a full-length series of innovations, which are then fed through the mode. Finally, the non-parametric method refers to the "original" bootstrap method as suggested by Efron (1979). Following this approach one does not assume a true data generating process, but instead resamples the observed data.

To recover the empirical distributions of the estimated parameters we conduct non-parametric circular block resampling, in the spirit of Efron (1979), and reconstruct multiple yield curve data samples from the original yield curve data. Alternatively, a semi-parametric bootstrapping technique could have been applied.[12] In principle, our assumed data generating process is the Nelson-Siegel model, cast in state space form. Hence, the observation equation is the Nelson-Siegel yield equation (4) and the state equation is a VAR(1) for the yield curve factors equation (11), which is similar to Diebold and Li (2006). Residuals are then observable at the level of the state and the observation equations. It is common that residuals from models applying yield curve data generate residuals that are strongly autocorrelated.[13] Consequently, a semi-parametric block-bootstrapping scheme could be used, where innovations to the state and observation equations are drawn, whereby new yield-curve data-samples would be generated. However, it is not clear what would be gained from using a semi-parametric approach in our setting, in particular, since we do not focus our hypothesis tests on pivot statistics but directly on the estimated parameters.[14]

[12]A parametric bootstrap is deemed unrealistic in the preset case due to the statistical properties of the data.

[13]Using co-integration analysis, as e.g. Campbell and Shiller (1991) the problem of high residual autocorellation is naturally removed.

[14]It seems that bootstrapping pivot statistics, i.e. statistics that do not depend on the parameters of the estimated model, generally provide better statistical results, than what is obtained when bootstrapping the parameters of the model directly (see e.g. Davidson and MacKinnon (2006)). It could therefore (possibly) be argued, that a semi-parametric approach would have the advantage of allowing us to recover the standard errors of the estimated parameters, by calculating the first derivatives to our model, and, as such, allow us to base hypothesis testing on pivot statistics. However, it is not entirely clear how this would be done in our two-stage estimation procedure, and further more, acknowledging that yield curve data is nearly I(1) series, it seems more than heroic to assume that such a scheme would provide reliable results. In this connection it should also be mentioned, that a semi-parametric resampling scheme would also imply an explicit assumption to be made about the correlation structure between innovations to the dynamic evolution of yield curve factors and the innovations to the yield equation. Such an explicit modelling of innovation term correlation is naturally subject to err as regards to the assumptions made. A non-parametric resampling scheme applied directly at the level of observed yields does

Our non-parametric procedure commences as follows. We denote by G the matrix of observed yield ratios with elements $y_{t,\tau}/y_{t-1,\tau}$ where $t = (2, \ldots, T)$ and $\tau = (1, \ldots, N)$. We first randomly select a starting yield curve y_k, where the index k is an integer drawn randomly from a discrete uniform distribution $[1, \ldots, T]$. The resulting k marks the random index value at which the starting yield curve is taken.

In a second step, blocks of length w are sampled from the matrix of yield ratios G using a circular scheme. The generic i-th block can be denoted by $\widetilde{g}_{z,i}$ where z is a random number from $[2, \ldots, T]$ denoting the first observation of the block and $i = 1 \ldots I$, where I is the maximum number of blocks drawn.[15] Note that, in the spirit of circular block-resampling, if the first observation of the block is T then a block is constructed using this observation and the first $w-1$ observations of G. A full data-sample of regenerated yield curve ratios $\widetilde{G}$ can then be constructed by vertical concatenation of the drawn data blocks $\widetilde{g}_{z,i}$ for $i = 1 \ldots I$.

Finally, a new data set of resampled yields can be constructed via:

$$\begin{cases} \widetilde{y}_1 & = y_k \\ \widetilde{y}_s & = \widetilde{y}_{s-1} \odot \{\widetilde{G}\}_s, \qquad s = 2, \ldots, S, \end{cases} \tag{14}$$

where $\{\widetilde{G}\}_s$ denotes the s^{th} row of the matrix of resampled ratios $\widetilde{G}$, and $\odot$ denotes element by element multiplication.

We choose to resample from yield ratios for two reasons. First, it ensures

not suffer from the same deficiency. Finally, it can be remarked that a semi-parametric resampling scheme would also not elevate the nuisance of resampled yield curves having negative realisations.

[15] We use $\sim$ to indicate the re-sampled variables.

positiveness of the resampled yields. Second, as reported in Table 1, yields are highly autocorrelated and close to $I(1)$. Therefore, one could resample from first differences, but as reported in Table 2, first differences of yields are highly autocorrelated and not variance-stationary. Yield ratios display better statistical properties regarding variance-stationarity, as can be seen by comparing the correlation coefficients for squared differences and ratios in Table 2. Block-bootstrapping is used to account for serial correlation in the yield curve ratios. Moreover, the circular scheme corrects the issue pertaining to the regular block bootstrap technique of not allocating equal sampling probability to observations located at the beginning and at the end of the data series.[16]

[TABLE 2 AROUND HERE]

A similar resampling technique has been proposed by Rebonato et al. (2005). They provide a detailed account for the desirable statistical features of this approach. In the present context we recall that the method ensures: (i) the exact asymptotic recovery of all the eigenvalues and eigenvectors of yields; (ii) the correct reproduction of the distribution of curvatures of the yield

[16]We thank V. Corradi for pointing out to us that resampling from ratios leads our resampled data to have a biased mean. However, since this bias can be shown to affect results only when the yield ratios deviate from unity, the bias problem can only marginally influence our results. In fact, the mean of yield ratios across all maturities in our data sample is close to unity; reflecting the near integratedness of yield curve data. In addition, it should be mentioned, that the obvious alternative, namely sampling from the first differences of yields, will also bias the mean of the resampled data. This bias would however not be introduced at the bootstrapping level, but when enforcing that nominal yields are only meaningful when they are positive. In practice, when resampling from first differences, one would need to discard all yield curve realisations having one or more resampled yield observations landing in the negative territory. Naturally, such a procedure would also bias the mean of the resampled series, and the bias would be of an unknown size.

curve across maturities; (iii) the correct qualitative recovery of the transition from super- to sub-linearity as the yield maturity is increased in the variance of n-day changes, and (iv) satisfactory accounting of the empirically-observed positive serial correlations in the yields.

To test hypotheses H_0^1 to H_0^4 we employ the following scheme:

(1) Construct a yield curve sample $\widetilde{y}$ following equation (14);

(2) Estimate the Nelson-Siegel yield curve factors $\widetilde{X}_t^{NS}$ on $\widetilde{y}$;

(3) Use $\widetilde{X}_t^{NS}$ to estimate the parameters $\widetilde{a}^{NA}$ and $\widetilde{b}^{NA}$ from the arbitrage-free model given in equations (7) - (13);

(4) Repeat steps 1 to 3, 1000 times to build a distribution for the parameter estimates $\hat{a}^{NA}$ and $\hat{b}^{NA}$;

(5) Construct confidence intervals for $\hat{a}^{NA}$ and $\hat{b}^{NA}$ using the sample quantiles of the empirical distribution of the estimated parameters.

Note that by fixing λ in step 2, the Nelson-Siegel factor loading structure remains unchanged from repetition to repetition. We set the block length equal to 48 observations (4 years of data), i.e. $w = 48$, and generate a total of 372 yield curve observations for each replication, i.e. $S = 372$.[17]

IV Results

This section presents three sets of results to help assess whether the Nelson-Siegel model is compatible with arbitrage-freeness when applied to US zero-

[17] All the results presented in the paper are robust to changes in the size of the block length (in a reasonable way).

coupon data. Our main result is a test of equality of the factor loadings of the affine arbitrage-free model (based on exogenous Nelson-Siegel yield curve factors), equations (7) - (13), to the factor loadings of the Nelson-Siegel model, equations (2) - (4), on the basis of the resampling technique outlined in section III. In addition we compare the in-sample and out-of-sample performance of the Nelson-Siegel model to the no-arbitrage model.

A Testing results

Using the resampling methodology outlined in section III, we generate empirical distributions for each factor loading of the no-arbitrage yield curve model in equation (10). Results are presented for each maturity covered by the original data sample. The Nelson-Siegel factor loading structure, in equations (2) and (3), is constant across all bootstrapped data sampled because λ is treated as a known parameter.[18] Hence, only the extracted Nelson-Siegel factors vary across the bootstrap samples.

Parameter estimates and corresponding empirical confidence intervals for the no-arbitrage model, equations (7) - (13), are shown in Table 3. The diagonal elements of the matrices holding the estimated autoregressive coefficients $\widehat{\Phi}$ and the covariance matrix of the VAR residuals $\widehat{\Sigma}$, in equation (11), are significantly different from zero at a 95 percent level of confidence.[19]

[18] The results presented in the paper are robust to changes in λ. We have performed the calculations for other values of λ, namely $\lambda = 0.08$, $\lambda = 0.045$, and $\lambda = 0.0996$, and the results for these values of λ are qualitatively the same as the ones presented in the paper.

[19] The results from the tests are robust to a diagonal specification of the VAR process for the factors in equation (11). However, in terms of forecasting the more parsimonius representation performs better. Since the main purpose of the paper is not to compare the forecasting performance of different model specifications and for readability, we stick to the original full formulation of the model.

In addition, the estimates of the first two elements of the (3×1) vector b_1^{NA} in equation (13), are also different from zero, judged at the same level of confidence. The estimate for a_1^{NA} is not significantly different from zero, which is in line with the zero intercept in the NS model.

[TABLE 3 AROUND HERE]

The estimated intercepts of the no-arbitrage model $\hat{a}^{NA}$, computed as in equations (7) - (8), are presented in Table 4, for each maturity covered by the original data. This table reports also the 95 percent confidence intervals, obtained from the resampling, and the Nelson-Siegel intercepts a^{NS}. Therefore, results in Table 4 allow for testing H_0^1 for the equality between the intercepts in the yield curve equations for the no-arbitrage and the Nelson-Siegel models. Tables 5 to 7 present the corresponding results that allow us to test H_0^2, H_0^3, and H_0^4, i.e. whether the corresponding yield curve factor loadings are equal. The empirical 95 percent confidence intervals are included in Tables 4, 5, 6 and 7. The upper and lower bounds of the confidence intervals are denoted by a subscript U L, respectively.

[TABLE 4 to 7 AROUND HERE]

Figure 4 gives a visual representation of the results contained in Tables 4 to 7. The figure shows the estimated no-arbitrage loadings, $\hat{a}^{NA}$ and $\hat{b}^{NA}$, with the relative 50 percent and 95 percent empirical confidence intervals obtained from resampling, as well as the parameter values for the Nelson-Siegel model, i.e. a^{NS} and b^{NS}, for comparison.

It is clear from Figure 4 that the empirical distributions are highly skewed for most of the maturities. Consider, for example, the plot for the intercept estimates (the top left plot in Figure 4) at maturity 120.

[FIGURE 4 AROUND HERE]

This non-normality of the distributions for the estimated no-arbitrage parameters, is further analyzed in Table 8. This table shows that all distributions display skewness, excess kurtosis, or both. Selected maturities are shown in Table 8, however, this result holds for all maturities included in the sample. We also perform the Jarque-Bera test for normality, and reject normality at a 95 percent confidence level for all maturities.

[TABLE 8 AROUND HERE]

Visual confirmation of the documented non-normality is provided by Figures 5 to 8. For a representative selection of maturities, these figures show the empirical distribution of the estimated no-arbitrage loadings, and a normal distribution approximation. In addition, the figures show the 95 percent confidence intervals computed as sample quantiles of the empirical distribution of the parameters and the normal approximation.

[FIGURE 5 to 8 AROUND HERE]

The non-normality of the empirical distributions for the bootstrapped intercepts $\hat{a}^{NA}$, and factor loadings $\hat{b}^{NA}$, indicates that the confidence intervals should be constructed using the sample quantiles of the empirical distribution.

By inspecting the tables, we reach the following conclusions for the tested hypotheses:

$$H_0^1 : a_\tau^{NA} = a_\tau^{NS} \equiv 0 \qquad \text{not rejected at a 95\% level of confidence,}$$
$$H_0^2 : b_\tau^{NA}(1) = b_\tau^{NS}(1) \qquad \text{not rejected at a 95\% level of confidence,}$$
$$H_0^3 : b_\tau^{NA}(2) = b_\tau^{NS}(2) \qquad \text{not rejected at a 95\% level of confidence,}$$
$$H_0^4 : b_\tau^{NA}(3) = b_\tau^{NS}(3) \qquad \text{not rejected at a 95\% level of confidence.}$$

The hypotheses H_0^1 through H_0^4 test the equality between each no-arbitrage factor loading and the corresponding Nelson-Siegel factor loading separately for each maturity. The last rows of Tables 4 - 7 show the F statistics for the null hypothesis that the corresponding no-arbitrage factor loadings are jointly equal to the Nelson-Siegel factor loadings for all maturities in the sample. To perform these tests we use the empirical variance-covariance matrix of the estimates obtained from the resampling. All the joint tests support the result of no statistical difference between the Nelson-Siegel and the no-arbitrage loadings. We have also computed the joint F test for the equality of all the NA and NS factor loadings across maturities. The test statistic is 0.25 and the 95 percent critical F-value with 72 and 300 degrees of freedom is 1.34. Therefore, we also cannot reject the hypothesis that the loading structures of the two models are equal.

For the test of the curvature parameter in H_0^4 an additional comment is warranted. As can be seen from Figure 4, the curvature parameter, at middle maturities, is the closest to violating the 95 percent confidence band, and this parameter thus constitutes the “weak point” of the Nelson-Siegel model in relation to the no-arbitrage constraints. This finding is in line with Bjork and Christensen (1999) who prove that a Nelson-Siegel type model with two additional curvature factors, each with its own λ, theoretically would be arbitrage-free. However, when acknowledging that Litterman and Scheinkman (1991) find that the curvature factor only accounts for approximately 2 percent of the variation of yields, and in the light of our results, one can question the significance of adding additional factors. Our empirical finding is also supported by the theoretical results in Christensen et al. (2007) who show that adding an additional term at very long maturities reconciles the dynamic Nelson-Siegel model with the affine arbitrage-free term structure models.

Using yield curve modeling for purposes other than relative pricing, as for example central bankers and fixed-income strategists do, one might be tempted to use the Nelson-Siegel model on the basis of its compatibility with arbitrage-freeness.

B In-sample comparison

To conduct an in-sample comparison of the two models, we estimate the Nelson-Siegel model in equations (2) - (4) and the no-arbitrage model in equations (7) - (13), where the latter model uses the yield curve factors extracted from the former. Measures of fit are displayed in Table 9.

A general observation is that both models fit data well: the means of the residuals for all maturities are close to zero and show low standard deviations. The root mean squared error, RMSE, and the mean absolute deviation, MAD, are also low and similar for both models.

More specifically, Table 9 shows that the averages of the residuals from the fitted Nelson-Siegel model, $\hat{\epsilon}^{NS}$, for the included maturities, are all lower than 16 basis points, in absolute value. In fact, the mean of the absolute residuals across maturities is 5 basis points, while the corresponding number for $\hat{\epsilon}^{NA}$ is 3 basis points. The 3 months maturity is the worst fitted maturity for the no-arbitrage model with a mean of the residuals of 8 basis points. For the Nelson-Siegel model the worst fitted maturity is the 1 month segment with a mean of the residuals close to -16 bp. Furthermore, the two models have the same amount of autocorrelation in the residuals. A similar observation is made for the Nelson-Siegel model alone by Diebold and Li (2006).

[TABLE 9 AROUND HERE]

Drawing a comparison on the basis of RMSE and MAD figures gives the conclusion that both models fit data equally well.

C Out-of-sample comparison

As a last comparison-check of the equivalence of the Nelson-Siegel model and the no-arbitrage counterpart, we perform an out-of-sample forecast experiment. In particular, we generate h-steps ahead iterative forecasts in the following way. First, the yield curve factors are projected forward using the estimated VAR parameters from equation (11)

$$\hat{X}^{NS}_{t+h|t} = \sum_{s=0}^{h-1} \hat{\Phi}^s \hat{\mu} + \hat{\Phi}^h \hat{X}^{NS}_t,$$

where $h \in \{1, 6, 12\}$ is the forecasting horizon in months. Second, out-of-sample forecasts are calculated for the two models, given the projected factors,

$$\hat{y}^{NS}_{t+h|t} = b^{NS} \hat{X}^{NS}_{t+h|t},$$
$$\hat{y}^{NA}_{t+h|t} = \widehat{a}^{NA}_{t} + \widehat{b}^{NA}_{t} \hat{X}^{NS}_{t+h|t},$$

where subscripts t on $\widehat{a}^{NA}_t$ and $\widehat{a}^{NA}_t$ indicate that parameters are estimated using data until time t. To evaluate the prediction accuracy at a given forecasting horizon, we use the mean squared forecast error, MSFE i.e. the average squared error over the evaluation period, between t_0 and t_1, for the h-months ahead forecast of the yield with maturity τ

$$MSFE(\tau, h, m) = \frac{1}{t_1 - t_0 + 1} \sum_{t=t_0}^{t_1} \left(\hat{y}^{m}_{t+h,\tau|t} - y_{t+h,\tau} \right)^2, \tag{15}$$

where $m \in \{NA, NS\}$ denotes the model.

The results presented are expressed as ratios of the MSFEs of the two models against the MSFE of a random walk. The random walk represents a naïve forecasting model that historically has proven very difficult to outperform especially at short forecasting horizons. The success of the random walk model in the area of yield curve forecasting is due to the high degree of persistence exhibited by observed yields. The random walk h-step ahead prediction, at time t, of the yield with maturity τ is

$$\hat{y}_{t+h,\tau|t} = y_{t,\tau}.$$

To produce the first set of forecasts, the model parameters are estimated on a sample defined from 1970:01 to 1993:01, and yields are forecasted for the chosen horizons, h. The data sample is then increased by one month and the parameters are re-estimated on the new data covering 1970:01 to 1993:02. Again, forecasts are produced for the forecasting horizons. This procedure is repeated for the full sample, generating forecasts on successively increasing data samples. The forecasting performances are then evaluated over the period 1994:01 to 2000:12 using the MSFE, as shown in equation (15).

Table 10 reports on the out-of-sample forecast performance of the Nelson-Siegel and the no-arbitrage model evaluated against the random walk forecasts.

[TABLE 10 AROUND HERE]

The well-known phenomenon of the good forecasting performance of the random walk model is observed for the 1 month forecasting horizon. For the 6 and 12 month forecasting horizons, the Nelson-Siegel model and the no-arbitrage counterpart generally perform better than the random walk model, as shown by all ratios being less than one (except for the 10 years maturity).

Turning now to the relative comparison of the no-arbitrage model against the Nelson-Siegel model, it can be concluded that they exhibit very similar forecasting performances. If we consider every maturity for each forecasting horizon as an individual observation, then there are in total 54 observations. In 24 of these cases the Nelson-Siegel model is better, in 23 cases the no-arbitrage model is better, and in the remaining 7 cases the models perform equally well. Even when one model is judged to be better than its competitor, the differences in the performance ratios are very small.

In summary, it can be concluded that there is no systematic pattern across maturities and forecasting horizons showing when one model is better than its competitor. Indeed, to formally compare the forecasting performance of the two models we compute the Diebold-Mariano statistic for each maturity and forecasting horizon. At a 5 percent level we do not reject the hypothesis that the no-arbitrage model and the Nelson-Siegel model forecast equally well for the majority of the forecasts. At this significance level we reject the hypothesis of equal forecasting performance for maturities 9 and 120 months in 1-step ahead forecasts and for maturity 96 months in 6- and 12-steps ahead forecasts. In these four cases the Diebold-Mariano tests shows that the forecasts produced by the Nelson-Siegel model are better at 5 percent level of significance. However, none of the models is able to outperform the other in any maturity for a significance level of 1 percent.

V Conclusion

In this paper we show that the model proposed by Nelson and Siegel (1987) is compatible with arbitrage-freeness, in the sense that the factor loadings from the model are not statistically different from those derived from an arbitrage-free model which uses the Nelson-Siegel factors as exogenous factors, at a 95 percent level of confidence.

In theory, the Nelson-Siegel model is not arbitrage-free as shown by Bjork and Christensen (1999). However, using US zero-coupon data from 1970 to 2000, a yield curve bootstrapping approach and the implied arbitrage-free factor loadings, we cannot reject the hypothesis that Nelson-Siegel factor loadings fulfill the no-arbitrage constraints, at a 95 percent confidence level. Furthermore, we show that the Nelson-Siegel model performs as well as the no-arbitrage counterpart in an out-of-sample forecasting experiment. Based on these results, we conclude that the Nelson-Siegel model is compatible with arbitrage-freeness. This finding implies that the factor loading structure of the Nelson-Siegel model is rich enough to capture the dynamic behavior of yield curve factors, when such factors are estimated on data generated by traders who rely on arbitrage-free models, as it is likely to be the case for US yield curve data. It is an interesting question whether our suggested methodology would lead to a rejection of the tested hypothesis if applied to a less well-organized bond market. We leave this question to be answered by future research.

Our conclusion is of relevance to fixed-income money managers and central banks in particular, since such organizations traditionally rely heavily on the Nelson-Siegel model for policy and strategic investment decisions.

Tables and Graphs

Table 1: Summary statistics of the US zero-coupon data

τ	mean	std dev	min	max	$\rho(1)$	$\rho(2)$	$\rho(3)$	$\rho(12)$
1	6.44	2.58	2.69	16.16	0.97*	0.93*	0.89*	0.69*
3	6.75	2.66	2.73	16.02	0.97*	0.94*	0.91*	0.71*
6	6.98	2.66	2.89	16.48	0.97*	0.94*	0.91*	0.73*
9	7.10	2.64	2.98	16.39	0.97*	0.94*	0.91*	0.73*
12	7.20	2.57	3.11	15.82	0.97*	0.94*	0.91*	0.74*
15	7.31	2.52	3.29	16.04	0.97*	0.94*	0.91*	0.75*
18	7.38	2.50	3.48	16.23	0.98*	0.94*	0.92*	0.75*
21	7.44	2.49	3.64	16.18	0.98*	0.95*	0.92*	0.76*
24	7.46	2.44	3.78	15.65	0.98*	0.94*	0.92*	0.75*
30	7.55	2.36	4.04	15.40	0.98*	0.95*	0.92*	0.76*
36	7.63	2.34	4.20	15.77	0.98*	0.95*	0.93*	0.77*
48	7.77	2.28	4.31	15.82	0.98*	0.95*	0.93*	0.78*
60	7.84	2.25	4.35	15.01	0.98*	0.96*	0.94*	0.79*
72	7.96	2.22	4.38	14.98	0.98*	0.96*	0.94*	0.80*
84	7.99	2.18	4.35	14.98	0.98*	0.96*	0.94*	0.78*
96	8.05	2.17	4.43	14.94	0.98*	0.96*	0.95*	0.81*
108	8.08	2.18	4.43	15.02	0.98*	0.96*	0.95*	0.81*
120	8.05	2.14	4.44	14.93	0.98*	0.96*	0.94*	0.78*

Descriptive statistics of monthly yields at different maturities, τ, for the sample from January 1970 to December 2000. $\rho(p)$ refers to the sample autocorrelation of the series at lag p and * denotes significance at 95 percent confidence level. Confidence intervals are computed according to Box and Jenkins (1976).

Table 2: Autocorrelations

Yield differences

τ	$\rho(1)$	$\rho(3)$	$\rho(12)$	$\rho^2(1)$	$\rho^2(3)$	$\rho^2(12)$
1	0.06	-0.07	-0.06	0.23*	0.08	0.08
3	0.12*	-0.05	-0.13*	0.34*	0.07	0.22*
6	0.16*	-0.09	-0.08	0.32*	0.09	0.20*
12	0.15*	-0.10	-0.05	0.16*	0.11*	0.13*
24	0.18*	-0.11*	0.00	0.21*	0.13*	0.13*
36	0.14*	-0.11*	0.03	0.12*	0.14*	0.14*
60	0.13*	-0.07	0.03	0.09	0.13*	0.13*
84	0.10	-0.09	-0.03	0.17*	0.22*	0.18*
120	0.10	-0.05	-0.03	0.15*	0.19*	0.23*

Yield ratios

τ	$\rho(1)$	$\rho(3)$	$\rho(12)$	$\rho^2(1)$	$\rho^2(3)$	$\rho^2(12)$
1	0.07	-0.05	0.10	0.23*	0.12*	0.02
3	0.11*	0.00	0.01	0.34*	0.10	0.16*
6	0.16*	0.00	0.04	0.25*	0.13*	0.13*
12	0.16*	-0.04	0.04	0.10	0.13*	0.07
24	0.16*	-0.07	0.03	0.06	0.12*	0.03
36	0.13*	-0.09	0.06	0.01	0.06	0.05
60	0.12*	-0.04	0.05	0.01	0.01	0.01
84	0.11*	-0.04	0.00	0.04	0.07	0.03
120	0.08	-0.03	0.00	0.03	0.06	0.06

Sample autocorrelations of first yield differences $\triangle y$, squared first yield differences $\triangle y^2$, yield ratios $\frac{y_t}{y_{t-1}}$ and squared demeaned yield ratios $\left(\frac{y_t}{y_{t-1}} - \bar{\mu}\right)^2$, for selected maturities τ, at lags 1, 3 and 12. $*$ denotes significance at 95 percent confidence level. Confidence intervals are computed according to Box and Jenkins (1976). $\rho(p)$ and $\rho^2(p)$ denote, respectively, the correlation of the variables and their squares, at lag p.

Table 3: Parameter estimates

Parameter	Estimate	$Q_{2.5}$	$Q_{97.5}$
$\hat{\mu}_1$	0.0001*	0.0000	0.0005
$\hat{\mu}_2$	0.0001	-0.0002	0.0006
$\hat{\mu}_3$	-0.0003	-0.0017	0.0005
$\hat{\Phi}_{11}$	0.991*	0.922	1.016
$\hat{\Phi}_{21}$	-0.028	-0.093	0.045
$\hat{\Phi}_{31}$	0.066	-0.121	0.310
$\hat{\Phi}_{12}$	0.026	-0.023	0.044
$\hat{\Phi}_{22}$	0.933*	0.888	1.021
$\hat{\Phi}_{32}$	0.038	-0.182	0.157
$\hat{\Phi}_{13}$	0.000	-0.022	0.034
$\hat{\Phi}_{23}$	0.036	-0.005	0.067
$\hat{\Phi}_{33}$	0.771*	0.774	0.974
$\hat{\Sigma}_{11}$	0.0003*	0.0001	0.0005
$\hat{\Sigma}_{21}$	-0.0001	-0.0003	0.0002
$\hat{\Sigma}_{31}$	-0.0002	-0.0013	0.0000
$\hat{\Sigma}_{22}$	0.0005*	0.0001	0.0010
$\hat{\Sigma}_{32}$	0.0000	-0.0005	0.0002
$\hat{\Sigma}_{33}$	0.0010*	0.0002	0.0018

Parameter estimates (continued)

Parameter	Estimate	$Q_{2.5}$	$Q_{97.5}$
$\hat{\lambda}_{0,1}$	0.182	-1.922	1.239
$\hat{\lambda}_{0,2}$	0.140	-1.646	1.183
$\hat{\lambda}_{0,3}$	-0.046	-1.047	2.551
$\hat{\lambda}_{1,11}$	-35.424	-247.627	214.559
$\hat{\lambda}_{1,21}$	-70.551	-390.503	162.438
$\hat{\lambda}_{1,31}$	70.755	-331.180	154.500
$\hat{\lambda}_{1,12}$	72.424	-478.827	296.723
$\hat{\lambda}_{1,22}$	-31.733	-208.627	384.491
$\hat{\lambda}_{1,32}$	96.701	-178.384	349.269
$\hat{\lambda}_{1,13}$	-117.191	-468.725	98.800
$\hat{\lambda}_{1,23}$	-101.960	-622.469	10.095
$\hat{\lambda}_{1,33}$	-10.912	-108.704	446.474
$\hat{a}_1^{NA}$	0.0000	-0.0009	0.0004
$\hat{b}_1^{NA}(1)$	0.982*	0.891	1.145
$\hat{b}_1^{NA}(2)$	0.932*	0.828	1.068
$\hat{b}_1^{NA}(3)$	0.001	-0.087	0.064

Estimated parameters of the no-arbitrage model in equations (7) to (13) with the 95 percent confidence intervals obtained by resampling. The confidence intervals [$Q_{2.5}$ $Q_{97.5}$] refer to the empirical 2.5 percent and 97.5 percent quantiles of the distributions of the parameters. A star * is used to indicate when a parameter estimate is significantly different from zero at a 95 percent level of confidence.

Table 4: Estimation results for a^{NA}

τ	a^{NS}	$\hat{a}^{NA}$	$\widetilde{a}_L^{NA}$	$\widetilde{a}_U^{NA}$
1	0	0.0000	-0.0009	0.0004
3	0	0.0000	-0.0004	0.0005
6	0	0.0000	-0.0002	0.0006
9	0	0.0001	-0.0003	0.0007
12	0	0.0001	-0.0003	0.0005
15	0	0.0000	-0.0003	0.0004
18	0	0.0000	-0.0003	0.0003
21	0	0.0000	-0.0004	0.0002
24	0	0.0000	-0.0005	0.0001
30	0	0.0000	-0.0006	0.0001
36	0	-0.0001	-0.0007	0.0001
48	0	-0.0001	-0.0007	0.0002
60	0	-0.0001	-0.0006	0.0002
72	0	0.0000	-0.0004	0.0002
84	0	0.0000	-0.0002	0.0003
96	0	0.0000	-0.0001	0.0005
108	0	0.0001	-0.0002	0.0008
120	0	0.0001	-0.0003	0.0011
F stat	0.0103 (1.00)			

Estimated intercepts from the no-arbitrage model $\hat{a}^{NA}$ with the 95 percent confidence intervals obtained from the resampling $[\widetilde{a}_L^{NA}\ \widetilde{a}_U^{NA}]$. The confidence intervals refer to the empirical 2.5 percent and 97.5 percent quantiles of the distribution of the parameters. The second column of the Table reports the Nelson-Siegel loadings. The last row reports the F statistics for the equality of the corresponding NA and NS intercepts jointly for all maturities ($H_0 : a_\tau^{NA} = a_\tau^{NS}$ for $\forall \tau$). The p-value (with 18 and 354 degrees of freedom) is reported in parentheses.

Table 5: Estimation results for $b^{NA}(1)$

τ	$b^{NS}(1)$	$\hat{b}^{NA}(1)$	$\widetilde{b}_L^{NA}(1)$	$\widetilde{b}_U^{NA}(1)$
1	1	0.98	0.89	1.14
3	1	0.99	0.89	1.07
6	1	0.99	0.88	1.05
9	1	1.00	0.89	1.05
12	1	1.00	0.90	1.06
15	1	1.00	0.91	1.05
18	1	1.00	0.91	1.06
21	1	1.00	0.94	1.06
24	1	1.00	0.95	1.08
30	1	1.01	0.95	1.10
36	1	1.01	0.95	1.11
48	1	1.00	0.94	1.12
60	1	1.00	0.93	1.11
72	1	1.00	0.94	1.08
84	1	1.00	0.94	1.06
96	1	1.00	0.90	1.04
108	1	0.99	0.86	1.05
120	1	0.99	0.81	1.07
F stat	0.0084 (1.00)			

Estimated loadings of the level factor from the no-arbitrage model $\hat{b}^{NA}(1)$ with the 95 percent confidence intervals obtained from the resampling $[\widetilde{b}_L^{NA}(1)\ \widetilde{b}_U^{NA}(1)]$. The confidence intervals refer to the empirical 2.5 percent and 97.5 percent quantiles of the distribution of the parameters. The second column of the Table reports the Nelson-Siegel loadings on the level. The last row reports the F statistics for the equality of the corresponding NA and NS loadings of the level jointly for all maturities ($H_0 : b_\tau^{NA}(1) = b_\tau^{NS}(1)$for $\forall\tau$). The p-value (with 18 and 354 degrees of freedom) is reported in parentheses.

Table 6: Estimation results for $b^{NA}(2)$

τ	$b^{NS}(2)$	$\hat{b}^{NA}(2)$	$\widetilde{b}_L^{NA}(2)$	$\widetilde{b}_U^{NA}(2)$
1	0.97	0.93	0.83	1.07
3	0.91	0.89	0.82	0.99
6	0.84	0.84	0.74	0.93
9	0.77	0.78	0.67	0.86
12	0.71	0.72	0.61	0.78
15	0.66	0.66	0.56	0.71
18	0.61	0.62	0.51	0.66
21	0.56	0.57	0.47	0.62
24	0.53	0.53	0.44	0.58
30	0.46	0.46	0.37	0.53
36	0.41	0.41	0.32	0.48
48	0.32	0.32	0.25	0.41
60	0.27	0.26	0.20	0.35
72	0.23	0.22	0.17	0.32
84	0.19	0.19	0.14	0.28
96	0.17	0.17	0.11	0.26
108	0.15	0.15	0.08	0.25
120	0.14	0.13	0.05	0.24
F stat	0.0362 (1.00)			

Estimated loadings of the slope factor from the no-arbitrage model $\hat{b}^{NA}(2)$ with the 95 percent confidence intervals obtained from the resampling $[\widetilde{b}_L^{NA}(2)\ \widetilde{b}_U^{NA}(2)]$. The confidence intervals refer to the empirical 2.5 percent and 97.5 percent quantiles of the distribution of the parameters. The second column of the Table reports the Nelson-Siegel loadings on the slope. The last row reports the F statistics for the equality of the corresponding NA and NS loadings of the slope jointly for all maturities ($H_0 : b_\tau^{NA}(2) = b_\tau^{NS}(2)$for $\forall\tau$). The p-value (with 18 and 354 degrees of freedom) is reported in parentheses.

Table 7: Estimation results for $b^{NA}(3)$

τ	$b^{NS}(3)$	$\hat{b}^{NA}(3)$	$\widetilde{b}_L^{NA}(3)$	$\widetilde{b}_U^{NA}(3)$
1	0.03	0.00	-0.09	0.06
3	0.08	0.10	0.03	0.16
6	0.14	0.19	0.10	0.25
9	0.19	0.24	0.13	0.28
12	0.23	0.26	0.15	0.29
15	0.25	0.27	0.17	0.30
18	0.27	0.28	0.18	0.30
21	0.29	0.28	0.19	0.31
24	0.29	0.27	0.19	0.31
30	0.30	0.26	0.20	0.31
36	0.29	0.25	0.20	0.30
48	0.27	0.23	0.19	0.29
60	0.24	0.22	0.17	0.27
72	0.21	0.20	0.17	0.25
84	0.19	0.19	0.15	0.24
96	0.17	0.19	0.14	0.23
108	0.15	0.18	0.12	0.23
120	0.14	0.18	0.09	0.22
F stat	0.4340 (0.98)			

Estimated loadings of the curvature factor from the no-arbitrage model $\hat{b}^{NA}(3)$ with the 95 percent confidence intervals obtained from the resampling $[\widetilde{b}_L^{NA}(3)\ \widetilde{b}_U^{NA}(3)]$. The confidence intervals refer to the empirical 2.5 percent and 97.5 percent quantiles of the distribution of the parameters. The second column of the Table reports the Nelson-Siegel loadings on the curvature. The last row reports the F statistics for the equality of the corresponding NA and NS loadings of the curvature jointly for all maturities ($H_0 : b_\tau^{NA}(3) = b_\tau^{NS}(3)$for $\forall\tau$). The p-value (with 18 and 354 degrees of freedom) is reported in parentheses.

Table 8: Summary statistics of the resampled loadings

Intercept $\widetilde{a}^{NA}$

τ	mean	st.dev.	skewness	kurtosis
3	0.00	0.00	1.07	18.02
12	0.00	0.00	0.46	7.68
24	0.00	0.00	-5.52	75.26
60	0.00	0.00	0.15	5.88
84	0.00	0.00	8.29	137.29
120	0.00	0.00	2.26	23.40

Loading of the level $\widetilde{b}^{NA}(1)$

τ	mean	st.dev.	skewness	kurtosis
3	0.99	0.04	-0.25	5.26
12	0.98	0.04	-0.74	9.18
24	1.01	0.03	0.92	16.65
60	1.02	0.04	0.06	7.95
84	1.00	0.03	-1.08	22.85
120	0.96	0.07	-0.96	7.76

Loading of the slope $\widetilde{b}^{NA}(2)$

τ	mean	st.dev.	skewness	kurtosis
3	0.91	0.04	0.26	7.09
12	0.71	0.04	-1.35	8.56
24	0.52	0.04	-0.91	17.57
60	0.27	0.04	1.46	24.18
84	0.20	0.04	1.16	25.60
120	0.14	0.05	-0.20	10.93

Loading of the curvature $\widetilde{b}^{NA}(3)$

τ	mean	st.dev.	skewness	kurtosis
3	0.09	0.03	1.08	6.71
12	0.24	0.03	-1.20	6.38
24	0.27	0.03	-2.65	14.58
60	0.22	0.02	-1.69	17.12
84	0.19	0.02	-1.95	22.93
120	0.16	0.03	-0.97	9.68

Summary statistics of the empirical distributions of the estimated no-arbitrage loadings obtained using resampled data.

Table 9: Measures of Fit

Residuals from the Nelson-Siegel model

τ	mean	st dev	min	max	RMSE	MAD	$\rho(1)$	$\rho(6)$	$\rho(12)$
1	-0.159	0.200	-1.046	0.387	0.200	0.040	0.513	0.332	0.443
3	0.027	0.114	-0.496	0.584	0.114	0.013	0.274	0.159	0.326
6	0.091	0.135	-0.412	0.680	0.135	0.018	0.543	0.346	0.471
12	0.046	0.122	-0.279	0.483	0.122	0.015	0.586	0.127	0.289
24	-0.040	0.073	-0.398	0.261	0.073	0.005	0.493	0.044	0.153
36	-0.066	0.090	-0.432	0.339	0.089	0.008	0.417	0.256	0.183
60	-0.053	0.096	-0.520	0.292	0.096	0.009	0.655	0.312	-0.037
84	0.006	0.097	-0.446	0.337	0.096	0.009	0.518	0.159	-0.083
120	0.002	0.140	-0.763	0.436	0.140	0.020	0.699	0.345	0.091

Residuals from no-arbitrage model

τ	Mean	st dev	min	max	RMSE	MAD	$\rho(1)$	$\rho(6)$	$\rho(12)$
1	0.000	0.168	-0.730	0.752	0.168	0.028	0.361	0.197	0.363
3	0.079	0.132	-0.512	0.815	0.132	0.018	0.448	0.218	0.311
6	0.058	0.134	-0.299	0.792	0.134	0.018	0.577	0.359	0.430
12	-0.019	0.109	-0.357	0.436	0.109	0.012	0.512	0.144	0.304
24	-0.040	0.072	-0.322	0.217	0.072	0.005	0.494	0.138	0.094
36	-0.017	0.089	-0.286	0.405	0.088	0.008	0.478	0.322	0.264
60	0.003	0.100	-0.332	0.378	0.100	0.010	0.687	0.350	0.099
84	0.019	0.097	-0.486	0.342	0.097	0.010	0.529	0.157	-0.068
120	-0.059	0.144	-0.798	0.380	0.144	0.021	0.706	0.466	0.251

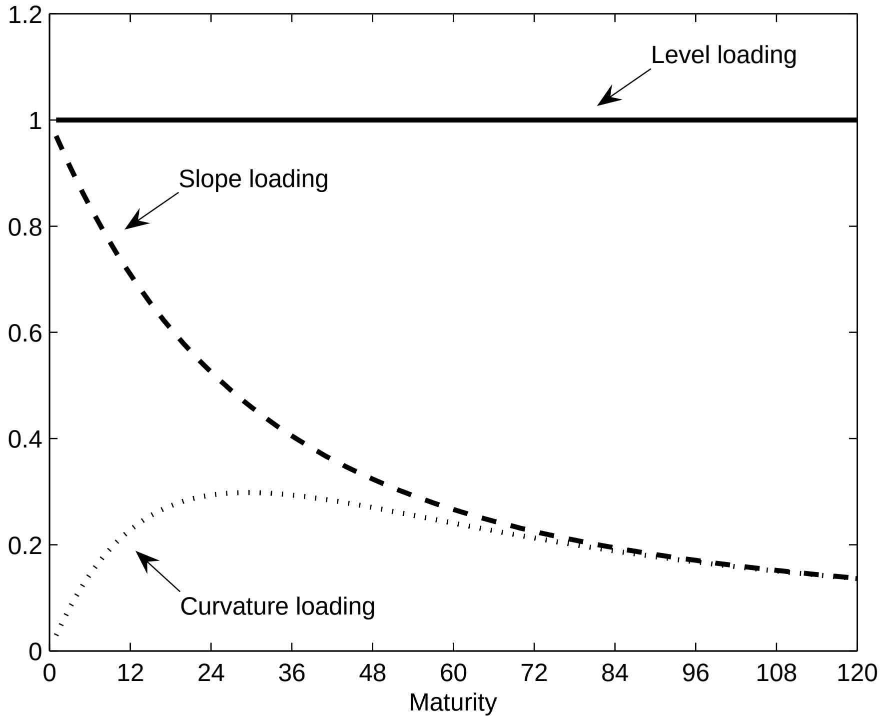
Summary statistics of residuals of the Nelson-Siegel and the no-arbitrage models. The Nelson-Siegel model is estimated according to equations (2) - (4). The no-arbitrage yield curve model is estimated according to equations (7) - (13). Statistics are shown for selected maturities, τ. RMSE is the root mean squared error and MAD is the mean absolute deviation. Autocorrelations are denoted by $\rho(p)$, where p is the lag.

Table 10: Out-of-sample performance

	1-m ahead		6-m ahead		12-m ahead	
τ	NS	NA	NS	NA	NS	NA
1	0.82	**0.67**	0.68	**0.56**	0.67	**0.59**
3	0.91	**0.89**	0.72	**0.70**	0.64	**0.62**
6	1.08	**1.07**	**0.80**	0.82	**0.65**	0.66
9	**1.06***	1.26	**0.79**	0.82	**0.63**	0.66
12	**1.01**	1.02	**0.79**	0.81	**0.63**	0.65
15	1.06	**1.00**	**0.78**	0.79	**0.63**	0.64
18	1.04	1.04	**0.79**	0.80	**0.64**	0.65
21	**1.06**	1.09	0.79	0.79	0.65	0.65
24	**1.09**	1.12	0.79	0.79	0.66	0.66
30	**1.04**	1.05	0.79	**0.77**	0.68	**0.67**
36	0.99	0.99	0.79	**0.77**	0.70	**0.68**
48	**0.98**	0.99	0.83	**0.80**	0.75	**0.73**
60	1.10	**1.04**	0.88	**0.85**	0.81	**0.78**
72	1.02	**1.01**	0.89	**0.88**	0.85	**0.83**
84	1.08	**1.07**	0.91	**0.90**	0.87	**0.86**
96	1.03	1.03	**0.92***	0.94	**0.91***	0.92
108	**1.04**	1.09	**0.94**	0.98	**0.93**	0.96
120	**1.08***	1.35	**1.01**	1.09	**1.00**	1.06

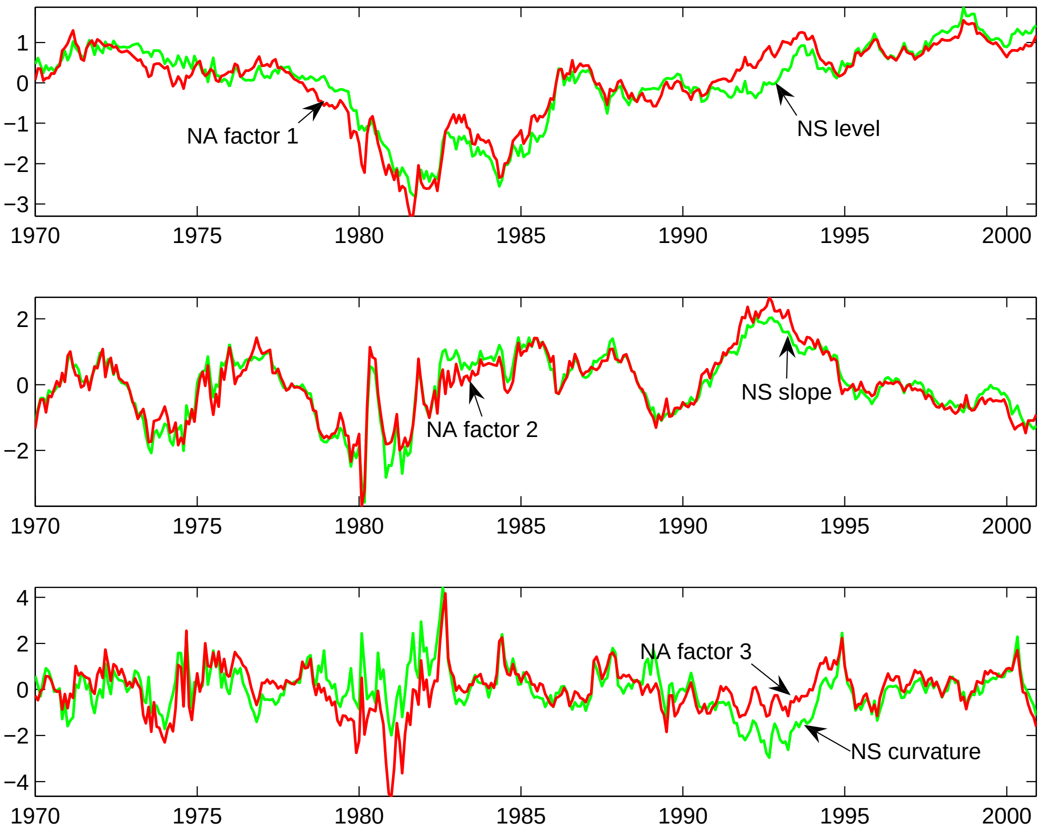
Ratios of the Mean Squared Forecast Error (MSFE) of the no-arbitrage model (NA) and the Nelson-Siegel model (NS) both measured against the performance of the random walk model. A ratio lower than 1 means that the MSFE for the respective model is lower than the forecast error generated by the random walk, and hence that the model performs better than the random walk model. The models are estimated on successively increasing data samples starting 1970:1 until the time the forecast is made, and expanded by one month each time a new set of forecasts are generated. Forecasts for horizons of 1, 6 and 12 months ahead are evaluated on the sample from 1994:1 to 2000:12. Bold entries in the table indicate superior performance of one model (NA or NS) against the other model. The star (*) indicates that the forecast is better at 5 % significance level according to the Diebold-Mariano test.

Figure 1: Nelson-Siegel factor loadings



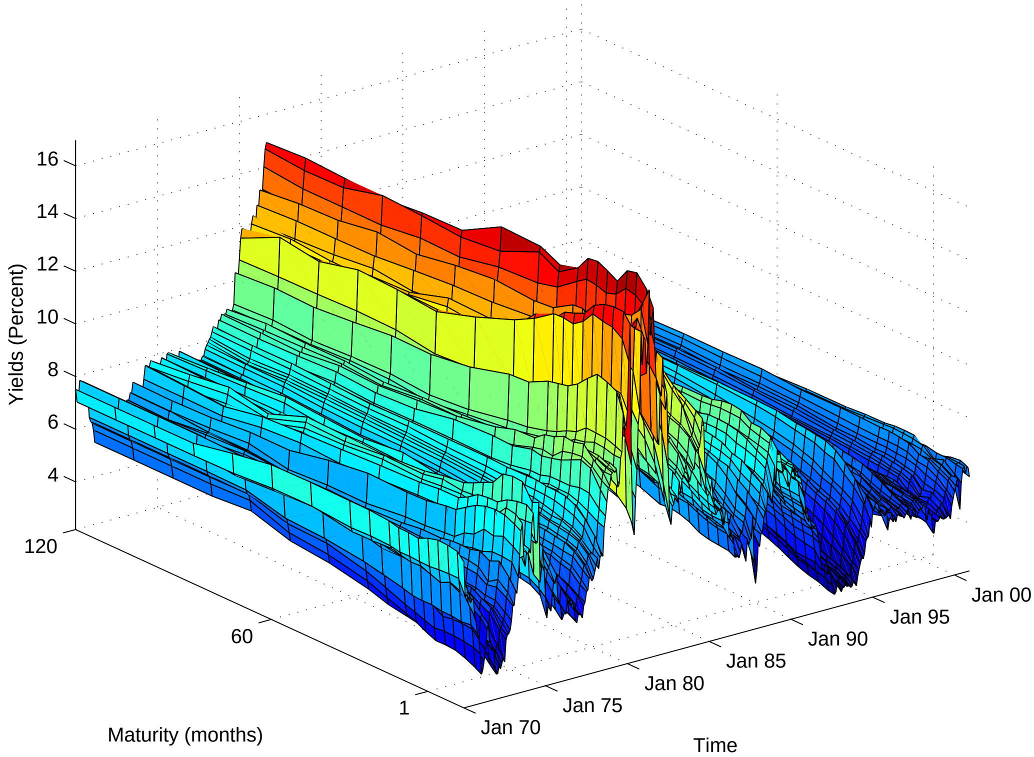
Nelson and Siegel (1987) factor loadings using the re-parameterized version of the model as presented by Diebold and Li (2006). The factor loadings b^{NS} are computed using $\lambda = 0.0609$ and equation (3).

Figure 2: No-Arbitrage Latent factors and Nelson and Siegel factors



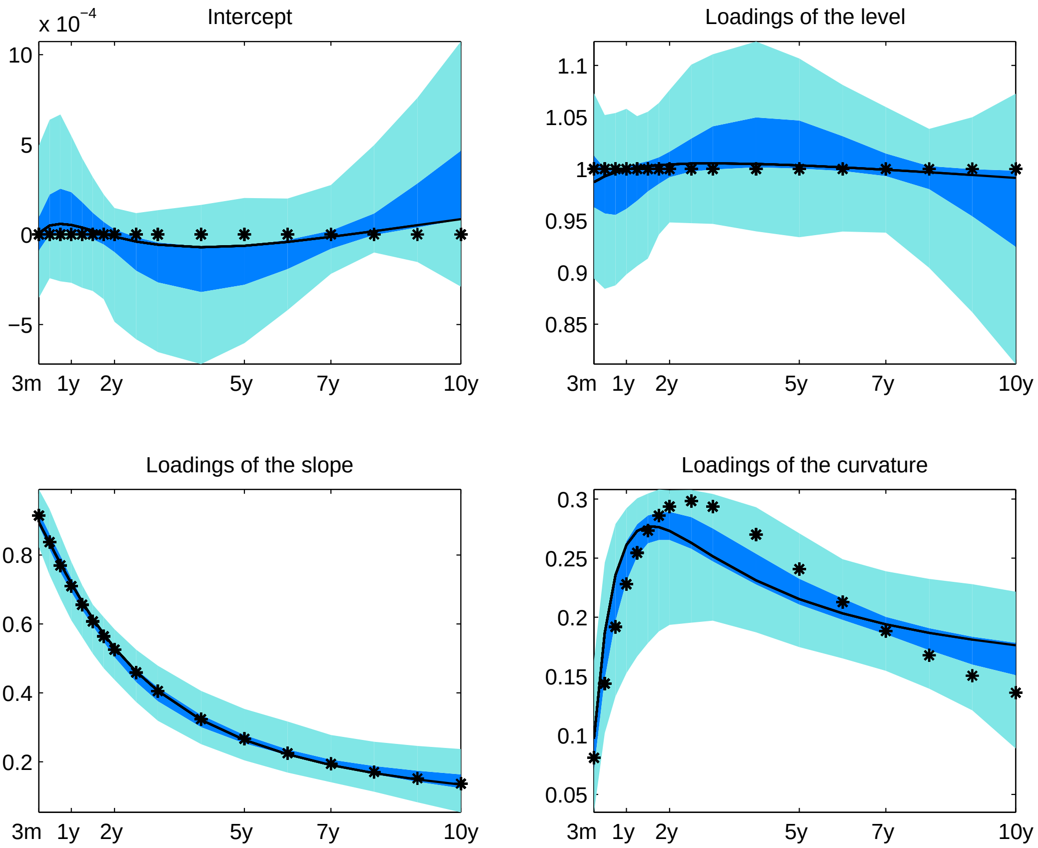
Extracted yield curve factors using US zero-coupon data observed at a monthly frequency and covering the period from 1970:1 to 2000:12. Factors are extracted from the Nelson-Siegel model and from the no-arbitrage model. "NS level" and "NA factor 1" refer to the first extracted factor from each model. The second and third extracted factors are correspondingly labeled "NS slope", "NA factor 2" and "NS curvature", "NA factor 3".

Figure 3: Zero-coupon yields data



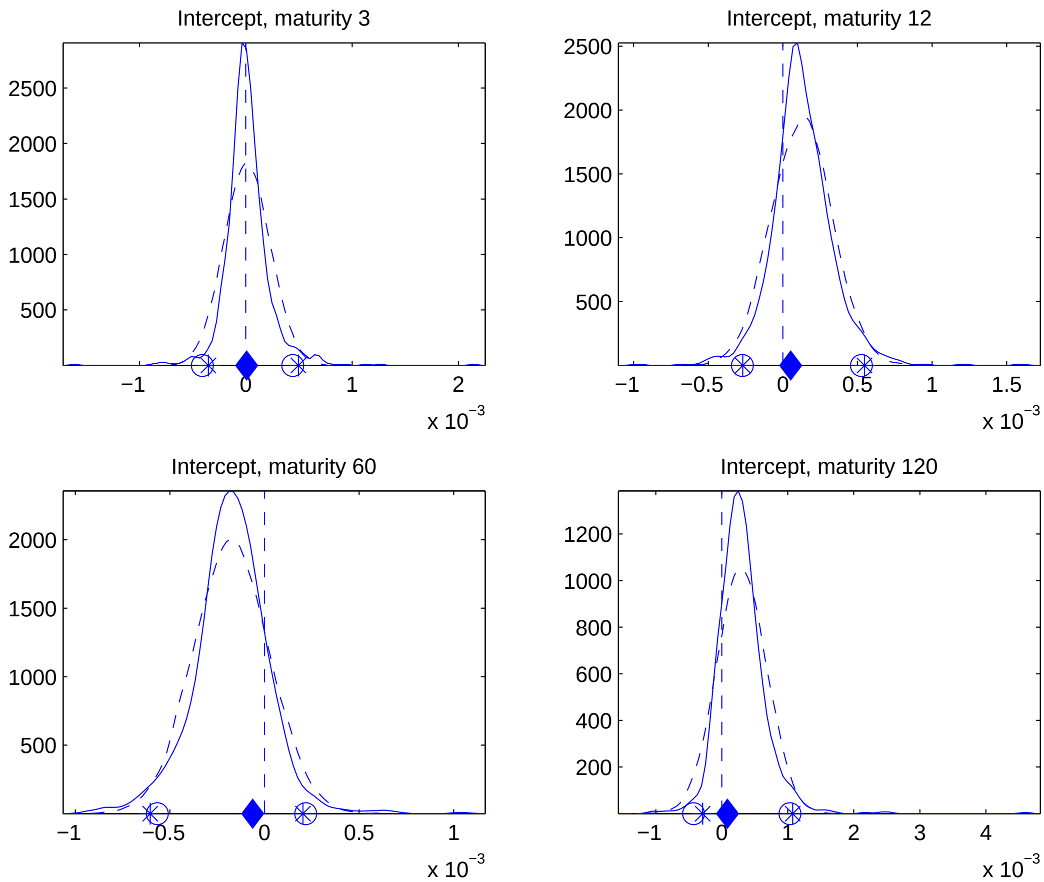
U.S. zero-coupon yield curve data observed at monthly frequency from 1970:1 to 2000:12 at maturities $1, 3, 6, 9, 12, 15, 18, 21, 24, 30, 36, 48, 60, 72, 84, 96, 108$ and 120 months.

Figure 4: No-Arbitrage loadings of the Nelson and Siegel factors



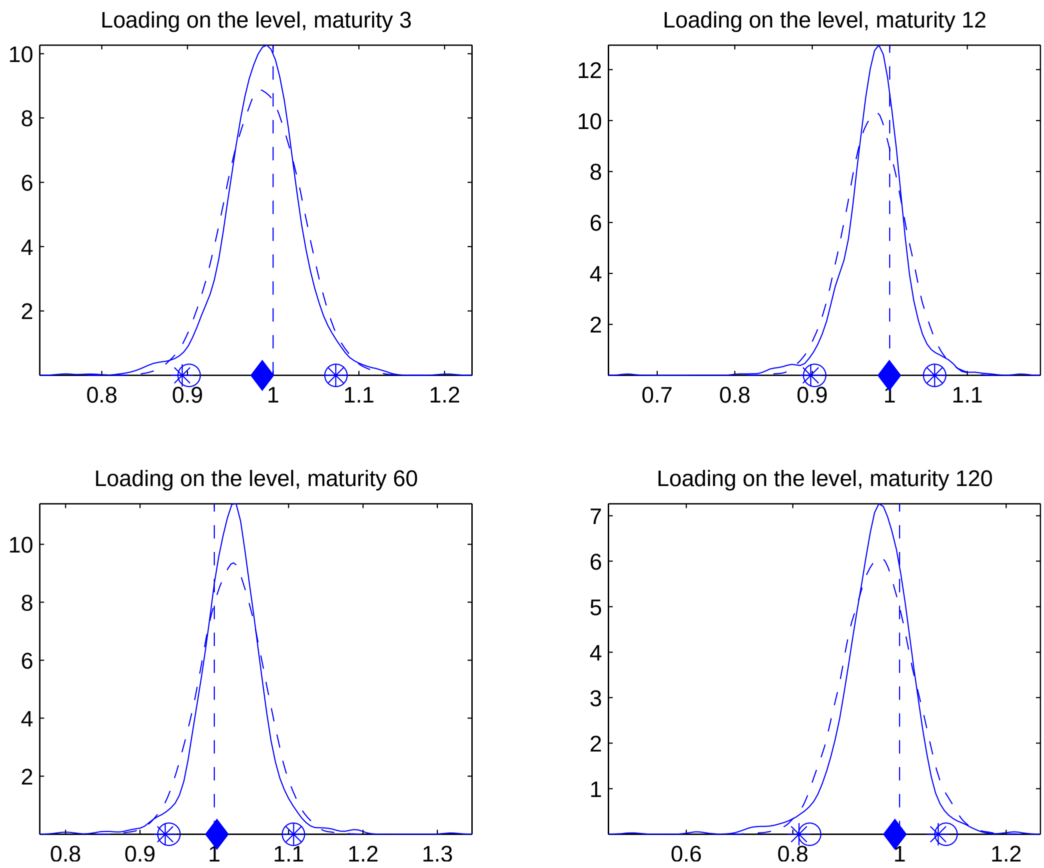
Estimated factor loadings and empirical 50 and 95 percent confidence intervals. Star * indicate the factor loadings from the Nelson-Siegel model, i.e. a^{NS} and b^{NS} in equations (2) and (3), while the continuous lines indicate the corresponding factor loadings estimated from the no-arbitrage model, i.e. $\widehat{a}^{NA}$ and $\widehat{b}^{NA}$ in equations (7) to (9). The distributions of the latter are obtained through resampling. The dark-shaded areas are the 50 percent confidence intervals, while the light-shaded areas show the 95 percent confidence intervals. These are computed as sample quantiles of the empirical distributions of the loadings obtained through resampling.

Figure 5: Distribution of the estimated loadings for a^{NA}



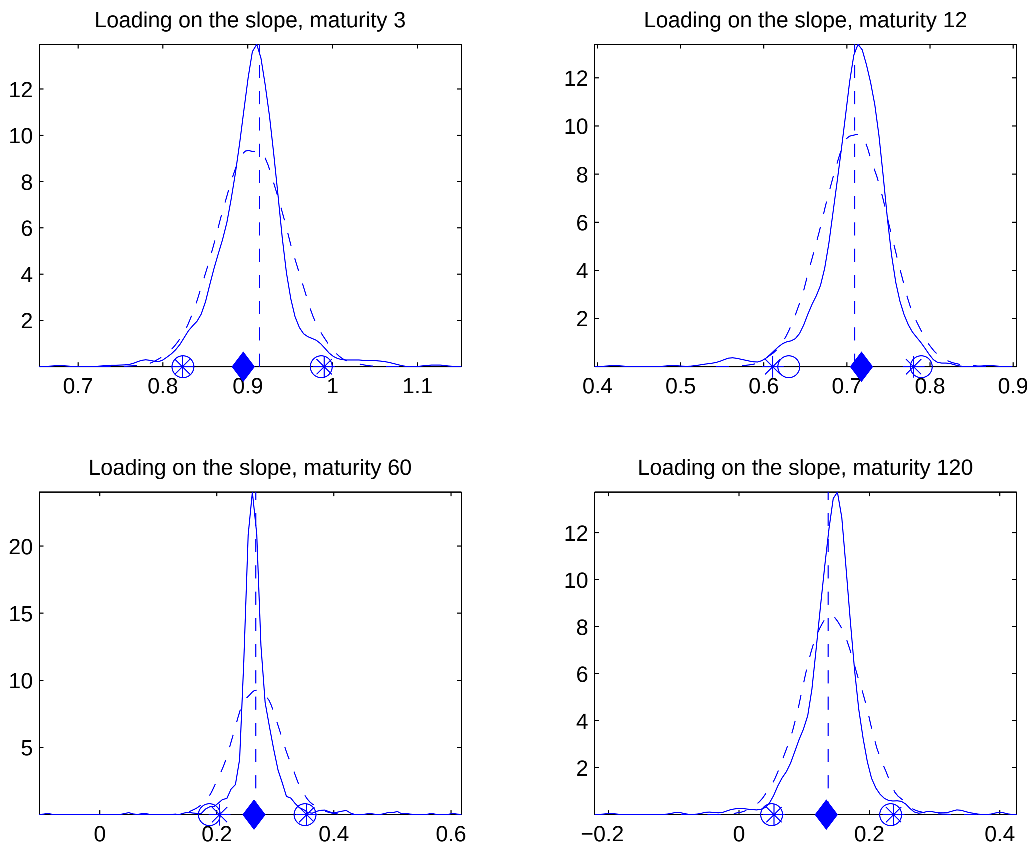
Empirical distributions, for selected maturities, of the no-arbitrage intercepts obtained from the resampling (continuous line), with the relative 95 percent confidence interval (asterisks) computed as sample quantiles. The dashed line is the Gaussian approximation with the relative 95 percent confidence intervals (circles). The diamonds are the estimated no-arbitrage intercepts and the dashed vertical line indicates the Nelson and Siegel intercepts.

Figure 6: Distribution of the estimated loadings for $b^{NA}(1)$



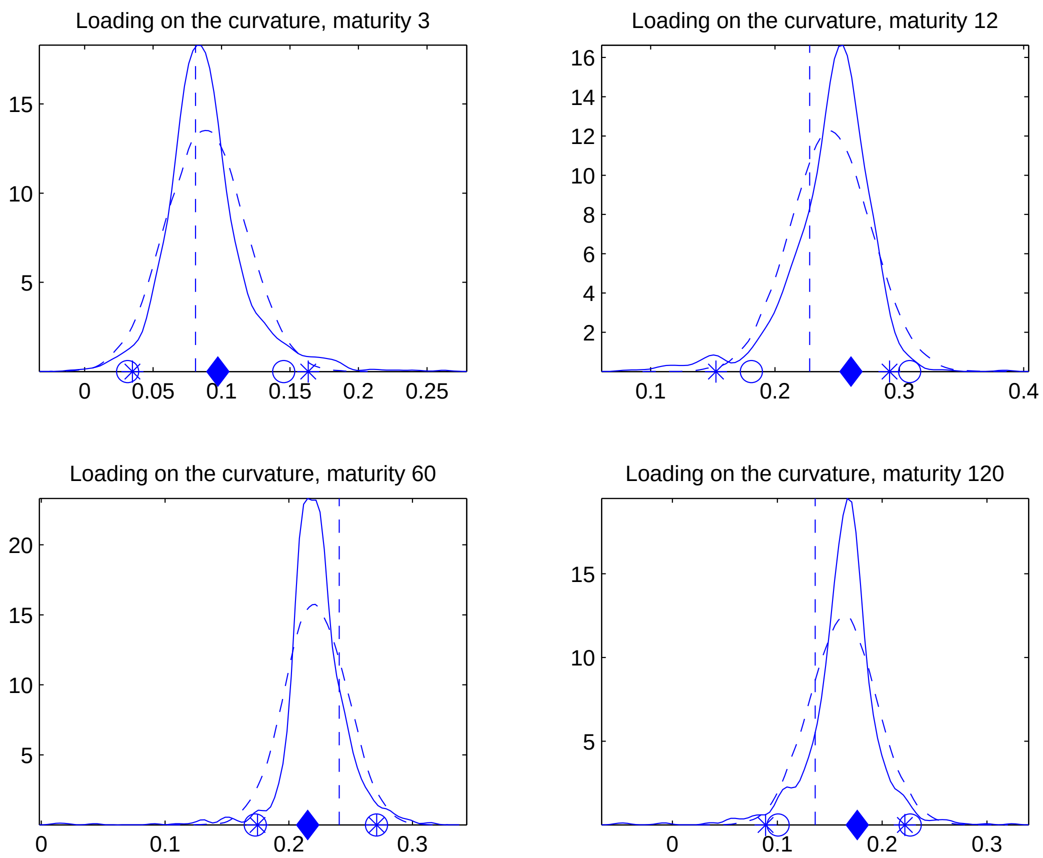
Empirical distributions, for selected maturities, of the no-arbitrage loadings of the level obtained from the re-sampling (continuous line), with the relative 95 percent confidence interval (asterisks) computed as sample quantiles. The dashed line is the Gaussian approximation with the relative 95 percent confidence intervals (circles). The diamonds are the estimated no-arbitrage loadings of the level and the dashed vertical line indicates the relative Nelson and Siegel loadings.

Figure 7: Distribution of the estimated loadings for $b^{NA}(2)$



Empirical distributions, for selected maturities, of the no-arbitrage loadings of the slope obtained from the re-sampling (continuous line), with the relative 95 percent confidence interval (asterisks) computed as sample quantiles. The dashed line is the Gaussian approximation with the relative 95 percent confidence intervals (circles). The diamonds are the estimated no-arbitrage loadings of the slope and the dashed vertical line indicates the relative Nelson and Siegel loadings.

Figure 8: Distribution of the estimated loadings for $b^{NA}(3)$



Empirical distributions, for selected maturities, of the no-arbitrage loadings of the curvature obtained from the re-sampling (continuous line), with the relative 95 percent confidence interval (asterisks) computed as sample quantiles. The dashed line is the Gaussian approximation with the relative 95 percent confidence intervals (circles). The diamonds are the estimated no-arbitrage loadings of the curvature and the dashed vertical line indicates the relative Nelson and Siegel loadings.

References

Ahn, Dong-Hyun, Robert F. Dittmar, and Ronald A. Gallant (2002) 'Quadratic Term Structure Models: Theory and Evidence.' *Review of Financial Studies* 15(1), 243–288

Ang, A., and M. Piazzesi (2003) 'A No-Arbitrage Vector Autoregression of Term Structure Dynamics with Macroeconomic and Latent Variables.' *Journal of Monetary Economics* 50(4), 745–787

Ang, A., G. Bekaert, and M. Wei (2007) 'The term structure of real rates and expected inflation.' *Journal of Finance, forthcoming*

Ang, A., M. Piazzesi, and M. Wei (2006) 'What Does the Yield Curve Tell us about GDP Growth.' *Journal of Econometrics* 131, 359–403

Bansal, R., and H. Zhou (2002) 'Term Structure of Interest Rates with Regime Shifts.' *Journal of Finance* 57, 1997–2043

Bernadell, C., J. Coche, and K. Nyholm (2005) 'Yield Curve Prediction for the Strategic Investor.' ECB Working Paper 472

BIS (2005) *Zero-Coupon Yield Curves: Technical Documentation* (Bank for International Settlements, Basle)

Bjork, T., and B.J. Christensen (1999) 'Interest Rate Dynamics and Consistent Forward Rate Curves.' *Mathematical Finance* 9, 323–348

Black, F., and P. Karasinski (1993) 'Bond and option pricing when the short rates are log-normal.' *Financial Analyst Journal* 47, 52–59

Black, F., E. Derman, and E. Toy (1990) 'A one-factor model of interest rates and its application to treasury bond options.' *Financial Analyst Journal* 46, 33–39

Box, G.E.P., and G.M. Jenkins (1976) *Time Series Analysis: Forecasting and Control* (San Francisco: Holden Day)

Campbell, J.Y., and R.J. Shiller (1991) 'Yield spreads and interest rate movements: A birds eye view.' *Review of Economic Studies* 58, 495–514

Chen, R.R., and L. Scott (1993) 'Maximum likelihood estimation for a multifactor equilibrium model of the term structure of interest rates.' *Journal of Fixed Income* 3, 14–31

Christensen, Jens, Francis Diebold, and Glenn Rudebusch (2007) 'The Affine Arbitrage-Free Class of Nelson-Siegel Term Structure Models.' FRB of San Francisco Working Paper No. 2007-20

Cox, J.C., J.E. Ingersoll, and S.A. Ross (1985) 'A theory of the term structure of interest rates.' *Econometrica* 53, 385–407

Dai, Q., and K. Singleton (2000) 'Specification analysis of affine term structure models.' *Journal of Finance* 55, 1943–1978

Davidson, R., and J.G. MacKinnon (2006) 'Bootstrap Methods in Econometrics.' In *Palgrave Handbooks of Econometrics: Vol.1 Econometric Theory,* ed. T. C. Mills and K. Patterson (Palgrave)

De Pooter, M., F. Ravazzolo, and D. van Dijk (2007) 'Predicting the Term Structure of Interest Rates.' Working Paper

Diebold, F. X., and C. Li (2006) 'Forecasting the term structure of government bond yields.' *Journal of Econometrics* 130, 337–364

Diebold, F.X., C Li, and V. Yue (2007) 'Global Yield Curve Dynamics and Interactions: A Generalized Nelson-Siegel Approach.' *Journal of Econometrics, forthcoming*

Diebold, F.X., G. D. Rudebusch, and S. B. Aruoba (2006a) 'The macroeconomy and the yield curve: A dynamic latent factor approach.' *Journal of Econometrics* 131, 309–338

Diebold, F.X., L Ji, and C. Li (2006b) 'A Three-Factor Yield Curve Model: Non-Affine Structure, Systematic Risk Sources, and Generalized Duration.' In *Macroeconomics, Finance and Econometrics: Essays in Memory of Albert Ando,* ed. L. R. Klein (Cheltenham, U.K.: Edward Elgar) pp. 240–274

Diebold, F.X., M. Piazzesi, and G. D. Rudebusch (2005) 'Modeling Bond Yields in Finance and Macroeconomics.' *American Economic Review* 95, 415–420

Duffee, G.R. (2002) 'Term premia and interest rate forecasts in affine models.' *Journal of Finance* 57, 405–443

—— (2006) 'Term Structure Estimation without Using Latent Factors.' *Journal of Financial Economics* 79, 507–536

Duffie, D., and R. Kan (1996) 'A yield-factor model of interest rates.' *Mathematical Finance* 6, 379–406

Efron, B. (1979) 'Bootstrap methods: Another look at the jackknife.' *Annals of Statistics* 7, 1–26

Litterman, R., and J. Scheinkman (1991) 'Common factors affecting bond returns.' *Journal of Fixed Income* 47, 129–1282

Merton, R. C. (1973) 'Theory of rational option pricing.' *Bell Journal of Economics and Management Science* 4, 141–183

Nelson, C.R., and A.F. Siegel (1987) 'Parsimonious modeling of yield curves.' *Journal of Business* 60, 473–89

Rebonato, R., S. Mahal, M.S. Joshi, L. Bucholz, and K. Nyholm (2005) 'Evolving Yield Curves in the Real-World Measures: A Semi-Parametric Approach.' *Journal of Risk* 7, 29–62

Soderlind, P., and L.O.E. Svensson (1997) 'New Techniques to Extract Market Expectations from Financial Instruments.' *Journal of Monetary Economics* 40, 383–429

Vasicek, O. (1977) 'An equilibrium characterization of the term structure.' *Journal of Financial Economics* 5, 177–188