Coexistence of Money and Higher-Return Assets—Again*

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1 Introduction

Every applied model designed to study monetary policy contains non interest-bearing money and higher-return assets. The higher-return assets may be capital, titles to capital, or government bonds. Therefore, in some way or other, such models confront the issue that Hicks [4] in 1935 said was the main challenge facing monetary theory: why do people hold money in the presence of higher-return assets? My main purpose is to describe a new theory of coexistence of money and higher-return assets. However, before doing that I will describe a new critique of what seems to be the main existing theory of coexistence—cash-in-advance models. That critique will, I hope, make you more open to consideration of a new theory.

2 A Simple and Stark Setting

Both the critique and the new theory can be exposited against the background of a simple and familiar setting: a pure-exchange, discrete-time world of infinitely-lived people who differ only in their periodic endowment streams.

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There is one perishable and divisible good per date. There are two, unit intervals of people, groups 1 and 2. Each member of group 1 has the time stream of endowments $(y, 0, y, 0, ...)$, while each member of group 2 has the stream $(0, y, 0, y, ...)$, where $y > 0$. Everyone has the same preferences represented by expected discounted utility with discount factor $\beta \in (0, 1)$ and period utility function $u : \mathbb{R}_+ \to \mathbb{R}$. The function u is strictly increasing, strictly concave, and differentiable, and $u'(0)$ is sufficiently high. I refer to this world as the alternating-endowment model.

In order to focus on the coexistence question involving outside assets, I assume throughout that there are informational restrictions that rule out private credit—private borrowing and lending. One restriction that suffices is that current actions are not remembered in the future. That implies that those who surrender the good voluntarily must receive outside assets in exchange.

There are two outside assets: money and one-period discount government bonds. The sequence of actions at a date is as follows. Each person begins a date with some money. Then the person can exchange money for bonds—each bond being a title to a unit of money at the end of the period—at an exogenous price $p \in (0, 1)$ in terms of money. (Imagine that there are vending machines maintained by the government which offer such bonds in exchange for money.) Then, *after* portfolios are chosen, there is trade involving assets and the good. Then, the bonds mature—automatically turn into money. The implied interest payments are financed by money creation (inflation).

For this world, I will say that the coexistence challenge is met if people retain some money after choosing portfolios. Put somewhat differently, the coexistence challenge is to model trade of the good for assets so that at least some people have an incentive to retain some money.

Although the coexistence challenge is severe in the above setting because the bonds so obviously dominate money in rate of return, there are at least two reasons to attempt to deal with it. First, and most important, almost every model used to study monetary policy has such dominance even if the assets are not titles to money. Second, assets that resemble the bonds in the above setting have at times appeared in actual economies. For example, during the First World War, both the U.S. and France issued small denomination, payable-to-the-bearer, small denomination bonds and in neither instance did the bonds drive out money (for the French experience, see Makinen and Woodward [9]).

3 Trading Posts and Cash-in-advance

So far as I know, the only rationale for cash-in-advance models when there are multiple assets is a Shapley-Shubik trading-post model and an equilibrium in that model with no activity at the posts at which assets other than money can be traded for goods. In a static Cournot-type quantity game for the trading-post model, inactivity of any given post is a Nash equilibrium because a single agent has no incentive to place quantity orders on an inactive post. This has been taken to be a rationale for assuming that people do not have an opportunity to trade securities for goods.

However, the fact that inactivity of any given post is a Nash equilibrium also implies that no trade at all is a Nash equilibrium. In part to eliminate such equilibria, Dubey and Shubik [1], in a static quantity-game version of the trading-post model, introduce a refinement which eliminates no trade: they assume that there are small exogenous offers (given from the outside) at each post and say that an equilibrium satisfies the refinement if it is a limit as those exogenous offers approach zero. Krishna [7] applies a version of that refinement in the above bond-money setting and gets results that I will now describe.

To put the above money-bond setting into the trading-post model, it is sufficient to consider two potentially active trading posts: one at which the good is traded for money (the money post) and one at which the good is traded for bonds (the bond post). People simultaneously choose non negative quantities of assets and of the good to offer at both posts. They make those choices after money-bond portfolios are chosen and subject to the constraints that offers cannot exceed pre-offer holdings. Because analyzing the Cournot quantity game is difficult in an inter-temporal setting, Krishna assumes that people face prices (at each post) and behave as price takers (see also Hayashi and Matsui [3]).

Krishna adopts the following version of the Dubey-Shubik refinement: there is an exogenous amount of the good, $\epsilon > 0$, given from outside, at each of the two posts at each date. Let p_{tm} be the date-t price of the good at the money post and let p_{tb} be that price at the bond post, where p_{tm} is in units of money per unit of the good and p_{tb} is in units of bonds at face value per unit of the good. In addition to the condition that each person choose money-bond portfolios and quantities to offer at the posts optimally subject to the implied sequence of price-taking budget constraints, the $\epsilon-$equilibrium conditions are

$$p_{tm} = \frac{M_t}{Y_{tm} + \epsilon} \text{ and } p_{tb} = \frac{B_t}{Y_{tb} + \epsilon}, \tag{1}$$

where M_t is the total offer of money at the money post, B_t is the total offer of bonds at the bond post, and Y_{tm} and Y_{tb} are the total offers of goods at each post, respectively. An $\epsilon = 0$ equilibrium satisfies the refinement if it is a limit of ϵ-equilibria as $\epsilon \to 0$.[1]

If $\epsilon = 0$ and if it is assumed that the bond post is inactive—meaning that people do not face a price at which they can sell bonds for the good or vice versa—then this model becomes a cash-in-advance model. And, as is well-known, it has equilibria that satisfy coexistence. For example, if $p \in (\beta, 1)$ and if initially all money is held in equal amounts by the people with a 0 endowment of the good, then there an equilibrium in which no one buys bonds, in which all the money is offered at each date, in which each person with endowment y consumes c_y and each person with endowment 0 consumes c_0 where these are the unique solution to $c_y + c_0 = y$ and $u'(c_y) = \beta u'(c_0)$, and in which $Y_{tm} = c_0$.

Krishna shows that this equilibrium does not satisfy the refinement. He shows that if $p < 1$ and $\epsilon > 0$, then there is no equilibrium with $Y_{tm} > 0$. The proof is a simple arbitrage argument, one which is consistent with the short-sales constraints of the trading-post model and one which makes no appeal to the special assumptions of the alternating endowment model. Suppose to the contrary that there is an equilibrium with $Y_{tm} > 0$. If so, then $p_{tm} \geq p_{tb}$ and $p_{tm} > 0$. The latter implies $M_t > 0$. But the former implies that any person whose offer contributes to making $M_t > 0$ would do better by using that money to buy bonds and spending the bonds on the bond post. Hence, there is no such equilibrium.

Krishna also shows that no such argument applies to activity at the bond post. For the alternating-endowment model, he shows that there is a stationary equilibrium with activity at the bond post that satisfies the refinement. With interest financed by inflation, that equilibrium has the same allocation as the cash-in-advance equilibrium described above.

The Dubey-Shubik refinement is not the only way to test the robustness of shutting down trade of bonds for goods. Howitt [5] considers models in which people in the model face costs of operating posts and can set up trading posts with bid and ask prices at each post. Somewhat surprisingly, no one has

[1] For the results to be stated, pointwise convergence suffices.

used such a model in a money-bond setting like that described here. With operating costs that are similar for money posts and bond posts, I doubt that there are sensible equilibria with activity at money posts. Another possibility is to study non-simultaneous play; it is well-known that the no-trade equilibrium in the Shapley-Shubik model is sensitive to the assumption of simultaneous play by all the agents.

I conclude from the above that competitive trade and coexistence are difficult to reconcile. The new theory I will describe departs from competitive trade by assuming, as a feature of the environment, that trade of goods for assets is pairwise between people—not pairwise in objects as in the trading-post model.

4 Cash-in-advance with a Twist

The new theory is due to Tao Zhu and me (see [14]). Although, as I will point out later, it can be applied to many different settings, I begin, as promised, by describing it for the alternating-endowment model. To get trade that is pairwise between people, I here assume that after portfolios are chosen, each person with a 0 endowment, a buyer, exogenously meets a randomly drawn person with endowment y, a seller. That is, each person is in one single-coincidence meeting at each date.[2] Zhu and I assume that there is no asymmetric information in the meeting regarding portfolios. The theory makes a *selection* from the *pairwise core* implied by the portfolios of the buyer and seller who meet.

To describe the theory, I begin by doing partial equilibrium in the sense of taking as given the way the buyer and seller value post-trade holdings of assets. After describing the partial equilibrium, I will say a bit about what we know about general equilibrium determination of those valuations and of portfolios.

Let us denote the pre-trade portfolio of the buyer by the pair (m_b, b_b) and that of the seller by (m_s, b_s), where, in each case, the first component is money holdings and the second is bond holdings at face value. Let (m'_b, b'_b) and (m'_s, b'_s) denote the respective post-trade portfolios. I assume that the buyer's payoff is given by $u(c_b)+v_b(m'_b+b'_b)$ and that of the seller by $u(c_s)+v_s(m'_s+b'_s)$, where v_b and v_s are strictly increasing and concave. Only the total value of

[2]A version with random meetings in which a person meets anyone at random would have similar consequences.

the portfolio appears in these payoffs because the bonds turn into money. The pairwise core can be defined in terms of the following problem.

Problem 1 *For given $L \geq 0$, choose $q \in [0, y]$ and nonnegative (m'_b, b'_b) and (m'_s, b'_s) to maximize $u(q) + v_b(m'_b + b'_b)$ subject to*

$$u(y - q) + v_s(m'_s + b'_s) \geq u(y) + v_s(m_s + b_s) + L, \tag{2}$$

and

$$(m'_b + b'_b) + (m'_s + b'_s) \leq (m_b + b_b) + (m_s + b_s). \tag{3}$$

Notice that this problem can be stated entirely in terms of pre- and post-trade asset totals. In other words, the composition of a given wealth total between money and bonds at face value is not payoff relevant. That suggests that we have to do something special—some might say, weird—to get coexistence. After all, at the portfolio stage, both people see that feasible asset holdings are larger the more bonds they buy.

As is standard, there are many core allocations, even in terms of q and asset totals. They range from giving all the gains from trade to the buyer, $L = 0$ in (2), to giving all the gains to the seller. This feature gives us scope to make a selection from the pairwise core that depends on the compositions of the portfolios brought into the meeting—despite the fact that those compositions are not payoff relevant. We do that in such a way as to give potential buyers an incentive to enter the meeting with some money. Although our selection is not the result of bargaining, it is equivalent to making the buyer's bargaining power a function of m_b and to assuming that as much of the trade as possible is accomplished through the transfer of money.

Our selection is the solution to the following two-step maximization problem. We call it *cash-in-advance with a twist* because step 1 has a cash-in-advance constraint and step 2 does not.

Problem 2 *Step 1 : Choose (q_1, x_1) to maximize $u(q_1) + v_b(m_b + b_b - x_1)$ subject to $x_1 \leq m_b$ and*

$$u(y - q_1) + v_s(m_s + b_s + x_1) \geq u(y) + v_s(m_s + b_s). \tag{4}$$

Step 2 : Choose (q, x) to maximize $u(y - q) + v_s(m_s + b_s + x)$ subject to

$$u(q) + v_b(m_b + b_b - x) \geq u(q_1) + v_b(m_b + b_b - x_1). \tag{5}$$

Notice that the step 1 choice determines the lower bound on the buyer's utility in step 2. The crucial properties of the solution to Problem 2 are as follows.

Proposition 1 *The solution to Problem* 2 *is in the pairwise core. Moreover, the solution satisfies* $(q - q_1, x - x_1) \geq (0, 0)$ *and is positive if and only if the constraint* $x_1 \leq m_b$ *is binding.*

The proof appears in [14]. The solution to problem 2 can be depicted in an Edgeworth Box diagram (see figure 1). Point A is the initial endowment. If the cash-in-advance constraint is not binding in step 1, then the outcome of that step is point B and step 2 is null. Otherwise, the outcome of step 1 is a point like C and the final outcome is point D.

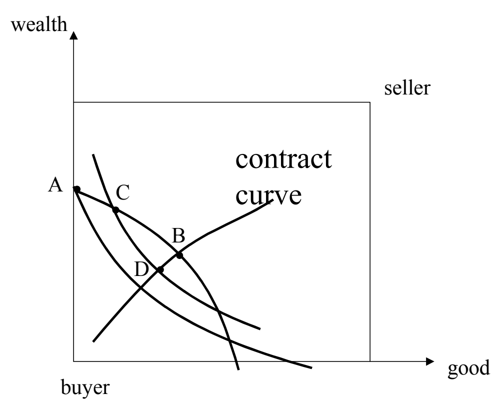

Figure 1: Alternative problem 2 solutions.

According to our theory, when people face the exogenous price p at which they can buy bonds, they look forward to trading according to the outcome of problem 2. In the alternating-endowment model, those with endowment y know they are sellers and, therefore, use all their money to buy bonds. Those who have endowment 0 know they are buyers. They face a trade-off. If they enter trade with no money, then they know they will get none of the gains from trade in the meeting. However, they are sacrificing wealth by holding on to money.

That trade-off suggests that the model will give coexistence. However, that can be formally verified only as part of a general equilibrium. Therefore, I now turn to a brief discussion of what I know about general equilibrium.

4.1 Existence of a steady state

Zhu and I can prove existence of a steady state for a version of the model with a general, but finite allowable set of individual asset holdings and for p, the discount on bonds, in the neighborhood of unity. The proof depends on an existence result for a money-only ($p = 1$) version of the model, an existence result that applies the methods in Zhu [13]. However, because a somewhat different background model is sued in [13], I will set out the claims as conjectures—even though they are well-founded claims.

Before setting out those conjectures, a few comments are in order. First, rather than finance interest payments by money creation literally, it is convenient to adopt an equivalent scheme: a proportional tax on end-of-period nominal wealth. That scheme allows us to hold the total amount of money constant. Let M denote that total at the start of a date, before bond purchases. We denote the set of possible individual money holdings at that time by $\mathbf{Z} = \{0, 1, 2, ..., Z\}$. The only assumptions we make about M and $\mathbf{Z}$ are that M is sufficiently large (which assures that the indivisibility of assets is not too severe) and that Z/M is sufficiently large (which assures that the measure of people at the upper bound is small). We also bound portfolios of money and bonds (measured at face value); if (m, b) denotes a person's portfolio of money and bonds, then we assume $m + b \leq Z$. In addition, because of the bound and the indivisibility of assets, we use versions of problems 1 and 2 with bounds and with lotteries. We also randomize whenever that is necessary, as it is when we tax nominal wealth to finance interest payments.

A steady state for this model can be defined as a pair of measures, π_y and π_0, and a pair of value functions, w_y and w_0, all of which pertain to the start of a period before bonds are purchased. All have domain $\mathbf{Z}$. The π's describe the distributions of money holdings for each group and the w's describe expected discounted utilities for each group. The equilibrium conditions are standard and include the condition that the tax levied on nominal wealth finances the interest payments implied by the bond purchases.

The first conjecture describes a money-only steady state.

Conjecture 1 *If $p = 1$, then there exists a steady state with strictly increas-*

ing and strictly concave value functions, w_y and w_0, and with full-support distributions, π_y and π_0.

With $p = 1$, this is a money-only model with take-it-or-leave-it offers by buyers. Because there is no exogenous source of uncertainty, it may seem surprising that there is a steady state with the heterogeneity of money holdings implied by the full-support property. There is, in fact, another steady state with degenerate distributions. However, that steady state has step-function value functions. As a consequence, it is not a limit of commodity steady states as the commodity value goes to zero (see Wallace and Zhu [12]). In contrast, the full-support steady state with its concave value functions can be shown to be such a limit. That is one reason to focus on full-support steady states.

Here is the existence claim.

Conjecture 2 *If p is sufficiently close to* 1, *then there exists a steady state with strictly increasing and strictly value functions, w_y and w_0, and with full-support distributions, π_y and π_0.*

And, finally, we have the coexistence claim.

Conjecture 3 *In any conjecture-2 steady state, there is a positive measure of people who enter the trade-for-goods stage of the model with money.*

The argument for coexistence does not use the bound on individual holdings and does not depend on p being near 1. Instead, it uses boundedness of the value functions and the full-support property to show that sufficiently rich buyers retain some money.

4.2 The selection as *an* equilibrium

Our defense of our theory is that it produces outcomes in the pairwise core. Alternatively, we can say that the conjecture-2 equilibrium outcome is weakly implementable and not subject to renegotiation by buyers and sellers in meetings.[3]

[3] By being weakly implementable, we mean that we can devise a game, consistent with trade being voluntary, which has a sub-game perfect equilibrium whose outcome is the general equilibrium of conjecture 2. However, the game either has a very limited strategy

However, the same can be said of many other outcomes. There are outcomes that do not distinguish between money and bonds. And there are outcomes that reverse the roles of bonds and money. There are even variants of problem 2 consistent with coexistence. We could assign some small positive gain from trade to the seller in step 1 and some small gain to the buyer in step 2. If those are small enough, then they should be consistent with coexistence. We adopt problem 2 rather than one of the variants of it because our existence argument depends on the money-only ($p = 1$) version being step 1 of that problem.

Zhu and I are not bothered by such multiplicity. In our view, multiplicity is inherent in the simple model we have described. People could coordinate on using the bonds and money as perfect substitutes, and, if they do, then the bonds drive out money. Multiplicity has a long history in monetary economics. In the preface to *Money and the Mechanism of Exchange*, Jevons listed what he said were "currency questions which press for solution:"

> Shall we count in pounds, or dollars, or francs, or marks? Shall we have gold or silver, or gold *and* silver, as the measure of value? Shall we employ a paper currency or a metallic one? (page viii).

Our theory says that against the background of pairwise trade, there is a multiplicity that includes coexistence of money and higher-return assets.

That multiplicity depends on pairwise trade or at least on small-group trade. One way to see that is to let the size of a meeting grow by replicating the buyer-seller pair in our meetings. If that is done and if we continue to require that outcomes be in the meeting-specific core, then coexistence necessarily disappears as the meeting size grows because the core converges to a competitive equilibrium. In our setting with three objects in each meeting—the good, money, and bonds—the way that bonds and money appear in payoffs implies that any competitive equilibrium has money and bonds trading one for one. And that, of course, is not consistent with coexistence.

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space–for example, say "yes" or "no" to the problem 2 outcome with "no" by either person implying no trade in the meeting—or has a rich strategy space but is a pure coordination game. Either way, the problem-2 trade would have to be a candidate outcome that is *on the table*.

4.3 Generality of the theory

Our theory has wide applicability. It obviously can be used to get imperfect substitutability between the monies of different countries—even under fixed exchange rates.[4] To adapt it to that purpose, let $p = 1$ and assume that the "bonds" do not automatically turn into money. Instead, let them have a distinct and permanent color so that they are a distinct money. Also, let the vending machines maintain a fixed exchange rate between the two monies by allowing people to buy or sell one money for the other. If in problem 2, we assume that the favored money is the seller's "home" money, then buyers who anticipate meeting a foreign seller will want to acquire foreign money.

The model can also be used with the exogenous price p replaced by competitive trade in assets. Indeed, a conjecture-2 steady state can be reinterpreted as a steady state in which the quantity of purchased bonds is exogenously supplied by the government to a competitive market in money and bonds. That reinterpretation suggests that the model can be applied to other kinds of assets, including real assets. However, it is important that any such competitive trade among assets be separated from trade of assets for goods. If not, then the pairwise or small-group aspect of trade involving assets and goods is lost.

4.4 Heterogeneity and tractability

There seems to be an intimate connection between pairwise meetings and extreme heterogeneity in the sense of full-support distributions. As noted above, the sensible steady state in the alternating-endowment model is one with full-support distributions even though the model has no exogenous source of uncertainty. Although there are models with pairwise meetings which avoid the heterogeneity, they make use of special assumptions (see, for example, [2], [8], and [11]).

Full-support heterogeneity comes at a price. It precludes the construction of simple examples and it makes computing equilibria somewhat difficult. However, because heterogeneous-agent models are being widely used in other areas of macroeconomics, I see no reason why they cannot be used in the study of monetary policy.

[4] A two-money precurser of the theory appears in [10]. There, however, portfolios are so special that either step 1 or step 2 of problem 2 is necessarily null.

5 Concluding remarks

I have posed the coexistence challenge in as stark a form as is imaginable. The standard applied model, a cash-in-advance model, can meet the coexistence challenge in that form. However, as I have suggested, it shuts down markets in a way that is not robust to various ways of modeling whether markets are active. Zhu and I [14] provide an alternative way to achieve coexistence. The alternative depends on trade occurring in two-person meetings. For such trade, it selects from the implied pairwise core so that what happens in a meeting depends on the compositions of the portfolios brought into the meeting—despite the fact that those compositions are payoff-irrelevant. If we, instead, insist that what happens should depend only on payoff-relevant aspects of the meeting, then we must depart from the stark way I have posed the coexistence challenge. Departures would somehow have to reinterpret the coexistence challenge so that the payoffs on other assets do not dominate money in so obvious a way.

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