Liquidity and the Welfare Cost of Inflation*

(Preliminary and Incomplete)

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Abstract

This paper studies the long run effects of monetary policy in a micro-founded model with trading frictions and endogenous market segmentation. Agents must pay a fixed cost to participate in a centralized liquidity market. By endogenizing the participation decision, this model endogenizes the responses of velocity, output, the degree of market segmentation, as well as the distribution of money. As inflation decreases, agents are induced to participate less frequently in the centralized liquidity market, leading to a lower velocity of money, a smaller liquidity market, fewer resources spent on market participation and higher heterogeneity in money holdings across agents. The welfare costs of inflation implied are different from previous papers in the literature since inflation can distort the agents consumption profile, affect market participation, and redistribute money holdings. The model provides a general framework that nests several existing search models as special cases for different specifications of the fixed cost.

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1 Introduction

This paper studies the long run effect of monetary policy in a micro-founded model with trading frictions and endogenous market segmentation. The study is motivated by the recent development in the literature of search theoretic models of monetary exchange. A recent paper by Lagos and Wright (2005) proposes a new framework for monetary policy analysis. Unlike reduced-form models, the Lagos-Wright model is micro-founded in the sense that the role of money is modeled explicitly. Unlike the existing search literature, the implied money distribution is degenerate and thus the model is analytically tractable. They calibrate the model to standard observations and find a much higher welfare cost of inflation than that in the literature (e.g. Lucas (2000)).

In a standard search model, it is technically challenging to keep track of the non-degenerate distribution of money holdings as a result of the heterogenous trading histories of agents in a decentralized market. Lagos and Wright solve this problem by assuming that agents have quasi-linear preferences and that agents trade in a centralized market periodically. These features imply a degenerate money distribution, thus the model becomes analytically tractable. The tractability, however, does come with a cost. By allowing agents to re-balance portfolio (at no cost) every period, the centralized market in Lagos-Wright model effectively provides an "insurance" function so that agents can undo all individual-specific trading shocks immediately. Special assumptions about preferences, timing and nature of meetings kill many potentially interesting properties of standard search models. First, by forcing the money distribution to be degenerate, the model precludes discussions of any distributional effects of monetary policy. Second, the introduction of the centralized market eliminates by assumption the insurance role of inflation. Third, unlike in existing search models, the manner of money injection does not matter. Finally, in a Lagos-Wright economy, money is perfectly neutral and the model is unable to generate any short run dynamics in response to an one-time money injection.

In general, we are interested in generalizing the Lagos-Wright model to answer the following questions: If participation in the centralized market is costly and endogenous, what are the short-run and long-run effects of monetary policy? How robust are the results of the Lagos-Wright model (e.g. welfare cost of inflation)? Can the model generate more realistic implications by endogenizing market participation (e.g. short run dynamics)? How is the relationship between inflation and the centralized market in providing insurance against individual-specific shocks? This paper will focus on the long-run properties of this model and relegate the short-run analysis to a companion paper (Chiu and Molico (2006)).

In particular, we assume that agents have to pay a fixed cost to participate in the centralized market. This assumption is to capture the idea that participation in organized trading, intermediation and financial market is costly. Moreover, it is motivated by the fact that not all households actively and frequently participate in these activities, and that the participation rate depends on the state of the economy (e.g. inflation and financial development). Because of this market participation friction, agents choose to attend the centralized market only infrequently and to keep an inventory of money for trading in the decentralized market. As a result, the centralized market provides only limited "insurance". By endogenizing the decision of participation, this model also endogenizes the responses of velocity, output, the degree of market segmentation as well as the monetary distribution. The welfare cost of inflation implied in this model is also different from the Lagos-Wright model because, in this model, inflation can distort the consumption profile, affect market participation, and redistribute money holdings. Another distinct feature of this model is that the manner of money injection matters. Money injected into the decentralized market provides insurance function by redistributing wealth from the money-rich to the money-poor. Money injected into the centralized market redistributes wealth from the non-participants to the participants.

Some progress has been made in the search theory literature to extend Lagos and Wright framework by assuming that agents can trade in the centralized market only infrequently. Berentsen, Camera and Waller (2005), Ennis (2005) and Williamson (2006) study an environment in which agents participate in the centralized market at an exogenous rate. This paper takes a further step to fully endogenize the participation in the centralized market because the participation decision should not be taken as invariant to policy intervention. This paper also generalizes the existing search literature by developing a framework that nests several existing search models as special cases. When the fixed cost, denoted as κ, is zero, the model reduces to Lagos and Wright (2005) in which the Friedman rule is optimal. When κ is infinite, the model reduces to Molico (2006) in which positive inflation can be welfare improving. Finally, this paper also contributes to the literature by integrating an endogenous market segmentation model (focusing on market participation frictions)[1] with a search-theoretic model (focusing on goods trading frictions).

The rest of the paper is organized as follows. Section 2 gives a brief preview of the model. Section 3 describes the environment. Section 4 defines equilibrium. Section 5 discusses the numerical algorithm and section 6 uses numerical examples to illustrates how the model works by studying the welfare cost of inflation. Section 6 concludes the paper.

2 Preview of the Model

To motivate why it is interesting to study a search model with endogenous market segmentation, we can first look at the properties of a standard money search model with divisible money (e.g. Molico (2006)) as well as the Lagos-Wright model (2005), as shown in Figure 1.

As shown in 1(a), agents in a standard search model have to trade in a decentralized market in which they are subject to random pairwise matchings. In that model, it is technically challenging to keep track of the monetary distribution over time. To see why, suppose agents enter the decentralized market with exactly the same money holdings, due to their heterogeneous trading histories, they will end up with a non-degenerate money distribution at the end of the day. Overtime, this monetary distribution will become very elaborate and

[1] Models with exogenous market segmentation include Grossman and Weiss (1983), Alvarez and Atkeson (1997) and Alvarez, Atkeson and Edmond (2003). Models with endogenous market segmentation include Alvarez, Atkeson and Kehoe (1999), Chiu (2005) and Khan and Thomas (2005). All these models impose an exogenous cash-in-advance constraint.

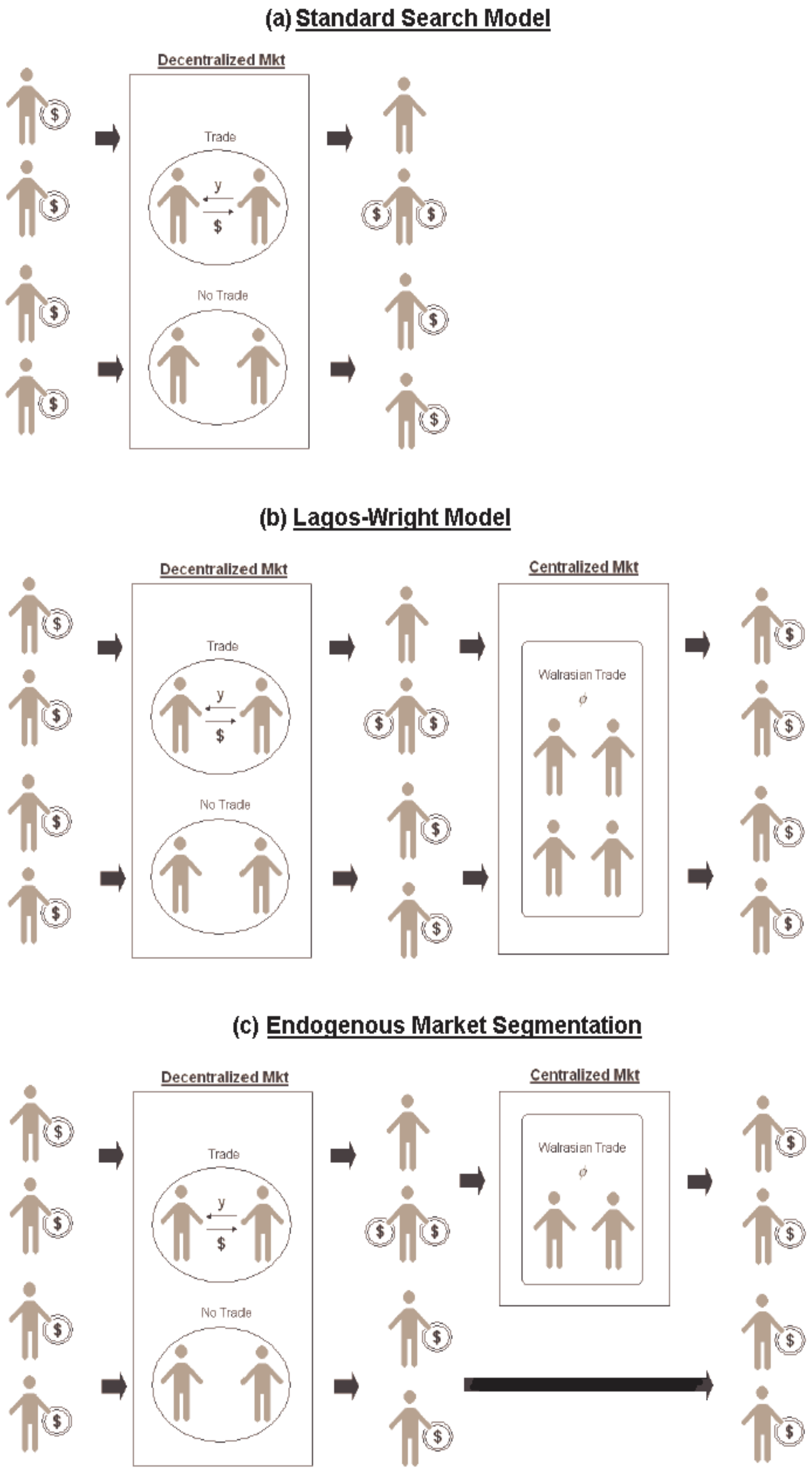

Figure 1: Model Comparison

analytically intractable.

To deal with this problem, as shown in 1(b), Lagos and Wright introduce a centralized general goods market into the model. After each round of decentralized trading, agents enter the centralized market in which they can re-adjust costlessly their portfolio by producing and trading general goods. By assuming agents have quasilinear preferences over the goods produced and consumed in this market, effectively they provide agents with a way of fully insuring against the uncertainty with respect to their money holdings in the decentralized market, and accordingly, the money distribution collapses again to a degenerate distribution[2]. While this modification makes the model analytically tractable, it however precludes discussions of any distributional effects of monetary policy. If one believes that an important and interesting aspect of monetary policy is the effects on the distribution of money holdings across economic agents, then this model may not be the best framework to analyze the full effects of monetary policy.

In this paper, we relax the assumption in Lagos and Wright that agents can enter the centralized market costlessly every period. Instead, agents have to pay a fixed cost, $\kappa \in [0, \infty]$, to use the centralized market. As a result, only a fraction of agents choose to participate in the centralized market : the market is segmented in the sense that, when money is injected into the centralized market, only a fraction of the agents are on the demand side of the transaction (1(c)). More importantly, this participation rate is a function of the inflation rate and the fixed cost. With a high inflation rate and a low fixed cost, agents choose to participate in the centralized market every period. So the model reduces to the Lagos-Wright specification, with a degenerate distribution. As inflation decreases or fixed cost rises, agents choose to participate in the centralized market less frequently, the distribution becomes more elaborate and the velocity of money decreases. Buyers in the decentralized market choose not to spend all of their money holdings but to keep inventories of money across period.

[2]Quasilinear preferences per se are not sufficient to guarantee that the money distribution is degenerate given that might involve some (very rich) agents producing negative quantities of the general good. However, as long as the efficient quantity is sufficiently large one can show that the non-negativity constraint on production will not be binding.

As the fixed cost goes to infinity, agents never attend the centralized market and the model reduces to the standard search model.

In this model, the welfare cost of inflation is quite different from earlier models. In the Lagos and Wright, inflation is undesirable because it makes decentralized trading more costly. In addition to this effect, there are three more channels through which inflation can affect welfare in this model. First, inflation induces agents to participate in the centralized market more frequently. Since participation is costly, inflation leads to waste of resources. Second, even if the participation rate were kept unchanged, agents still choose to spend their money faster to avoid the inflation tax, making the consumption profile less smooth. Third, inflation can redistribute money holdings. The exact effect of this redistribution effect depends on the manner of money injection. If money is injected into the decentralized market by means of lump-sum transfers then there is a redistribution from the money-rich to the money-poor. If the same injection is made into the centralized market, then there is a redistribution from the non-participants to the market participants.

3 The Model

Time is discrete and denoted by $t = 0, 1, 2, ...$ There are two types of non-storable commodities: general and special goods. The economy consists of a continuum $[0, 1]$ of agents. The per-period utility of an agent is given by

$$U(X_t) - C(Y_t) + u(x_t) - c(y_t),$$

where $U(X)$ denotes the utility of consuming X units of the general good, $C(Y)$ denotes the disutility of producing Y units of the general good, $u(x)$ denotes the utility of consuming x units of the special good, and $c(y)$ denotes the disutility of producing y units of the special good. We assume that $u(\cdot)$ and $U(\cdot)$ are twice continuously differentiable, strictly increasing, concave (with u strictly concave), and satisfy $U(0) = u(0) = 0$, $U'(\bar{X}) = 1$ for some $\bar{X} > 0$,

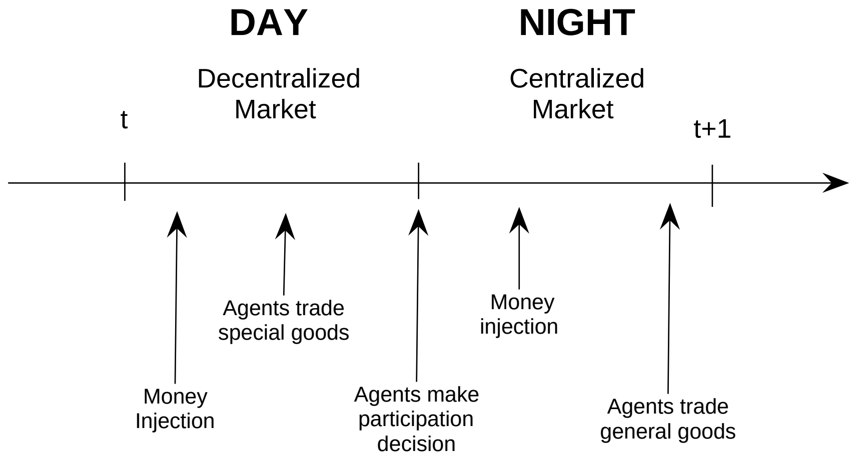

Figure 2: Time line

and $u'(\bar{x}) = 1$ for some $\bar{x} > 0$. Also, $C(Y) = Y$ and $c(y) = y$. Agents discount the future at discount factor $\beta \in (0, 1)$.

In this economy, there is an additional, perfectly divisible, and costlessly storable object which cannot be produced or consumed by any private individual, called *fiat money*. Agents can hold any non-negative amount of money $\hat{m} \in \mathbb{R}_+$. The money stock at the beginning of period t is denoted M_t. In what follows we express all nominal variables as fraction of the beginning of the period money supply (before any of the current period's money transfers which we will describe below), $m \equiv \frac{\hat{m}}{M}$. Let $\nu_t : \mathfrak{B}_{\mathbb{R}_+} \to [0, 1]$ denote the probability measure associated with the money (as a fraction of the beginning of period money supply) distribution at the beginning of period t, where $\mathfrak{B}_{\mathbb{R}_+}$ denotes the Borel subsets of $\mathbb{R}_+$.

Each period is divided into two subperiods: day and night. In the day time, there is a decentralized market for trading special goods. In the night time, there is a centralized market for trading general goods (see Figure 2).

As in standard search-theoretical models of money, in the decentralized market, agents are subject to trading frictions modeled as pairwise random matching. To generate the need for trade, we assume that agents cannot consume their own production of special goods. To generate the use of money, we assume that the probability of having a double coincidence of wants meeting is zero and that all trading histories are private information.[3] The probability that an agent consumes something his/her match partner produces is $\sigma \in [0, \frac{1}{2}]$. Similarly, the probability that an agent produces something that his/her match partner consumes is σ. Therefore, with a probability $1 - 2\sigma$, trading partners do not want each other's goods. When two individuals meet and one consumes the good the other produces, they bargain over the amount of output and the amount of money to be traded. Let $q_t(m_b, m_s; \nu_t) \geq 0$ be the amount of output and $d_t(m_b, m_s; \nu_t) \geq 0$ the amount of money determined by the bargaining process at date t between a buyer with money holdings m_b and a seller with m_s, when the probability measure at the beginning of the period is ν_t. In particular, the terms-of-trade are assumed to be determined by take-it-or-leave-it offers by the buyers.[4] Let $\omega_t : \mathfrak{B}_{\mathbb{R}_+} \rightarrow [0, 1]$ denote the probability measure over money holdings at the entrance of the centralized market (after trade in the decentralized market).

At night after the decentralized market closes, there is a Walrasian market for the general good that opens. Participation in that market is costly. It is assumed that, at the beginning of each night, each agent i draws a random fixed cost κ_t^i (in units of the general good). The cost κ is assumed to be i.i.d. across time and agents with uniform distribution over the support $[0, \bar{\kappa}]$. Given the individual's realization of the fixed cost, an agent must decide whether or not to participate in the market. Agents take the price of money in terms of the general good in that market, ϕ_t, as given. If an agent chooses not to participate in the centralized market, he/she consumes zero amount of the general good in autarky. If the

[3]For money to be valued it is only required that in some meetings there is no double coincident of wants. For simplicity, we focus on purely monetary trades and, by assumption, preclude the possibility of barter in the decentralized market.

[4]More generally, we could consider that the terms of trade were determined by the solution of a generalized Nash-bargaining problem as in Lagos-Wright. As shown in that paper if the seller has some bargaining power additional distortions exist that would imply higher welfare costs of inflation. The same would be true here. In that sense, we provide a lower bound for the welfare costs of inflation.

agent decides to participate he/she must decide how much of the general good to consume and produce, and how much money holdings to carry into the decentralized market the next day.

Given the environment, the only feasible trades during the day are the exchange of special goods for money and at night barter in general goods or the exchange of general goods for money.

The money stock is assumed to grow at a constant growth rate $\mu = \frac{M_t}{M_{t-1}}$ for all t. Money growth is accomplished via money transfers at the entrance of each market. Agents receive a monetary transfer at the beginning of the decentralized market, $\tau_d(m, \nu_t)$ (as in Lagos-Wright or Molico). In addition, centralized market participants receive monetary transfers at the entrance of the centralized market, $\tau_c(m, \omega_t)$, before trade occurs. We assume that the monetary transfers (monetary policy rule) are such that rate of monetary growth in each market is constant over time. Let the rate of monetary growth in the decentralized market be

$$\mu_d \equiv \int_0^\infty [m + \tau_d(m, \nu_t)]\, \nu_t(m) dm. \tag{1}$$

Define the rate of growth in the centralized market to be $\mu_c \equiv \frac{\mu}{\mu_d}$. The details of the monetary injection in the centralized market will be described below.

This concludes the description of the environment. In what follows, we will gradually build towards the definition of equilibrium.

4 Equilibrium

In this section we define a recursive equilibrium for this economy. We begin by describing the individual and aggregate state variables. An individual's state variable consists of his/her money holdings (as a fraction of the beginning of the period money supply). The aggregate state variable is, in turn, defined as the current probability measure over money holdings. Thus, at the beginning of the period an individual's state is described by the pair (m, ν), and at the entrance of the centralized market by (m, ω). Agents take as given the law of motion of the aggregate state variable defined by $\nu' = H_\nu(\omega)$ and $\omega = H_\omega(\nu)$ which we will describe in detail below, where prime denotes the future period.[5] Also, agents take as given the price of money in units of the general good in the centralized market, ϕ, as a function of the current aggregate state, $\phi : \Lambda \to \mathbb{R}_+ \setminus \{0\}$, where Λ denotes the space of probability measures over $\mathfrak{B}_{\mathbb{R}_+}$.[6] Finally, agents take as given the monetary policy rules (transfers) $\tau_d : \mathbb{R}_+ \times \Lambda \to \mathbb{R}$ and $\tau_c : \mathbb{R}_+ \times \Lambda \to \mathbb{R}$.

4.1 The Centralized Market

In what follows we describe the value functions of participants and non-participants in the centralized market, and then step back to examine the entry decision of an agent after the realization of the fixed cost in order to derive the value function at the entrance of the centralized market.

Consider the expected lifetime utility of an agent that after incurring the fixed cost participates in the centralized market, $W^1(m, \omega)$, where m is the money balance held by the agent normalized by the beginning-of-the-period money stock. Given the price of money, $\phi(\omega)$, and the monetary policy rules, the value function is given by

$$
\begin{aligned}
& W^1(m,\omega) = \max_{X,Y,m' \geq 0} U(X) - Y + \beta V(m', \nu') \\
& s.t. \\
& Y \geq X + \phi(\omega)[m'\mu - m - \tau_c(m,\omega)] \\
& \nu' = H_\nu(\omega), \qquad (2)
\end{aligned}
$$

where $V(m, \nu)$ is the value function for an agent at the beginning of the day with money

[5] Equivalently, define the law of motion of by $\nu' = H(\nu) \equiv H_\nu(H_\omega(\nu))$.

[6] Note that by restricting ϕ to be strictly positive, we focus on only monetary equilibrium in which money has value.

balances m when the aggregate state is ν.[7] Given the individual state m and aggregate state ω, an agent chooses the optimal amounts of the general good consumption (X), the general good production (Y), as well as the money holding at the entrance of the next decentralized market (m'). The budget constraint simply states that the expenditure on consumption and on net money purchase is no greater than the income from production.

The expected lifetime utility of an agent not participating in the centralized market with money holding m is given by

$$W^0(m, \omega) = \beta V\left(\frac{m}{\mu_c}, H_\nu(\omega)\right). \tag{3}$$

Non-participants consume and produce nothing and their money balance (as a fraction of the beginning of the period money supply) declines at the rate of money growth.

We now consider the decision of whether or not to participate in the centralized market. Consider the case of an agent at the entrance of the centralized market with money holdings m when the aggregate state is ω that draws a fixed cost κ. The agent will participate in the centralized market as long as

$$W^1(m, \omega) - \kappa > W^0(m, \omega).$$

Define $\hat{\kappa}(m, \omega)$ to be the threshold value such that any agent with state (m, ω) that draws a cost κ chooses to participate if $\kappa < \hat{\kappa}(m, \omega)$. The threshold function $\hat{\kappa} : \mathbb{R}_+ \times \Lambda \to [0, \bar{\kappa}]$ is defined as

$$\hat{\kappa}(m, \omega) = \min\{\max\{0, W^1(m, \omega) - W^0(m, \omega)\}, \bar{\kappa}\}. \tag{4}$$

Given this threshold function we can define the value function for an agent at the entrance

[7] In what follows, we will assume that $V(\cdot, \omega)$ and $\tau_c(\cdot, \omega)$ are continuous functions. By the *Theorem of the Maximum*, $W^1(\cdot, \omega)$ is a continuous function and the set of optimizers is a nonempty, compact-value, and an u.h.c. correspondence. By the *Measurable Selection Theorem*, define $m'(m, \omega)$ to be a measurable section of such correspondence.

of the centralized market (before drawing the fixed cost) as

$$W(m,\omega) = \int_0^{\hat{\kappa}(m,\omega)} \left[W^1(m,\omega) - \kappa\right] \frac{1}{\bar{\kappa}}\, d\kappa + \int_{\hat{\kappa}(m,\omega)}^{\bar{\kappa}} W^0(m,\omega) \frac{1}{\bar{\kappa}}\, d\kappa,$$

or simplifying,

$$W(m,\omega) = \left[W^1(m,\omega) - \frac{\hat{\kappa}(m,\omega)}{2}\right] \frac{\hat{\kappa}(m,\omega)}{\bar{\kappa}} + W^0(m,\omega)\left[1 - \frac{\hat{\kappa}(m,\omega)}{\bar{\kappa}}\right]. \tag{5}$$

The fraction of agents participating in the centralized market is given by

$$f(\omega) = \int_0^{\infty} \frac{\hat{\kappa}(m,\omega)}{\bar{\kappa}}\, \omega(dm).$$

The monetary policy rule $\tau_c(m,\omega)$ must satisfy

$$\int_0^{\infty} \tau_c(m,\omega) \frac{\hat{\kappa}(m,\omega)}{\bar{\kappa}}\, \omega(dm) = \mu_c - 1. \tag{6}$$

Also, in equilibrium, choices of money holdings, $m'(m,\omega)$, satisfy the money market clearing condition,

$$\int_0^{\infty} \frac{\hat{\kappa}(m,\omega)}{\bar{\kappa}}\, m'(m,\omega)\, \omega(dm) = \int_0^{\infty} \frac{\hat{\kappa}(m,\omega)}{\bar{\kappa}}\, m\, \omega(dm) + \mu_c - 1. \tag{7}$$

This simply states that, the total amount of money taken from the centralized market into next period's decentralized market (l.h.s.) is equal to the total amount brought into the centralized market (r.h.s.), which consists of the money holdings of the participants at the entrance of the market as well as the government money injection into that market.

4.1.1 A Simple Example

Here, we consider a simple example to illustrate how the centralized market works. For simplicity, we assume that the non-negativity constraint on Y in the participants' problem is not binding. Define X^* to be the quantity such that $U'(X) = 1$ with $U(\cdot)$ strictly concave, that is, the ex-ante efficient quantity. If X^* is sufficiently large it can be shown that the constraint $Y \geq 0$ will not be binding for any agent. In this case, participants can fully insure against the idiosyncratic risk faced in the decentralized market and the optimal general good consumption choice can be easily shown to be X^* for all agents. Furthermore, it can be shown that the optimal money balance carried into the next period will be given by $m^* = \arg\max_{m'}[-\phi m'\mu + \beta V(m')]$, independently of an agent's initial money holdings (see Lagos and Wright for a detailed discussion). Furthermore assume that the monetary transfer is either proportional to an agent's money holdings or constant (lump-sum).

As a result, the value function for a participant will be linear in m and can be written as

$$W^1(m, \omega) = U(X^*) - X^* + \phi(\omega)\left[m + \tau_c(m, \omega)\right] - \phi(\omega)m^*\mu + \beta V(m^*, H_\nu(\omega)).$$

The threshold function is then given by

$$\hat{\kappa}(m, \omega) = U(X^*) - X^* + \phi(\omega)\left[m + \tau_c(m, \omega)\right] - \phi(\omega)m^*\mu + \beta V\left(m^*, H_\nu(\omega)\right) - \beta V\left(\frac{m}{\mu}, H_\nu(\omega)\right).$$

Concavity of V implies that W_0 is a concave function as shown in Figure 3(a). In this example, agents with low money holdings will always choose to pay the fixed cost to sell the general good for liquidity in the centralized market. Also, agents with high money holdings will always choose to pay the fixed cost to purchase the general good in the centralized market. The threshold function is given by the $\hat{\kappa}(m)$ curve in Figure 3(b).

4.2 The Decentralized Market

We now consider the bargaining problem of an agent in the decentralized market. Consider a single coincidence meeting when a buyer holds a money balance m_b and a seller holds a balance m_s, after the decentralized market's money injection, when the aggregate state is ν. We assume that the buyer makes a take-it-or-leave-it offer to the seller. That is, he/she

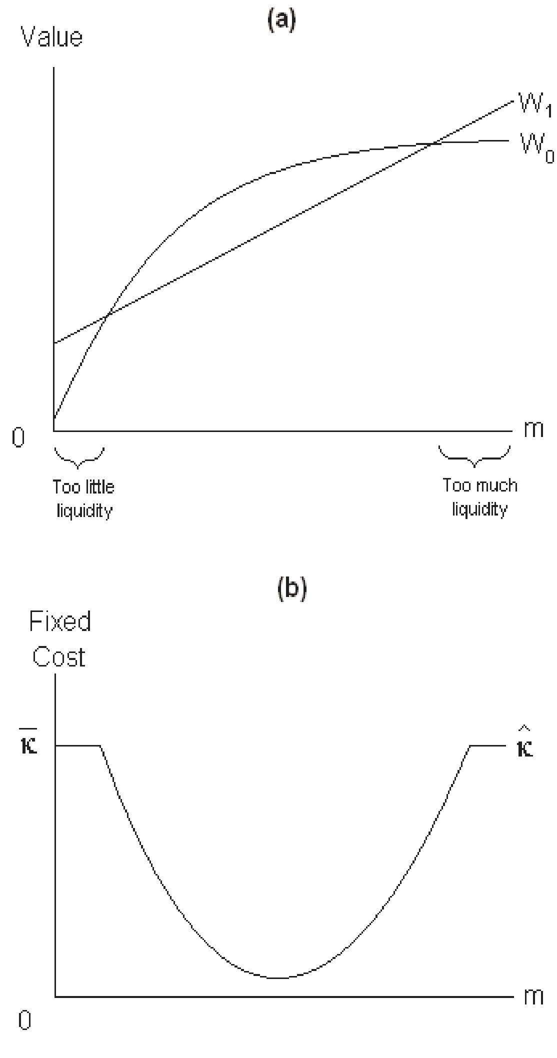

Figure 3: (a) $W_0(m, \omega)$ and $W_1(m, \omega)$, (b) Threshold Fixed Cost

proposes a trade of an amount of money, d, for a quantity of special good, q, that solve the following problem:

$$\max_{q\geq 0,\, 0\leq d\leq m_b} u(q) + W(m_b - d, H_\omega(\nu)) \tag{8}$$

subject to

$$-q + W(m_s + d, H_\omega(\nu)) \quad = \quad W(m_s, H_\omega(\nu)).$$

Or, equivalently, by replacing the latter constraint into the objective function,

$$\max_{0\leq d\leq m_b} u[W(m_s + d, H_\omega(\nu)) - W(m_s, H_\omega(\nu))] + W(m_b - d, H_\omega(\nu)).$$

The buyer makes an offer to maximize his/her surplus subject to making the seller indifferent between trading and not trading. Note that, given that $W(\cdot, H_\omega(\nu))$ is a continuous function, the objective function of the bargaining problem is continuous. Also, the set $d \in [0, m_b]$ is non-empty and compact. Thus, by the *Theorem of the Maximum* and the *Measurable Selection Theorem* the set of optimizers is a non-empty, compact-valued, and u.h.c. correspondence and admits a measurable selection. Define $d(m_b, m_s, \nu)$ to be such selection. The function $q(m_b, m_s, \nu)$ can then be obtained from the seller's participation constraint.

The expected lifetime utility of an agent that enters the period with money balance m (before the decentralized market money injection) is given by

$$\begin{aligned} V(m,\nu) = (1-\sigma)W[m + \tau_d(m,\nu), H_\omega(\nu)] + \sigma \int_0^\infty \{u[q(m + \tau_d(m,\nu), m_s, \nu)] \\ + W\{m + \tau_d(m,\nu) - d[m + \tau_d(m,\nu), m_s, \nu], H_\omega(\nu)\}\}\, \nu(dm_s). \end{aligned} \tag{9}$$

The first term is the value for an agent that either is a seller, with probability σ, and thus has a zero net surplus from trade, or meets no one. The second term is the expected value of being a buyer.

4.3 Laws of Motions

Before defining a recursive equilibrium for this economy, we describe the laws of motion $\nu' = H_\nu(\omega)$ and $\omega = H_\omega(\nu)$. We begin by describing the evolution of the aggregate state from the beginning of the centralized market to the beginning of the next decentralized market, H_ν. Define the function $\Pi : \mathbb{R}_+ \times \mathfrak{B}_{\mathbb{R}_+} \to [0, 1]$ to be

$$\Pi(m, B; \omega) = \begin{cases} \frac{\hat{\kappa}(m,\omega)}{\bar{\kappa}}, & \text{if } \frac{m}{\mu} \in B \text{ and } m'(m, \omega) \notin B; \\ 1 - \frac{\hat{\kappa}(m,\omega)}{\bar{\kappa}}, & \text{if } \frac{m}{\mu} \notin B \text{ and } m'(m, \omega) \in B; \\ 1, & \text{if } \frac{m}{\mu} \in B \text{ and } m'(m, \omega) \in B; \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Given that, for each m, $\Pi(m, \cdot; \omega)$ is a probability measure on $(\mathbb{R}_+, \mathfrak{B}_{\mathbb{R}_+})$, and, for each $B \in \mathfrak{B}_{\mathbb{R}_+}$, $\Pi(\cdot, B; \omega)$ is a $\mathfrak{B}_{\mathbb{R}_+}$-measurable function, Π is a well defined transition function. Then, the law of motion $H_\nu(\cdot)$ can be defined as

$$\nu'(B) = H_\nu(\omega)(B) \equiv \int_0^\infty \Pi(m, B; \omega)\, \omega(dm) \ \ \forall B \in \mathfrak{B}_{\mathbb{R}_+}.$$

We now describe the evolution of the aggregate state from the beginning of the decentralized market to the beginning of the centralized market. Let $T = \{buyer, seller, neither\}$ and define the space $(T, \mathfrak{T})$, where $\mathfrak{T}$ is the σ-algebra. Define the probability measure $\tau : \mathfrak{T} \to [0, 1]$, with $\tau(buyer) = \tau(seller) = \sigma$, and $\tau(neither) = 1 - 2\sigma$. Then, $(T, \mathfrak{T}, \tau)$ is a measure space. Define an *event* to be a pair $e = (t, m)$, where $t \in T$ and $m \in \mathbb{R}_+$. Intuitively, t denotes an agent's trading status and m the money holdings of his current trading partner. Let $(E, \mathfrak{E})$ be the space of such *events*, where $E = T \times \mathbb{R}_+$ and $\mathfrak{E} = \mathfrak{T} \times \mathfrak{B}_{\mathbb{R}_+}$. Furthermore, let $\xi : \mathfrak{E} \to [0, 1]$ be the product probability measure. Define the mapping

$\gamma(m, e) : \mathbb{R}_+ \times E \to \mathbb{R}_+$, where

$$\gamma(m,e) = \begin{cases} m + \tau_d(m,\nu) - d[m + \tau_d(m,\nu), \cdot, H_\omega(\nu)], & \text{if } e = (buyer, \cdot\,); \\ m + \tau_d(m,\nu) + d[\cdot, m + \tau_d(m,\nu), H_\omega(\nu)], & \text{if } e = (seller, \cdot\,); \\ m + \tau_d(m,\nu), & \text{otherwise.} \end{cases}$$

We can now define $P : \mathbb{R}_+ \times \mathfrak{B}_{\mathbb{R}_+} \to [0,1]$ to be

$$P(m, B; \nu) \equiv \xi(\{e \in E | \gamma(m,e) \in B\}).$$

Again, P is a well defined transition function.[8] Then,

$$\omega(B) = H_\omega(\nu)(B) \equiv \int_0^\infty P(m, B; \nu)\ \nu(dm)\ \ \forall B \in \mathfrak{B}_{\mathbb{R}_+}$$

Finally, we can describe the law of motion of the aggregate state over the two markets as

$$\nu'(B) = H(\nu)(B) \equiv \int_0^\infty \int_0^\infty \Pi[\bar{m}, B; H_\omega(\nu)]\ P(m, d\bar{m}; \nu)]\ \nu(dm)\ \ \forall B \in \mathfrak{B}_{\mathbb{R}_+}. \tag{11}$$

4.4 Recursive Equilibrium

We are finally ready to define a recursive equilibrium for this economy.

Definition 1 (Recursive Equilibrium) *A recursive equilibrium is a list of:*

Pricing function: $\phi : \Lambda \to \mathbb{R}_+ \backslash \{0\}$;

Monetary Policy Functions: $\tau_d : \mathbb{R}_+ \times \Lambda \to \mathbb{R}$ and $\tau_c : \mathbb{R}_+ \times \Lambda \to \mathbb{R}$;

Law of motion: $H : \Lambda \to \Lambda$;

Value functions: $V : \mathbb{R}_+ \times \Lambda \to \mathbb{R}$ and $W : \mathbb{R}_+ \times \Lambda \to \mathbb{R}$;

Policy functions: $X : \mathbb{R}_+ \times \Lambda \to \mathbb{R}_+$, $Y : \mathbb{R}_+ \times \Lambda \to \mathbb{R}_+$, and $m' : \mathbb{R}_+ \times \Lambda \to \mathbb{R}_+$;

Terms of Trade: $q : \mathbb{R}_+ \times \mathbb{R}_+ \times \Lambda \to \mathbb{R}_+$ and $d : \mathbb{R}_+ \times \mathbb{R}_+ \times \Lambda \to \mathbb{R}_+$;

[8] By construction, for each m, $P(m, \cdot\,; \nu)$ is a probability measure on $(\mathbb{R}_+, \mathfrak{B}_{\mathbb{R}_+})$. Furthermore, given the measurability of $d(\cdot\,, \cdot\,; \nu)$, $P(\cdot\,, B; \nu)$ is a $\mathfrak{B}_{\mathbb{R}_+}$-measurable function.

such that:

1. given the pricing function, the monetary policy functions, the law of motion, the terms of trade, and the policy functions, the value functions satisfy the functional equations (2-5) and (9);

2. given the value functions, the pricing function, the monetary policy functions, and the law of motion of the aggregate state, the policy functions solve (2);

3. given the value functions, the terms of trade solve (8);

4. given the terms of trade and the monetary policy functions, the law of motion of the aggregate state is defined by (11);

5. given the value functions, the monetary policy functions satisfy (1) and (6);

6. the centralized market clearing condition, (7), is satisfied.

In the remainder of the paper we will only focus on stationary equilibria, where, $\nu = H(\nu)$.

5 Numerical Algorithm

In this section we briefly present the numerical algorithm developed for finding stationary monetary equilibria of the model and discuss some computational considerations. The basic strategy of the algorithm is to iterate on a mapping defined by the value function equations (5) and (9) and the law of motion of the aggregate state given by equation (11). Special care is taken in keeping track of the distribution of wealth and its composition across iterations. In particular, we keep track of a large sample of agent's money balance and use non-parametric density estimation methods. A Fortran 90 version of the code is available from the authors by request. We begin the algorithm at the entrance of the centralized market.

A brief description of the algorithm follows:

Step 1. Given an initial guess for the distribution of money holding at the entrance of the centralized market, draw a large sample of agent's money balance.[9]

[9]In all the numerical exercises we use a sample of 10 000 agents.

Step 2. Define a grid on the state space of money holdings and an initial guess for the value function at the entrance of the decentralized market, $V^0(m)$, by defining the value of the function at the gridpoints and using interpolation methods to evaluate the function at any other point.[10]

Step 3. Given the sample of money holding at the entrance of the centralized market and the value function at the entrance of the decentralized market, find the market clearing price and participation rate by solving the centralized market problem (4) for all agents in the sample and iterating on ϕ and f, given an initial guess, until the market clears.

Step 4. Given these, the function $W(.)$ is given by (5) .

Step 5. Given the market clearing price, update the money holding of the agents by solving their optimization problem. The distribution of money holding at the decentralized market is estimated using Gaussian kernel non-parametric density estimation methods.[11]

Step 6. Given the value function W(.) and the distribution of money holdings at the entrance of the decentralized market, update the value function $V(m)$ by using the mapping defined by equation (9) to compute its value at the new gridpoints and re-estimating the interpolant coefficients.

Step 7. For each individual on the sample, update their money holding by simulating their meetings to derive the distribution at the entrance of the centralized market.

Repeat steps 3 to 7 until convergence is achieved.

[10]We use a grid of 30 gridpoints unevenly spread so as to capture well the change in concavity of the value function. We experimented with increasing the number and location of the gridpoints without significant quantitative or qualitative implications for our results. An Akima interpolation method from the IMSL fortran routines was used to keep track of all functions.

[11]To deal with the fact that the money holdings choices of the participants might imply the existence of mass points in the distribution we introduce a very small perturbation (a find a penny- lose a penny assumption) in their optimal choice to smooth the distribution allowing the usage of the Gaussian kernel estimation method.

6 Numerical Results

In what follows we use the numerical algorithm presented in the last section to find and characterize stationary equilibria of the model. In particular, we characterize the typical features of a stationary equilibrium of the model and illustrate the potential effects of inflation by considering different types of monetary injections.

In the exercises that follow, we interpret the centralized market as a pure liquidity market and assume that $U(X) = X$. This implies that the only reason agents might choose to participate in this market is to re-balance their portfolios. As such, this can be interpreted as purely a financial market.[12] Furthermore, given our interpretation,we ignore the value added in this market for the computation of aggregate output and velocity.

We adopt the following functional form for the utility of consumption in the decentralized market:

$$u(q) = \frac{(q+b)^{1-\eta} - b^{1-\eta}}{1-\eta}$$

Our objective is to parameterize the model in order to match the velocity of money (or alternatively, the demand for money) implied by the data. Note however that, in the model, the velocity of money is affected by several parameters. In particular, it is affected by the curvature parameter η, by the arrival rate σ, the choice of the length of a period (or equivalently, β), and the fraction of agents that participate in the liquidity market (and thus on the fixed cost). Furthermore, most of these parameters are not observable. In the absence of other clear targets, the parameters are not perfectly identifiable from the data. As such, in the exercises that follow we fix some of the parameters and later conduct some sensitivity analysis. For the exercises below we set $b \simeq 0$ and $\eta = 0.99$, that is that the utility function is close to *log*. We maximize the probability of trading in the decentralized market by setting $\sigma = 0.5$, and define the length of a period to be one month. We set the

[12]Under appropriate assumptions to preclude their circulation in the decentralized market, one could introduce nominal bonds in this market and interpret the trades as financial asset trades. In that world, in equilibrium, agents are indifferent between using bonds or trading the general good.

Table 1: Parameter Values

Parameter	Value
σ	0.5
β	0.9967
b	0.0001
η	0.99
$\bar{\kappa}$	0.06

discount factor to $\beta = 0.9967$ implying an annual real interest rate of 4 percent. Given the length of the period, we choose $\bar{\kappa}$ such that the level of velocity matches the average level of consumption velocity of money among O.E.C.D. countries (see Figure 5).[13] Table 1 summarizes the parameter values used.

Per-period velocity is measured by

$$\nu = \sigma \int_0^\infty \int_0^\infty d(m, \tilde{m}, \nu)\ \nu(dm)\ \nu(d\tilde{m}).$$

6.1 Monetary Injections into the Decentralized Market

Here, we consider the case where monetary injections are made solely into the decentralized market via lump-sum transfers. In this case $\tau_d(m, \nu) = \mu - 1 = \mu_d - 1$ and $\tau_c(m, \nu) = 0$ for all (m, ν).

We begin by characterizing a stationary equilibrium. Figure (4) shows the participation function and money distribution for the case of zero inflation. The participation function gives the probability with which an agent, with a given money holdings after trade in the decentralized market, will participate in the centralized market. First, note that agents that are poor enough will always choose to participate in the centralized market. Also, for high enough money holdings the probability will converge to one. Given the assumption

[13] We use I.F.S. data on consumption and monetary aggregates to compute for each country the average velocity from 1970 to 2000. According to the I.F.S. definition, money "comprises transferable demand deposits of monetary authorities and deposit money banks, other than those of the central government, and currency outside banks plus, where applicable, private sector demand deposits with the postal checking system and with the Treasury".

of linear utility in the centralized market, every agent that participates in the centralized market will choose to bring the same amount of money into the decentralized market. This feature generates the spikes (mass points) of the distribution. Given that not all agents will participate in the centralized market in a given period and the randomness in trading opportunities in the decentralized market, the distribution of money will, in general, be non-degenerate. The discreteness in the choice of whether or not to participate in the centralized market will, in general, imply that the value functions will not be concave, which generates the "wiggles" in the participation function. It is interesting to note that, unlike in Molico (2006), poor agents will choose to spend all their money in the decentralized market. This is due to the fact that in the presence of the liquidity market they do not need to self-insure by keeping some money holdings. In general, the willingness of relatively poor agents to spend money is higher and they will hold less money balances for precautionary motives than in Molico (2006). This allows us to match the velocity of money in the data which was not possible in Molico (2006) since in that environment the only way of insuring was to carry large precautionary money balances. Note also that, for 0% inflation, agents that re-balance their portfolios will bring into the decentralized market approximately enough liquidity for three purchases.

We now consider the effects of inflation on the stationary monetary equilibrium. Table 2 reports the outcome of the stationary equilibrium for annualized inflation rates of 0%, 5%, 10%, and 20%. With higher inflation, the price of money in terms of the general good in the centralized market, in general, goes down. Agents choose to economize on their money holdings by participating in the liquidity market more frequently, leading to a higher participation rate. As a result, the velocity of money rises and the total fixed cost of participation increases. Figure (5) shows the consumption velocity predicted by the model and the data. Our model implication is consistent with the positive relationship between velocity and inflation in the cross country data.

We can also measure the welfare cost of inflation by deriving how much consumption

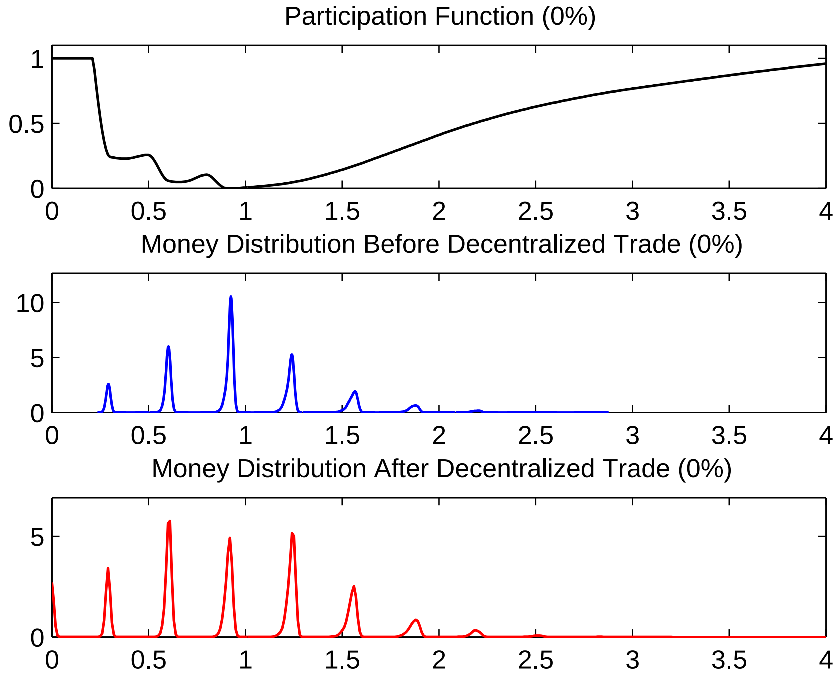

Figure 4: Participation Function and Money Distribution (0% inflation)

Table 2: Effect of Inflation

	0%	5%	10%	20%
Price of Money (ϕ)	3.08	2.20	1.89	1.34
Participation Rate (f)	14%	25%	32%	46%
Annualized Velocity	1.90	2.70	3.03	4.18
Total Fixed Cost (% Annual GDP)	0.23%	0.30%	0.34%	0.47%
Welfare Cost (% C)	0%	0.48%	1.01%	2.21%

agents would be willing to sacrifice to reduce inflation to 0%. Using $\hat{\kappa}_\mu$, q_μ, d_μ, ν_μ and ω_μ to denote the functions and distributions in an equilibrium with inflation rate $\mu - 1$, we can define the average expected value with inflation rate $\mu - 1$ as

$$\bar{U}(\mu) = (1-\beta)^{-1}\{\sigma \int_0^\infty \int_0^\infty [u(q_\mu(m,\tilde{m},\nu_\mu)) - q_\mu(m,\tilde{m},\nu_\mu)]\, \nu_\mu(dm)\, \nu_\mu(d\tilde{m}) - \int_0^\infty \frac{\hat{\kappa}_\mu(m,\omega_\mu)^2}{2\bar{\kappa}}\, \omega_\mu(dm)\}.$$

Then the welfare cost of having money growth rate μ relative to zero inflation is given by $1 - \Delta_0(\mu)$ where $\Delta_0(\mu)$ solves

$$\bar{U}(\mu) = (1-\beta)^{-1}\{\sigma \int_0^\infty \int_0^\infty [u[q_0(m,\tilde{m},\nu_0)\Delta_0(\mu)] - q_0(m,\tilde{m},\nu_0)]\ \nu_0(dm)\ \nu_0(d\tilde{m}) - \int_0^\infty \frac{\hat{\kappa}_0(m,\omega_0)^2}{2\bar{\kappa}}\ \omega_0(dm)\}.$$

To focus on one number, we find that the welfare cost of 10% inflation is $1 - \Delta_0(10\%) = 1.01\%$ of consumption. This number is slightly higher than Lucas (2000) estimate and in line with the findings of Lagos and Wright for the same pricing mechanism as we use in this paper. It is important to notice however that the welfare costs are not linear in inflation. In this model, for low rates of inflation, the redistributive role of lump-sum transfers (insurance role) leads to smaller welfare costs of inflation, while for higher inflation rates the opposite is true.

Figure (6) illustrates the responses of the participation functions and money distributions to inflation. As inflation goes up, the expected duration of time between two consecutive trips to the liquidity market reduces. At 5% inflation, an agent bring into the decentralized market approximately enough liquidity for two purchases. At 20% inflation, they bring enough liquidity for only one purchase. As a result, unlike in Lagos and Wright, the money distribution is a function of the monetary policy: a higher money growth rate implies the distribution is less heterogenous.

6.2 Liquidity Market and Welfare

This subsection compares the welfare effect of inflation under different assumptions regarding the liquidity market. In figure (7), we fix the parameter values for (σ, β, b, η) and examine different specifications for the participation cost. We consider three different environments in which the participation in the centralized liquidity market is either (i) infinitely costly,

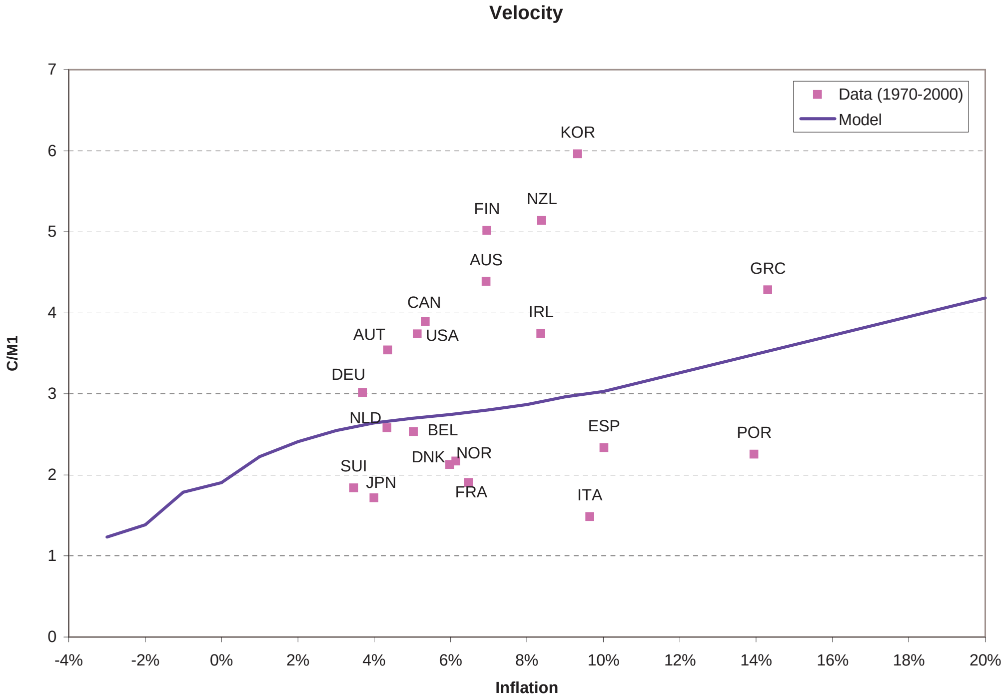

Figure 5: Velocity of Money as a Function of Inflation

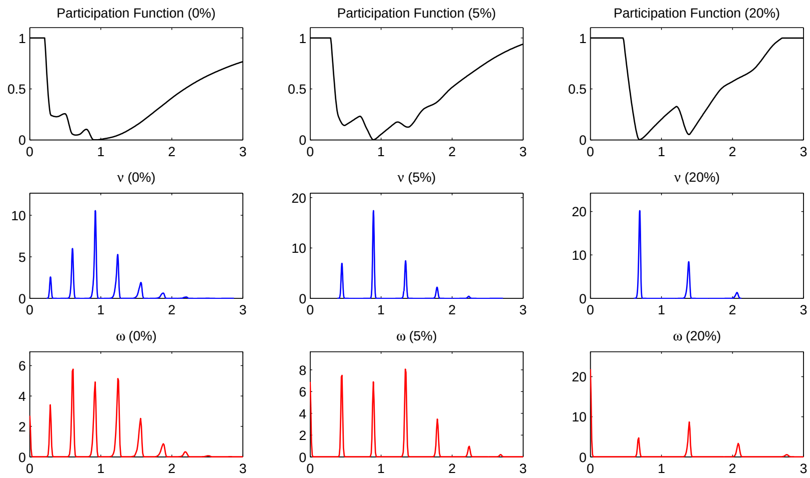

Figure 6: Participation and Distribution for Different Inflation

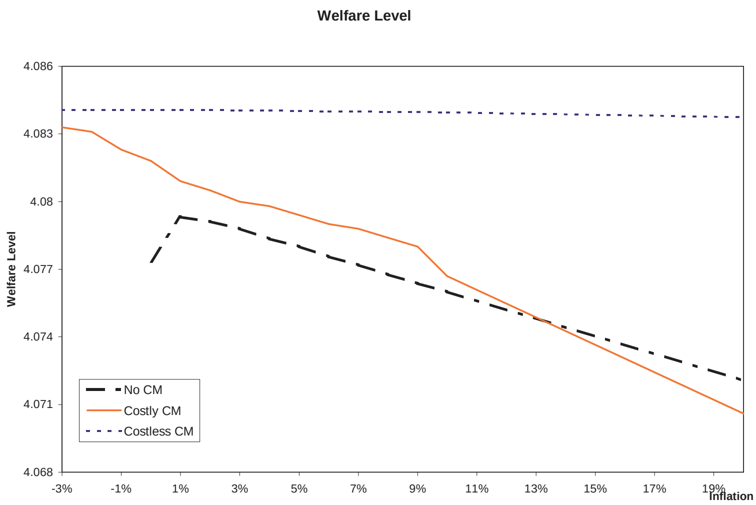

Figure 7: Welfare Level and Inflation: Model Comparison

(ii) costless, or (iii) finitely costly.[14]

(i) No liquidity market

If participating in the liquidity market is infinitely costly for all agents, then the model becomes Molico (2006). In this case, the money distribution is non-degenerate and thus inflation has redistribution effect. Note that, the welfare effect of inflation is non-monotonic: a small positive inflation can improve welfare but further increases in inflation will lower the welfare level.

(ii) Costless liquidity market

If the cost of participation is zero for all agents, then the model becomes Lagos and Wright. In this case, agents participate in the liquidity market every period and the distribution becomes degenerate. With our parameter values, inflation has only a small negative effect on welfare[15]. Apparently, the average welfare level is lower for agents living in economy (i) than in economy (ii). What is the value of a liquidity market? Or equivalently, how much consumption are agents willing to sacrifice in order to move from economy (i) to economy (ii)? In general, the value of the liquidity market depends on the inflation rate

[14] Deflation for economy (i) is not yet examined.

[15] Note that the welfare cost of inflation reported here for an economy with a costless liquidity market is smaller than those reported in Lagos and Wright (2005) because the parameter values are not chosen to calibrate their model.

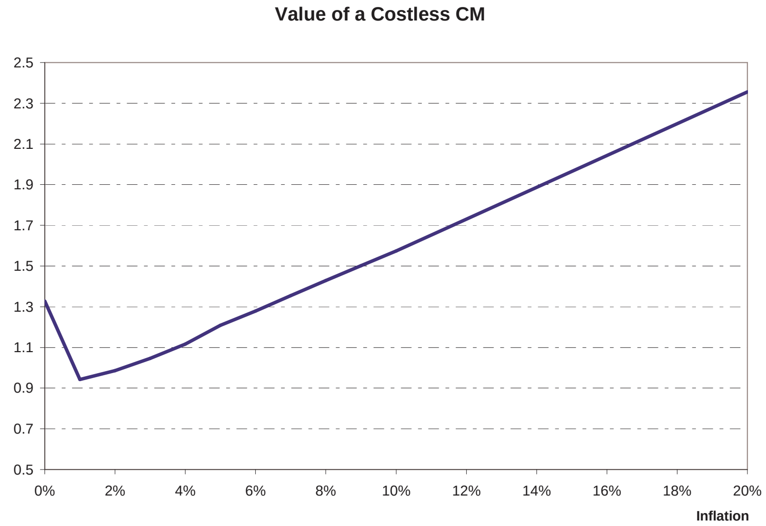

Figure 8: Value of a costless centralized liquidity market

(Figure (8)). The experiment suggests that, for zero inflation, agents are willing to give up 1.3% of their consumption every period to live in an economy with a costless liquidity market. With a small positive inflation, the value of a liquidity market goes down because the insurance function provided by inflation reduces the need for the liquidity market. For high inflation, this value goes up again. This non-monotonic relation highlights the idea that moderate inflation and the liquidity market are alternative insurance mechanism in a world with idiosyncratic trading risks.

(iii) Costly liquidity market

If the access to the liquidity market is costly, as in our model, then agents participate in the liquidity market periodically. As shown in figure (7), the average welfare is always lower than in a world with a costless liquidity market. One interesting finding is that, when the inflation is low, the welfare level is higher in economy (iii) than in economy (i). But for high inflation, the relation reverses. This is related to the fact that, when agents decide to participate in the liquidity market, they do not internalize the external effect on other money holders through the price of money. In this case, the use of a costly liquidity market lowers average welfare due to "over-participation".

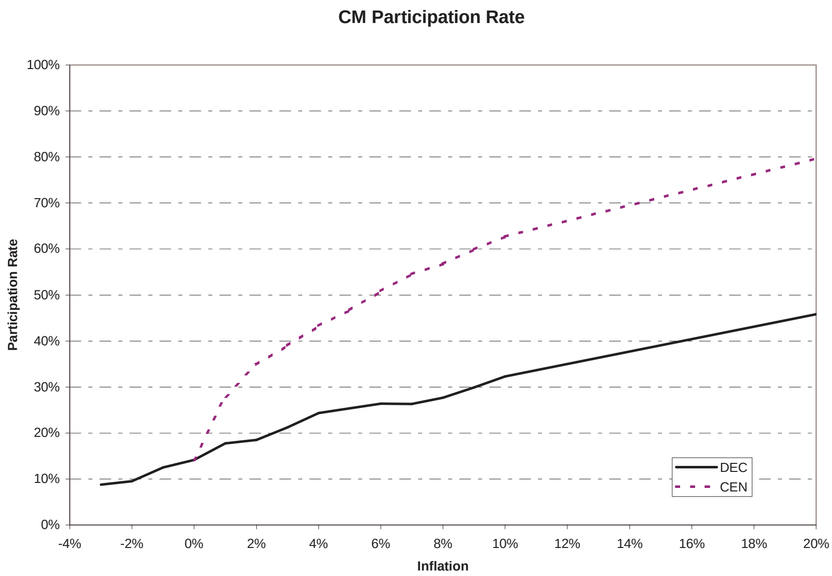

Figure 9: Participation Rate

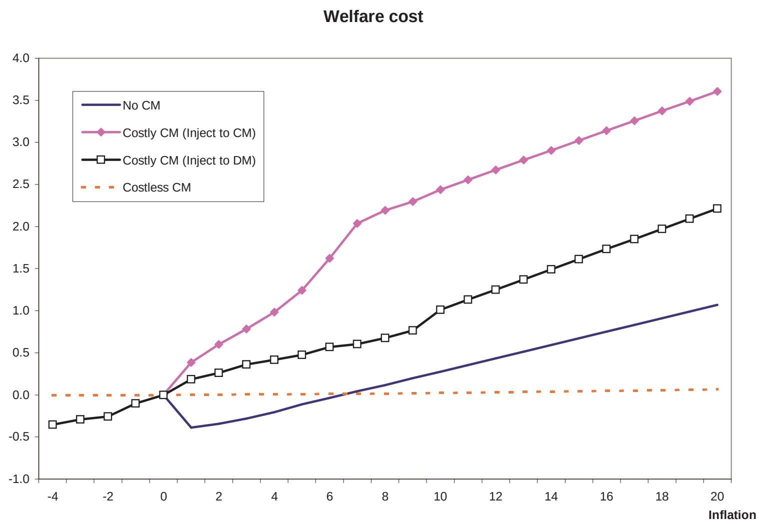

Figure 10: Welfare Cost of Inflation

6.3 Monetary Injections into the Centralized Market

This section considers the case when money is injected to the centralized market via lump sum transfers to participants. In this case $\tau_c(m, \nu) = \mu - 1 = \mu_c - 1$ and $\tau_d(m, \nu) = 0$ for all (m, ν). Money injection in this fashion works like a nominal subsidy to liquidity market participation and leads to redistribution between participants and non-participants. As shown in figure (9), for positive inflation, money injection to the centralized market can induce higher participation rate than injection to the decentralized market.[16]

Figure (10) reports the welfare costs of inflation for different models using the same set of parameter values for $(\sigma,\ \beta,\ b,\ \eta)$. When the fixed cost is zero for all agents, the welfare cost is small and linearly increasing in the inflation. When the fixed cost is infinite for all agents, the welfare cost is a non-monotonic function of the inflation rate. When the fixed cost is finite and random, the welfare cost is a non-linear increasing function of the inflation rate. The welfare cost is higher for centralized market injection than for decentralized market injection.

7 What's Next?

The centralized market in Lagos and Wright (2005) actually performs two roles: (1) centralized, non-monetary trading of general consumption goods, and (2) re-balancing of money holding. The setup described above focuses on the second role by interpreting the centralized market as a liquidity market. We are working on a version of the model with two centralized markets, one for each role.

[16] The case for deflation is not yet examined.

8 Conclusion

This paper develops a micro-founded model of monetary exchange that integrates the monetary search literature (focusing on goods trading friction) and the market segmentation literature (focusing on market participation friction). Unlike in the Lagos-Wright setting, agents have to pay a fixed cost to participate in the centralized market. As a result, agents choose to participate in the centralized market infrequently, leading to a non-degenerate money distribution. By endogenizing the decision of participation, this model also endogenizes the responses of velocity, output, degree of market segmentation as well as the monetary distribution. The welfare cost of inflation implied in this model is also different from the Lagos-Wright model because inflation can distort the consumption profile, affect the market participation, and redistribute money holdings. This paper also extends and generalizes the existing search literature by developing a general framework that nests several existing search models as special cases.

The framework developed in this paper is promising not just for long run but also for short run analysis. Unlike the Lagos-Wright model, this model can generate short run non-neutrality due to trading friction and market participation friction. Moreover, the money distribution and the degree of market segmentation respond endogenously over time to monetary shocks. The authors has make some progress on this direction of research in a companion paper.

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