

# Dynamic Debt Runs\*

Zhiguo He<sup>†</sup>      Wei Xiong<sup>‡</sup>

August 6, 2010

## Abstract

Firms commonly spread out their debt expirations across time to reduce the liquidity risk generated by large quantities of debt expiring at the same time. By doing so, they introduce a dynamic coordination problem. In deciding whether to rollover his debt, each maturing creditor is concerned about the rollover decisions of other creditors whose debt matures during his next contract period. We develop a model with a time-varying firm fundamental and a staggered debt structure to analyze this problem. We derive a unique threshold equilibrium with fear of a firm's future rollover risk driving preemptive runs. Our model characterizes fundamental volatility, asset illiquidity, reliability of credit lines, and debt maturity as determinants of such dynamic runs.

JEL classification: G01, G20

Keywords: Rollover Risk, Debt Crisis, Dynamic Coordination

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\*We thank Viral Acharya, Max Bruche, Markus Brunnermeier, Peter DeMarzo, Doug Diamond, Itay Goldstein, Milton Harris, Christopher House, Patrick Kehoe, Thorsten Koepl, Arvind Krishnamurthy, Pete Kyle, Guido Lorenzoni, Stephen Morris, Jonathan Parker, Darius Palia, Adriano Rampini, Michael Roberts, Andrew Robinson, David Romer, Alp Simsek, Hyun Song Shin, Andrei Shleifer, Amir Sufi, Bilge Yilmaz, and seminar participants at 2010 AEA/AFA Meetings, 2009 Bank of Canada Workshop on the Financial System, 2010 Bank of Portugal Conference on Monetary Economics, Chicago, Columbia, Cornell, DePaul, 2009 FESAMES, Federal Reserve Board, Harvard/MIT, HKMA/BIS, IMF, Kellogg, 2009 Mitsui Finance Symposium at Michigan, 2009 NBER Summer Institute workshops in Monetary Economics and in Capital Markets and the Economy, New York Fed, Philadelphia Fed, Maryland, Minnesota, Notre Dame, Princeton, Texas-Austin, Wharton, Stanford, and 2009 University of Michigan Conference on the Financial Crisis for helpful discussion and comments.

<sup>†</sup>University of Chicago, Booth School of Business. Email: zhiguo.he@chicagobooth.edu.

<sup>‡</sup>Princeton University and NBER. Email: wxiong@princeton.edu.

# 1 Introduction

Runs by creditors on non-bank financial institutions, such as investment banks, special investment vehicles, conduits, and hedge funds, are widely regarded as one of the causes of the credit crisis of 2007-2008.<sup>1</sup> The freeze of the U.S. asset backed commercial paper (ABCP) markets in 2007 provided a vivid illustration of runs on the financial institutions. Prompted by concerns about the mounting delinquencies of subprime mortgages, outstanding ABCP fell by a staggering \$400 billion (one third of the existing amount) during the second half of 2007.<sup>2</sup> The classic bank-run model of Diamond and Dybvig (1983) has been frequently used to describe and analyze panic runs in different situations, including those in the recent credit crisis. Their model studies the problem faced by bank depositors, who must simultaneously decide whether to withdraw their demand deposits. In particular, the model highlights the externality of each depositor's withdrawal decision on other creditors, and shows that it can lead to a self-fulfilling bank-run equilibrium. In this equilibrium, each depositor chooses to withdraw and the bank fails even though it is still solvent.

Different from banks, non-bank financial institutions are mostly financed by short-term debt contracts, such as commercial paper and repo transactions. While demand deposits allow bank depositors to run at any time, debt contracts lock in creditors until contract expirations. Interestingly, financial institutions recognize the potential liquidity risk from having large quantities of debt expiring at the same time, and, in practice, spread out their debt expirations across time.<sup>3</sup> By doing so, they reduce the risk generated by the Diamond-Dybvig type of coordination problem between creditors whose debt contracts mature at the same time. But some of the financial institutions (e.g., Bear Stearns) nevertheless experienced severe runs during the recent credit crisis. Their experience raises many questions about determinants of runs as well as the roles played by central banks in bailing out financial institutions facing runs. These questions motivate a model to incorporate the dynamic nature of runs.

This paper develops a continuous-time model to analyze how runs occur on firms with

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<sup>1</sup>See comments of various regulators and researchers, e.g., Bernanke (2008), Brunnermeier (2009), Cox (2008), Gorton (2008), Krishnamurthy (2009), and Shin (2009).

<sup>2</sup>See Covitz, Liang, and Suarez (2009).

<sup>3</sup>For example, on February 10, 2009, the data from Bloomberg show that Morgan Stanley, one of the major U.S. investment banks, had short-term debt (with maturities less than 1.5 years) expiring on almost every day throughout February and March 2009. If we sum up the total value of Morgan Stanley's expiring short-term debt in each week, the values for the following five weeks are 62 million, 324 million, 339 million, 239 million, and 457 million, respectively. The Federal Reserve Release also shows that the commercial paper issued by financial firms in aggregate has maturities well spread out over time.

staggered debt structures. In our model, a firm finances its long-term asset holding by rolling over short-term debt with a continuum of small creditors. The firm uses a staggered debt structure, in which debt expirations are uniformly spread out across time. This structure implies that the fraction of debt maturing in a short period is small. Thus, each maturing creditor does not need to worry much about the rollover decisions of other maturing creditors at the same time. However, he faces the risk that the firm could fail during his next contract period if future maturing creditors choose not to roll over their debt contracts. Because of this so-called rollover risk, he needs to coordinate his rollover decision with future maturing creditors.

To facilitate our analysis, we also make two assumptions on the asset side. First, the firm asset is illiquid. When some maturing creditors choose to run and the firm fails to raise new funds to repay them, it has to prematurely liquidate the asset at a fire-sale price equal to a fraction of its fundamental value. Second, the firm's asset fundamental is time-varying and every creditor observes the same public information about its current value. This assumption is realistic as assets held by financial institutions are mostly financial securities whose fundamentals change over time and are largely observable by the public. Furthermore, we also allow the firm to have some imperfect credit lines, which sustain the firm under runs until it breaks at a random Poisson time.

We derive in closed form a unique threshold equilibrium, in which each maturing creditor chooses to run on the firm if the firm fundamental falls below a certain endogenously determined threshold. To protect himself against the firm's future rollover risk caused by other creditors, each maturing creditor will choose to roll over his debt if and only if the current fundamental provides a sufficient safety margin. Each creditor's optimal threshold choice depends on that of others—if a creditor anticipates that the creditors maturing during his next contract period are more likely to run (i.e., using a higher rollover threshold), he has a greater incentive to run ahead of them (i.e., using an even higher threshold) when he gets the chance now. In this way, creditors engage in a preemptive “rat race,” which leads each creditor to choose a rollover threshold substantially higher than he would in the absence of the coordination problem. As a reflection of this rat race, when the firm's fundamental volatility is sufficiently high and the reliability of credit lines is low, creditors would choose to run on the firm even if its current liquidation value (i.e., the asset fundamental after the fire-sale discount) is sufficient to pay back its liability. This outcome is intriguing because such a strong fundamental precludes bank-run equilibria in the static Diamond-Dybvig setting. Our model thus prompts more attention on runs driven by fear of future rollover risk.

The emergence of the unique threshold equilibrium derived in our model builds on insights developed in the coordination-game literature. In the widely used global games models, which were initially proposed by Carlsson and van Damme (1993) and later popularized by Morris and Shin (1998), agents possess noisy private information about some fundamental variables and need to simultaneously choose their actions in the presence of strategic complementarities. The heterogeneity in their information allows them to coordinate their synchronous actions and reach a unique equilibrium. In our model, creditors share the same information about the firm fundamental, but make rollover decisions at different times. As such, the time-varying firm fundamental allows them to coordinate their asynchronous actions. This equilibrium selection insight builds on Frankel and Pauzner (2000), who show that in dynamic coordination games, fundamental shocks can act as a coordination device for agents who choose actions at different times.

By analyzing this unique equilibrium, we show that time-varying fundamentals not only contribute to the equilibrium selection, but also play a key role in driving creditors' preemptive runs. In particular, we characterize a set of economic determinants of runs. First, firms with deteriorating fundamentals are more likely to experience runs. Second, the more illiquid a firm's asset is, the more likely are runs on the firm. This is because a deeper discount of the firm's asset in the secondary market exposes each creditor to a greater loss in the event of a forced liquidation in the future. These model implications are consistent with the widely held views that both deteriorating fundamentals and asset illiquidity are important drivers of many historical financial crises (see the brief literature review in the next subsection).

Third, higher fundamental volatility increases a firm's exposure to runs. This is because higher volatility makes the firm's fundamental more likely to hit below the other creditors' rollover threshold during a creditor's contract period, thus motivating him to use a higher rollover threshold.

Fourth, when fundamental volatility is low, credit lines can mitigate the creditors' incentives to run by sustaining the firm under the runs for a certain period during which a creditor's contract may mature. Surprisingly, when fundamental volatility is sufficiently high, credit lines can exacerbate the creditors' incentives to run because fundamental volatility can cause the firm fundamental to severely deteriorate during the period that the firm survives by its credit lines. This result suggests that uncertain bailout effort by central banks during financial crises (which acts like imperfect credit lines) can be counter productive in deterring runs.

Fifth, under a wide range of parameter values, firms with shorter debt maturities are

more exposed to runs, because they face greater rollover risk. The last three implications are especially relevant to the recent credit crisis as the sudden rise of market volatility, financial firms' excessive use of short-term debt, and central banks' uncertain bailout effort interacted with each other and contributed to instability of the financial system.

The paper is organized as follows. The following subsection briefly reviews the related literature. Section 2 describes the model setup. We derive a unique debt-run equilibrium in Section 3 and analyze the determinants of the equilibrium rollover threshold in Section 4. Finally, we conclude in Section 5. All technical proofs are given in the Appendix.

## 1.1 The Related Literature

Our model complements Rochet and Vives (2004) and Goldstein and Pauzner (2005) in integrating two distinct and long-standing views about runs. The first view, advocated by Friedman and Schwartz (1963) and Kindleberger (1978), attributes many historical banking crises to unwarranted panics by arguing that the banks that were forced to liquidate in such episodes were illiquid rather than insolvent. The alternative view, proposed by Mitchell (1941) and others, suggests that runs occur when depositors have fundamental concerns about the health of banks. Each of these views has motivated a body of theoretical models of bank runs.<sup>4</sup> See Gorton and Winton (2003) and Allen and Gale (2007) for two recent reviews of the history of financial crises and different theories of runs.

In our model, creditors worry about that future fundamental deterioration may lead to insolvency. When creditors respond to insolvency risk by failing to roll over their maturing debt, they create rollover risk. Rollover risk is self-reinforcing: creditors respond by choosing to run at even higher fundamental thresholds. Rochet and Vives (2004) and Goldstein and Pauzner (2005) adopt the global games framework to extend the static Diamond-Dybvig type of bank run setting. In their models, the bank fundamentals are unobservable and depositors use noisy private signals to coordinate their simultaneous withdrawal decisions. Inefficient coordination between depositors leads to inefficient runs on weak but solvent banks. These models also capture the tendency of runs to occur on banks with weaker fundamentals and with more illiquid asset holdings. In contrast, by incorporating a time-varying firm fundamental and a staggered debt structure, our model highlights fundamental volatility, reliability of credit lines, and debt maturity structure as additional determinants of runs.

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<sup>4</sup>Diamond and Dybvig (1983), Postlewaite and Vives (1987), Peck and Shell (2003), and Caballero and Krishnamurthy (2008) offer models of panic-driven runs, while Bryant (1980), Gorton (1988), Chari and Jagannathan (1988), Jacklin and Bhattacharya (1988), and Allen and Gale (1998) focus on the fundamental risk of bank loans and the depositors' signal extraction problem in driving runs.

Our paper also adds to the growing literature on the implications of dynamic coordination problems for a range of economic issues. Our model differs from the existing studies not only in the economic context, but also in the key economic insight. Abreu and Brunnermeier (2003) develop a bubble-attack model, in which non-common knowledge induced by arbitrageurs' sequential awareness of the existence of a price bubble leads each arbitrageur to delay his attack. Chamley (2003) shows that speculators can learn from observing an exchange rate within a pegged band in the previous periods about whether their mass is sufficiently large for a successful attack. Angeletos, Hellwig, and Pavan (2007) develop a dynamic global games model to study the roles played by learning and public information in equilibrium selection and in driving equilibrium dynamics. These models all feature a constant fundamental and focus on the roles of non-common knowledge and endogenous learning. Our model assumes common knowledge among creditors about the firm fundamental. This feature insulates our model from various concerns about the robustness of global games models with private information.<sup>5</sup>

Guimaraes (2006) and Plantin and Shin (2008) build on the equilibrium selection insight of Frankel and Pauzner (2000) to study coordinated currency attacks and speculative dynamics in carry trades. In both models, time-varying fundamentals, together with frictions that prevent investors from instantaneously changing their investment positions in a currency, allow investors to coordinate around a unique equilibrium. Guimaraes highlights that small frictions can cause a long delay in investors' attacks on an overvalued currency, while Plantin and Shin focus on funding externalities created by carry trades and the resulting large negative movements in exchange rate dynamics.

In contrast to the aforementioned dynamic coordination models, our model highlights a new role of fundamental fluctuations (a key variable in standard economic models) in driving firms' financial instability. That is, in the presence of the lock-in effect of debt maturity and concave payoffs of debt contracts, fundamental fluctuations motivate each creditor to run ahead of others.<sup>6</sup>

Our paper echoes several recent studies on the financial instability created by short-term debt. Acharya, Gale, and Yorulmazer (2009) also study financial institutions' rollover risk

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<sup>5</sup>For example, Weinstein and Yildiz (2007) point out that different information structures can lead to different equilibrium outcomes, while Angeletos and Werning (2006) and Hellwig, Mukherji, and Tsyvinski (2006) show that multiple equilibria may arise if sufficient information is revealed through market price.

<sup>6</sup>The ability of our model to characterize economic determinants of panic runs differentiates it from Cole and Kehoe (2000). They model the trigger of a country's self-fulfilling debt crisis by an exogenous random sunspot when the country's fundamental is inside a crisis zone where the Diamond-Dybvig type of multiple self-fulfilling equilibria exist.

and show that fast rollover frequency can lead to diminishing debt capacity. Brunnermeier and Oehmke (2009) study the conflict between long-term and short-term creditors and show that this conflict can motivate all creditors to demand short-term debt. He and Xiong (2009) analyze the role played by market illiquidity and short-term debt in exacerbating the conflict between debt and equity holders. Morris and Shin (2009) build a global games model to analyze the illiquidity component of financial institutions’ credit risk. In contrast, our model focuses on preemptive runs caused by creditors’ fear of a firm’s future rollover risk.

## 2 Model

We consider a continuous-time model with an infinite horizon. A firm invests in a long-term asset by rolling over short-term debt. One can interpret this firm as any firm, either financial or non-financial. Our model is perhaps more appealing for financial firms because they tend to have higher leverage and more short-term debt. To make debt runs a relevant concern, we assume that the capital markets are imperfect in the following sense: the firm cannot find a single creditor with “deep pockets” to finance all of its debt and has to rely on a continuum of small creditors. The firm spreads its debt expirations uniformly across time. Then, if some of the maturing creditors choose not to roll over their debt and the firm fails to raise new funds from its imperfect credit lines to pay off them, the firm is bankrupt and has to liquidate its asset in an illiquid secondary market at a discount.

### 2.1 Asset

We normalize the firm’s asset holding to be 1 unit. The firm borrows \$1 at time 0 to acquire its asset. Once the asset is in place, it generates a constant stream of cash flow, i.e.,  $rdt$  over the time interval  $[t, t + dt]$ . At a random time  $\tau_\phi$ , which arrives according to a Poisson process with intensity  $\phi > 0$ , the asset matures with a final payoff. An important advantage of assuming a random asset maturity with a Poisson process is that at any point before the maturity, the expected remaining time-to-maturity is always  $1/\phi$ .

The asset’s final payoff is equal to the time- $\tau_\phi$  value of a stochastic process  $y_t$ , which follows a geometric Brownian motion with constant drift  $\mu$  and volatility  $\sigma > 0$ :

$$\frac{dy_t}{y_t} = \mu dt + \sigma dZ_t,$$

where  $\{Z_t\}$  is a standard Brownian motion. We assume that the value of the fundamental process is publicly observable at any time.

Taken together, the firm asset generates a constant cash flow of  $r dt$  before  $\tau_\phi$  and a final value of  $y_{\tau_\phi}$  at  $\tau_\phi$ . Then, by assuming that agents in this economy (including the firm creditors) are risk-neutral and have a discount rate of  $\rho > 0$ , we can compute the fundamental value of the firm asset as its expected discounted future cash flows:

$$F(y_t) = E_t \left[ \int_t^{\tau_\phi} e^{-\rho(s-t)} r ds + e^{-\rho(\tau_\phi-t)} y_{\tau_\phi} \right] = \frac{r}{\rho + \phi} + \frac{\phi}{\rho + \phi - \mu} y_t, \quad (1)$$

where the two components,  $\frac{r}{\rho + \phi}$  and  $\frac{\phi}{\rho + \phi - \mu} y_t$ , correspond to the present values of the asset's constant cash flow and final payoff, respectively. Since the asset's fundamental value increases linearly with  $y_t$ , we will conveniently refer to  $y_t$  as the firm fundamental.

The assumption that the firm fundamental is time-varying is natural. It is somewhat strong to assume that the fundamental is publicly observable. This assumption mainly serves to insulate our model from further complications caused by agents' private information about the firm fundamental. In fact, our model would stay intact if we assume that the fundamental is unobservable and instead all agents only observe the same noisy public signals.

## 2.2 Debt Financing

The firm finances its asset holding by issuing short-term debt. A key contributing factor to the recent credit crisis was the excessive use of short-term debt, such as commercial paper and repos, by financial institutions in the preceding period, e.g., Brunnermeier (2009), Gorton (2008), Krishnamurthy (2009), and Shin (2009). Why do firms use short-term debt? Short-term debt is a natural response of outside creditors to a variety of agency problems inside the firm, e.g., Calomiris and Kahn (1991) and Diamond and Rajan (2009). By choosing short-term financing, creditors keep the option to pull out if they discover that firm managers are pursuing value-destroying projects.<sup>7</sup> The short commitment period also makes short-term debt less information sensitive and thus less exposed to adverse-selection problems, e.g., Gorton and Pennacchi (1990). As a result, short-term debt also has a lower financing cost. In this paper, we take a realistic debt structure as given in order to maintain the simplicity of the model and focus on the coordination problem generated by short-term debt.<sup>8</sup>

We emphasize an important feature of real-life firms' debt structure: firms tend to spread out their debt expirations over time to reduce liquidity risk (see evidence given in Footnote

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<sup>7</sup>See Kashyap, Rajan, and Stein (2008) for a recent review of this agency literature and capital regulation issues related to the recent financial crisis.

<sup>8</sup>Cheng and Milbradt (2009) extend our model to allow the firm manager to freely switch between two projects, a good one with high drift and low volatility and an inferior one with low drift and high volatility. They show that the use of short-term debt can discipline the manager from choosing the inferior project when the firm fundamental is high.



3). In this way, they avoid having to roll over a large fraction of their debt on a single day. Specifically, we assume that the firm finances its asset holding by issuing one unit of debt divided uniformly among a continuum of small creditors with measure 1. The promised interest rate is  $r$  so that the cash flow from the asset exactly pays off the interest payment until the asset matures or until the firm is forced to liquidate the asset prematurely.<sup>9</sup> Following the staggered-pricing model of Calvo (1983) and the credit-risk model of Leland (1998), we assume that each debt contract lasts for a random period, which ends upon the arrival of an independent Poisson shock with intensity  $\delta > 0$ . In other words, the duration of each debt contract has an exponential distribution. Once the contract expires, the creditor chooses whether to roll over the debt or to run. The maturity shocks are independent across creditors so that each creditor expects some other creditors' contracts to mature before his. He is thus exposed to the firm's rollover risk.

In aggregate, the firm has a fixed fraction  $\delta dt$  of its debt maturing over  $[t, t + dt]$ , where the parameter  $\delta$  represents the firm's rollover frequency. The random maturity assumption simplifies the complication of keeping track of the remaining maturities of individual contracts, because at any time before the maturity the expected remaining maturity of each contract is always  $1/\delta$ . By matching  $1/\delta$  with the fixed maturity of a real-life debt contract, this assumption captures the first order effect of debt maturity when a creditor makes his rollover decision.<sup>10</sup>

## 2.3 Runs and Liquidation

When the maturing creditors choose to run, they expose the firm to bankruptcy risk if it cannot raise new funds to repay the running creditors. The firm would be extremely frail if a single creditor's run would cause it to fail. To prevent this, we allow the firm to draw on

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<sup>9</sup>To focus on the coordination problem between creditors, we also take the interest payment of the firm debt as given. One might argue that when facing rollover difficulties, the firm can attract the maturing creditors by promising higher interest rates. However, doing so dilutes the stakes of other creditors in the firm and would motivate earlier maturing creditors to demand higher interest rates preemptively, similar to the preemptive runs highlighted in our model. In other words, promising higher interest rates could become a self-enforcing tightening mechanism on the firm, instead of a way to bail it out. We will leave a more elaborate analysis of this effect for future research.

<sup>10</sup>This assumption also generates an artificial second-order effect: If the debt contracts have a fixed maturity, a creditor, after rolling over his contract, will go to the end of the maturity queue. The random maturity assumption makes it possible for the creditor to be released early and therefore to run before other creditors when the asset fundamental deteriorates. This possibility makes the creditor less worried about the firm's rollover risk than he would be if the debt contract had a fixed maturity. This in turn makes him more likely to roll over his debt. We have verified this outcome by numerically analyzing a variation of our model with fixed debt maturity. Thus, by assuming random debt maturity, our model underestimates the firm's rollover risk.

pre-committed credit lines from other institutions or the government. However, the credit lines are imperfect, as the issuing institutions may experience their own financial distresses and the government may face political pressures against supporting distressed firms. As a result, persistent runs will eventually cause the firm to fail.

More specifically, over a short time interval  $[t, t + dt]$ ,  $\delta dt$  fraction of the firm's debt contracts mature. If these creditors choose to run, the firm will draw on its credit lines to raise new funds to pay off the running creditors. We assume that with probability  $\theta \delta dt$ , the issuer of the firm's credit lines fails to provide liquidity and the firm is thus forced into liquidation. The parameter  $\theta > 0$  measures the unreliability of the firm's credit lines. The higher the value of  $\theta$ , the less reliable the firm's credit lines, and therefore the more likely the firm will be forced into liquidation given the same creditor outflow rate. With probability  $1 - \theta \delta dt$ , the firm is able to raise new funds through the credit lines to pay off the running creditors. For simplicity, we assume that the new funds raised from the credit lines have the same debt contract as the existing ones. Taken together, if every maturing creditor chooses to run, the firm fails with Poisson intensity  $\theta \delta$ , i.e., it survives on average for a period of  $\frac{1}{\theta \delta}$ .<sup>11</sup>

Once the firm fails to raise new funds to pay off the running creditors, it falls into bankruptcy and has to liquidate its asset in an illiquid secondary market. We assume that the firm can only recover a fraction  $\alpha \in (0, 1)$  of its fundamental value. That is, the firm obtains a fire-sale price of

$$\tilde{L}(y_t) = \alpha F(y_t) = L + ly_t, \quad (2)$$

where

$$L = \frac{\alpha r}{\rho + \phi} \quad \text{and} \quad l = \frac{\alpha \phi}{\rho + \phi - \mu}. \quad (3)$$

For simplicity, we rule out partial liquidations in this model.

The liquidation value will then be used to pay off all creditors on an equal basis. In other words, both the running creditors and the other creditors who are locked in by their current contracts get the same payoff  $\min(\tilde{L}(y), 1)$ . Ex ante, each creditor's expected payoff from

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<sup>11</sup>The imperfect credit lines are realistic as credit lines were frequently withdrawn by issuers during the recent credit crisis, either because they also faced funding problems or because they were concerned about future funding problems and thus chose to hoard liquidity. Regarding the runs in the asset-backed commercial paper (ABCP) market in 2007, Covitz, Liang, and Suarez (2009) find that across different ABCP programs, the reliability of their credit lines is also an important determinant of the likelihood of runs. One could also interpret  $\theta$  as inversely related to the firm's cash reserve. If the firm has more cash reserves, it can survive the creditors' runs for a longer period. Since outside creditors usually cannot directly observe the balance of a firm's cash reserve, from their perspective the failure of the firm under creditors' runs will occur at a random time.

choosing run is still 1 because the probability of the firm failure  $\theta\delta dt$  is in a higher  $dt$  order.<sup>12</sup>

Due to the staggered debt structure in our continuous-time setting, the fraction of maturing creditors over a small time interval (i.e.,  $\delta dt$ ) is small. This implies that an individual creditor's running decision is not affected by the concurrent decisions of other maturing creditors. This feature insulates our model from the Diamond-Dybvig type of static coordination problem, in which agents make simultaneous decisions, and instead allows us to focus on the coordination problem between creditors whose contracts mature at different times.

Our model implicitly assumes that once in distress, the firm cannot raise more capital by issuing new equity. This assumption is consistent with the existence of information asymmetry between the firm and outside equity holders, and with the existence of conflict of interest between debt and equity holders.<sup>13</sup>

## 2.4 Parameter Restrictions

To make our analysis meaningful, we impose several parameter restrictions. First, we bound the interest payment by

$$\rho < r < \rho + \phi. \quad (4)$$

The first part  $r > \rho$  makes the interest payment attractive to the creditors, who have a discount rate of  $\rho$ . The second part  $r < \rho + \phi$  rules out the scenario where the interest payment is so attractive that rollover becomes the dominant strategy even when the firm fundamental  $y_t$  is close to zero. Essentially, this condition ensures the existence of the lower dominance region in which each creditor's dominant strategy is to run if the firm fundamental  $y_t$  is sufficiently low.

Second, we limit the growth rate of the firm fundamental by

$$\mu < \rho + \phi. \quad (5)$$

Otherwise, the firm's fundamental value in equation (1) would explode.

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<sup>12</sup>This observation implies that in our model the sharing rule in the event of bankruptcy is inconsequential. We can also assume that during bankruptcy those maturing creditors who have chosen to run get a full payoff 1, while the remaining creditors who are locked in by their current contracts get  $\min(\tilde{L}(y), 1)$ . This alternative assumption gives a greater incentive for maturing creditors to run. However, since the probability of the firm failure is  $\theta\delta dt$ , the difference in incentive is negligible.

<sup>13</sup>When a firm faces liquidity problems in the debt market, equity holders could find it optimal not to inject more equity. By injecting equity they bear all the financial burden of keeping the firm from bankruptcy, but the benefit is shared by both debt and equity holders. See He and Xiong (2009) for a formal analysis of this distortion in short-term debt crises.

Third, we also limit the premature liquidation recovery rate of the firm asset:

$$\alpha < \frac{1}{\frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu}}, \quad (6)$$

so that  $L + l < 1$  (see equation (3)). Under this condition, the asset liquidation value is not enough to pay off all the creditors when  $y_t = 1$ . This condition is sufficient for ensuring that each creditor is concerned about the firm's future rollover risk when the firm fundamental  $y_t$  is in an intermediate region.

Finally, we assume that the parameter  $\theta$  is sufficiently high:

$$\theta > \frac{\phi}{\delta(1-L-l)}, \quad (7)$$

so that the firm faces a serious bankruptcy probability when some creditors choose to run.

### 3 The Debt-Run Equilibrium

Given the firm's asset and financing structures described in the previous section, we now analyze the debt-run equilibrium. We limit our attention to monotone equilibria, equilibria in which each creditor's rollover strategy is monotonic with respect to the firm fundamental  $y_t$  (i.e., to roll over if and only if the firm fundamental is above a threshold). In making his rollover decision, a creditor rationally anticipates that once he rolls over the debt, he faces the firm's rollover risk during his contract period. This is because volatility could cause the firm fundamental to fall below the other creditors' rollover thresholds. As a result, the creditor's optimal rollover threshold depends on the other creditors' threshold choices.

In this section, we first set up an individual creditor's optimization problem in choosing his optimal threshold. We then construct a unique monotone equilibrium in closed form. Finally, we establish two benchmark settings for analyzing the debt-run equilibrium.

#### 3.1 An Individual Creditor's Problem

We first analyze the optimal rollover decision of an individual creditor by taking as given that all other creditors use a monotone strategy with a rollover threshold  $y_*$  (i.e., other creditors will roll over their debt if and only if the firm fundamental is above  $y_*$  when their debt contracts mature). During the creditor's contract period, his value function depends directly on the firm fundamental  $y_t$ , and indirectly on the other creditors' rollover threshold  $y_*$ . Since the creditor's future payoff is proportional to the unit of debt he holds, we denote  $V(y_t; y_*)$  as the creditor's value function normalized by the debt unit.

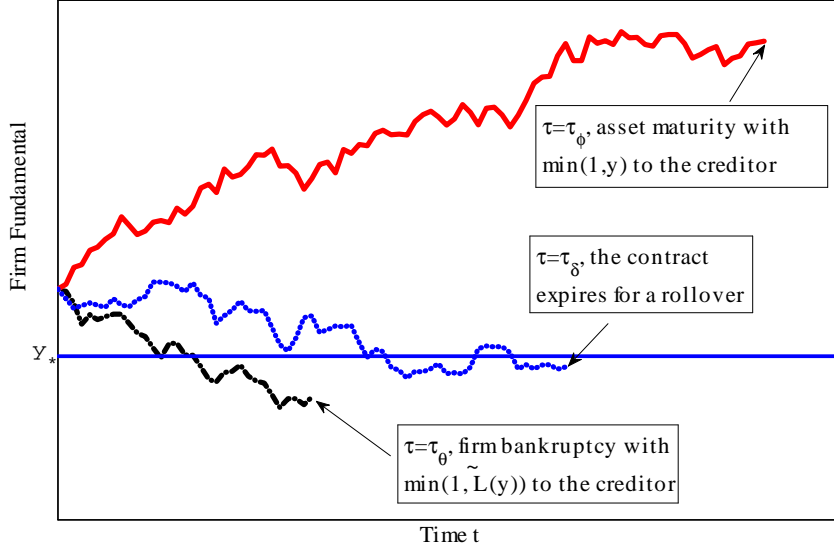


Figure 1: Three possible outcomes for a creditor.

For each unit of debt, the creditor receives a stream of interest payments  $r$  until

$$\tau = \min(\tau_\phi, \tau_\delta, \tau_\theta),$$

which is the earliest of the following three events, illustrated in Figure 1 at the end of three different fundamental paths. On the top path, the firm stays alive until its asset matures at  $\tau_\phi$ . At this time, the creditor gets a final payoff of  $\min(1, y_{\tau_\phi})$ , i.e., the face value 1 if the asset's maturity payoff  $y_{\tau_\phi}$  is sufficient to pay all the debt, and  $y_{\tau_\phi}$  otherwise. The possibility that the asset's maturity value may be insufficient to pay off the debt represents the firm's insolvency risk. On the bottom path, the firm fundamental drops below the other creditors' rollover threshold and the firm is eventually forced to liquidate its asset prematurely at  $\tau_\theta$ . At this time, the creditor gets  $\min(1, L + ly_{\tau_\theta})$ . This outcome represents the firm's rollover risk. On the middle path, the firm stays alive (although its fundamental dips below the other creditors' rollover threshold) until  $\tau_\delta$  when the creditor's contract expires. At this time, the creditor has an option, i.e., he can choose whether to roll over depending on whether the continuation value  $V(y_{\tau_\delta}; y_*)$  is higher than getting the one dollar back.

Due to risk neutrality, the individual creditor's value function is given by

$$\begin{aligned}
 V(y_t; y_*) = & E_t \left\{ \int_t^\tau e^{-\rho(s-t)} r ds + e^{-\rho(\tau-t)} \left[ \min(1, y_\tau) \mathbf{1}_{\{\tau=\tau_\phi\}} \right. \right. \\
 & \left. \left. + \min(1, L + ly_\tau) \mathbf{1}_{\{\tau=\tau_\theta\}} + \max_{\text{rollover or run}} \{V(y_\tau; y_*), 1\} \mathbf{1}_{\{\tau=\tau_\delta\}} \right] \right\}, \tag{8}
 \end{aligned}$$

where  $\mathbf{1}_{\{\cdot\}}$  is an indicator function that takes a value of 1 if the statement in the bracket is true or zero otherwise. The individual creditor's future payoff during his contract period depends on other creditors' rollover choices because other creditors' runs might force the firm to liquidate its asset prematurely, as illustrated by the bottom path of Figure 1. This dependence gives rise to strategic complementarities in the creditors' rollover decisions, and thus creates a coordination problem between creditors whose contracts mature at different times.

By considering the change in the creditor's continuation value over a small time interval  $[t, t + dt]$ , we can derive his Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} \rho V(y_t; y_*) &= \mu y_t V_y + \frac{\sigma^2}{2} y_t^2 V_{yy} + r + \phi [\min(1, y_t) - V(y_t; y_*)] \\ &\quad + \theta \delta \mathbf{1}_{\{y_t < y_*\}} [\min(L + l y_t, 1) - V(y_t; y_*)] + \delta \max_{\text{rollover or run}} \{0, 1 - V(y_t; y_*)\}. \end{aligned} \quad (9)$$

The left-hand side term  $\rho V(y_t; y_*)$  represents the creditor's required return. This term should be equal to the expected increment in his continuation value, as summarized by the terms on the right-hand side.

- The first two terms  $\mu y_t V_y + \frac{\sigma^2}{2} y_t^2 V_{yy}$  capture the expected change in the continuation value caused by the fluctuation in the firm fundamental  $y_t$ .
- The third term  $r$  is the interest payment per unit of time.

The next three terms capture the three events illustrated in Figure 1:

- The fourth term  $\phi [\min(1, y_t) - V(y_t; y_*)]$  captures the possibility that the asset matures during the time interval, which occurs with probability  $\phi dt$  and generates an impact of  $\min(1, y_t) - V(y_t; y_*)$  on the creditor's continuation value.
- The fifth term  $\theta \delta \mathbf{1}_{\{y_t < y_*\}} [\min(L + l y_t, 1) - V(y_t; y_*)]$  represents the expected effect of premature liquidation from other creditors' runs, which occurs with probability  $\theta \delta \mathbf{1}_{\{y_t < y_*\}} dt$  (other maturing creditors will run only if  $y_t < y_*$ ) and generates an impact of  $\min(L + l y_t, 1) - V(y_t; y_*)$  on the creditor's continuation value.
- The last term  $\delta \max_{\text{rollover or run}} \{0, 1 - V(y_t; y_*)\}$  captures the expected effect from the creditor's own contract expiration, which arrives with probability  $\delta dt$ . Upon its arrival, the creditor chooses whether to roll over or to run:  $\max_{\text{rollover or run}} \{0, 1 - V(y_t; y_*)\}$ .

It is obvious that a maturing creditor will choose to roll over his contract if and only if  $V(y_t; y_*) > 1$ , and to run otherwise. This implies that if the value function  $V$  only crosses 1 at a single point  $y'$ , i.e.,  $V(y'; y_*) = 1$ , then  $y'$  is the creditor's optimal threshold.

**Externality on Future Maturing Creditors** The rollover decision of current-period maturing creditors affects not only their own payoffs, but also future maturing creditors'. In particular, their decision to run adds to the firm's bankruptcy probability and thus imposes an implicit cost on future maturing creditors. Since they do not internalize the cost on others, this externality is the ultimate source of debt runs in our model. To see this point precisely, we summarize the payoff (or continuation value) of the current-period maturing creditors and future maturing creditors depending on the choice of the current-period maturing creditors in Table 1. For simplicity, we treat all the current-period maturing creditors as one identity in this illustration.

Table 1. Externality on future maturing creditors.

Choice of current-period maturing creditors	Run		Rollover
Possible firm outcomes	failed	survived	survived
Probability	$\theta\delta dt$	$1 - \theta\delta dt$	1
Payoff of current-period maturing creditors	$\tilde{L}(y)$	1	$V(y)$
Payoff of future maturing creditors	$\tilde{L}(y)$	$V(y)$	$V(y)$

The maturing creditors will choose run if  $1 \cdot (1 - \theta\delta dt) + \tilde{L}(y) \cdot \theta\delta dt > V(y)$ , which is  $V(y) < 1$  after ignoring the higher order  $dt$  term. Their runs reduce the remaining creditors' continuation value by

$$V(y) - \left[ V(y) \cdot (1 - \theta\delta dt) + \tilde{L}(y) \cdot \theta\delta dt \right] = \left[ V(y) - \tilde{L}(y) \right] \theta\delta dt.$$

While this effect is of the  $dt$  order, a creditor needs to bear the accumulative externality effect of all maturing creditors before him, which, in expectation, could be significant.<sup>14</sup>

### 3.2 The Unique Monotone Equilibrium

We first focus our attention on symmetric monotone equilibria, and then show that there cannot be any asymmetric monotone equilibrium. In a symmetric monotone equilibrium, each creditor's optimal threshold choice  $y'$  must be equal to the other creditors' threshold  $y_*$ . Thus, we obtain the condition for determining the equilibrium threshold:

$$V(y_*; y_*) = 1.$$

<sup>14</sup>The current-period maturing creditors' runs also impose externalities on each other. But this effect is one time and of the  $dt$  order, thus can be ignored.

We employ a guess-and-verify approach to derive a unique monotone equilibrium in four steps.<sup>15</sup> First, we derive an individual creditor's value function  $V(y_t; y_*)$  from the HJB equation in (9) by assuming that every creditor (including the creditor under consideration) uses the same monotone strategy with a rollover threshold  $y_*$ . Due to the terms  $\min(1, y_t)$  and  $\min(L + ly_t, 1)$  in (9), the value function depends on the value of  $y_*$  in three cases:

1. If  $y_* < 1$ ,

$$V(y_t; y_*) = \begin{cases} \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\phi+\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_t + A_1 y_t^{\eta_1} & \text{when } 0 < y_t \leq y_* \\ \frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu} y_t + A_2 y_t^{-\gamma_2} + A_3 y_t^{\eta_2} & \text{when } y_* < y_t \leq 1 \ ; \\ \frac{r+\phi}{\rho+\phi} + A_4 y_t^{-\gamma_2} & \text{when } y_t > 1 \end{cases}$$

2. If  $1 \leq y_* < \frac{1-L}{l}$ ,

$$V(y_t; y_*) = \begin{cases} \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\phi+\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_t + B_1 y_t^{\eta_1} & \text{when } 0 < y_t \leq 1 \\ \frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_t + B_2 y_t^{-\gamma_1} + B_3 y_t^{\eta_1} & \text{when } 1 < y_t \leq y_* \ ; \\ \frac{r+\phi}{\rho+\phi} + B_4 y_t^{-\gamma_2} & \text{when } y_t > y_* \end{cases}$$

3. If  $y_* \geq \frac{1-L}{l}$ ,

$$V(y_t; y_*) = \begin{cases} \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\phi+\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_t + C_1 y_t^{\eta_1} & \text{when } 0 < y_t \leq 1 \\ \frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_t + C_2 y_t^{-\gamma_1} + C_3 y_t^{\eta_1} & \text{when } 1 < y_t \leq \frac{1-L}{l} \\ \frac{r+\phi+\theta\delta+\delta}{\rho+\phi+(1+\theta)\delta} + C_4 y_t^{-\gamma_1} + C_5 y_t^{\eta_1} & \text{when } \frac{1-L}{l} < y_t \leq y_* \\ \frac{r+\phi}{\rho+\phi} + C_6 y_t^{-\gamma_2} & \text{when } y_t > y_* \end{cases}$$

The coefficients  $\eta_1, \eta_2, \gamma_1, \gamma_2, A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, C_1, C_2, C_3, C_4, C_5$ , and  $C_6$  are given in Appendix A.1 and are expressions of the model parameters and  $y_*$ .

Second, based on the derived value function, we show that there exists a unique fixed point  $y_*$  such that  $V(y_*; y_*) = 1$ . Third, we prove the optimality of the threshold  $y_*$  for any

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<sup>15</sup>Our model is substantially different from the standard dynamic coordination game frameworks. In our model, each creditor's flow payoff from the debt contract (interest payment  $r$  and possible asset maturity payoff  $\min(y, 1)$ ) does not exhibit any strategic complementarity. Instead, strategic complementarities emerge from the implicit dependence of a creditor's continuation value function on other creditors' rollover decisions (equation (8)). The standard game frameworks, e.g., Frankel and Pauzner (2000) and Burdzy, Frankel, and Pauzner (2001), typically specify strategic complementarity in agents' flow payoffs, i.e., an agent's payoff in a given period is higher if his current-period strategy overlaps with that of a greater fraction of the population. This important difference in model framework prevents us from readily applying the method of iterated deletion of dominated strategies used by Burdzy, Frankel, and Pauzner (2001). Instead, we derive the equilibrium by invoking a guess-and-verify approach.



individual creditor, i.e.,  $V(y; y_*) > 1$  for  $y > y_*$  and  $V(y; y_*) < 1$  for  $y < y_*$ . Finally, we show that there cannot be any asymmetric monotone equilibrium.

We summarize the main results in the following theorem.

**Theorem 1** *There exists a unique monotone equilibrium, in which each maturing creditor chooses to roll over his debt if  $y_t$  is above the threshold  $y_*$  and to run otherwise. The equilibrium threshold  $y_*$  is uniquely determined by the condition that  $V(y_*, y_*) = 1$ .*

The Diamond-Dybvig model features multiple self-fulfilling equilibria. What leads to the unique equilibrium in our model? To understand this issue, first note the existence of lower and upper dominance regions. When the firm fundamental  $y_t$  is sufficiently low (i.e., close to zero), an individual creditor's dominant strategy is run (lower dominance region). This is because even if all other creditors choose to roll over in the future, the expected asset payoff at the maturity plus the interest payments before the asset maturity are not as attractive as getting one dollar back now. On the other hand, when the firm fundamental  $y_t$  is sufficiently high (i.e., close to infinity), the creditor's dominant strategy is rollover (upper dominance region). Even if all other creditors choose to run in the future, the firm's liquidation value is sufficient to pay off the debt in the event of a forced liquidation.

When the firm fundamental is in the intermediate region between the two dominance regions, self-fulfilling multiple equilibria could arise if creditors make synchronous rollover decisions or if the firm fundamental stays constant over time. In an earlier version of this paper, which is listed as NBER working paper 15482, we derive several variations of our model. In one of the variations (which is also described in Section 3.3.2), all the debt contracts mature at the same time and the creditors simultaneously decide whether to roll over into new perpetual contracts, which last until the firm asset matures. In another variation, the firm still uses a staggered debt structure, but its fundamental stays constant over time. In both variations, self-fulfilling multiple equilibria arise when the firm fundamental is in an intermediate region. This outcome suggests that the unique equilibrium derived in our model is a joint effect of the staggered debt structure and the time-varying fundamental.

The intuition works as follows. Since the firm fundamental varies over time, different creditors face different fundamentals when making their rollover decisions. The current fundamental allows each maturing creditor to assess the firm's future rollover risk, thus coordinating his rollover decision with future maturing creditors. In other words, a unique (subgame perfect) equilibrium emerges, because anticipation of future creditors' uniquely determined rollover strategy inside the dominance regions allows the creditors to induce

their optimal strategy inside the intermediate region.<sup>16</sup> This key insight follows Frankel and Pauzner (2000) and Burdzy, Frankel, and Pauzner (2001), who show that in dynamic coordination games with strategic complementarities, random fundamental shocks allow agents to coordinate their asynchronous actions and induce a unique equilibrium.

### 3.3 Two Useful Benchmarks

In this subsection, we establish two benchmarks for our analysis of the debt-run equilibrium.

#### 3.3.1 The Single-Creditor Benchmark

First, we consider a setting that is otherwise identical to the main model except that a single creditor holds all the debt of the firm. The single creditor faces a contract period which expires upon the arrival of a Poisson shock with intensity  $\delta$ . When the contract expires, the single creditor decides whether to roll over the debt for another random contract period or not. If he decides not to roll over, the firm is forced into a premature liquidation. In this event, the creditor's payoff is  $\min(L + ly_t, 1)$ . Because the single creditor does not need to worry about the firm's future rollover risk with other creditors, his rollover decision is free of the coordination problem with other creditors. As a result, he would internalize the cost of a premature liquidation. The following proposition shows that he will always roll over his debt if the liquidation cost is sufficiently high.

**Proposition 2** *Suppose that a single creditor finances all the debt of the firm. If the cost of a premature liquidation is sufficiently high, i.e.,  $\alpha$  is sufficiently low, then the single creditor will always roll over his debt.*

The different outcomes between this benchmark setting and the main model highlight the externality of each creditor's run on other creditors as the key friction that drives the debt runs in our model.

#### 3.3.2 The Static-Rollover Benchmark

Next, we consider another setting where all the creditors roll over their debt only once and at the same time. Suppose that the firm's debt contracts all expire at time 0, and the current

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<sup>16</sup>This mechanism is analogous to that in the global games models developed by Carlsson and van Damme (1993) and Morris and Shin (1998). In the global games models, agents possess noisy signals about a fundamental variable and each agent uses his private signal to form expectations of other agents' signals and simultaneous actions. In our model, creditors have the same information but make their rollover decisions at different times. Since the firm fundamental is time-varying and persistent, the current fundamental allows each maturing creditor to form expectations of future maturing creditors' rollover decisions.

firm fundamental is  $y_0$ . At this time, each creditor decides whether to run or to roll over into a perpetual debt contract lasting until the firm asset matures at  $\tau_\phi$ . We also assume that if all creditors choose to run, the firm might fail with a probability of  $\theta_s \in (0, 1)$ . Because all creditors simultaneously choose their rollover decisions at time 0 and the firm does not face any future rollover risk, this setting closely resembles that in the Diamond-Dybvig model and thus serves as a benchmark to evaluate the effects of rollover risk in the main model.

**Proposition 3** *Assume the aforementioned setting. Then, if  $y_0 > y_h \equiv \frac{1-L}{l}$  (the upper dominance region), an individual creditor's dominant strategy is to roll over; if  $y_0 < y_l$  (the lower dominance region), where  $y_l$  is a value less than  $y_h$ , the creditor's dominant strategy is to run; if  $y_0 \in [y_l, y_h]$  (the intermediate region), the creditor's optimal choice depends on the others', i.e., it is optimal to run if the others choose to run and it is optimal to roll over if the others choose to roll over.*

Proposition 3 shows that when creditors make simultaneous rollover decisions and face no future rollover risk, multiple self-fulfilling equilibria emerge if the firm fundamental is in an intermediate region. In particular, creditors choose to run only if the fundamental is below a critical level  $\frac{1-L}{l}$ . In contrast, we will show in Section 4.3 that when creditors make asynchronous rollover decisions, the equilibrium rollover threshold derived in Theorem 1 could be higher than  $\frac{1-L}{l}$ . This result highlights the severity of creditors' fear of the firm's future rollover risk.

## 4 Determinants of Equilibrium Rollover Threshold

Despite the absence of self-fulfilling multiple equilibria in our model, preemptive debt runs can still occur through a rat race between the creditors in choosing higher and higher rollover thresholds. In this section, we analyze the effects of this rat race and the determinants of the equilibrium rollover threshold.

For illustration, we will use a set of baseline values for the model parameters:

$$\rho = 5\%, \quad r = 10\%, \quad \delta = 10, \quad \phi = 0.2, \quad \theta = 2, \quad \mu = 5\%, \quad \sigma = 10\%, \quad \alpha = 60\%. \quad (10)$$

The creditors have a discount rate  $\rho = 5\%$ . The firm asset generates a constant stream of cash flow at a rate of 10% per annum, which is paid out to the creditors as interest payments. The interest payments are attractive since the interest rate  $r$  is much higher than the creditors' discount rate  $\rho$ . We choose the firm's rollover frequency  $\delta$  to be 10, which implies an average

debt maturity of about 37 days ( $365/\delta$ ). This implied maturity matches the average maturity of outstanding asset-backed commercial paper in February 2009 (Federal Reserve Release).  $\phi = 0.2$  implies that the firm asset on average lasts for 5 years ( $1/\phi$ ), which is much longer than the debt maturity and resembles the typical duration of a mortgage bond.  $\theta = 2$  means that conditional on every maturing creditor choosing to run, the firm can survive on average for 18 days ( $1/\theta\delta$ ).<sup>17</sup> The firm fundamental  $y_t$  has a growth rate of  $\mu = 5\%$  per annum and a volatility of  $\sigma = 10\%$  per annum. Finally, when the firm liquidates its asset prematurely, it only recovers  $\alpha = 60\%$  of the asset's fundamental value. This implies that  $L = 0.24$  and  $l = 0.6$  in equation (3). Under these baseline parameters, the equilibrium rollover threshold is  $y_* = 1.19$ , at which the firm's fundamental value is  $F(y_*) = 1.59$ .

## 4.1 Liquidation Recovery Rate

We first illustrate the key rat race mechanism in determining the equilibrium rollover threshold. Consider the following thought experiment. Suppose that initially the liquidation recovery rate of the firm asset is  $\alpha_h$ , and, correspondingly, every creditor uses an equilibrium threshold level  $y_{*,0}$ . Unexpectedly, at a certain time, all creditors find out that the recovery rate drops to a lower level  $\alpha_l < \alpha_h$ . What would the new equilibrium threshold be?

Let's start with an individual creditor's threshold choice, which depends on others' choice. Suppose that all the other creditors still use the original threshold  $y_{*,0}$ . Then, by solving the HJB equation in (9), we can derive the creditor's optimal threshold  $y_{*,1}$ , which is higher than  $y_{*,0}$  because the lower liquidation value generates a greater expected loss to the creditor in the event that the firm is forced into a premature liquidation during his contract period. Of course, each creditor will go through this same calculation and choose a new threshold. If all creditors choose the threshold  $y_{*,1}$ , then an individual creditor's optimal threshold as the best response to  $y_{*,1}$  would be  $y_{*,2}$ , another level even higher than  $y_{*,1}$ . If all creditors choose  $y_{*,2}$ , then each creditor would go through another round of updating, and so on and so forth. Figure 2 illustrates this updating process until it eventually converges to a fixed point  $y_{*,\infty}$ , the new equilibrium threshold.

The difference between the threshold levels  $y_{*,1}$  and  $y_{*,0}$  represents the necessary safety margin a creditor would demand in response to the reduced asset liquidation value if other creditors' rollover strategies stay the same. This increase in threshold is eventually amplified to a much larger increase  $y_{*,\infty} - y_{*,0}$  through the rat race between creditors. This amplification

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<sup>17</sup>This  $\theta$  value is rather modest relative to the recent experience of Bear Stearns, which lasted for 3 days under the runs of its creditors and clients before a forced sale to JP Morgan in March 2008, e.g., Cox (2008).

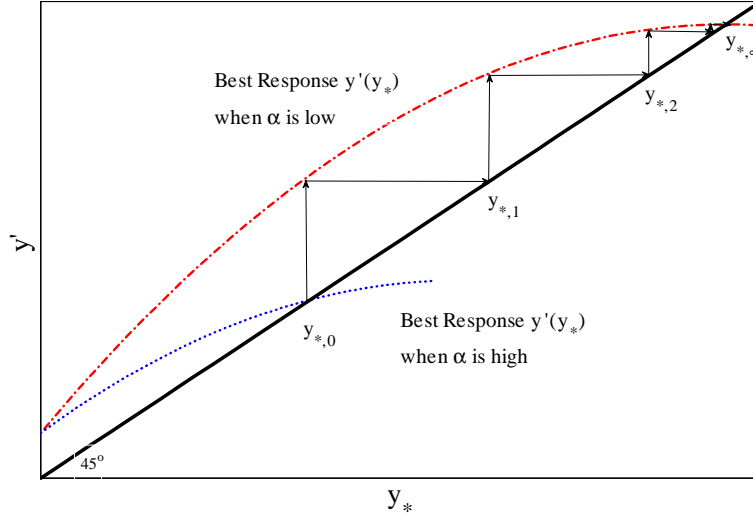


Figure 2: An illustration of the rat race between creditors in choosing rollover thresholds.

mechanism is a reflection of the externality of each creditor's running decision on other creditors and directly drives the debt runs in our model.

We now analyze the magnitude of the change in the equilibrium rollover threshold as we vary  $\alpha$  from its baseline value of 0.6. We measure the threshold by the fundamental value of the firm asset at  $y_*$ ,  $F(y_*) = \frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu}y_*$ , which is directly comparable to the firm's outstanding liability, 1.

In Figure 3, the flat thin solid line represents the equilibrium threshold  $F(y_{*,0}) = 1.59$  when  $\alpha$  takes the baseline value 0.6. The thick solid line shows that as  $\alpha$  deviates from its baseline value of 0.6 and decreases from 0.7 to 0.3,  $F(y_{*,\infty})$  rises monotonically from 1.36 to 3.18. Note that  $F(y_{*,\infty})$  is always above 1. As each maturing creditor only holds a partial stake in the firm, it makes sense for him to run and get his money back before the firm's fundamental value drops below the outstanding liability. This is because he does not internalize the cost of his run on the whole firm.

Moreover, the equilibrium threshold decreases with  $\alpha$  because a lower liquidation value increases the expected loss to each creditor in the event of a forced liquidation. We formally prove this result in the following proposition:

**Proposition 4** *The equilibrium rollover threshold  $y_*$  decreases with the firm's premature liquidation recovery rate  $\alpha$ .*

To illustrate the magnitude of the aforementioned amplification effect, we further decompose  $F(y_{*,\infty}) - F(y_{*,0})$ , the effect of an  $\alpha$  change on  $F(y_*)$ , into two components. The

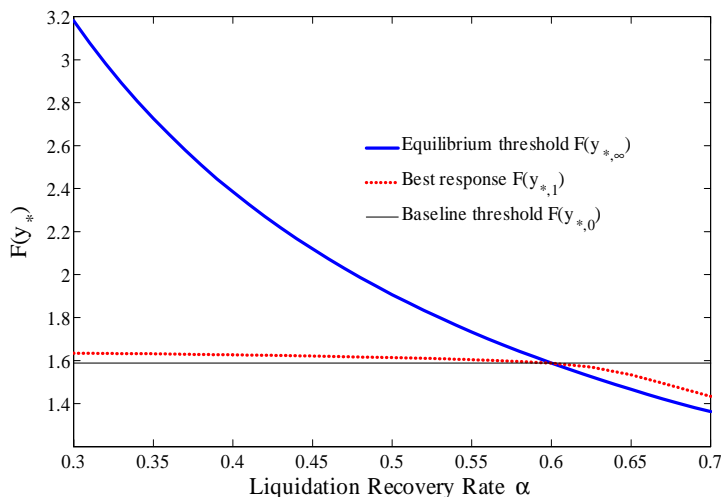


Figure 3: The equilibrium rollover threshold vs the liquidation recovery rate  $\alpha$ . This figure uses the baseline parameters in (10). The threshold is measured in the firm’s fundamental value  $F(y_*)$ . The thin solid line is the baseline threshold level,  $F(y_{*,0})$ . The thick solid line plots the equilibrium threshold  $F(y_{*,\infty})$ , as  $\alpha$  deviates from its baseline value. The dashed line plots a creditor’s best response  $F(y_{*,1})$  to the change in  $\alpha$  while fixing other creditors’ threshold at  $F(y_{*,0})$ .

dashed line in Figure 3 plots the best response of a creditor in the absence of the rat race between creditors. Suppose  $\alpha$  drops unexpectedly from its baseline level 0.6 to 0.4. After the drop in  $\alpha$ , by solving the HJB equation in (9) numerically, we find that an individual creditor will choose an optimal threshold  $F(y_{*,1}) = 1.63$  (on the dashed line) if the other creditors’ rollover threshold is fixed at the baseline level  $F(y_{*,0}) = 1.59$  (the thin solid line). The difference  $F(y_{*,1}) - F(y_{*,0}) = 0.04$  represents the safety margin necessary to compensate the creditor for the increased expected bankruptcy loss in the absence of the rat race. Of course, once we take into account the rat race, each creditor ends up choosing a higher equilibrium threshold of  $F(y_{*,\infty}) = 2.38$  (on the thick solid line). The difference  $F(y_{*,\infty}) - F(y_{*,1})$  represents the amplification effect of the rat race, which is about 20 times the effect without the rat race. This decomposition shows that in the absence of the rat race between creditors, a change in  $\alpha$  only has a rather modest effect on the equilibrium threshold choice, but the rat race dramatically amplifies this effect.

Taken together, Figure 3 shows that a change in asset illiquidity can have a large effect on the firm’s financial stability.

**Spillover and Systemic Risk** Federal Reserve Chairman Ben Bernanke (2008) describes the potential systemic risk following the collapse of Bear Stearns as the key reason that led

the Fed to open the discount window to every major investment bank. We can readily extend our model to include multiple firms holding similar assets to analyze this type of systemic risk triggered by creditors' panic runs on one firm. As these firms face the same downward sloping demand curve for their assets in an illiquid secondary market, the liquidation recovery rate  $\alpha$  of each firm depends on other firms' liquidation, e.g., Shleifer and Vishny (1992). Suppose that one firm, say Bear Stearns, suffers idiosyncratic negative shocks to its fundamental. As a result, when this firm experiences runs by its creditors and needs to liquidate its asset, the liquidation potentially pushes down the liquidation values of other firms. This in turn increases the losses of other firms' creditors in the event that their firms are forced into liquidation. Thus, through this liquidation-value channel, panic runs spill over to these firms as their creditors now have greater incentives to run, even if there is no fundamental deterioration in these firms.<sup>18</sup> The possibility of other firms experiencing runs also feeds back to the creditors of the initial firm in distress, creating even greater incentives to run. In this way, a rat race to exit risky debt is underway not just between creditors of one firm, but also between creditors of all firms holding similar assets.<sup>19</sup> Thus, market liquidity evaporates and systemic risk becomes imminent.

## 4.2 Fundamental Volatility

Fundamental volatility  $\sigma$  affects an individual creditor's optimal rollover threshold through several channels. We can intuitively discuss these channels through various terms in the creditor's value function in equation (8). First, when the firm's fundamental volatility increases, its insolvency risk, which is reflected by the term  $\min(1, y_\tau) \mathbf{1}_{\{\tau=\tau_\phi\}}$ , rises because it becomes more likely that the firm's asset value at the asset maturity could be insufficient to pay off its liability. The increased insolvency risk prompts each creditor to use a higher rollover threshold. Second, a higher volatility also increases the firm's rollover risk through the term  $\min(1, L + ly_\tau) \mathbf{1}_{\{\tau=\tau_\theta\}}$  (i.e., other creditors might choose to run and cause the firm to fail before the creditor's debt matures.) More precisely, through a rat race similar to the one described in the previous subsection, imperfect coordination between creditors causes each creditor to choose an even higher threshold to protect himself against other creditors' runs in the future. Third, once the creditor's debt matures, he has the option to

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<sup>18</sup>This spillover mechanism is complementary to the existing ones proposed by Allen and Gale (2000) through the interbank lending channel and by Kyle and Xiong (2001) through the wealth effect of financial intermediaries.

<sup>19</sup>This mechanism is closely related to market runs analyzed by Bernardo and Welch (2004) and Morris and Shin (2004a), who treat each firm's credit constraint as exogenously imposed.

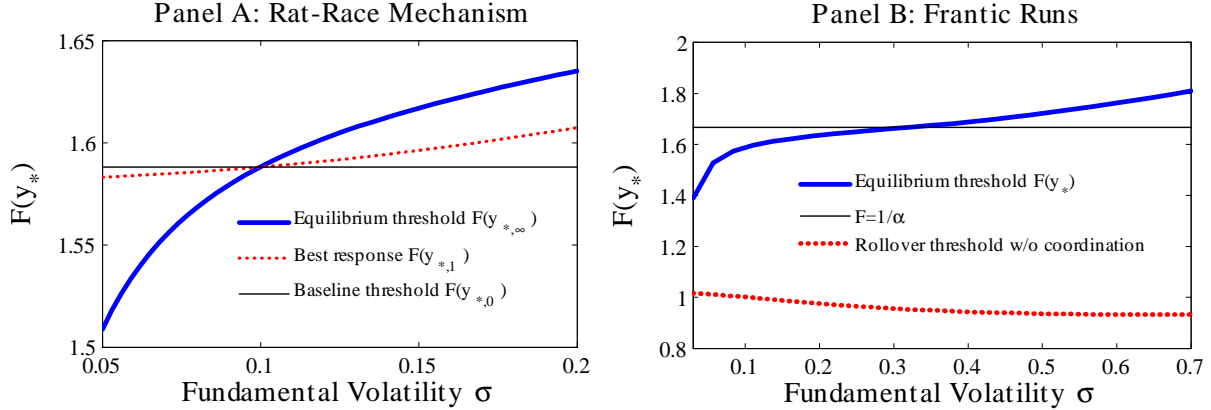


Figure 4: The equilibrium rollover threshold vs the fundamental volatility  $\sigma$ . This figure uses the baseline parameters in (10). The threshold is measured in the firm’s fundamental value  $F(y_*)$ . In panel A, the thin solid line is the baseline threshold level  $F(y_{*,0})$ , the thick solid line plots the equilibrium threshold  $F(y_{*,\infty})$  as  $\sigma$  deviates from its baseline value, and the dashed line plots a creditor’s best response  $F(y_{*,1})$  to the change in  $\sigma$  while fixing other creditors’ threshold at  $F(y_{*,0})$ . In Panel B, the thick solid line plots the equilibrium threshold  $F(y_*)$ , the thin solid line gives the benchmark level  $\frac{1}{\alpha}$ , and the dashed line plots the rollover threshold  $F(y_s)$  in the absence of any coordination problem.

roll over his debt and take advantage of the debt’s high interest payments if the firm fundamental is sufficiently strong. Through this embedded option, which is reflected by the term  $\max_{\text{rollover or run}} \{V(y_\tau; y_*), 1\} \mathbf{1}_{\{\tau=\tau_\delta\}}$ , a higher fundamental volatility motivates the creditor to choose a lower rollover threshold. The effect of the embedded option works in an opposite direction to those of the insolvency risk and rollover risk.

Figure 4 illustrates the net effect of these three channels. In Panel A, as  $\sigma$  deviates from its baseline value of 10% and increases from 5% to 20%, the equilibrium rollover threshold  $F(y_*)$  (the thick line) increases from 1.51 to 1.63. We can formally prove that the equilibrium threshold increases with  $\sigma$  if the firm’s credit lines are sufficiently unreliable, i.e.,  $\theta$  is sufficiently high. Under this condition, the firm would easily fail under a run, and consequently the embedded-option channel becomes dominated by the other two channels. In fact, our numerical exercises show that this result also holds when  $\theta$  takes a modest value.

**Proposition 5** *Suppose that  $\theta$  is sufficiently high. Then, the equilibrium rollover threshold  $y_*$  increases with the firm’s fundamental volatility  $\sigma$ .*

To illustrate the effect of the rat race, we also plot an individual creditor’s best response  $F(y_{*,1})$  to the change in  $\sigma$  (the dashed line) while fixing the other creditors’ threshold at the



baseline level  $F(y_{*,0}) = 1.59$  when  $\sigma$  takes its baseline level 10%. When  $\sigma$  rises above its baseline level, the increase  $F(y_{*,1}) - F(y_{*,0})$  represents the safety margin that the creditor would demand to protect himself against the increased rollover risk in the absence of the rat race between creditors. As  $\sigma$  varies from 5% to 20%,  $F(y_{*,1})$  increases from 1.51 to 1.63. Relative to the dashed line, the thick solid line shows that the range of the equilibrium threshold  $F(y_{*,\infty})$  is wider. For instance, when we increase  $\sigma$  from 10% to 15%, an individual creditor will only raise his threshold by 0.01, from  $F(y_{*,0}) = 1.59$  to  $F(y_{*,1}) = 1.60$ , if the other creditors' threshold is fixed at 1.59. However, after taking into account the rat race between creditors, each would use a new equilibrium threshold of 1.62, which implies that the rat race amplifies the effect of the volatility increase by 200%. Overall, the plot in Panel A shows that an increase in fundamental volatility can significantly raise the equilibrium rollover threshold.<sup>20</sup>

**Frantic Runs** Recall that in Section 3.3.1 we considered a static-rollover benchmark, in which all creditors make simultaneous rollover decisions and face no future rollover risk. This setting closely resembles the Diamond-Dybvig model. Proposition 3 shows that in this setting panic runs occur only when the firm fundamental is below the critical level  $y_h = \frac{1-L}{l}$ . At this level, the firm's fundamental value is  $F(y_h) = \frac{1}{\alpha}$ , which means that the firm can pay off its liability even after liquidating its asset at the fire-sale price. In the absence of future rollover risk, this strong fundamental is sufficient to preclude any creditor's concern about other creditors' concurrent rollover decisions, and thus provides a useful benchmark to evaluate the severity of dynamic runs.

Panel B of Figure 4 plots the equilibrium rollover threshold against fundamental volatility  $\sigma$  over a wider range than Panel A. Once  $\sigma$  rises above 30%, the equilibrium rollover threshold  $F(y_{*,\infty})$  (the thick solid line) surpasses  $1/\alpha$  (the thin solid line). That is, even though the firm is so well capitalized that it can pay back its liability after a forced liquidation, creditors are still not assured and may choose to run.

What drives this type of frantic runs? The driving force is exactly the firm's future rollover risk. The capacity for the firm's liquidation value to pay back its liability now is not a guarantee for future periods since the liquidation value may fall with the fundamental later. As a result, a maturing creditor is still worried that during his next contract period,

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<sup>20</sup>One might argue that analyzing the effect of fundamental uncertainty in static bank-run models based on the global games framework could lead to a similar insight. Such an analysis requires a specific information structure for agents holding private information, which, as we discussed in footnote 5, is crucial for the results.

other creditors might choose to run and cause the firm to fail. The firm’s liquidation value at that time may not be sufficient to pay off its liability. When this concern becomes sufficiently strong, he chooses to run despite the firm’s strong fundamental now. Figure 4 shows that this occurs when the firm’s fundamental is sufficiently volatile.

One might argue that as fundamental volatility becomes large, the firm’s insolvency risk also rises. To further highlight that the frantic runs are not simply driven by insolvency risk, we also consider an otherwise identical firm financed by a single large creditor, based on the setting described in Section 3.3.1. Suppose that a small creditor also holds a negligible fraction of the firm’s debt. Since the large creditor will always roll over his debt (Proposition 2), the small creditor’s rollover threshold choice is only affected by the firm’s insolvency risk and his embedded option in the firm.<sup>21</sup> As shown by the dashed line in Panel B of Figure 4, his rollover threshold decreases with the firm’s fundamental volatility, suggesting that the embedded-option effect dominates the insolvency-risk effect. The contrast between this line and the thick solid line confirms that the frantic runs are indeed driven by the creditors’ fear of the firm’s future rollover risk.

The emergence of such frantic runs demonstrates the severe effect of dynamic runs. In practice, firms spread out their debt expirations across time to reduce the liquidity risk of having to roll over large quantities of debt at the same time. But, the staggered debt structure also introduces a dynamic coordination problem between creditors. Our model shows this dynamic coordination problem can cause each creditor’s rollover threshold to be higher than the benchmark level that is sufficient to preclude any run in the static bank-run setting considered by Diamond and Dybvig (1983). This outcome thus prompts more attention on this type of dynamic runs.<sup>22</sup>

### 4.3 Credit Lines

Credit lines temporarily sustain the firm under the creditors’ runs. A common intuition suggests that the stronger the firm’s credit lines, the less likely creditors will choose to run. In other words, the creditors’ equilibrium rollover threshold should decrease with the reliability of the firm’s credit lines  $\frac{1}{\theta\delta}$ . We now examine this intuition.

<sup>21</sup>The value of his debt is given by equation (8) with the rollover risk term  $\min(1, L + ly_\tau) \mathbf{1}_{\{\tau=\tau_\theta\}}$  removed.

<sup>22</sup>Rollover risk could also arise from rolling over synchronous short-term debt contracts. In that setting, each creditor faces two types of coordination problems on a rollover date: one is the static problem with concurrent creditors and the other is the dynamic problem with future creditors. The presence of these two types of coordination problems makes it difficult to isolate the effect of each of them. Nevertheless, comparing the rollover risk generated by synchronous and asynchronous debt structures is an interesting and challenging topic for future research.

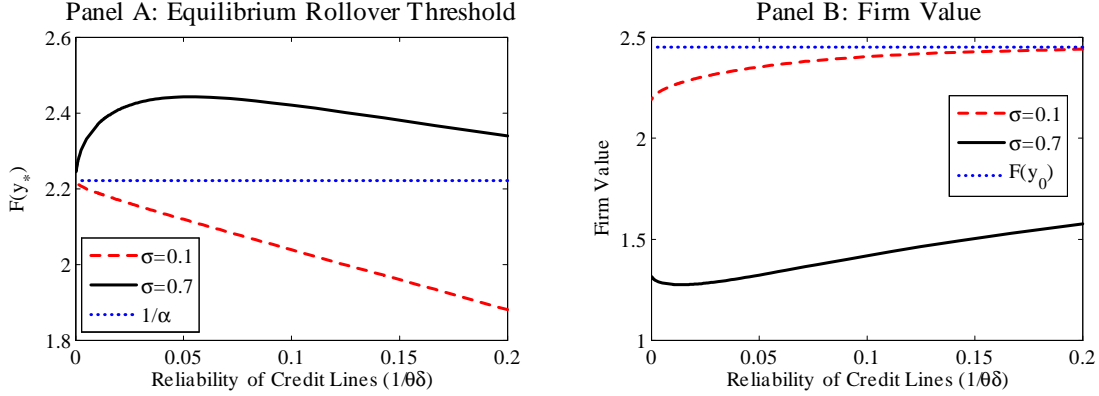


Figure 5: Equilibrium effects of credit lines. This figure uses the baseline parameters in (10). In panel A, the threshold is measured in the firm’s fundamental value  $F(y_*)$ . The solid and dashed lines plot the equilibrium threshold with respect to the quality of the firm’s credit lines  $1/\theta\delta$  when its fundamental volatility  $\sigma = 70\%$  and  $10\%$ , while the dotted line gives the benchmark level  $F(y_*) = 1/\alpha$ . In Panel B, the solid and dashed lines plot the firm value when  $\sigma = 70\%$  and  $10\%$ , conditional on the creditors’ equilibrium rollover threshold.

Figure 5 illustrates the equilibrium effects of credit lines. Panel A plots the equilibrium rollover threshold with respect to the reliability of the firm’s credit lines. The dotted flat line gives the benchmark level  $F(y_h) = \frac{1}{\alpha}$ , while the dashed and solid lines correspond to the equilibrium threshold for two different levels of fundamental volatility  $\sigma = 10\%$  and  $70\%$ . When the firm does not have any credit line (i.e.,  $\frac{1}{\theta\delta} \rightarrow 0$ ), the equilibrium rollover threshold collapses to the benchmark level  $F(y_h) = \frac{1}{\alpha}$  regardless of the value of fundamental volatility. This shows that the presence of credit lines is important to the occurrence of frantic runs discussed earlier.

When fundamental volatility is low ( $\sigma = 10\%$ ), the dashed line shows that the equilibrium rollover threshold decreases with the reliability of credit lines. This is consistent with the standard intuition that credit lines deter runs. However, when volatility is high ( $\sigma = 70\%$ ), the solid line shows that this relationship becomes non-monotonic—the rollover threshold initially increases with the reliability of credit lines and then decreases with it. The increasing segment of the relationship is intriguing as it indicates that stronger credit lines can prompt creditors to run earlier.

This non-monotonic pattern arises due to a joint effect of fundamental volatility and credit lines. On one hand, the credit lines can sustain the firm under the creditors’ runs for a certain period, during which a creditor’s contract may mature; on the other hand, during this period, fundamental volatility can cause the firm fundamental to deteriorate to

far below its current level when the firm eventually fails under the creditors' persistent runs. The former effect mitigates the creditors' incentives to run, while the latter effect exacerbates the incentives. In particular, the latter effect increases with fundamental volatility due to the concavity of debt payoff. Thus, when fundamental volatility is sufficiently high, the latter effect dominates the former effect, and, as a result, more reliable credit lines prompt creditors to run earlier. This joint effect of fundamental volatility and credit lines also underlies the occurrence of frantic runs discussed earlier.

Panel B of Figure 5 further plots the firm value with respect to the reliability of credit lines. The firm value is calculated by the expectation of its future discounted cash flows, given the creditors' equilibrium rollover threshold and the firm's bankruptcy probability. The plot shows that when fundamental volatility is high ( $\sigma = 70\%$ ) and the reliability of credit lines is low, the firm value can decrease with the reliability of credit lines.

To link our model to the recent credit crisis, one can broadly interpret the credit lines in the model as measures taken by central banks to temporarily bail out financially distressed financial firms. As political pressures can cause central banks to abandon their bailout effort, the reliability of any bail program is uncertain, as captured by the random breakdown of credit lines in our model. While many policy makers and pundits have advocated for various government bailout programs during the crisis, our model suggests that when fundamental volatility is sufficiently high, a bailout program with uncertain reliability can be counterproductive as it exacerbates rather than mitigates creditors' incentives to run.

#### 4.4 Rollover Frequency

The firm's rollover frequency  $\delta$  is another key determinant of its rollover risk. As  $\delta$  increases, each creditor's contract period, which has an expected duration of  $1/\delta$ , gets shorter. This generates two opposing effects on the equilibrium. First, each individual creditor is locked in for a shorter period. As a result, his embedded option on the firm is more valuable as he has more flexibility to pull out if the firm fundamental deteriorates. The increased embedded-option value makes the creditor more willing to roll over his debt, i.e., to choose a lower rollover threshold. On the other hand, a higher  $\delta$  also means that the other creditors are locked in for a shorter period. As a result, during the creditor's contract period, the firm is more susceptible to the rollover risk created by the other creditors. The increased rollover risk therefore motivates him to choose a higher rollover threshold. The equilibrium threshold  $y_*$  trades off the embedded-option effect and the rollover-risk effect.

Figure 6 plots the equilibrium rollover threshold (the thick solid line) as we vary  $\delta$  from

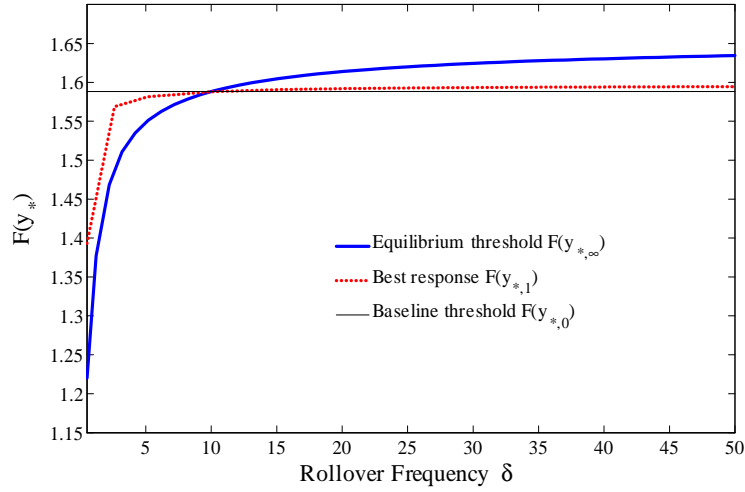


Figure 6: The equilibrium rollover threshold vs the rollover frequency  $\delta$ . This figure uses the baseline parameters in (10). The threshold is measured in the firm's fundamental value  $F(y_*)$ . The thin solid line is the baseline threshold level  $F(y_{*,0})$ . The thick solid line plots the equilibrium threshold  $F(y_{*,\infty})$  as  $\delta$  deviates from its baseline value. The dashed line plots a creditor's best response  $F(y_{*,1})$  to the change in  $\delta$  while fixing the other creditors' threshold at  $F(y_{*,0})$ .

its baseline value of 10 to a range between 0.2 to 50, along with an individual creditor's best response (the dashed line) to the  $\delta$  change while fixing the other creditors' rollover threshold at the baseline level of 1.59. As  $\delta$  increases from 0.2 to 50, the equilibrium rollover threshold  $F(y_*)$  increases from 1.15 to 1.64. This monotonically increasing pattern in  $F(y_*)$  suggests that the rollover risk effect dominates the embedded-option effect in this illustration. In unreported numerical analysis, we also find that this holds true over a wide range of parameter values. The embedded-option effect becomes dominant only when  $\theta$  is low, i.e., the firm's credit lines are sufficiently reliable and the firm's rollover risk is modest.<sup>23</sup>

We again observe a dramatic amplification effect caused by the rat race among the creditors in choosing higher and higher thresholds. For instance, consider raising  $\delta$  from the baseline level 10 to 50, i.e., shortening the debt maturity from 27 days to about 1 week. An individual creditor would slightly increase his rollover threshold by 0.005 in the absence of the rat race, while the new equilibrium threshold is higher by 0.05, implying that the rat race amplifies the effect of the  $\delta$  increase by a factor of 10.

<sup>23</sup>To be precise, fixing the other parameters in (10), this happens when  $\theta$  is lower than 0.1, i.e., conditional on each maturing creditor choosing to run, the firm can survive on average for one year ( $1/\theta\delta$ ).

**Creditors' Maturity Preference** Our discussion earlier suggests that each creditor would prefer a shorter debt maturity so that he has more flexibility to pull out of a firm before others if the firm fundamental deteriorates later. We formally derive the following proposition:

**Proposition 6** *Fixing the other creditors' rollover frequency, each creditor's value function increases with his own rollover frequency.*

This proposition suggests that in the absence of any commitment device like debt covenants or regulatory requirement, the firm could use shortening debt maturity as a survival tool when creditors refuse to roll over their maturing debt. Our earlier analysis suggests that this would happen when the firm fundamental falls, when the fundamental volatility rises, or when the firm asset becomes more illiquid. In fact, this is consistent with the dramatic shortening in the maturity structure of commercial paper issuance around mid-September 2008, e.g., Krishnamurthy (2009).<sup>24</sup> However, our analysis also shows that as other creditors' debt maturity becomes shorter, each creditor has a greater incentive to run because he anticipates the firm's greater rollover risk in the future. Thus, maturity shortening can act as a self-enforcing tightening mechanism to push the firm closer and closer to bankruptcy.<sup>25</sup>

**Credit Risk** The standard credit modeling approach, following the classic structural model of Merton (1974), focuses on insolvency risk (i.e., the risk that a firm's asset value could fall below its liability) as the only source of credit risk (i.e., the risk that a firm defaults on its debt). However, our model shows that even before the firm becomes insolvent, fear of the firm's future rollover risk can cause creditors to run on the firm. Thus, rollover risk is an important source of credit risk.

To illustrate this effect, we examine the credit spread of a hypothetical infinitesimal zero-coupon bond issued by the firm analyzed in our model. The bond has a face value of 1 and a fixed maturity  $T$ . We specify the bond payoff so that it precisely captures the firm's credit risk before time  $T$ .<sup>26</sup> The credit spread is the difference between the bond yield and the yield

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<sup>24</sup>The issuance of commercial paper with maturities less than 9 days increased by more than 50%, replacing maturities over 9 days. Anecdotally, much of the shortening was in fact to overnight paper.

<sup>25</sup>Brunnermeier and Oehmke (2009) formally analyze the equilibrium maturity choice in a setting different than ours. They show that the conflict of interest between short-term and long-term creditors leads to a maturity rat race, through which the firm ends up with excessive short-term debt in equilibrium.

<sup>26</sup>The bond payoff depends on three scenarios: 1) if the firm's asset matures before  $T$  and before any forced liquidation, the bond pays  $\min(y_{\tau_\phi}, 1)$ ; 2) if a forced liquidation occurs before  $T$  and before the asset matures, the bond pays  $\min(L + ly_{\tau_\phi}, 1)$ ; 3) otherwise, the bond pays 1.

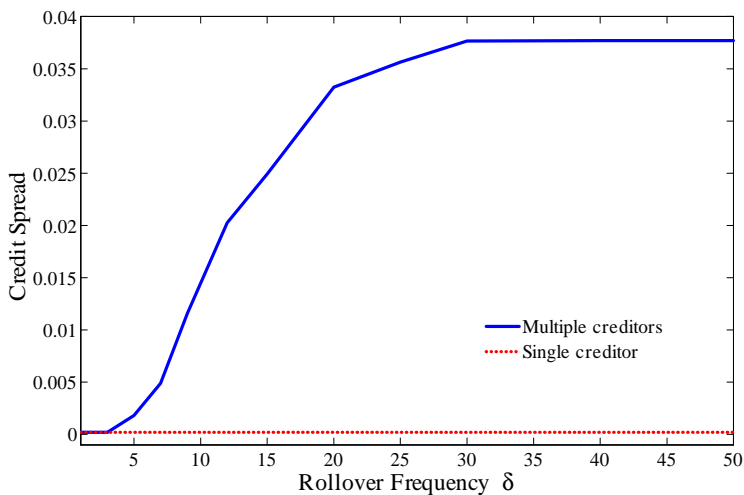


Figure 7: Credit spread vs debt rollover frequency. This figure uses the baseline parameters in (10),  $y_0 = 1.25$ , and  $T = 0.25$ . The solid line plots the credit spread of a firm financed by a continuum of small creditors. The dashed line plots the credit spread of a firm financed by a single creditor.

of a risk-free bond with the same maturity.<sup>27</sup> To provide a benchmark, we also introduce another firm identical in all other dimensions except that it is financed by a single creditor with deep pockets. As the single creditor will always roll over his debt (Proposition 2), this firm has no rollover risk.

We numerically calculate the value of this hypothetical bond with  $y_0 = 1.25$ ,  $T = 0.25$  (3 months) for both firms, and plot the credit spreads with respect to their debt rollover frequency  $\delta$  in Figure 7. The difference between these two credit spreads measures the contribution of rollover risk to the credit risk of the firm with multiple creditors. The credit spread of the firm with a single creditor is independent of  $\delta$ . However, the credit spread of the firm with multiple creditors increases sharply from less than 2 basis points to over 380 basis points as  $\delta$  increases from 1 to 50 (i.e., from once every one year to once every week). While this illustration is simplistic, it nevertheless shows that rollover risk can be a substantial part of a firm’s credit risk.

In conjunction with our earlier analysis of rollover risk, this exercise suggests that the firm’s corporate bond spreads depend not only on common measures of credit risk, such as fundamental risk and leverage, but also on its asset illiquidity and debt maturity structure. Our model thus corroborates Morris and Shin (2004b, 2009) who also point out that the

<sup>27</sup>Because of the three possible scenarios, the risky bond could provide a payoff before its maturity  $T$ . For a fair comparison, we also impose the same timing of payoff on the risk-free bond, which has a value of  $\frac{\phi}{\rho+\phi} + \frac{\rho}{\rho+\phi}e^{-(\rho+\phi)T}$ . Then, the yield of the risk-free bond is  $\beta_{riskfree} = -\frac{1}{T} \ln \left( \frac{\phi}{\rho+\phi} + \frac{\rho}{\rho+\phi}e^{-(\rho+\phi)T} \right)$ .

coordination problem between creditors can have an important effect on firms' credit risk. They develop models with two periods by using the global games framework. In contrast, our continuous-time setting is easier to integrate with the standard credit risk models.

## 5 Conclusion

In this paper, we develop a dynamic model of panic runs by creditors on a firm, which invests in an illiquid asset by rolling over staggered short-term debt contracts. Our model highlights that the firm's rollover risk is intertwined with its fundamental risk. In particular, fear of the firm's future rollover risk could motivate each creditor to preemptively run ahead of others even when the firm is still fundamentally healthy. Our model characterizes deteriorating fundamentals, asset illiquidity, fundamental volatility, reliability of credit lines, and debt maturity as important determinants of such dynamic runs. In particular, when fundamental volatility is sufficiently high, our model shows that creditors may choose to run even if the current firm fundamental is higher than the benchmark level that is sufficient to preclude any run in the static bank-run setting considered by Diamond and Dybvig (1983), and that imperfect credit lines (or government bailout programs) can be counter productive in deterring runs.

## A Appendix

### A.1 Proof of Theorem 1

Using the HJB equation in (9), we first construct an individual creditor's value function by utilizing the fact that in any symmetric equilibrium all creditors (including this individual creditor) use the same monotone strategy with threshold  $y_*$ . The equilibrium threshold must then be the solution to the equation  $V(y_*; y_*) = 1$ . Of course, individual optimality requires that  $V(y; y_*) > 1$  for  $y > y_*$  and  $V(y; y_*) < 1$  for  $y < y_*$ , a condition that we will verify in Lemma 9. Later in Lemma 10 we also show that there does not exist any asymmetric threshold equilibrium.

When all creditors use the same threshold  $y_*$ , the HJB equation (9) becomes

- If  $y < y_*$ ,

$$0 = \frac{\sigma^2}{2} y^2 V_{yy} + \mu y V_y - [\rho + \phi + (\theta + 1) \delta] V(y; y_*) + \phi \min(1, y) + \theta \delta \min(L + ly, 1) + r + \delta; \quad (11)$$

- If  $y \geq y_*$ ,

$$0 = \frac{\sigma^2}{2} y^2 V_{yy} + \mu y V_y - (\rho + \phi) V(y; y_*) + \phi \min(1, y) + r. \quad (12)$$



The value function has to satisfy these two differential equations and be continuous and differentiable at the boundary point  $y_*$ . In solving these differential equations, we need to introduce the two roots to the first fundamental equation for (11):

$$\frac{1}{2}\sigma^2 x(x-1) + \mu x - [\rho + \phi + (1+\theta)\delta] = 0, \quad (13)$$

which are

$$-\gamma_1 = -\frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2[\rho + \phi + (1+\theta)\delta]}}{\sigma^2} < 0, \quad (14)$$

and

$$\eta_1 = -\frac{\mu - \frac{1}{2}\sigma^2 - \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2[\rho + \phi + (1+\theta)\delta]}}{\sigma^2} > 1; \quad (15)$$

and the two roots to the second fundamental equation for (12):

$$\frac{1}{2}\sigma^2 x(x-1) + \mu x - (\rho + \phi) = 0, \quad (16)$$

which are

$$-\gamma_2 = -\frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2(\rho + \phi)}}{\sigma^2} < 0, \quad (17)$$

and

$$\eta_2 = -\frac{\mu - \frac{1}{2}\sigma^2 - \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2(\rho + \phi)}}{\sigma^2} > 1. \quad (18)$$

We summarize the constructed value function below.

**Lemma 7** *Given the equilibrium rollover threshold  $y_*$ , the value function of an individual creditor is given by the following three cases:*

1. If  $y_* < 1$ ,

$$V(y; y_*) = \begin{cases} \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\phi+\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu}y + A_1 y^{\eta_1} & \text{when } 0 < y \leq y_* \\ \frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu}y + A_2 y^{-\gamma_2} + A_3 y^{\eta_2} & \text{when } y_* < y \leq 1 \\ \frac{r+\phi}{\rho+\phi} + A_4 y^{-\gamma_2} & \text{when } 1 < y \end{cases}. \quad (19)$$

The four coefficients  $A_1$ ,  $A_2$ ,  $A_3$ , and  $A_4$  are given by

$$\begin{aligned} A_1 &= \frac{[H_3\gamma_2 + H_1] - y_*^{-\eta_2}(\gamma_2 H_4 + H_2 y_*)}{(\eta_1 + \gamma_2) y_*^{\eta_1 - \eta_2}}, \\ A_2 &= \frac{y_*^{\gamma_2}}{\eta_2 + \gamma_2} [\eta_2 H_4 - H_2 y_* + A_1 (\eta_2 - \eta_1) y_*^{\eta_1}], \\ A_3 &= \frac{y_*^{-\eta_2}}{\eta_2 + \gamma_2} [\gamma_2 H_4 + H_2 y_* + A_1 (\eta_1 + \gamma_2) y_*^{\eta_1}], \\ &= \frac{1}{\eta_2 + \gamma_2} [H_3\gamma_2 + H_1], \\ A_4 &= A_2 - \frac{1}{\eta_2 + \gamma_2} [H_3\eta_2 - H_1], \end{aligned}$$

where

$$\begin{aligned}
H_1 &= -\frac{\phi}{\rho + \phi - \mu}, \\
H_2 &= \frac{\theta \delta l (\rho + \phi - \mu) - \phi (1 + \theta) \delta}{(\rho + \phi + (1 + \theta) \delta - \mu) (\rho + \phi - \mu)}, \\
H_3 &= -\frac{\phi \mu}{(\rho + \phi) (\rho + \phi - \mu)}, \\
H_4 &= \frac{r + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} - \frac{r}{\rho + \phi} + H_2 y_*.
\end{aligned}$$

2. If  $1 < y_* \leq \frac{1-L}{l}$ ,

$$V(y; y_*) = \begin{cases} \frac{r + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} + \frac{\phi + \theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y + B_1 y^{\eta_1} & \text{when } y \leq 1, \\ \frac{r + \phi + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} + \frac{\theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y + B_2 y^{-\gamma_1} + B_3 y^{\eta_1} & \text{when } 1 < y \leq y_*, \\ \frac{r + \phi}{\rho + \phi} + B_4 y^{-\gamma_2} & \text{when } y_* < y. \end{cases} \quad (20)$$

The four coefficients  $B_1$ ,  $B_2$ ,  $B_3$ , and  $B_4$  are given by

$$\begin{aligned}
B_1 &= B_3 - \frac{M_2 \gamma_1 + M_1}{\eta_1 + \gamma_1}, \\
B_2 &= \frac{M_2 \eta_1 - M_1}{\eta_1 + \gamma_1} < 0, \\
B_3 &= \frac{(\gamma_1 - \gamma_2) B_2 (y_*)^{-\gamma_1} + \gamma_2 M_3 - \frac{\theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y_*}{(\eta_1 + \gamma_2) y_*^{\eta_1}}, \\
B_4 &= \frac{\eta_1 + \gamma_1}{\eta_1 + \gamma_2} B_2 y_*^{\gamma_2 - \gamma_1} + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} \frac{\theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y_*^{\gamma_2 + 1}, \\
&\quad - \frac{\eta_1}{\eta_1 + \gamma_2} \left[ \frac{r + \phi}{\rho + \phi} - \frac{r + \phi + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} \right] y_*^{\gamma_2}, \\
&= \frac{(\eta_1 + \gamma_1) B_2 y_*^{-\gamma_1} - \eta_1 M_3 - \frac{\theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y_*}{(\eta_1 + \gamma_2) y_*^{-\gamma_2}},
\end{aligned}$$

where

$$\begin{aligned}
M_1 &= \frac{\phi}{\rho + \phi + (1 + \theta) \delta - \mu}, \\
M_2 &= \frac{\phi \mu}{(\rho + \phi + (1 + \theta) \delta) (\rho + \phi + (1 + \theta) \delta - \mu)}, \\
M_3 &= \frac{r + \phi}{\rho + \phi} - \frac{r + \phi + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} - \frac{\theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y_*.
\end{aligned}$$

3. If  $y_* > \frac{1-L}{l}$ ,

$$V(y; y_*) = \begin{cases} \frac{r + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} + \frac{\phi + \theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y + C_1 y^{\eta_1} & \text{when } y \leq 1, \\ \frac{r + \phi + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} + \frac{\theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y + C_2 y^{-\gamma_1} + C_3 y^{\eta_1} & \text{when } 1 < y \leq \frac{1-L}{l}, \\ \frac{r + \phi + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} + C_4 y^{-\gamma_1} + C_5 y^{\eta_1} & \text{when } \frac{1-L}{l} < y \leq y_*, \\ \frac{r + \phi}{\rho + \phi} + C_6 y^{-\gamma_2} & \text{when } y > y_*. \end{cases} \quad (21)$$

The six coefficients  $C_1, C_2, C_3, C_4, C_5$  and  $C_6$  are given by

$$\begin{aligned}
C_1 &= C_3 - \frac{K_4\gamma_1 + K_5}{\eta_1 + \gamma_1}, \\
C_2 &= \frac{K_4\eta_1 - K_5}{\eta_1 + \gamma_1}, \\
C_3 &= C_5 + \frac{K_2\gamma_1 - K_3\frac{1-L}{l}}{(\eta_1 + \gamma_1)\left(\frac{1-L}{l}\right)^{\eta_1}}, \\
C_4 &= C_2 - \frac{K_2\eta_1 + K_3\frac{1-L}{l}}{(\eta_1 + \gamma_1)\left(\frac{1-L}{l}\right)^{-\gamma_1}}, \\
C_5 &= \frac{(\gamma_1 - \gamma_2)C_4y_*^{-\gamma_1} - \gamma_2K_1}{(\eta_1 + \gamma_2)y_*^{\eta_1}}, \\
C_6 &= \frac{(\eta_1 + \gamma_1)C_4y_*^{-\gamma_1} + \eta_1K_1}{(\eta_1 + \gamma_2)y_*^{-\gamma_2}},
\end{aligned}$$

where

$$\begin{aligned}
K_1 &= \frac{r + \phi + \theta\delta + \delta}{\rho + \phi + (1 + \theta)\delta} - \frac{r + \phi}{\rho + \phi}, \\
K_2 &= \frac{\theta\delta(1-L)}{\rho + \phi + (1 + \theta)\delta} - \frac{\theta\delta(1-L)}{\rho + \phi + (1 + \theta)\delta - \mu}, \\
K_3 &= \frac{\theta\delta l}{\rho + \phi + (1 + \theta)\delta - \mu}, \\
K_4 &= \frac{\phi}{\rho + \phi + (1 + \theta)\delta - \mu} - \frac{\phi}{\rho + \phi + (1 + \theta)\delta}, \\
K_5 &= \frac{\phi}{\rho + \phi + (1 + \theta)\delta - \mu}.
\end{aligned}$$

**Proof.** We can derive the three cases listed above using the same method. For illustration we just solve the first case with  $y_* < 1$ . Depending on the value of  $y$ , we have the following three scenarios.

- If  $0 < y \leq y_*$  :

$$\frac{\sigma^2}{2}y^2V_{yy} + \mu yV_y - [\rho + \phi + (1 + \theta)\delta]V(y) + (\phi + \theta\delta l)y + r + \theta\delta L + \delta = 0.$$

The general solution of this differential equation is given in the first line of equation (19) with the coefficient  $A_1$  to be determined by the boundary conditions. Note that to ensure the value of  $V$  is finite as  $y$  approaches zero, we have ruled out another power solution  $y^{-\gamma_1}$  of the equation .

- If  $y_* < y \leq 1$  :

$$\frac{\sigma^2}{2}y^2V_{yy} + \mu yV_y - (\rho + \phi)V(y) + \phi y + r = 0.$$

The general solution of this differential equation is given in the second line of equation (19) with the coefficients  $A_2$  and  $A_3$  to be determined by the boundary conditions.

- If  $y > 1$  :

$$\frac{\sigma^2}{2}y^2V_{yy} + \mu yV_y - (\rho + \phi)V(y) + r + \phi = 0.$$

The general solution of this differential equation is given in the third line of equation (19) with the coefficient  $A_4$  to be determined by the boundary conditions. Note that to ensure the value of  $V$  is finite as  $y$  approaches infinity, we have ruled out another power solution  $y^{\eta_2}$  of the equation.

To determine the four coefficients  $A_1$ ,  $A_2$ ,  $A_3$ , and  $A_4$ , we have four boundary conditions at  $y = y_*$  and 1, i.e., the value function  $V(y)$  must be continuous (value-matching) and differentiable (smooth-pasting) at these two points. Solving these boundary conditions leads to the coefficients given in Lemma 7. ■

Based on the value function derived in Lemma 7, we now show that there exists a unique threshold  $y_*$  that satisfies the equilibrium condition.

**Lemma 8** *There exists a unique  $y_*$  such that*

$$V(y_*; y_*) = 1.$$

**Proof.** Define

$$W(y) \equiv V(y; y).$$

We need to show that there is a unique  $y_*$  such that  $W(y_*) = 1$ .

We first show that  $W(y)$  is monotonically increasing when  $y < 1$ . In this case, we can directly extract the value of  $W(y)$  from equation (19), which, by neglecting terms independent of  $y$ , is

$$W(y) = \left[ -H_1 + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} H_2 \right] y + \frac{[H_3 \gamma_2 + H_1]}{\eta_1 + \gamma_2} y^{\eta_2}.$$

Note that

$$\begin{aligned} \frac{dW(y)}{dy} &= -H_1 + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} H_2 + \frac{[H_3 \gamma_2 + H_1]}{\eta_1 + \gamma_2} \eta_2 y^{\eta_2 - 1} \\ &> -H_1 + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} H_2 + \frac{[H_3 \gamma_2 + H_1]}{\eta_1 + \gamma_2} \eta_2 \\ &= \frac{\eta_1 - 1}{\eta_1 + \gamma_2} (H_2 - H_1) + \frac{\eta_2 - \gamma_2 - 1}{\eta_1 + \gamma_2} H_1 + \frac{\gamma_2 \eta_2}{\eta_1 + \gamma_2} H_3, \end{aligned}$$

where the second inequality is due to the fact that  $H_3 < 0$  and  $H_1 < 0$  (defined in Lemma 7).

In the first term above,

$$H_2 - H_1 = \frac{\theta \delta l + \phi}{\rho + \phi + (1 + \theta) \delta - \mu}$$

is positive according to the parameter restriction in (5). For the second term, note that  $\eta_2 - \gamma_2 - 1 = -2 \frac{\mu}{\sigma^2}$ . Then after some algebraic substitutions (note that  $\gamma_2 \eta_2 = \frac{2(\rho + \phi)}{\sigma^2}$ ), the sum of the second and third terms is

$$-2 \frac{\mu}{\sigma^2} \frac{1}{\eta_1 + \gamma_2} H_1 + \frac{\gamma_2 \eta_2}{\eta_1 + \gamma_2} H_3 = 0.$$

Thus,  $\frac{dW(y)}{dy} > 0$ .

We now show that  $W(y)$  is monotonically increasing when  $1 < y \leq \frac{1-L}{l}$ . Equation (20) implies that

$$\begin{aligned} W(y) &= \frac{r + \phi}{\rho + \phi} + B_4 y^{-\gamma_2} \\ &= \frac{M_2 \eta_1 - M_1}{\eta_1 + \gamma_2} y^{-\gamma_1} + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} \frac{\theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y \\ &\quad + \frac{\gamma_2}{\eta_1 + \gamma_2} \frac{r + \phi}{\rho + \phi} + \frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \phi + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta}. \end{aligned}$$

We now show  $\eta_1 < \frac{M_1}{M_2} = \frac{\rho + \phi + (1 + \theta)\delta}{\mu}$ . Plugging  $x = \frac{\rho + \phi + (1 + \theta)\delta}{\mu}$  into the first fundamental equation (13), we find that the value is positive, which implies that  $\eta_1 < \frac{M_1}{M_2}$ . Therefore  $M_2\eta_1 - M_1 < 0$ , and the first term is increasing in  $y$ . Because  $\eta_1 > 1$ , the second term is increasing in  $y$ . As a result,  $W(y)$  is increasing in  $y$ .

Similarly we can show that  $W(y)$  is increasing in  $y$  for  $y > \frac{1-L}{l}$ . Equation (21) implies that

$$W(y) = \frac{r + \phi}{\rho + \phi} + C_6 y^{-\gamma_2} = \frac{\gamma_2}{\eta_1 + \gamma_2} \frac{r + \phi}{\rho + \phi} + \frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \phi + \theta\delta L + \delta}{\rho + \phi + (1 + \theta)\delta} + \frac{(\eta_1 + \gamma_1) C_4 y^{-\gamma_1}}{\eta_1 + \gamma_2}.$$

Since  $\frac{K_5}{K_4} = \frac{K_3 \frac{1-L}{l}}{-K_2} = \frac{\rho + \phi + (1 + \theta)\delta}{\mu} = M_1/M_2$ , we have

$$\begin{aligned} \frac{(\eta_1 + \gamma_1) C_4 y^{-\gamma_1}}{\eta_1 + \gamma_2} &= \frac{K_4 \eta_1 - K_5 + \frac{-K_2 \eta_1 - K_3 \frac{1-L}{l}}{(\frac{1-L}{l})^{-\gamma_1}}}{\eta_1 + \gamma_2} y^{-\gamma_1} \\ &= \frac{\eta_1 - M_1/M_2}{\eta_1 + \gamma_2} \left( K_4 y^{-\gamma_1} + (-K_2) \left( \frac{ly}{1-L} \right)^{-\gamma_1} \right). \end{aligned}$$

Therefore, because  $\eta_1 - M_1/M_2 < 0$  as shown in the case of  $1 < y \leq \frac{1-L}{l}$ , and we can check that  $K_4 > 0$  and  $-K_2 > 0$ ,  $W(y)$  is strictly increasing.

Next, we need to ensure that  $W(0) < 1$ . Equation (19) implies that

$$W(0) = \frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \theta\delta L + \delta}{\rho + \phi + (1 + \theta)\delta} + \frac{\gamma_2}{\eta_1 + \gamma_2} \frac{r}{\rho + \phi}.$$

The parameter restriction in (4) ensures that

$$\frac{r + \theta\delta L + \delta}{\rho + \phi + (1 + \theta)\delta} < 1 \quad \text{and} \quad \frac{r}{\rho + \phi} < 1,$$

thus,  $W(0) < 1$ .

Finally note that under our parameter restrictions in (4) and (6) we have

$$W(\infty) = \frac{\gamma_2}{\eta_1 + \gamma_2} \frac{r + \phi}{\rho + \phi} + \frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \phi + \theta\delta L + \delta}{\rho + \phi + (1 + \theta)\delta} > 1.$$

Because  $W(y)$  is continuous and monotonically increasing, and because  $W(0) < 1$  and  $W(\infty) > 1$ , there exists a unique  $y_*$  such that  $W(y_*) = 1$ . ■

Lemma 8 implies that there can be at most one symmetric monotone equilibrium. Next, we verify that a monotone strategy with the threshold level determined in Lemma 8 is indeed optimal for an individual creditor if every other creditor uses this threshold.

**Lemma 9** *If every other creditor uses a monotone strategy with a threshold  $y_*$  identified in Lemma 8, then the same strategy is also optimal for an individual creditor.*

**Proof.** To show that the value function constructed in Lemma 7 is indeed optimal for an individual creditor, i.e., the value function solves the HJB equation (9), we need to verify that  $V(y; y_*) > 1$  for  $y > y_*$  and  $V(y; y_*) < 1$  for  $y < y_*$ . By construction in Lemma 7,  $V(0; y_*) = \frac{r + \theta\delta L + \delta}{\rho + \phi + (1 + \theta)\delta} < 1$  and  $V(\infty; y_*) = \frac{r + \phi}{\rho + \phi} > 1$ . We just need to show that  $V(y; y_*)$ , as a function of  $y$ , only crosses 1 once at  $y_*$ . Later in this proof we simply write  $V(y; y_*)$  as  $V(y)$ .

*We first consider the case where  $y_* < 1$ .*

We prove by contradiction. Suppose that  $V(y)$  also crosses 1 at another point below  $y_*$ . Then, there exists  $y_1 < y_* < 1$  such that

$$V(y_1) > V(y_*) = 1, V'(y_1) = 0, \text{ and } V''(y_1) < 0.$$

Using the differential equation (11), we have

$$\begin{aligned} V(y_1) &= \frac{\frac{1}{2}\sigma^2 y_1^2 V_{yy}(y_1) + \phi \min(1, y_1) + \theta\delta(L + ly_1) + r + \delta}{\rho + \phi + (\theta + 1)\delta} \\ &< \frac{(\phi + \theta\delta l)y_1 + \theta\delta L + r + \delta}{\rho + \phi + (\theta + 1)\delta} < \frac{\phi + \theta\delta l + \theta\delta L + r + \delta}{\rho + \phi + (\theta + 1)\delta} < 1. \end{aligned}$$

The last inequality is implied by the parameter restrictions in (4) and (7). This is a contradiction with  $V(y_1) > 1$ . Thus,  $V(y)$  cannot cross 1 at any  $y$  below  $y_*$ .

Next, we show that  $V(y)$  is monotonic in the region  $y \geq y_*$ . Suppose that  $V(y)$  is non-monotone, then there exist two points  $y_* \leq y_1 < y_2$  such that

$$V(y_1) > V(y_2), V'(y_1) = V'(y_2) = 0, \text{ and } V''(y_1) < 0 < V''(y_2).$$

(If, say,  $y_1$  happens to be on the break point 1 where the second derivative is not necessary continuous, then take the point as  $1+$  as  $V''(1+)$  has to be negative. The same caveat applies to the case where  $y_1 = y_*$ .) According to the differential equation (12), we have

$$\begin{aligned} V(y_1) &= \frac{\frac{1}{2}\sigma^2 y_1^2 V_{yy}(y_1) + r + \phi \min(1, y_1)}{\rho + \phi} \\ &> \frac{\frac{1}{2}\sigma^2 y_2^2 V_{yy}(y_2) + r + \phi \min(1, y_2)}{\rho + \phi} = V(y_2), \end{aligned}$$

which is a contradiction.

We next consider the case where  $y_* \geq 1$ . We do not separate the two cases of  $1 < y_* \leq \frac{1-L}{l}$  and  $y_* > \frac{1-L}{l}$ , as the following proof applies to both.

The expression in equation (20) or (21) implies that  $V(y)$  has to approach  $\frac{r+\phi}{\rho+\phi}$  from below (because  $\frac{r+\phi}{\rho+\phi}$  is the debt holder's highest possible payoff), thus  $B_4$  or  $C_6$  is strictly negative. This implies that  $V(y)$  is increasing on  $[y_*, \infty)$ , and

$$V'(y_*) > 0.$$

Now consider the region  $[0, y_*)$ , it is easy to check that  $V'(0) > 0$ . Therefore, if  $V(y)$  is not monotonic on  $[0, y_*)$ , there must exist two points  $y_1 < y_2$  such that

$$V(y_1) > V(y_2), V'(y_1) = V'(y_2) = 0, \text{ and } V''(y_1) < 0 < V''(y_2).$$

According to the HJB equation, we have

$$\begin{aligned} V(y_1) &= \frac{\frac{1}{2}\sigma^2 y_1^2 V_{yy}(y_1) + r + \phi \min(1, y_1) + \delta[1 + \theta \min(L + ly_1, 1)]}{\rho + \phi + (1 + \theta)\delta} \\ &< \frac{\frac{1}{2}\sigma^2 y_2^2 V_{yy}(y_2) + r + \phi \min(1, y_2) + \delta[1 + \theta \min(L + ly_2, 1)]}{\rho + \phi + (1 + \theta)\delta} = V(y_2), \end{aligned}$$

which is a contradiction. Thus,  $V(y)$  is also monotonically increasing on  $[0, y_*)$ .

To summarize, we have shown that  $V(y)$  only crosses 1 once at  $y_*$ . Thus, it is optimal for an individual creditor to roll over his debt if  $y > y_*$  and to run if  $y < y_*$ . ■

Finally, we prove that there does not exist any asymmetric monotone equilibrium.

**Lemma 10** *There does not exist any asymmetric monotone equilibrium in which creditors choose different rollover thresholds.*

**Proof.** We prove by contradiction. Suppose that there exists an asymmetric monotone equilibrium. Then, there exist at least two groups of creditors who use two different monotone strategies with thresholds  $y_{*,1} < y_{*,2}$ . For creditors who use the threshold  $y_{i,*}$ , we denote their value function as  $V^i(y)$ . At the corresponding thresholds, we must have

$$V^1(y_{1,*}) = V^2(y_{2,*}) = 1.$$

Moreover, we must have

$$V^1(y_{2,*}) = V^2(y_{1,*}) = 1,$$

because each creditor is free to switch between these two strategies. Then for all  $y \in [y_{1,*}, y_{2,*}]$ , we must have  $V^1(y) = V^2(y) = 1$ . Otherwise the threshold strategies cannot be optimal. This implies that each creditor is indifferent between choosing any threshold in  $[y_{1,*}, y_{2,*}]$ . Denote by  $\zeta(y)$  the measure of creditors who use a threshold lower than  $y \in [y_{1,*}, y_{2,*}]$ . Then,  $V^i$  has to satisfy the HJB equation in this region:

$$\begin{aligned} \rho V^i(y) &= \mu y V_y + \frac{\sigma^2}{2} y^2 V_{yy} + r + \phi [\min(1, y) - V^i(y)] \\ &\quad + \theta \delta \zeta(y) [\min(L + ly, 1) - V^i(y)] + \delta \max\{1 - V^i(y), 0\}. \end{aligned}$$

Since  $V^i(y) = 1$  for any  $y \in [y_{1,*}, y_{2,*}]$ , we have

$$\rho = r + \phi [\min(1, y) - 1] + \theta \delta \zeta(y) [\min(L + ly, 1) - 1].$$

Note that  $\zeta(y)$  is non-decreasing in  $y$  because it is a distribution function. Since both  $\min(1, y)$  and  $\min(L + ly, 1)$  are also non-decreasing in  $y$ , the only possibility that the above equation holds is that  $L + ly > 1$  and  $y > 1$  for  $y \in [y_{1,*}, y_{2,*}]$ . Then,  $\rho = r$  has to hold. This contradicts the parameter restriction that  $\rho < r$  in (4). ■

## A.2 Proof of Proposition 2

We use a guess-and-verify approach. We first construct the single creditor's value function if he always chooses to roll over the debt, and then verify that this value function is higher than the payoff  $\min(L + ly, 1)$  from run if  $\alpha$  is sufficiently low.

Denote the single creditor's value function as  $V^s(y_t)$ . We can simply modify the HJB equation in (9) to get the following one:

$$\rho V^s = \mu y V_y^s + \frac{\sigma^2}{2} y^2 V_{yy}^s + r + \phi [\min(1, y) - V^s] + \delta \max_{\text{rollover or run}} \{0, \min(L + ly, 1) - V^s\}.$$

If the single creditor always chooses to roll over, this equation becomes

$$(\rho + \phi) V^s(y) = \frac{\sigma^2}{2} y^2 V_{yy}^s + \mu y V_y^s + \phi \min(1, y) + r.$$

This equation is identical to the equation for  $U$  in Appendix A.3, and therefore admits the same solution expressed in equation (22). The fact that  $D_1$  and  $D_2$  are negative implies that  $V^s(y)$  is globally concave. With the same condition (23) so that the liquidation cost is sufficiently large, we have

$$V^s\left(\frac{1-L}{l}\right) > 1.$$

Since  $V^s(y)$  is increasing in  $y$ ,  $V^s(y) > \min(L + ly, 1)$  for  $y > \frac{1-L}{l}$ . For  $0 < y < \frac{1-L}{l}$ , note that  $V^s(y) > L + ly$  hold for both end points, i.e.,  $V^s(0) > L$  and  $V^s(\frac{1-L}{l}) > 1$ . Because  $V^s(y)$  is concave and  $L + ly$  is linear,  $V^s(y)$  is always above  $L + ly$  in the region  $y \in (0, \frac{1-L}{l})$ . Thus,  $V^s(y) > \min(L + ly, 1)$  always holds. That is, the single creditor will always choose to roll over.

### A.3 Proof of Proposition 3

As mentioned in the main text, in this modified synchronous setting the firm's debt contracts all expire at time 0. At this time, each creditor decides whether to run or to roll over into a perpetual debt contract lasting until the firm asset matures at  $\tau_\phi$ . If all creditors choose to run, we assume that there is a probability  $\theta_s \in (0, 1)$  that the firm cannot find new creditors to replace the outgoing ones and is forced into a premature liquidation.<sup>28</sup> The current firm fundamental is  $y_0$ .

We first derive an individual creditor's value function  $U(y)$  if the firm survives the creditors' rollover decisions at time 0 and thus will be able to stay until the asset maturity at  $\tau_\phi$ .  $U(y)$  satisfies the following differential equation:

$$\rho U = \mu y U_y + \frac{1}{2} \sigma^2 y^2 U_{yy} + \phi [\min(1, y) - U] + r.$$

It is direct to solve this differential equation:

$$U(y) = \begin{cases} \frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu} y + D_1 y^{\eta_2} & \text{if } 0 < y < 1 \\ \frac{r+\phi}{\rho+\phi} + D_2 y^{-\gamma_2} & \text{if } y > 1 \end{cases}, \quad (22)$$

where

$$D_1 = -\frac{\frac{\phi}{\rho+\phi-\mu} + \gamma_2 \frac{\phi\mu}{(\rho+\phi-\mu)(\rho+\phi)}}{\eta_2 + \gamma_2}$$

$$D_2 = \frac{-\frac{\phi}{\rho+\phi-\mu} + \eta_2 \frac{\phi\mu}{(\rho+\phi-\mu)(\rho+\phi)}}{\eta_2 + \gamma_2}.$$

$D_1$  and  $D_2$  are constant and independent of the liquidation recovery parameter  $\alpha$ . Because  $U(y)$  is dominated by the fundamental value of the bank asset,  $U(y) < \frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu} y$ . This implies that  $D_1 < 0$ . In addition, since  $U(\infty) = \frac{r+\phi}{\rho+\phi}$ ,  $D_2 < 0$  and  $U(y)$  approaches  $\frac{r+\phi}{\rho+\phi}$  from below. Therefore  $U(y)$  is a monotonically increasing function with

$$U(0) = \frac{r}{r+\phi} < 1 \quad \text{and} \quad U(\infty) = \frac{r+\phi}{\rho+\phi} > 1.$$

Then the intermediate value theorem implies that there exists  $y_l > 0$  such that  $U(y_l) = 1$ .

Define  $y_h \equiv \frac{1-L}{l}$ . According to the parameter restriction (6),  $y_h > 1$ . We impose the following condition so that a premature liquidation is sufficiently costly, i.e.,  $\alpha$  is sufficiently small:

$$\alpha < \frac{\rho + \phi - \mu}{\phi \left[ \left( D_2 \frac{r-\rho}{r+\phi} \right)^{\frac{1}{\gamma_2}} + \frac{r(\rho+\phi-\mu)}{\phi(\rho+\phi)} \right]}. \quad (23)$$

<sup>28</sup>In this synchronous rollover setting, the liquidation probability parameter  $\theta_s$  has to be inside  $(0, 1)$ , while the liquidation intensity parameter  $\theta$  in the main model can be higher than 1 (conditional on creditors' runs the liquidation probability over  $(t, t + dt)$  is  $\theta \delta dt$ ).



This condition is analogous to the parameter restriction (6) in our main model. Given this condition and that  $\frac{1-L}{l} = \frac{\rho+\phi-\mu}{\phi\alpha} - \frac{r(\rho+\phi-\mu)}{\phi(\rho+\phi)}$ , we have

$$U\left(\frac{1-L}{l}\right) = \frac{r+\phi}{\rho+\phi} + D_2\left(\frac{1-L}{l}\right)^{-\gamma_2} > 1,$$

which further implies that  $y_l < y_h = \frac{1-L}{l}$ .

Next, we show that if  $y_0 > y_h$ , then it is optimal for an individual creditor to roll over, even if all the other creditors choose to run (so that the liquidation probability is  $\theta_s$ ). Note that the liquidation value of the bank asset is sufficient to pay off all the creditors because  $L + ly_0 > 1$ . Thus, the creditor's expected payoff from choosing run is  $\theta_s + (1 - \theta_s) = 1$ . His expected payoff from choosing rollover is  $\theta_s + (1 - \theta_s)U(y_0)$ , which is higher than the expected payoff from choosing run.

Next, we show that if  $y_0 < y_l$ , then it is optimal for an individual creditor to run even if all the other creditors choose rollover. In this case, the bank will always survive no matter what the individual creditor's decision is. If he chooses to run, he gets a payoff of 1, while if he chooses to roll over, his continuation value function is  $U(y_0) < 1$ . Thus, it is optimal for the creditor to run.

Finally, we consider the case when  $y_0 \in [y_l, y_h]$ . If all the other creditors choose to roll over, then an individual creditor's payoff from run is 1, while his continuation value function is  $U(y_0) > 1$ . Thus it is optimal for him to roll over too. If all the other creditors choose to run, then his expected payoff from run is  $\theta_s(L + ly_0) + (1 - \theta_s)$ . His expected payoff from choosing rollover is  $(1 - \theta_s)U(y_0)$ , because once the bank is forced into a premature liquidation, the liquidation value of the bank asset is not sufficient to pay off the other outgoing creditors and the creditor who chooses rollover gets zero. Therefore we need to ensure that  $\theta_s(L + ly_0) > (1 - \theta_s)(U(y_0) - 1)$ . Analogous to the parameter restriction (7) of our main model, we impose a parameter restriction on  $\theta_s$  so that it is sufficiently large:

$$\frac{\theta_s}{1 - \theta_s} > \frac{1}{L} \frac{r - \rho}{\rho + \phi}.$$

Then, because  $U(y_0) - 1 < \frac{r+\phi}{\rho+\phi} - 1 = \frac{r-\rho}{\rho+\phi}$ , we have  $(1 - \theta_s)(U(y_0) - 1) < (1 - \theta_s) \frac{r-\rho}{\rho+\phi} < \theta_s L < \theta_s(L + ly_0)$ . As a result, it is optimal for the creditor to run with other creditors.

## A.4 Proof of Proposition 4

Note that  $y_*$  is determined by the condition that  $W(y_*) = V(y_*; y_*) = 1$ . Theorem 1 implies that if  $y_* > \frac{1-L}{l}$ , it is determined by the following implicit function:

$$\begin{aligned} 1 = W(y_*) &= \frac{\eta_1 - M_1/M_2}{\eta_1 + \gamma_2} \left( K_4 y_*^{-\gamma_1} + (-K_2) \left( \frac{ly_*}{1-L} \right)^{-\gamma_1} \right) \\ &+ \frac{\gamma_2}{(\eta_1 + \gamma_2)} \frac{r + \phi}{\rho + \phi} + \frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \phi + \theta\delta L + \delta}{\rho + \phi + (1 + \theta)\delta}, \end{aligned} \quad (24)$$

where  $L = \frac{\alpha r}{\rho + \phi}$  and  $l = \frac{\alpha\phi}{\rho + \phi - \mu}$  increase with  $\alpha$ , and  $M_1/M_2$ , and  $K_4$  are independent of  $\alpha$ . By the implicit function theorem,  $\frac{dy_*}{d\alpha} = -\frac{\partial W/\partial\alpha}{\partial W/\partial y_*}$ . Since we have shown that  $\partial W/\partial y_* > 0$  in Lemma 8, to prove the claim we need to show that  $\partial W/\partial\alpha > 0$ . There are two terms in  $W$  that involve  $\alpha$ : 1) because  $-K_2 = \frac{\mu\theta\delta(1-L)}{(\rho+\phi+(1+\theta)\delta)(\rho+\phi+(1+\theta)\delta)}$ , the second term in the first bracket is proportional to  $-\frac{(1-L)^{1+\gamma_1}}{l^{\gamma_1}}$ , which is increasing in  $\alpha$ ; and 2) the second term  $\frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \phi + \theta\delta L + \delta}{\rho + \phi + (1 + \theta)\delta}$  in the second line is increasing in  $\alpha$ . Therefore  $\partial W/\partial\alpha > 0$ , and  $\frac{dy_*}{d\alpha} < 0$ .

When  $1 < y_* \leq \frac{1-L}{l}$ , it is determined by the following implicit function:

$$1 = W(y_*) = \frac{M_2 \eta_1 - M_1}{\eta_1 + \gamma_2} y_*^{-\gamma_1} + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} \frac{\theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y_* + \frac{\gamma_2}{\eta_1 + \gamma_2} \frac{r + \phi}{\rho + \phi} + \frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \phi + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta}. \quad (25)$$

Therefore

$$\partial W / \partial \alpha = \eta_1 \frac{\theta \delta \frac{r}{\rho + \phi}}{\rho + \phi + (1 + \theta) \delta} + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} \frac{\theta \delta \frac{\phi}{\rho + \phi - \mu}}{\rho + \phi + (1 + \theta) \delta - \mu} y_* > 0, \quad (26)$$

which implies  $\frac{dy_*}{d\alpha} > 0$ .

When  $y_* < 1$ , it is determined by the following implicit function:

$$W(y_*) = \frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} + \frac{\gamma_2}{\eta_1 + \gamma_2} \frac{r}{\rho + \phi} + \frac{[H_3 \gamma_2 + H_1]}{(\eta_1 + \gamma_2)} y_*^{\eta_2} + \left[ \frac{\eta_1 - 1}{\eta_1 + \gamma_2} \frac{\theta \delta l + \phi}{\rho + \phi + (1 + \theta) \delta - \mu} + \frac{1 + \gamma_2}{(\eta_1 + \gamma_2)} \frac{\phi}{(\rho + \phi - \mu)} \right] y_* = 1,$$

where  $H_3$  and  $H_1$  are independent of  $\alpha$ . Then

$$\partial W / \partial \alpha = \frac{\eta_1}{\eta_1 + \gamma_2} \frac{\theta \delta \frac{r}{\rho + \phi}}{\rho + \phi + (1 + \theta) \delta} + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} \frac{\theta \delta \frac{\phi}{\rho + \phi - \mu}}{\rho + \phi + (1 + \theta) \delta - \mu} > 0. \quad (27)$$

Taken together, the equilibrium rollover threshold  $y_*$  decreases with  $\alpha$ .

## A.5 Proof of Proposition 5

First note that as  $\theta \rightarrow \infty$ ,  $y_* \rightarrow \frac{1-L}{l} > 1$ . The reason follows. As  $\theta \rightarrow \infty$ , the firm fails immediately after creditors start to run on the firm. Thus, the rollover risk term  $\min(1, L + ly_\tau) \mathbf{1}_{\{\tau = \tau_\theta\}}$  in equation (8) is replaced by a boundary condition that when  $y = y_*$ ,  $V(y, y_*) = L + ly_*$ . It is direct to see that the equilibrium condition  $V(y_*, y_*) = 1$  implies that  $y_* = \frac{1-L}{l}$  is the unique equilibrium threshold.

Then, by the continuity of  $y_*$  with respect to  $\theta$ , if  $\theta$  is sufficiently high,  $y_* > 1$ . Our numerical exercises also show that this holds true over a wide range of parameter values. Thus, we will focus on showing that  $y_*$  increases with  $\sigma^2$  in the range where  $y_* > 1$ .

Since  $y_*$  is determined by the implicit function  $W(y_*) = V(y_*, y_*) = 1$ , to show that  $y_*$  increases with  $\sigma^2$ , we only need to verify that  $\frac{\partial W(y)}{\partial \sigma^2} < 0$ .

We first note several inequalities. Directly from condition (5), we have  $\frac{\partial \eta_i}{\partial \sigma^2} < 0$  and  $\frac{\partial \gamma_i}{\partial \sigma^2} < 0$  for  $i = 1, 2$ . Moreover, by using the definitions of  $\eta_1$  in (15) and  $\gamma_2$  in (17), we can also show that

$$\frac{\partial \left( \frac{\gamma_2}{\eta_1 + \gamma_2} \right)}{\partial \sigma^2} < 0. \quad (28)$$

We now consider the case where  $1 < y_* \leq \frac{1-L}{l}$ . Based on  $W(y)$  given in equation (25), we have

$$\begin{aligned} \frac{\partial W(y)}{\partial \sigma^2} &= \frac{\partial \left( \frac{\eta_1 - M_1/M_2}{\eta_1 + \gamma_2} \right)}{\partial \sigma^2} M_2 y^{-\gamma_1} + \frac{\partial \left( \frac{-1}{\eta_1 + \gamma_2} \right)}{\partial \sigma^2} \frac{\theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y \\ &+ \frac{\eta_1 - M_1/M_2}{\eta_1 + \gamma_2} M_2 y^{-\gamma_1} \ln y \frac{\partial(-\gamma_1)}{\partial \sigma^2} + \frac{\partial \left( \frac{\gamma_2}{\eta_1 + \gamma_2} \right)}{\partial \sigma^2} \left( \frac{r + \phi}{\rho + \phi} - \frac{r + \phi + \theta \delta (L + ly) + \delta}{\rho + \phi + (1 + \theta) \delta} \right). \end{aligned}$$

As  $\frac{r+\phi}{\rho+\phi} - \frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} > 0$ , inequality in (28) implies that the last term is negative. Also,  $\frac{\partial\left(\frac{-1}{\eta_1+\gamma_2}\right)}{\partial\sigma^2} < 0$  implies that the second term is negative. Moreover, because  $M_2\eta_1 - M_1 < 0$  (shown in the proof of Lemma 8), and  $\frac{\partial(-\gamma_1)}{\partial\sigma^2} > 0$ , the third term is negative. Finally, note that when  $\theta$  is sufficiently large,  $\eta_1$  and  $\gamma_2$  are in the order of  $\theta^{0.5}$ . Since  $M_1/M_2 = \frac{\rho+\phi+(1+\theta)\delta}{\mu}$ , the first part of the second term  $\frac{\partial\left(\frac{\eta_1-M_1/M_2}{\eta_1+\gamma_2}\right)}{\partial\sigma^2}$  is approximately equal to  $-\frac{\partial\left(\frac{1}{\eta_1+\gamma_2}\right)}{\partial\sigma^2}M_1/M_2$ , which is negative. Taken together,  $\frac{\partial W(y)}{\partial\sigma^2} < 0$ .

We now consider the case where  $y_* > \frac{1-L}{l}$ . Based on  $W(y)$  in equation (24), we have

$$\begin{aligned} \frac{\partial W(y)}{\partial\sigma^2} &= \frac{\partial\left(\frac{\gamma_2}{\eta_1+\gamma_2}\right)}{\partial\sigma^2} \left(\frac{r+\phi}{\rho+\phi} - \frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta}\right) + \frac{\partial\left(\frac{\eta_1-M_1/M_2}{\eta_1+\gamma_2}\right)}{\partial\sigma^2} \left(K_4 y^{-\gamma_1} + (-K_2) \left(\frac{ly}{1-L}\right)^{-\gamma_1}\right) \\ &\quad + \frac{\eta_1 - M_1/M_2}{\eta_1 + \gamma_2} \frac{\partial\left(K_4 y^{-\gamma_1} + (-K_2) \left(\frac{ly}{1-L}\right)^{-\gamma_1}\right)}{\partial\sigma^2}. \end{aligned}$$

Using arguments similar to those presented in the previous case, it is easy to show that every term in this expression is negative. Thus,  $\frac{\partial W(y)}{\partial\sigma^2} < 0$ . This concludes the proof.

## A.6 Proof of Proposition 6

We distinguish between an individual creditor  $i$ 's rollover frequency  $\delta_i$  and other creditors' rollover frequency  $\delta_{-i}$ . We can rewrite the individual creditor's HJB equation for his value function  $V^i$ :

$$\begin{aligned} \rho V^i(y_t; y_*) &= \mu y_t V_y^i + \frac{\sigma^2}{2} y_t^2 V_{yy}^i + r + \phi [\min(1, y_t) - V(y_t; y_*)] \\ &\quad + \theta \delta_{-i} \mathbf{1}_{\{y < y_*\}} [\min(L + ly_t, 1) - V(y_t; y_*)] + \delta_i \max_{\text{rollover or run}} \{1 - V(y_t; y_*), 0\}. \end{aligned} \quad (29)$$

Suppose that we increase  $\delta_i$  from  $\delta$  to  $\delta' > \delta$ . We need to show that the creditor  $i$ 's value function with parameter  $\delta'$  is strictly higher than that with parameter  $\delta$ . To facilitate the comparison, we consider a new problem, in which the creditor's contract expires with rate  $\delta'$ , but he is only allowed to withdraw at his contract expiration if an independent random variable  $X = 1$ . This variable  $X$  can take values of 1 or 0 with probabilities of  $\lambda = \delta/\delta' < 1$  and  $1 - \lambda$ , respectively. This random variable effectively reduces the creditor's release rate to  $\delta$ . Thus, in this constrained problem with parameter  $\delta'$ , the creditor has the same value function as in the unconstrained problem with parameter  $\delta$ .

Next, consider the creditor's value function in the unconstrained problem (or,  $\lambda = 1$  always) with parameter  $\delta'$ , which should be strictly higher than that in the constrained problem. This is because if the creditor is allowed to withdraw when  $X = 0$  and  $y_t < y_*$ , his value function is strictly increased even if he keeps the same threshold. Then, it is obvious that the creditor's value function in the unconstrained problem with parameter  $\delta'$  is strictly higher than that in the same problem with parameter  $\delta$ .

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