A THEORY OF HOUSING COLLATERAL, CONSUMPTION INSURANCE AND RISK PREMIA

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ABSTRACT

In a model with housing collateral, a decrease in house prices reduces the collateral value of housing, increases household exposure to idiosyncratic risk, and increases the conditional market price of risk. This collateral mechanism can *quantitatively* replicate the conditional and the cross-sectional variation in risk premia on stocks for reasonable parameter values. The increase of the conditional equity premium and Sharpe ratio when collateral is scarce in the model matches the increase observed in US data. The model also generates a return spread of value firms over growth firms of the magnitude observed in the data, because the term structure of consumption strip risk premia is downward sloping.

Introduction

Recent asset pricing research has documented a number of striking differences in expected returns: between equity and bonds (Mehra and Prescott (1985)), between equity at different points in time (Lettau and Ludvigson (2003)), between size- and value-weighted portfolios (Fama and French (1992)), and between bonds of different maturities (Backus and Zin (1994)). To explain these differences in expected returns, we build a dynamic general equilibrium model that approximates the frictions inhibiting risk-sharing in an advanced economy like the US.

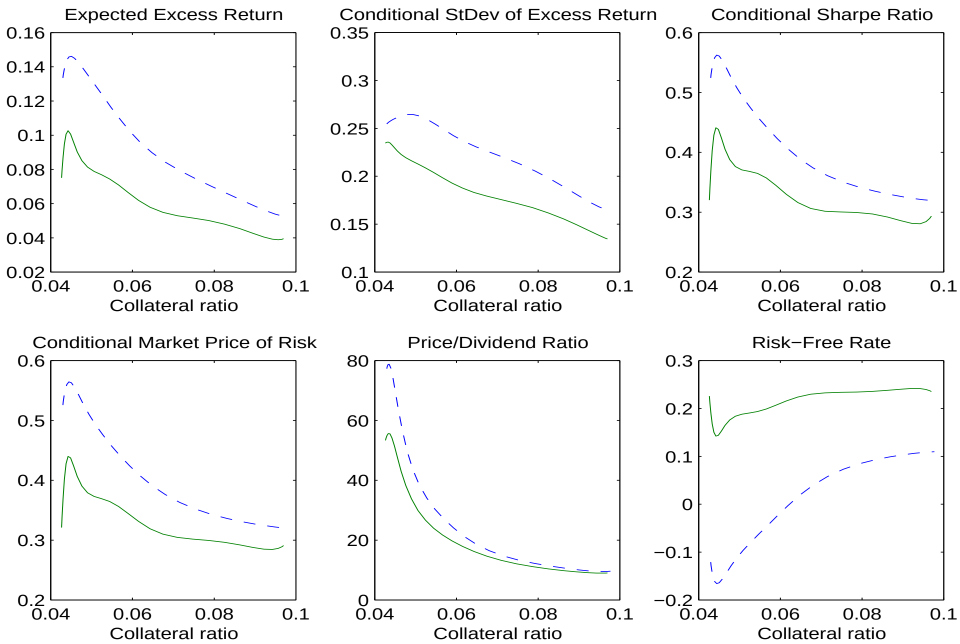
The households in our model trade a complete menu of assets, but they face solvency constraints because they can forget their debts without exclusion from trading. Instead, they lose their collateral assets. The solvency constraints are just tight enough to prevent default in equilibrium. The housing stock is the only net source of collateral in the model.

A decrease in the amount of housing collateral increases the limited commitment deviations from complete risk sharing outcomes in prices and quantities. Actual risk-sharing patterns between US metropolitan areas, explored in Lustig and VanNieuwerburgh (2004a), lend support to the collateral mechanism, while Campbell and Cocco (2004) provide similar evidence from UK household data. This paper focuses on intertemporal prices rather than quantities. Its main objective is to demonstrate that the endogenous time variation in the amount of housing collateral can *quantitatively* account for three of the differences in expected returns: between equity and the risk-free asset, between equity at different points in time, and between size and value portfolios. This complements the work by Lustig and VanNieuwerburgh (2004b), who provide empirical evidence for the collateral mechanism in US stock returns, but do not actually solve a general equilibrium model.

The benchmark Lucas (1978) model implies small and roughly constant stock risk premia over time and little or no risk premium variation in the cross-section, because US consumption growth is not volatile and close to i.i.d. over time. However, in the data the conditional risk premia in stock markets vary substantially over time. The excess returns on stocks are predictable (Lamont (1998)), especially at longer holding periods, and much of the variation in price-dividend ratios of stocks in US data is due to changes in expected returns rather than changes in expected dividend growth (Campbell and Shiller (1988)). Several explanations of the time-variation in risk premia have been proposed, most of them are preference-based or aggregate consumption-based. We take a different route. We chose to leave the aggregate consumption process and the preferences unchanged, but focus on the risk sharing technology instead.

Our model predicts high and volatile equity premia, as well as high Sharpe ratios, when collateral is scarce. The Sharpe ratio itself is equally volatile. Figure 1 shows that the model's expected excess return on equity, its conditional standard deviation, the conditional market price of risk and the conditional Sharpe ratio are decreasing functions of the amount of housing collateral available in the economy. We document similar dynamics for the excess returns and the Sharpe ratio in US data. The model also implies that the ratio of housing collateral wealth to total wealth, henceforth the *housing collateral ratio*, predicts future excess returns. Lustig and VanNieuwerburgh (2004b) document this predictability pattern in US stock returns.

Figure 1. Summary Conditional Asset Pricing Moments.
The first row plots the expected excess return on a claim to aggregate consumption (panel 1), its conditional standard deviation (panel 2) and its Sharpe ratio (panel 3). The second row plots the conditional market price of risk (panel 4), the conditional price-dividend ratio (panel 5) and the risk-free rate (panel 6). All series are averaged over histories and plotted against the housing collateral ratio. In each panel, the full line denotes the conditional moments, conditional on observing a low aggregate consumption growth rate tomorrow, whereas the dotted line denotes the moments conditional on observing a high aggregate consumption growth rate. The model simulation uses the benchmark calibration, discussed in section 2.2.



Risk premia also vary substantially across different securities. According to Fama and French (1992), value stocks earn returns that are on average six percent higher than growth stocks; this premium is of the same size as the equity premium. If value stocks are short duration stocks, a value premium implies that the term structure of consumption strip risk premia is downward sloping (Lettau and Wachter (2004)). Our model replicates this feature of the data.[1] In our model, bad aggregate shocks lower the conditional Sharpe ratio and this makes long duration stocks less rather than more risky.

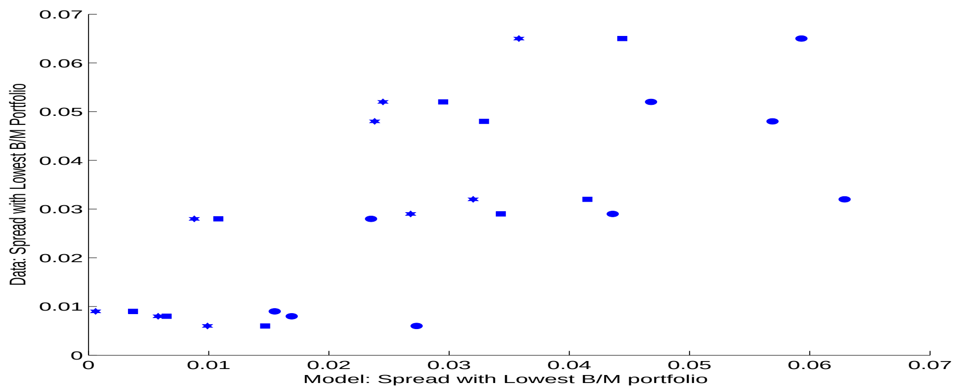
The housing collateral model generates a large value premium and higher Sharpe ratios for value stocks than growth stocks. We show this in two different ways. First, following Lustig and VanNieuwerburgh (2004b), we regress excess returns on value decile portfolios on the empirical counterparts of the model's risk factors. We then simulate artificial book-to-market decile excess returns by using these empirical factor loadings inside the model. The model produces risk premia spreads between the extreme value and growth portfolios of 6 percent per year, as high as in the data. Figure 2 plots the model-generated spreads on these B/M-sorted portfolios on the horizontal axis against the same spreads in the data on the vertical axis. Moreover, the Sharpe ratio on value portfolios is twice as high as the Sharpe ratio on growth portfolios, just as in the data. Value stocks are risky investments because their returns are more correlated with aggregate consumption growth shocks when the collateral ratio is low. This does not explain where the factor loadings come from. In a second exercise we show that the model generates a decreasing term structure of consumption strip risk premia. The model generates a value premium because short duration assets, such as value stocks, are more risky than long duration assets, such as growth stocks. The return spread between a basket of consumption strips with a duration of 5 years ('value') and a basket with a duration of 40 years ('growth') is 6 percent. The slope of the term structure increases when the collateral ratio is low and it flattens when the collateral ratio is high. This explains why value stock returns have a bigger loading on the collateral ratio interacted with consumption growth.

It has been understood for more two decades that frictions are needed to bring the consumption-based capital asset pricing model closer to the data, but incomplete markets with exogenous borrowing constraints, short sales constraints or transaction costs (e.g. Telmer (1993) and Heaton and Lucas (1996)), failed to deliver sufficiently large deviations from the benchmark model. The goal of our paper is to show how endogenous, state-contingent borrowing constraints interact with shocks

[1] Lettau and Wachter (2004) note that if bad news that increases the growth rate of marginal utility today, also increases the conditional Sharpe ratio, and if the conditional Sharpe ratio is positively autocorrelated, then the term structure of consumption risk premia is upward sloping. That is why the standard habit model (Campbell and Cochrane (1999)) implies a growth premium.

Figure 2. Spreads in B/M deciles

Returns in excess of the return on the first decile portfolio in the model (horizontal axis) vs. in the data for the 9 other deciles (vertical axis). The stock returns on the book-to-market deciles are from Kenneth French's web site, annual data for 1945-2003. The filled dots are the spreads computed using collateral measure myfa (fixed asset-based), the squares use a second collateral measure myrw (residential-wealth-based), and the stars use a third collateral measure mymo (mortgage-based). The model simulation uses the benchmark calibration, discussed in section 2.2.



in the housing market to deliver plausible asset pricing predictions. The equilibrium changes in the value of the housing stock perturb the risk sharing technology, and modify the effect the household's inability to commit to allocations and prices. For a plausibly calibrated housing collateral process, we significantly improve on the predictions of the canonical CCAPM, both in terms of the time-series variation in conditional asset pricing moments and the cross-sectional variation in returns.

Two channels in the model deliver time variation in the conditional market price of risk, at different frequencies. First, a drop in the housing collateral ratio adversely affects the risk sharing technology that enables households to insulate their consumption from labor income shocks. This makes households demand a higher price to bear risk in times with low housing collateral. This is the source of low frequency variation in the market price of risk. Second, households are more exposed to binding collateral constraints when the cross-sectional dispersion of labor income shocks increases. This typically occurs when aggregate consumption growth is low, as documented in Storesletten, Telmer, and Yaron (2004). A series of large aggregate consumption growth shocks increases the conditional standard deviation of the stochastic discount factor, because a larger fraction of households is more severely constrained when a negative aggregate consumption growth shock arrives. Immediately after a negative aggregate consumption growth shock, the conditional market price of risk drops. The left tail of the net wealth distribution has been erased and a much smaller mass of agents will be constrained when the next low aggregate consumption growth shock arrives. This is the source of high frequency variation in the market price of risk, also present in Lustig (2003). Only the combination of both can explain the time-series variation in asset prices and the cross-section of value-portfolio returns.

However, the same collateral mechanism and wealth distribution dynamics that close the gap between the model and the data, both in the time-series and the cross-section of equity risk premia, impute too much volatility to the risk-free rate. Introducing recursive utility helps to smooth interest rates, but does not resolve this problem altogether. Finally, the model predicts too high a correlation between housing returns and stock returns, a prediction shared with other asset pricing models. The lack of correlation in the data remains a stylized fact that is hard to account for.

Related Literature The model includes a second channel that transmits housing shocks to asset prices: non-separable preferences. If utility is non-separable in non-durable consumption and housing services, households want to hedge against shocks to the share of housing consumption in total consumption. This introduces composition risk, which is the focus of recent work by Piazzesi, Schneider, and Tuzel (2004), Yogo (2003), and earlier work by Flavin (2001). When housing services and consumption are complements, and rental price growth is positively correlated with returns, then households command a larger risk premium.[2] Our collateral effect does not hinge on the non-separability of preferences. Instead, it relies on imperfect consumption insurance among heterogenous households induced by occasionally binding collateral constraints. We explore the parameter space to quantify the relative effects of the composition risk mechanism and the collateral mechanism. We find that only the latter can quantitatively account for the observed volatility in conditional asset pricing moments, the cross-sectional variation in expected returns; composition risk can match high equity premia only if the implied rental price volatility is off by an order and magnitude, and it does not generate time variation in the conditional risk premia.[3]

The households in this economy trade contingent claims to insure against labor income risk. These claims have to be fully backed by the value of their housing wealth. As in Lustig (2003), we allow households to forget their debts. The new feature of our model is that each household

[2]Dunn and Singleton (1986) and Eichenbaum and Hansen (1990) report substantial evidence against the null of separability in a representative agent model with durable and non-durable consumption, but they conclude that introducing durables does not help in reducing the pricing errors for stocks.

[3]Piazzesi, Schneider, and Tuzel (2004) introduce heteroscedasticity in the housing expenditure share to obtain variation in the conditional market price of risk and predictability of excess returns on equity.

owns part of the housing stock. Housing provides utility services and collateral services. When a household chooses to forget its debts, it loses all its housing wealth but its labor income is protected from creditors. The household is not excluded from trading. The lack of commitment gives rise to collateral constraints. Their tightness depends on the abundance of housing collateral. We measure this by the housing collateral ratio: the ratio of collateralizable housing wealth to total wealth. An increase in the housing collateral ratio increases the scope for risk sharing, and it decreases the conditional dispersion of consumption growth across households.

We model the outside option as bankruptcy with loss of all collateral assets. In Kehoe and Levine (1993), Krueger (2000), Krueger and Perri (2003), and Kehoe and Perri (2002), limited commitment is also the source of incomplete risk-sharing across US households and across countries respectively, but the outside option upon default is exclusion from all future risk sharing arrangements. In our model, all promises are backed by all collateral assets. Geanakoplos and Zame (2000) and Kubler and Schmedders (2003) consider a different environment in which individual assets collateralize individual promises in an incomplete markets economy. Our paper is the first to argue that the housing collateral mechanism can quantitatively replicate the variation in conditional moments of asset prices observed in the data. Our emphasis on housing as the collateral asset, rather than financial assets, reflects two features of the US economy: the participation rate is much higher than for equity (two-thirds of households own their own house), and the aggregate value of housing in the US is roughly double the value of equity owned by households (Flow of Funds data).

We organize the paper as follows. Section 1 describes the environment, characterizes equilibrium allocations and defines the pricing kernel. Section 2 discusses the model computation and calibration. Asset pricing results from a simulation of the model are discussed in section 3, 4 and 5. Section 6 concludes. Section 7 contains all figures and tables. The appendix contains some details of the model (A.1, A.3), proofs of the propositions (A.2) and a description of the data (A.4). The results for an economy where preferences are recursive, rather than additive are available in a separate appendix. That appendix also contains auxiliary tables and derivations.[4]

[4] It can be downloaded from *http://www.econ.ucla.edu/people/faculty/Lustig.html.*

1. Model

This section starts with a complete description of the environment in section 1.1. The next section (1.2), sets up the household problem in a time zero trading environment, and characterizes equilibrium allocations using stochastic consumption weight processes. The growth rate of an *aggregate* consumption weight process drives the consumption growth of the unconstrained households. These unconstrained households price random payoffs. Section 1.3 explains how the dynamics of the collateral ratio affect equilibrium allocations. To gain intuition, section 1.4 describes the dynamics in a simple two-agent economy. Section 1.5 introduces sequential trading and discusses conditions under which these equilibria coincide with time zero trading equilibria.

We consider the simplest model of housing markets that delivers variation in the amount of housing collateral relative to total wealth. The housing market has efficient rental markets or spot markets for housing services. Ownership and consumption of housing are completely separated. We calibrate the persistence of the consumption/housing expenditure ratio to the data. Variation in the expenditure ratio changes the value of the housing tree relative to the value of the other, non-durable consumption tree.

1.1. Environment

This economy is populated by a continuum of infinitely lived households. The structure of uncertainty is twofold. $s = (y, z)$ is an event that consists of a household-specific component $y \in Y$ and an aggregate component $z \in Z$. These events take on values on a discrete grid $S = Y \times Z$. We use $s^t = (y^t, z^t)$ to denote the history of events. S^t denotes the set of possible histories up until time t. s follows a Markov process with transition probabilities π that obey:

$$\pi(z'|z) = \sum_{y' \in Y} \pi(y', z'|y, z) \ \forall z \in Z, y \in Y.$$

Because of the law of large numbers, $\pi_z(y)$ denotes both the fraction of households drawing y when the aggregate event is z and the probability that a given household is in state y when the aggregate state is z.

We use $\{x\}$ to denote an infinite stream $\left\{x_t(s^t)\right\}_{t=0}^{\infty}$. There are two types of commodities in this economy: a consumption good and housing services. These consumption goods cannot be stored.

We let $\{c(\theta_0, s_0)\}$ denote the stream of consumption and we let $\{h(\theta_0, s_0)\}$ denote the stream of housing services of a household of type (θ_0, s_0). The households rank consumption streams according to the criterion:

$$U(\{c\}, \{h\}) = \sum_{s^t|s_0} \sum_{t=0}^{\infty} \delta^t \pi(s^t|s_0) u\left(c_t(\theta_0, s^t), h_t(\theta_0, s^t)\right), \tag{1}$$

where δ is the time discount factor. The households have power utility over a CES-composite consumption good:

$$u(c_t, h_t) = \frac{\left[c_t^{\frac{\varepsilon-1}{\varepsilon}} + \psi h_t^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{(1-\gamma)\varepsilon}{\varepsilon-1}}}{1-\gamma}.$$

The parameter $\psi > 0$ converts the housing stock into a service flow, γ captures the degree of relative risk aversion, and ε is the intratemporal elasticity of substitution between non-durable consumption and housing services.[5]

The aggregate endowment of the non-durable consumption good is denoted $\{e\}$. The growth rate of the aggregate endowment depends only on the current aggregate state: $e_{t+1}(z^{t+1}) = \lambda(z_{t+1})e_t(z^t)$. Aggregate consumption c^a equals the aggregate endowment e. Each of the households is endowed with a claim to a labor income stream $\{\eta\}$. The labor income share $\hat{\eta}(y_t, z_t)$ only depends on the current state of nature. The level of labor income is given by $\eta(y_t, z^t) = \hat{\eta}(y_t, z_t)e(z^t)$. The aggregate endowment is the sum of the individual endowments:

$$\sum_{y' \in Y} \pi_z(y')\hat{\eta}_t(y', z) = 1, \; \forall z, t \geq 0.$$

The aggregate endowment of housing services is denoted $\{h^a\}$. Rather than specifying a process for $\{h^a\}$, we specify a process for the expenditure ratio of non-durable to housing services consumption $\{r\}$, where $r(z^t) = \frac{c^a(z^t)}{\rho(z^t)h^a(z^t)}$ and $\rho(z^t)$ denote the relative price of a unit of housing services. We use R to denote the ergodic set for the process r.

We let $p_t(s^t|s_0)$ denote the state price deflator. It is the price of a unit non-durable consumption to be delivered in state s^t, in units of time zero consumption. Finally, $\Pi_{s^t}[\{d\}]$ denotes the price of claim to $\{d\}$ in units of s^t consumption, $\Pi_{s^t}[\{d\}] = \sum_{\tau=t} \sum_{s^\tau|s^t}^{\infty} \left[p_\tau\left(s^\tau|s^t\right) d_\tau\left(s^\tau|s^t\right)\right]$.

[5] The preferences belong to the class of homothetic power utility functions of Eichenbaum and Hansen (1990). Special cases are separability ($\varepsilon = \gamma^{-1}$) and Cobb-Douglas preferences ($\varepsilon = 1$).

Markets open only at time zero. Households purchase a complete, state-contingent consumption plan $\{c(\theta_0, s_0), h(\theta_0, s_0)\}$ subject to a single, time zero budget constraint:

$$\Pi_{s_0}\left[\{c(\theta_0, s_0) + \rho h(\theta_0, s_0)\}\right] \leqslant \theta_0 + \Pi_{s_0}\left[\{\eta\}\right], \tag{2}$$

where θ_0 is the initial non-labor wealth. We use Θ_0 to denote the initial distribution of non-labor wealth holdings.

Solvency Constraints Households can default on their debts. When the household defaults, it keeps its labor income in all future periods. The household is not excluded from trading, even in the same period. However, all collateral wealth is taken away. As a result, the markets impose a solvency constraints that keep the households from defaulting. There is one such constraint for each node s^t on the household's trades:

$$\Pi_{s^t}\left[\{c(\theta_0, s_0) + \rho h(\theta_0, s_0)\}\right] \geq \Pi_{s^t}\left[\{\eta\}\right]. \tag{3}$$

As shown by Lustig (2003), imposing these solvency constraints is equivalent to imposing participation constraints when agents cannot be excluded from trading. The constraints depend on the state prices $\{p\}$, as well as the rental price $\{\rho\}$. Changes in equilibrium prices determine the tightness of the constraints, as we describe in the next section.

1.2. Equilibrium Prices and Allocations

We define an equilibrium with all trading at time zero and we characterize the equilibrium allocations. These Kehoe and Levine (1993) equilibria are essentially Arrow-Debreu equilibria. Appendix A.1 fills in the details.

Definition. *For given initial state z_0 and for given distribution Θ_0, an equilibrium consists of prices $\left\{p_t(s^t|s_0), \rho(z^t|z_0)\right\}$ and allocations $\left\{c_t(\theta_0, s^t), h_t(\theta_0, s^t)\right\}$ such that*

- *Given prices, the allocations solve the household's problem of maximizing (1) subject to (2) and (3) (except possibly on a set of measure zero).*

- *Markets clear for all t, z^t:*

$$\sum_{y^t} \int c_t(\theta_0, y^t, z^t) d\Theta_0 \frac{\pi(y^t, z^t | y_0, z_0)}{\pi(z^t|z_0)} = c_t^a(z_t) = e_t(z^t), \tag{4}$$

$$\sum_{y^t} \int h_t(\theta_0, y^t, z^t) d\Theta_0 \frac{\pi(y^t, z^t | y_0, z_0)}{\pi(z^t|z_0)} = h_t^a(z_t). \tag{5}$$

To determine the equilibrium consumption of households, it is helpful to examine the dual of the household maximization problem. Let $U_0(\{c\}, \{h\})$ denote the total utility from consuming $\{c\}$ and $\{h\}$. For given prices $\{p, \rho\}$ a household with label (w_0, s_0) minimizes the cost $C(\cdot)$ of delivering initial utility w_0 to itself:

$$\begin{aligned} C(w_0, s_0) &= \min_{\{c,h\}} \left(c_0(w_0, s_0) + h_0(w_0, s_0)\rho_0(s_0)\right) \\ &+ \sum_{s^t} p(s^t|s_0) \left(c_t(w_0, s^t|s_0) + h_t(w_0, s^t|s_0)\rho_t(z^t|z_0)\right) \end{aligned}$$

subject to the initial promised utility constraint: $U_0(\{c\}, \{h\}) \geq w_0$, and the solvency constraints (3), one for each node s^t. The initial promised value w_0 is determined such that the household spends its entire initial wealth: $C(w_0, s_0) = \theta_0 + \Pi_{s_0}[\{\eta\}]$. There is a monotone relationship between θ_0 and w_0.

Stochastic Consumption Weights Let $\{\gamma(\theta_0, s_0)\}$ denote the sequence of multipliers on the solvency constraints (3) imposed on household (θ_0, s_0). We define $\xi_t(\theta_0, s^t)$ to be household (θ_0, s_0)'s cumulative Lagrange multiplier:

$$\xi_t(\theta_0, s^t) = \ell(\theta_0, s_0) + \sum_{\tau=0}^{t} \sum_{s^\tau \preceq s^t} \gamma_\tau(\theta_0, s^\tau).$$

We refer to $\xi_t(\theta_0, s^t)$ as the *consumption weight* in state s^t for household (θ_0, s_0) in the dual problem. The initial weight $\ell(\theta_0, s_0)$ is the inverse of the Lagrange multiplier on the initial promised utility constraint, $\xi_0(\theta_0, s_0) = \ell$. $\{\xi(\theta_0, s_0)\}$ is a non-decreasing stochastic process.

The process $\xi_t^a(z^t)$ is the aggregate weight where $\xi_t^a(z^t)$ is $\sum_{y^t} \int \xi_t(\ell, y^t, z^t) d\Phi_0 \frac{\pi(y^t, z^t|y_0, z_0)}{\pi(z^t|z_0)}$. Φ_0 is the initial cross-sectional distribution over $\ell(\theta_0, s_0)$, implied by the initial wealth distribution Θ_0. $\xi_t^a(z^t)$ summarizes to what extent solvency constraints bind *on average*.

If the solvency constraint does not bind, the household's weight remains unchanged. When the constraint binds, its weight increases to a cutoff level $\ell^c(y_t, z^t)$ that depends only on the current event y_t.

$$\begin{aligned}\xi_t &= \xi_{t-1} \text{ if } \xi_{t-1} > \ell^c(y_t, z^t), \\ \xi_t &= \ell^c(y_t, z^t) \text{ otherwise.}\end{aligned}$$

This imputes limited memory to the allocations: a household's individual history y^{t-1} is erased whenever it switches to a state with binding constraints. This is the amnesia property, present in many endogenously incomplete markets models (Ljungqvist and Sargent (2004)).

Risk-Sharing Rule There is a mapping from the multipliers at s^t to the equilibrium allocations of both commodities. We refer to this mapping as the risk-sharing rule. This rule follows from the optimality conditions of the dual household problem and the market clearing conditions. Henceforth, we express individual-specific variables as functions of (ℓ, s^t) rather than (θ_0, s^t). We conjecture a linear risk sharing rule: the consumption share and the housing services share is a function of the household's own consumption weight and an aggregate sum of these weights:

$$c_t(\ell, s^t) = \frac{\xi_t(\ell, s^t)^{\frac{1}{\gamma}}}{\xi_t^a(z^t)} c_t^a(z^t) \text{ and } h_t(\ell, s^t) = \frac{\xi_t(\ell, s^t)^{\frac{1}{\gamma}}}{\xi_t^a(z^t)} h_t^a(z^t). \tag{6}$$

It is easy to verify that this rule satisfies the first order condition for non-durable and housing services consumption and the market clearing conditions.

When a household switches to a state with a binding constraint, its consumption share increases. Everywhere else, its consumption share is drifting downwards at the rate $\Delta \log \xi_{t+1}^a$. Shocks to $\xi_t^a(z^t)$ reflect aggregate shocks to the wealth distribution. Because they follow from an inability to insure against labor income shocks, they can be interpreted as liquidity shocks.

The perfect commitment environment is a benchmark for understanding this risk sharing rule. Because households are never constrained, the individual weight stays constant and is equal to the initial consumption weight: $\xi_t(\ell, s^t) = \xi_0(\ell, s_0) = \ell$. The aggregate weight process reflects the initial wealth distribution and is constant: $\xi_t^a(z^t) = \xi_0^a(z_0) = \int \ell(y_0, z_0)^{\frac{1}{\gamma}} d\Phi_0(z_0)$. Consumption shares are constant; consumption levels only move with aggregate consumption. There is full insurance.

Rental Prices In equilibrium, all households equate the ratio of marginal utilities for housing services and for consumption:

$$\rho_t(z^t) = \psi \left(\frac{h_t(\ell, s^t)}{c_t(\ell, s^t)} \right)^{\frac{-1}{\varepsilon}} = \psi \left(\frac{h_t^a(z^t)}{c_t^a(z^t)} \right)^{\frac{-1}{\varepsilon}}.$$

The price of rental services is a function of the aggregate history z^t only. As a result, all households have the same equilibrium expenditure ratio $r_t(\ell, s^t) = r_t(z^t) = \frac{c_t^a}{\rho_t h_t^a}$.

Stochastic Discount Factor For pricing purposes there is a stand-in consumer whose preferences are defined over "twisted" aggregate non-durable and housing services consumption processes:

$$\tilde{U}(\{c^a\}, \{h^a\}) = U \left(\left\{ \frac{c^a}{\xi_t^a} \right\}, \left\{ \frac{h^a}{\xi_t^a} \right\} \right)$$

In each state, the payoffs are priced by the household with the highest IMRS. This maximum is attained for the unconstrained households. If not, there would be an arbitrage opportunity. The risk sharing rule (6) can be used to compute the unconstrained households' intertemporal marginal rate of substitution (IMRS). The implied stochastic discount factor (SDF) is:

$$m_{t+1} = m_{t+1}^a \left(\frac{\xi_{t+1}^a}{\xi_t^a} \right)^{\gamma}, \tag{7}$$

The SDF consists of two parts. First, without the collateral constraints, ours is a representative agent economy: $m_{t+1} = m_{t+1}^a$. If utility is non-separable, the housing market introduces a novel risk factor: shocks to the non-housing expenditure share.

$$m_{t+1}^a = \delta \left(\frac{c_{t+1}^a}{c_t^a} \right)^{-\gamma} \left(\frac{\alpha_{t+1}^a}{\alpha_t^a} \right)^{\frac{\varepsilon\gamma - 1}{\varepsilon - 1}}.$$

where α^a is the aggregate non-durable expenditure share: $\alpha_t^a = \frac{c_t^a}{c_t^a + \rho_t h_t^a} = \frac{r_t}{1+r_t}$. This is the IMRS of the representative agent economy with non-separable preferences who consumes the aggregate non-durable and housing services endowment. When preferences are separable ($\varepsilon = \gamma^{-1}$), $m_{t+1}^a = \delta \left(\frac{c_{t+1}^a}{c_t^a} \right)^{-\gamma}$ is the SDF of the Lucas (1978) economy.

The second part of the SDF is the growth rate of the aggregate consumption weight process ξ_{t+1}^a, raised to the power γ. It reflects the risk of binding solvency constraints. When many households are severely constrained in state z^{t+1}, that state's price increases, because the unconstrained households experience high marginal utility growth: $\xi^a(z^{t+1}) > \xi^a(z^t)$. When nobody is constrained, the aggregate consumption weight process stays constant, $\xi^a(z^{t+1}) = \xi^a(z^t)$, and the representative agent SDF re-emerges. The risk of binding solvency constraints endogenously creates *heteroscedasticity* in the SDF.

As pointed out, the SDF equals the IMRS of those households that are unconstrained between period t and period $t+1$. However, this does not imply that the value of a claim to a non-negative dividend stream $\{d_t\}$ is the maximum valuation across agents, using the individual IMRS. In fact, the price exceeds the maximum valuation across agents if the constraints bind in equilibrium. The reason is that the identity of the households who price the asset switches over time. Only if an asset were not collateralizeable would its price equal the maximum valuation across agents. The same SDF prices all payoffs, including the rental income from houses.

No arbitrage implies that the return on a security j, R^j_{t+1}, satisfies $E_t[m_{t+1}R^j_{t+1}] = 1$. We model the return on the market as a levered claim to the aggregate non-durable endowment.

1.3. Collateral Supply

The tightness of the constraints depends on the ratio of aggregate housing wealth to total aggregate wealth. To see this, we aggregate the solvency constraints across households and define the housing collateral ratio $my\left(z^t\right)$:

$$my(z^t) = \frac{\Pi_{z^t}\left[\{h^a\rho\}\right]}{\Pi_{z^t}\left[\{c^a + h^a\rho\}\right]} = \frac{\Pi_{z^t}\left[\left\{\frac{c^a}{r}\right\}\right]}{\Pi_{z^t}\left[\left\{c^a\left(1+\frac{1}{r}\right)\right\}\right]}. \tag{8}$$

The numerator is the value of aggregate housing wealth. It equals the price of a claim to the aggregate housing dividend $\Pi_{z^t}\left[\{\rho h^a\}\right]$. If r is constant, the collateral ratio equals $\frac{1}{1+r}$. More generally, the second equality shows that persistent variation in the non-durable expenditure ratio r gives rise to persistent variation in my and in the amount of risk sharing that can be sustained. So, the housing collateral ratio indexes the risk-sharing capacity of the economy. We formalize this notion in the next propositions.

If the total consumption claim is valued sufficiently high, then perfect risk sharing can be sustained. We denote the price of a claim under perfect risk-sharing as $\Pi^*[\{\cdot\}]$.

Proposition 1. *Perfect risk sharing can be sustained if and only if*

$$\Pi_z^* \left[\left\{ c^a \left(1 + \frac{1}{r} \right) \right\} \right] \geq \Pi_{z,y}^* \left[\{ \eta(y,z) \} \right] \text{ for all } (y,z) \in (Y,Z) \text{ and for all } r \text{ in } R$$

If this condition is satisfied, each household can consume the average endowment without violating its solvency constraint. If not, perfect risk sharing is not feasible.

The following proposition states that an economy with more housing collateral (lower r) has lower cutoff weights, allowing for more consumption smoothing. An increase in the supply of collateral (lower r) brings the cutoff rules closer to their lower bound of zero. In the limit perfect risk-sharing obtains. Conversely, a decrease in the supply of collateral (higher r) brings the cutoff rules closer to their upper bound: the labor income shares. In the limit, as the collateral disappears altogether, the households revert to autarky. The following proposition makes this point more formally.

Proposition 2. *Assume utility is separable. Consider 2 economies with $r^1_\tau(z^\tau) < r^2_\tau(z^\tau)$ for all $z^\tau \geq z^t$. Then the cutoff rules satisfy $\ell^{1,c}(y_t, z^t) \leq \ell^{2,c}(y_t, z^t)$. As $r_\tau(z^\tau) \to \infty$ for all $z^\tau \geq z^t$, $\ell^c(y_t, z^t) \to \hat{\eta}(y_t, z^t)$. Conversely, as $r_\tau(z^\tau) \to 0$ for all $z^\tau \geq z^t$, $\ell^c(y_t, z^t) \to 0$*

Perturbations of the r process affect the equilibrium aggregate weight process. An economy with a uniformly higher r process (less collateral) has higher liquidity shocks and lower interest rates on average.

Corollary 1. *Consider 2 economies with $r^1_t(z^t) < r^2_t(z^t)$ for all z^t. Fix the distribution of initial multipliers across economies: $\Phi^1_0(z_0) = \Phi^2_0(z_0)$. Then $\left\{ \xi^{a,1}_t(z^t) \right\} \leq \left\{ \xi^{a,2}_t(z^t) \right\}$ and the state prices are higher on average in the second economy.*

The proposition and corollary illustrate the mechanism that underlies the time-variation in the equilibrium market price of aggregate risk by comparing two economies with different collateral processes $\{r\}$. In section 2, we calibrate the evolution of the expenditure ratio r, simulate the model and investigate the *equilibrium* changes in the conditional moments of the aggregate weight process.

1.4. Two-agent Example

To highlight the collateral mechanism we compare equilibria across economies with different, but constant, collateral ratios in a two-agent version of our economy. We focus on a simple setup with two idiosyncratic income states $y = (hi, lo)$, no aggregate uncertainty, and separable preferences. The aggregate endowment is constant and the expenditure ratio r is fixed. This simple model provides us with a laboratory in which to explore the changes in equilibrium allocations and prices in response to a change in the expenditure ratio r. It provides intuition for the full model, calibrated and solved in the remainder of the paper.

Cutoff Rule The consumption dynamics are similar to those in Kocherlakota (1996) and Alvarez and Jermann (2001a). The equilibrium consumption allocations can be characterized with a simple cutoff rule. The single state variable is the consumption share of the first agent c_1. We can solve for a lower bound $\underline{c}_1(y)$ and an upper bound $\overline{c}_1(y)$ on agent 1's consumption in each state y. The consumption share for agent 1 in the next period, c_1', is found by applying the following cutoff rule:

$$\begin{aligned}
\text{if } \underline{c}_1(y) &< c_1 < \overline{c}_1(y),\ c_1' = c_1 \\
\text{if } \underline{c}_1(y) &> c_1,\ c_1' = \underline{c}_1(y) \\
\text{if } \overline{c}_1(y) &< c_1,\ c_1' = \overline{c}_1(y).
\end{aligned}$$

The bounds are where the solvency constraint (3) holds with equality for agent 1 and agent 2 respectively:

$$\begin{aligned}
\Pi_1(\underline{c}_1(y), y) &= \underline{c}_1(1 + \frac{1}{r}) + \delta \sum_{y'} \pi(y'|y) \left[\min\left(\frac{c_1'}{c_1}, \frac{1-c_1'}{1-c_1}\right)\right]^{-\gamma} \Pi_1(c_1', y') \\
&= \Pi_1^{aut}(y) \text{ for all } y \in (lo, hi) \\
\Pi_2(\overline{c}_1(y), y) &= (1 - \overline{c}_1)(1 + \frac{1}{r}) + \delta \sum_{y'} \pi(y'|y) \left[\min\left(\frac{c_1'}{c_1}, \frac{1-c_1'}{1-c_1}\right)\right]^{-\gamma} \Pi_2(c_1', y') \\
&= \Pi_2^{aut}(y) \text{ for all } y \in (lo, hi),
\end{aligned}$$

where c_1' is determined by the cutoff rule. The value of claim to the labor endowment $\Pi_i^{aut}(y)$ is similarly defined as:

$$\Pi_i^{aut}(y) = \eta^i(y) + \delta \sum_{y'} \pi(y'|y) \left[\min\left(\frac{c_1'}{c_1}, \frac{1-c_1'}{1-c_1} \right) \right]^{-\gamma} \Pi_i^{aut}(y'). \tag{9}$$

This is a complete description of the equilibrium allocations. Perfect risk sharing is feasible (proposition 1) when the intersection of the two intervals $[\underline{c}_1(y), \overline{c}_1(y)]$ for $y = lo$ and $y = hi$ is non-empty. If it is non-empty, it has to contain a consumption share of 0.5. That is, it needs to satisfy the following condition:

$$\Pi^* \left[\{\eta^1\} \right] \leq \frac{1}{2} \left(1 + \frac{1}{r} \right) \Pi^* \left[\{1\} \right].$$

Ergodic Set Unless perfect risk sharing is feasible, one of the constraint binds in each state and the ergodic set for consumption and promised values $(\hat{c}, \widehat{\Pi})$ has mass on two points only. If the initial consumption share of agent 1 is outside of $(\underline{c}_1(lo), \underline{c}_1(hi))$, it will revert to this interval after one new, different shock. We use $\hat{c}(hi) = \underline{c}_1(hi)$ and $\hat{c}(lo) = \overline{c}_1(lo)$ to denote the ergodic consumption values, and $\widehat{\Pi}(hi) = \underline{\Pi}_1(hi)$ and $\widehat{\Pi}(lo) = \overline{\Pi}_1(lo)$ to denote the ergodic promised values. Each agent is constrained only when its endowment share η^i is high. This produces a system of six equations in six unknowns that fully characterizes a solution. The first two equations are promised value equations:

$$\widehat{\Pi}(y) = \widehat{c}(y)(1 + \frac{1}{r}) + \delta \sum_{y'} \pi(y'|y) \left[\min\left(\frac{\widehat{c}(y')}{\widehat{c}(y)}, \frac{1-\widehat{c}(y')}{1-\widehat{c}(y)} \right) \right]^{-\gamma} \widehat{\Pi}(y') \text{ for } y = lo, hi.$$

The third and fourth equations are $\widehat{\Pi}(hi) = \Pi_1^{aut}(hi)$ and $\underline{c}(lo) = 1 - \underline{c}(hi)$. Finally, there are two autarchic value equations in equation (9) that need to be solved.

The Collateral Effect When a household is constrained, its consumption share jumps up. The size of the jump depends on the level of the non-housing expenditure ratio r. Holding intertemporal prices constant, the system of equations above immediately implies that

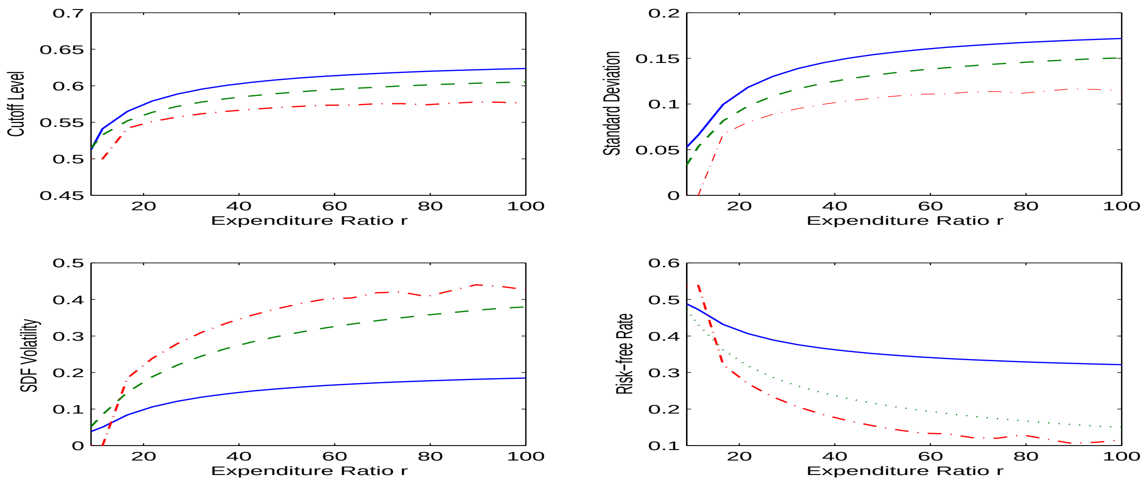
$$\frac{\partial \underline{c}_1(y)}{\partial r} > 0 \textit{ and } \frac{\partial \overline{c}_1(y)}{\partial r} < 0.$$

For given intertemporal prices, the cutoff weights become tighter as the non-durable expenditure ratio r increases. Because the lower bound on consumption shares increases and the upper bound decreases, there is less risk-sharing as the housing collateral ratio decreases (proposition 2). Of course this implies intertemporal prices cannot be constant. The new equilibrium pricing functional Π' satisfies $p'_t(y^t|y_0) \geq p_t(y^t|y_0)$ for all y^t, or interest rates are lower in the economy with less collateral (corollary 1). To see the net effect on allocations, we solve a numerical example.

Numerical Example The time discount factor δ is set to .65[6] and the risk aversion coefficient γ varies from 1 to 3. We choose labor income shares $\eta^1 = (.65, .35)$ with Markov transition matrix.[7] The collateral effect is summarized in Figure 3. It plots the cutoff consumption share, the standard deviation of consumption growth for agent one, the volatility of the SDF, and the risk-free rate against the expenditure ratio r.

In the limit, as more collateral is available (r decreases), the solvency constraints no longer bind and perfect risk sharing is feasible. The consumption share tends to 0.5. On the other hand, as collateral decreases (r increases), housing contributes no collateral in the limit and only autarchy is feasible: $\underline{c}_1(hi) = \widehat{\eta}^1(hi) = 0.65$, and $\overline{c}_1(lo) = \widehat{\eta}^1(lo) = 0.35$ (panel 1).

Figure 3. Risk Sharing and Expenditure Ratio in Two-Agent Economy
The discount factor is .65 and γ ranges from 1 (full line) to 2 (dotted line) and 3 (dash-dotted line). The first panel plots the cutoff consumption level $\underline{c}_1(y)$. The second panel plots the standard deviation of consumption growth, the third panel plots the volatility of the SDF and the fourth panel plots the risk-free rate.



As the collateral ratio decreases, the standard deviation of household consumption growth increases (panel 2) and the risk-free rate decreases monotonically (panel 4). The size of the aggregate

[6] In a two-agent economy, we need highly impatient households to get the constraints to bind.

[7] The transition matrix is $\Pi = \begin{pmatrix} .75 & .25 \\ .25 & .75 \end{pmatrix}$.

weight shock in the second part of the SDF depends on the collateral ratio. The process r governs the size of the jump in consumption shares when an agent switches from the low to the high income state. When r is high enough, there is no collateral and the liquidity shocks reach their maximum value, while, if r is low enough, consumption shares are constant and there are no liquidity shocks. Comparing one economy with low r to another with high r, low r means abundant collateral and small liquidity shocks, while a high current expenditure ratio r means low collateral and large liquidity shocks. The volatility of the SDF is higher in the high r economy, because larger jumps occur when the agents switch between hi and lo (panel 3).

Aggregate Uncertainty Next, we add aggregate uncertainty by adding two aggregate states (re, ex). We pick $\widehat{\eta}^1(hi, re) > \widehat{\eta}^1(hi, ex)$ and $\widehat{\eta}^1(lo, re) < \widehat{\eta}^1(lo, ex)$, while keeping the aggregate endowment constant. The increase in the dispersion of idiosyncratic shocks when aggregate consumption growth is low (re) delivers larger liquidity shocks when aggregate consumption growth is low and this effect is stronger in economies with low collateral ratios. This mechanism helps to generate large risk premia and time-varying risk premia on stocks, if the housing collateral ratio varies endogenously.[8]

1.5. Sequential Trading

This section describes a sequential trading arrangement that relies on more standard collateral constraints to prevent default. Using results by Alvarez and Jermann (2000), we argue that the equilibrium with time zero trading of section 1.2 can be mapped into an equilibrium sequential trading.

The financial markets are complete. Households trade a complete set of contingent claims a in forward markets. $a_t(\ell, s^t, s')$ is a promise made by agent (ℓ, s_0) to deliver one of unit the consumption good if event s' is realized in the next period. These claims trade at a price $q_t(s^t, s')$. All prices are quoted in units of the non-durable consumption good. The rental price is ρ_t; $p^h_t(z^t)$ denotes the (asset) price of the housing stock.

[8]In a model without housing, Lustig (2003) shows that these two-agent economies cannot generate high enough Sharpe ratios. That is why we chose to work with a continuum of households (section 2). For realistic values of the time discount factor ($\delta = .95$), we will be able to generate meaningful Sharpe ratios.

Household Problem The households maximize their utility (1) by choosing consumption and trades in contingent claims, subject to the budget constraints, the collateral constraints and a standard transversality condition. At the start of the period, the household purchases goods in the spot market $c_t(\ell, s^t)$, rental services in the rental market $h_t^r(\ell, s^t)$, contingent claims in the financial market and shares in the housing stock $h_{t+1}^o(\ell, s^t)$ subject to a wealth constraint:

$$c_t(\ell, s^t) + \rho_t(z^t)h_t^r(\ell, s^t) + \sum_{s'} q_t(s^t, s')a_t(\ell, s^t, s') + p_t^h(s^t)h_{t+1}^o(\ell, s^t) \leq W_t(\ell, s^t).$$

Next period wealth is:

$$W_{t+1}(\ell, s^t, s') = \eta_{t+1}(s^t, s') + a_t(\ell, s^t, s') + h_{t+1}^o(\ell, s^t)\left[p_{t+1}^h(s^t, s') + \rho_{t+1}(s^t, s')\right].$$

All of a household's state-contingent promises are backed by the cum-dividend value of its housing h_{t+1}^o, owned at the end of period t. In each node s^t, households face a separate *collateral constraint* for each event s':

$$-a_t(\ell, s^t, s') \leq h_{t+1}^o(\ell, s^t)\left[p_{t+1}^h(s^{t+1}) + \rho_{t+1}(s^{t+1})\right], \text{ for all } s^t, s'. \tag{10}$$

The collateral constraints prevent bankruptcy because households are not allowed to promise to repay more in any given state than the cum-dividend value of the housing stock in that state.

Competitive Equilibrium

Definition. *Given a distribution over initial wealth and endowments Θ_0, a competitive equilibrium is a feasible allocation $\left\{c(\ell, s^t), h^r(\ell, s^t), a(\ell, s^t), h^o(\ell, s^t)\right\}$ and a price vector $\left\{q, p^h, \rho\right\}$ such that (1) for given prices and initial wealth, the allocation solves each household's maximization problem and (2) the markets for the consumption good, the housing services, the contingent claims and housing shares clear.*

The equilibria in the economy with sequential trading are equivalent to the time zero Kehoe and Levine (1993) equilibria, if the equilibrium interest rates are high enough.

Proposition 3. *If the interest rates are high enough, the sequential equilibrium allocations can be supported as a Kehoe-Levine equilibrium.*

The proof is in appendix A.3 and follows Alvarez and Jermann (2000).

To show the equivalence, we define the market state price $p_t(z^t)$, the price at time 0 of a unit of consumption to be delivered at node z^t, as the product of the Arrow prices for the events along a path z^t:

$$p_t(z^t) = q_{t-1}(z^{t-1}, z')q_{t-2}(z^{t-1}) \dots q_0(z^1).$$

By iterating forward on the collateral constraints in (10), substituting for the time 0 budget constraint, and imposing a no-arbitrage condition on $\{p^h\}$, the sequence of collateral constraints can be restated as the solvency constraint in equation (3). It is a non-negativity constraint on net wealth in every history.

2. Computation

To solve the model numerically, we rely on an approximation of the growth rate of the aggregate weight process $g_t(z^t) = \frac{\xi_t^a(z^t)}{\xi_{t-1}^a(z^{t-1})}$ using a truncated history of aggregate shocks. This is discussed in section 2.1. In 2.2, we fully calibrate the model. We simulate the model and discuss the results in sections 3, 4, and 5.

2.1. Approximating Stationary Equilibria

In general, the aggregate weight process depends on the entire history of shocks z^∞. To avoid the curse of dimensionality, we truncate aggregate histories (Lustig (2003)). Households do not keep track of the entire aggregate history, only the last k lags: $z_t^k = (z_t, z_{t-1}, \cdots, z_{t-k})$ and the current expenditure ratio $r_t(z^t)$. The current expenditure ratio r_t contains additional information not present in the truncated history z^k, namely r_{t-k}.

For a household starting the period with weight $\xi \in L$, the policy function $l(y', z'; \xi, r, z^k) : L \times R \times Z^k \rightarrow \mathbb{R}$ produces the new individual weight in state (y', z'). There is one policy function $l(\cdot)$ for each pair $(y', z') \in Y \times Z$. The policy function $g^*(z'; r, z^k) : R \times Z^k \rightarrow \mathbb{R}$ forecasts the aggregate weight shock when moving to state z' after history (z^k, r).

Definition. *A stationary stochastic equilibrium is a joint distribution over individual weights, individual endowments, current housing - endowment ratio, truncated aggregate histories, a time invariant distribution $\Phi^*_{(r,z^k)}(\xi, y)$, and updating rules $l(\cdot)$ and $g^*(\cdot)$. For each $(z^{k\prime}, z^k)$ with $z^{k\prime} =$*

$\left(z', z^k\right)$

$$\Phi^*_{(r,z^{k\prime})} = \sum_{z^k} \pi(z^{k\prime}|z^k) \int Q\left(\xi, y, r, z^k\right) \Phi^*_{(r,z^k)}(d\xi \times dy)$$

where $Q\left(\xi, y, r, z^k\right)$ *is the transition function induced by the policy functions.*

The forecast of the aggregate weight shock is given by:

$$g^*(z'; r, z^k) = \sum_{y' \in Y} \int l(y', z'; \xi, r, z^k)^{\frac{1}{\gamma}} \Phi^*_{r,z^k}(d\xi \times dy) \frac{\pi(y', z'|y, z)}{\pi(z'|z)}, \tag{11}$$

for each z'. Intertemporal prices are pinned down by the stochastic discount factor in equation (7), using the forecasted shock $g^*(\cdot)$ as an approximation to the actual $g(\cdot)$. At the end of each period, we store $\left(\frac{l(y',z';\xi,r,z^k)^{\frac{1}{\gamma}}}{g^*(z';r,z^k)}\right)^{\gamma}$ as the household's identifying label ξ. This rescaling keeps the state variables stationary.

For any given realization $\{z\}$, the actual aggregate weight shock $g(\cdot)$ differs from the forecast $g^*(\cdot)$ because the distribution over individual weights and endowments $\Phi^*(\cdot)$ differs from the actual distribution $\Phi(\cdot)$, which depends on z^∞. The definition of stationary equilibrium implies that, *on average across aggregate histories*, $\Phi^*(\cdot) = \Phi(\cdot)$, and markets clear. That is, for every aggregate state z', the allocation error

$$c^a(z'; r, z^k) - c^a(z'; z^\infty) = \frac{g^*(z'; r, z^k) - g(z'; r, z^\infty)}{g^*(z'; r, z^k)} \tag{12}$$

is *on average* zero.[9] As k increases, the approximation error decreases because market clearing holds on average in long histories.

Algorithm We compute the approximating equilibrium as follows. The aggregate weight shock process is initialized at the full insurance value ($g^* = 1$) and the corresponding stochastic discount factor is computed. The cutoff rule for the individual weight shocks ensure that the solvency constraints hold with equality. Then the economy is simulated by drawing $\{z_t\}_{t=1}^T$ for $T = 10,000$ and $\{y_t\}_{t=1}^T$ for a cross-section of $5,000$ households. For each truncated history, we compute the sample mean of the aggregate weight shock $\{g_t^*(z', r, z^k)\}_{t=1}^T$ and the resulting stochastic discount factor $\{m_t^*(z', r, z^k)\}_{t=1}^T$. A new cut-off rule is computed with these new forecasts. These two steps

[9]There is an *exact* aggregation result if aggregate uncertainty is i.i.d., with k=0. See Lustig (2003) for a proof in a model without housing.

are iterated on until convergence.

We use $k = 5$ lags in all our computations. The percentage allocation errors provide a clear measure of the closeness to the actual equilibrium. For our benchmark calibration, the average error in equation 12 in a simulation of 10,000 periods is 0.0011 with standard deviation .0035. The largest error in absolute value is 0.0282.

2.2. Calibration

In this section we calibrate the model. Our benchmark parametrization, as well as the other parameters we consider for sensitivity analysis are summarized in table 1.

Income Process The first driving force in the model is the Markov process for the aggregate non-durable endowment process. It contains an aggregate and an idiosyncratic component.

The aggregate endowment growth process is taken from Mehra and Prescott (1985) and replicates the several moments of aggregate consumption growth in the 1871-1975 data. The growth rate of the aggregate endowment, λ, follows an autoregressive process:

$$\lambda_t(z_t) = \rho_\lambda \lambda_{t-1}(z_{t-1}) + \varepsilon_t,$$

with $\rho_\lambda = -.14, E(\lambda) = .0183$ and $\sigma(\lambda) = .0357$. We discretize the AR(1) process with two aggregate growth states $z = (ex, re) = [1.04, .96]$ and an aggregate state transition matrix $[.83, .17; .69, .31]$. The implied ratio of the probability of a high aggregate endowment growth state to the probability of a low aggregate endowment growth state is 2.65. The unconditional probability of a low endowment growth state is 27.4 percent.

The calibration of a heteroscedastic labor income process is taken from Storesletten, Telmer, and Yaron (2004). They conclude that the volatility of idiosyncratic labor income shocks in the US more than doubles in recessions. Log labor income shares follow an AR(1) with autocorrelation of .92 and a conditional variance of .181 in low and .0467 in high aggregate endowment growth states. Again the AR(1) process is discretized into a two-state Markov chain. The resulting individual income states are $\left(\eta^1(hi, ex), \eta^1(lo, ex)\right) = [.6578, .3422]$ in the high and $\left(\eta^1(hi, re), \eta^1(lo, re)\right) = [.7952, .2048]$ in the low aggregate endowment growth state.[10] We refer to the counter-cyclical labor

[10]The one difference with the Storesletten, Telmer, and Yaron (2004) calibration is that recessions are shorter in

income share dispersion as the Constantinides and Duffie (1996) effect.

Expenditure Ratio Following Piazzesi, Schneider, and Tuzel (2004), we specify an autoregressive process for the expenditure ratio r. The aggregate expenditure depends on the aggregate endowment growth:

$$\log r_{t+1} = \bar{r} + \rho_r \log r_t + b_r \lambda_{t+1} + \sigma_r \nu_{t+1}, \tag{13}$$

where ν_{t+1} is an i.i.d. standard normal process with mean zero, orthogonal to λ_{t+1}. In our benchmark calibration we set $\rho_r = .96$, $b_r = .93$ and $\sigma_r = .03$. We discretize the process for $\log(r)$ as a five-state Markov process. The parameter values are close to the estimates of (13) we find in US National Income and Products Accounts Data.[11] A second calibration switches off the effect of λ on $\log(r)$: $\rho_r = .96$, $b_r = 0$. Both calibrations fix $\sigma_r = .03$. We choose the constant $\bar{r}$ to match the average housing expenditure share of 19 percent in the National Income and Product Accounts data for 1929-2003.

Average Housing Collateral ratio We scale up aggregate income to simultaneously match the average expenditure share of housing services of 19 percent and the average ratio of housing wealth to total wealth 5 percent (benchmark). The first one matters for the composition effect on asset prices, the second for the collateral effect. In the model, the ratio of the aggregate non-durable endowment e_t to the aggregate non-durable consumption c_t^a is 1. The empirical counterpart to e_t is compensation of employees. The empirical counterpart to c_t^a is consumption expenditures on non-durables and services excluding housing services. On average between 1929-2003, the ratio of the former to the latter is 1.17. We use this factor to scale up labor income η in the model. The rescaling implies an average housing collateral ratio of 5 percent. Jorgenson and Fraumeni (1989) estimate human wealth to be 93 percent of total wealth. We investigate the sensitivity of our results by also considering an economy with ten percent collateral.

our calibration. In their paper the economy is in the low aggregate endowment growth state 50 percent of the time. That implies that the unconditional variance of our labor income process is lower.

[11] Panel A of table 9 in a separate appendix shows estimates for ρ_r and b_r that are consistent across samples and data sources. In periods of high aggregate consumption growth, the expenditure ratio increases. Alternatively, we could have calibrated a persistent process for the rental price $\log(\rho_t)$. Panel B shows that rental prices increase in response to a positive aggregate consumption growth in the post-war sample.

Preference Parameters In the benchmark calibration we use additive utility with $\delta = .95$, $\gamma = 8$, $\varepsilon = .05$. We fix $\psi = 1$ throughout.[12] We choose these parameters to match key unconditional asset pricing moments discussed in section 5.

We also compute the model for $\gamma \in [2, 10]$ and $\varepsilon \in [.05, .75]$. A choice for the parameter ε implies a choice for the volatility of rental prices:

$$\sigma(\Delta \log \rho_{t+1}) = \left| \frac{1}{\varepsilon - 1} \right| \sigma(\Delta \log r_{t+1}). \tag{14}$$

In NIPA data (1930-2002), the left hand side is .046 and the right-hand side is .041. The implied ε is .098. A choice for ε too close to one implies excessive rental price volatility. We take $\varepsilon = .05$ as our benchmark and explore parameter values $\varepsilon \leq .75$.

Table 1
Parameter Calibration

Parameter	Benchmark	Sensitivity Analysis
γ	8	[2,3,...,9,10]
ε	.05	[.15,.25,...,.65,.75]
ψ	1	·
$\bar{r}$	4.26	·
ρ_r	.96	·
b_r	.93	0
σ_r	.03	·
$E[my]$	.05	.10
λ	[1.04,.96]	·
η	[.6578,.7952,.3422,.2048]	[.6935,.6935,.3065,.3065]
κ	1	3

Market Return We assume that financial assets are in zero net supply.[13] Because of complete markets we price them as redundant securities. We define the market return as the return on a levered claim to the aggregate consumption process $\{c_t^a\}$. In the data, dividends are more volatile

[12]Note that $\frac{-cu_{cc}}{u_c} = \alpha_t^a \gamma + (1 - \alpha_t^a)\frac{1}{\varepsilon}$. The degree of relative risk aversion is a linear combination of γ and ε with weights depending on the non-durable expenditure share $\alpha_t^a = \frac{c_t}{\rho_t h_t + c_t}$. In all calibrations $\alpha^a = .81$ on average. In the simulation, the degree of risk aversion is relatively stable.

[13]Allowing for financial assets in positive net supply will not qualitatively affect the dynamics of our model, but it will increase the amount of collateral. As such, it increases the mean ratio of collateralizeable wealth to total wealth. In the data, mortgages and home equity lines of credit represent the large majority of household credit market instruments (70%, household sector balance sheet, Flow of Funds data). For these two reasons, the omission of financial wealth as collateral does not seem critical.

than aggregate consumption.[14] We denote the return on a levered claim to aggregate consumption growth R^l and choose leverage parameter $\kappa = 3$, where $\sigma(\Delta \log d_{t+1}) = \kappa\sigma(\Delta \log c^a_{t+1})$. We also price a non-levered claim on the aggregate consumption stream. We denote the corresponding return R^c.

Our objective is twofold: (1) to quantitatively assess the variation in *conditional* asset pricing moments, conditional on the housing collateral ratio and compare it to the data (section 3), and (2) to assess the cross-sectional variation in returns, i.e. can we generate a return spread between value and growth portfolio returns of the magnitude observed in the data (section 4)? The last section checks the unconditional moments against the data (section 5).

3. Conditional Asset Pricing Moments

Before we explore the conditional risk premia in our model, we illustrate the dynamics of the aggregate weight shocks. There are two distinct forces at work: at business cycle frequencies the shocks to the distribution of weights or the wealth distribution, and, at lower frequencies, there is the variation in the collateral ratio. Both of these are crucial.

3.1. SDF dynamics

Figure 4 plots the aggregate consumption growth shocks and the aggregate weight shocks; the size of the aggregate weight shocks is determined by the size of the left tail of the distribution:

$$\xi^a_t(z^t) = \sum_{y^t} \int \xi_t(\ell, y^t, z^t) \frac{\pi(z^t, y^t|z_0, y_0)}{\pi(z^t|z_0)} d\Phi_0 \tag{15}$$

$$= \sum_{y^t} \int_{\ell^c(y_t, z^t)} \xi_{t-1}(\ell, y^t, z^t) \frac{\pi(z^t, y^t|z_0, y_0)}{\pi(z^t|z_0)} d\Phi_0 \tag{16}$$

$$+ \sum_{y^t} \int^{\ell^c(y_t, z^t)} \ell^c(y_t, z^t) \frac{\pi(z^t, y^t|z_0, y_0)}{\pi(z^t|z_0)} d\Phi_0 \tag{17}$$

Cyclicality of the Weight Shocks What drives the cyclicality of these shocks? First, a larger fraction of agents draws higher labor income shares $\widehat{\eta}(y, z)$ when aggregate consumption growth is low. As a result of the persistence of labor income shocks, the household cutoff levels are higher

[14] For the period 1930-2003, the volatility of annual nominal dividend growth is 14.3%, whereas the volatility of annual nominal consumption growth (non-durables and services excluding housing services) is 5.7%. The ratio is 2.5.

in low aggregate consumption growth states, $l^c(y_t, z^{t-1}, re) > l^c(y_t, z^{t-1}, ex)$, and this increases the size of the aggregate weight shock $\Delta \log \xi^a_{t+1}(z^t, re) > \Delta \log \xi^a_{t+1}(z^t, ex)$ and the SDF in low aggregate consumption growth states (Figure 4).[15] In addition, low aggregate consumption growth states are short-lived in our model and more agents are constrained in these states as a result.

Heteroscedasticity of the Weight Shocks Figure 4 also reveals that the shocks are larger after long series of high aggregate consumption growth realizations. That is because more households are in the left tail of the wealth distribution with low weights relative to the cutoff values: When the bad aggregate shock arrives, these will be constrained. This gives rise to substantial heteroscedasticity in the stochastic discount factor, as is apparent from Figure 4.

Figure 4. Aggregate Consumption Growth Shocks and The Stochastic Discount Factor. Benchmark model calibration with risk aversion 8 and 5 percent collateral. One hundred period model simulation. Shaded bars indicate periods with low aggregate consumption growth. The dotted line denotes the Stochastic Discount Factor m_{t+1}.

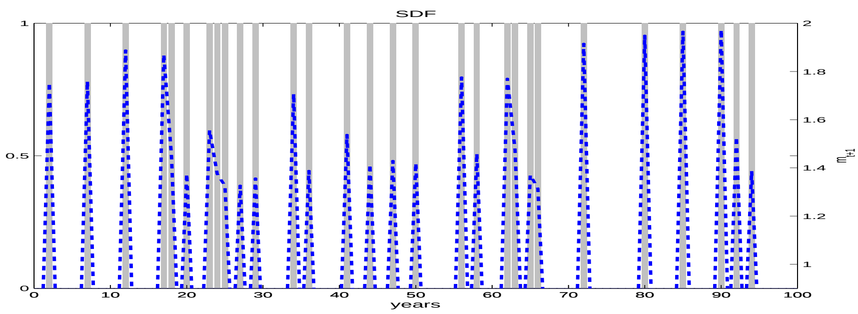
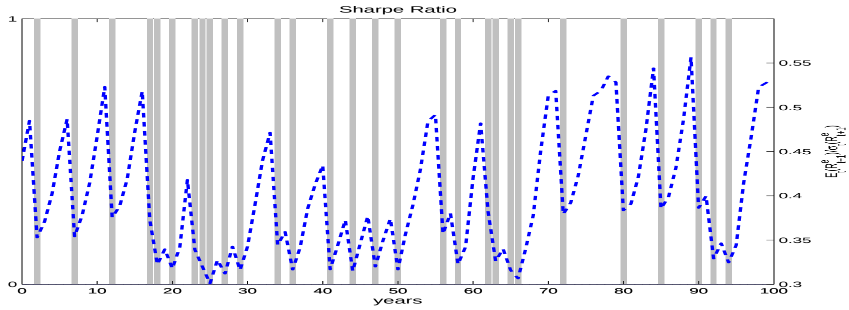


Figure 5 plots the Sharpe ratio for a simulation of the model and indicates periods with negative consumption growth by shaded bars. The Sharpe ratio increases during periods of high aggregate consumption growth and falls after a negative aggregate consumption growth realization. There is history dependence. The decline is bigger the longer was the preceding expansion.

The key to this positive correlation is the combination of history dependence in the wealth dynamics and the persistence of the housing collateral ratio. In every successive period of high aggregate consumption growth, the wealth distribution becomes more condensed because fewer households are constrained and the unconstrained households' consumption share drifts down. A low consumption growth shock after a long period of expansion leads to a very large aggregate

[15] Constantinides and Duffie (1996) build a negative correlation between the dispersion of consumption growth across households and aggregate stock returns in their model to generate large risk premia, drawing on earlier work by Mankiw (1986). The model of Lustig (2003) is a different version of this.

Figure 5. Aggregate Consumption Growth Shocks and Sharpe Ratio on Equity.
Benchmark model calibration with risk aversion 8 and 5 percent collateral. One hundred period model simulation. Shaded bars indicate periods with low aggregate consumption growth. The dotted line denotes the Sharpe ratio on a non-levered claim to aggregate consumption $E_t[R^{c,e}]/\sigma_t[R^{c,e}]$.



weight shock and a high SDF, as documented in a similar model without housing collateral by Lustig (2003). Many households are constrained and their consumption share jumps up. This shock wipes out the left tail of the wealth distribution.

Collateral Mechanism There is another source of heteroscedasticity: the endogenous movements in the collateral ratio; it is this paper's contribution. Movements in the expenditure ratio and endogenous movements in the stochastic discount factor drive movements in the housing collateral ratio. When collateral is scarce, households' solvency constraints bind more frequently. This mechanism is the focus of section 3.2.

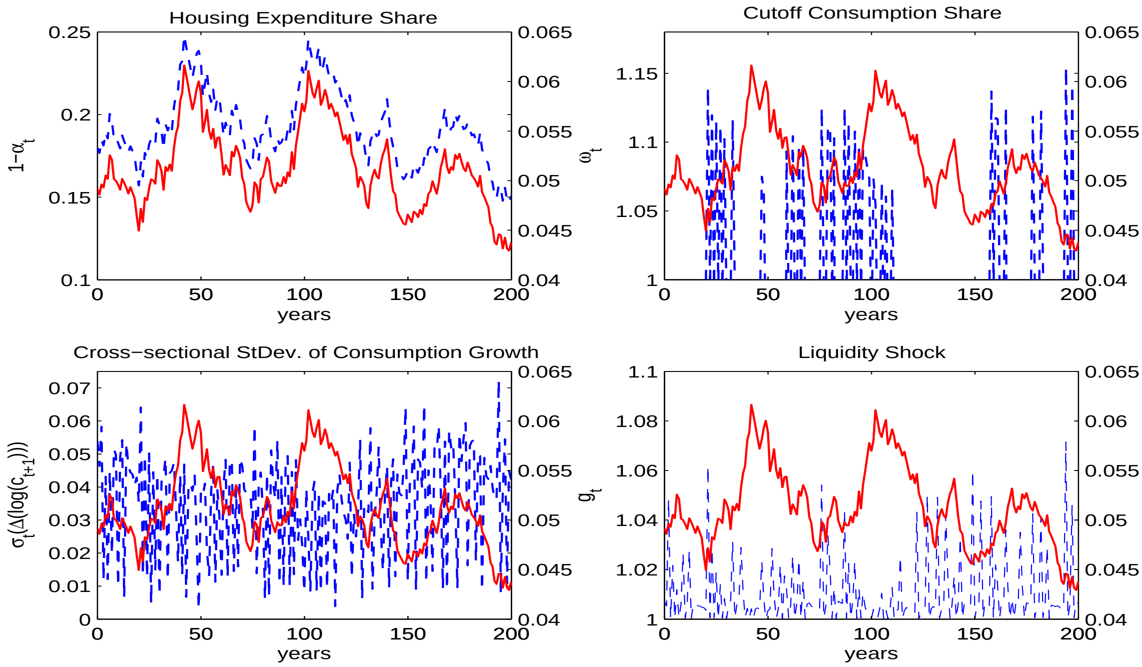
The first main prediction of the model is that asset prices behave differently in episodes of high collateral and in episodes of collateral scarcity. Expected excess returns are low and the risk-free rate stable in periods when my is high. However, when collateral is scarce, the equity premium and the Sharpe ratio are high. Movement in the housing collateral ratio induces substantial variation in the Sharpe ratio. In this section we do a quantitative assessment of this time-variation and show that is consistent with the data. Unconditional asset pricing moments obscure this time-variation because they average over different collateral regimes. In contrast, the representative agent model misses the time-variation in the Sharpe ratio completely.

3.2. Shocks to the Risk Sharing Technology

The shocks to the collateral ratio come from shocks to the housing endowment. Panel 1 of figure 6 shows the housing collateral ratio my (full line, right axis) together with the ratio of housing services consumption to total consumption $1-\alpha_t^a = \frac{1}{1+r_t}$ (dotted line, left axis). It is a typical two hundred period window of a long simulation of the benchmark model. The housing collateral ratio is the closely correlated with the housing expenditure share. It is also a persistent process.

Figure 6. Risk Sharing and Collateral Ratio

The average collateral share is 5 percent, the discount factor is .95 and the coefficient of risk aversion is 8. The first panel plots the housing expenditure share $1-\alpha_t^a$ (dotted line). The second panel is the cutoff level consumption share at which the solvency constraints hold with equality (dotted line). The third panel is the cross-sectional standard deviation of consumption growth across households (dotted line). The fourth panel is the aggregate weight shock g_{t+1} (dotted line). The full line in each panel is the collateral ratio my, the ratio of housing wealth to total wealth (right axis). The graphs display a two hundred period model simulation.



Panel 2 illustrates the collateral mechanism. It plots the cutoff consumption share, which is the consumption share at which the solvency constraint holds with equality (dotted line). This is the consumption share of a constrained household. The household's average income share is normalized to one. The consumption share jumps to the cutoff level when the household runs into a binding constraint. This happens when its income share switches from the low to the high idiosyncratic state. The graph shows that the consumption share for constrained households is bigger when collateral is scarce (my is low, full line). For example, in period 195, the consumption share is 15 percent above its mean of one, whereas in period 50, the consumption share is only 7 percent above its mean.

When collateral is scarce, the solvency constraints bind more severely. The consumption share of the constrained households jumps up, while the unconstrained households' consumption share decreases precipitously. As a result, the cross-sectional standard deviation of consumption growth increases. In times of collateral scarcity, there is less risk-sharing. Panel 3 of figure 6 plots the consumption growth dispersion (dashed line, left axis) against the housing collateral ratio my (full line, right axis). Even though the consumption shares change in important ways when collateral constraints bind, the *unconditional* volatility of consumption growth for an individual household is moderate. In our benchmark model it is less than 10% of the unconditional volatility of individual income growth. There is a considerable amount of risk-sharing.

The aggregate weight shock $g_{t+1} = \frac{\xi^a_{t+1}}{\xi^a_t}$, which we refer to as the liquidity shock, governs the rate at which the consumption share of the unconstrained agents decreases. Panel 4 plots g^γ (dotted line) against the housing collateral ratio (full line). In times of collateral scarcity, the constraints bind more tightly and this is reflected in a large liquidity shock. For example, in period 50 or 110, the liquidity shock is close to one, whereas in period 195 it is 1.07. The stochastic discount factor is high and more volatile in such periods. When housing collateral is abundant, the aggregate weight shock is close to 1 and our model's stochastic discount factor reduces to the one in the representative agent economy. We turn to the effects on asset prices next.

3.3. Conditional Risk Premia

Figure 1 in the introduction summarizes the findings for conditional asset pricing moments. The top row plots the conditional mean, standard deviation and Sharpe ratio on a claim to aggregate consumption, averaged over histories of the aggregate state z^k, against the housing collateral ratio. In each panel, the solid line plots the conditional moments, conditional on observing a low aggregate consumption growth rate tomorrow ($\lambda(z') = .96$), the dashed line is conditional on observing a high aggregate consumption growth rate ($\lambda(z') = 1.04$). On average, the equity premium is 9 percent higher when collateral is scarce ($my = .04$) than when it is abundant ($my = .10$), conditional on being in a boom tomorrow. It is 6 percent higher conditional on being in a recession. Stock returns are up to ten percent more volatile when collateral is scarce (panel 2). The conditional Sharpe ratio, conditional on being in a boom tomorrow, is .6 when collateral is scarce and .3 when collateral is abundant (panel 3). The bottom row displays the conditional market price of risk $\sigma_t[m_{t+1}]/E_t[m_{t+1}]$, an upper bound on the Sharpe ratio (in panel 4), the conditional price dividend ratio (panel 5) and the risk-free rate (panel 6). The price-dividend ratio is high when collateral is scarce. The demand for insurance against binding solvency constraints drives up the price stocks. So, the model simultaneously generates a high equity premium and a high price-dividend ratio because the risk-free rate is very low when collateral is scarce (panel 6).

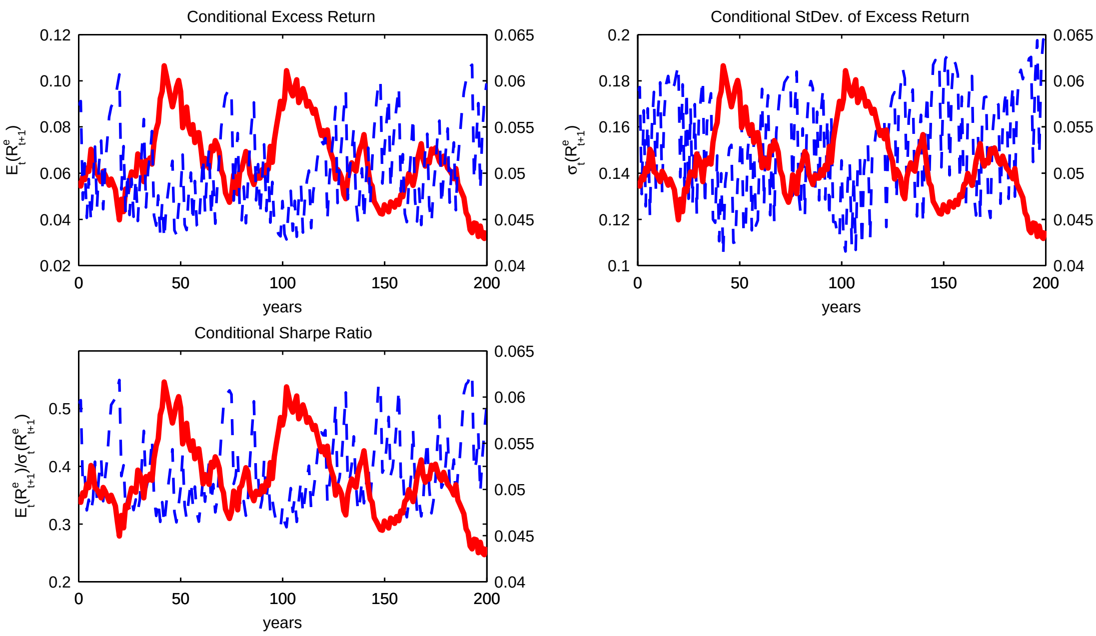
Time Series The conditional expected return on stocks in excess of the risk-free rate is higher in periods of collateral scarcity. Zooming in on the same 200 simulation periods of the benchmark calibration that were used in figure 6, the first panel of figure 7 displays the conditional expected excess return on a non-levered claim to aggregate consumption (dotted line, left axis). The conditional equity premium is below 4 percent when the housing collateral ratio is high (for example in period 110), and almost 11 percent when my is low (for example in period 190).

The second panel of figure 7 shows that the conditional volatility of the excess return on the consumption claim (dotted line, left axis) is 10 percent when collateral is abundant (period 110) and doubles to 20 percent when collateral is scarce (period 195). Excess returns are much less volatile when collateral is abundant. The net result of the collateral mechanism is a Sharpe ratio that is higher in times of collateral scarcity. The third panel of figure 7 plots the Sharpe ratio on the stock return against the housing collateral ratio (dotted line, left axis). It is 0.3 in period 110 and almost 0.6 in period 195.

US data To evaluate our model against the data, we estimate the Sharpe ratio on annual data from 1927-1992 and compare it to the variation in the Sharpe ratio generated by the model. The conditional mean return is the projection of the excess return on the housing collateral ratio, the dividend yield and the ratio of aggregate labor income to consumption, all of which have been shown to forecast annual returns. Likewise, the conditional volatility is the projection of the standard deviation of intra-year monthly returns on the same predictors. Using the projection coefficient estimates we form the Sharpe ratio as the ratio of the predicted excess returns and predicted volatility. Table 2 shows the estimation results for 1 year returns (column 1), but also

Figure 7. Conditional Asset Pricing Moments and Collateral Ratio

The average collateral share is 5 percent, the discount factor is .95 and the coefficient of risk aversion is 8. The first panel plots the expected excess return $R_t^{c,e}$ on a non-levered claim to aggregate consumption (dotted line), the second panel is the conditional standard deviation of the excess return $R_t^{c,e}$ (dotted line), and the third panel is the Sharpe ratio on a non-levered claim to aggregate consumption (dotted line). The full line in each panel is the collateral ratio my, the ratio of housing wealth to total wealth (right axis). The graphs display a two hundred period model simulation.



for 5 year and 10-year cumulative excess returns. The last three lines indicate the mean and standard deviation of the Sharpe ratio as well as its correlation with the housing collateral ratio. In the estimation, the standard deviation of the Sharpe ratio on 1, 5 and 10 year cumulative excess returns is .10, .18, and .20. Lettau and Ludvigson (2003) do a similar exercise for quarterly excess returns between 1952:4 and 2000:4. Their estimate of the unconditional standard deviation of the Sharpe ratio is .45.

In the model, the unconditional standard deviation of the Sharpe ratio is .40, .42, .40 for 1, 5 and 10 year cumulative excess returns on a non-levered consumption claim. Other models have a hard time generating this volatility. For example, the unconditional standard deviation of the Sharpe ratio is .09 for the Campbell and Cochrane (1999) model and the consumption volatility model of Lettau and Ludvigson (2003). The volatility of the Sharpe ratio in the representative agent model is even smaller.

Furthermore, the correlation between the Sharpe ratio and the measure of collateral scarcity $\widetilde{my}$

Table 2
Long-Term Sharpe Ratios in Data.

Parameter estimates for $R^e_{t+1} = b_0 + b_1 R_t + b_2 dp_t + b_3 lc_t + b_4 \widetilde{my}_t + \varepsilon_{t+1}$ and $Vol_{t+1} = a_0 + a_1 dp_t + a_2 lc_t + a_3 \widetilde{my}_t + a_4 Vol_t + a_5 Vol_{t-1}$. The variables dp, lc and $\widetilde{my}$ are the dividend yield, the labor income-consumption ratio and the housing collateral scarcity measure based on value of mortgages. In particular $\tilde{my}_t = max(my_t) - my_t/(max(my_t) - min(my_t))$. R^e denotes the value weighted market return in excess of a 1 month T-bill return. Vol_t is the standard deviation of the 12 monthly returns in year t. $R1, R5, R10$ denote the 1-year, 5-year and 10-year ahead cumulative excess returns. The estimation is by GMM with the OLS normal conditions as moment conditions. Standard errors are Newey-West with lag length 3. The estimation period is the longest common sample: 1927-1992. The last three rows indicate the sample mean, sample standard deviation of the predicted Sharpe ratio and sample correlation between the Sharpe ratio and the scarcity of housing collateral. The predicted Sharpe ratio is the predicted mean excess return divided by its predicted standard deviation.

	R1	Vol1	R5	Vol5	R10	Vol10
constant	-.24	.09	.71	-.004	.76	.05
(s.e.)	(.34)	(.05)	(0.28)	(.04)	(.42)	(.04)
lag ret	.04		.74		.76	
(s.e.)	(.13)		(.13)		(.08)	
dp	1.07	.33	-1.20	.40	.26	.12
(s.e.)	(2.00)	(.27)	(3.16)	(.22)	(3.00)	(.21)
lc	.22	-.07	-.76	.03	-.92	-.03
(s.e.)	(.32)	(.05)	(.33)	(.03)	(.45)	(.04)
$\widetilde{my}$	.02	-.01	.48	-.04	.70	.01
(s.e.)	(0.20)	(.02)	(.20)	(.02)	(.23)	(.02)
lag vol		.51		.96		.80
(s.e.)		(.20)		(.12)		(.12)
2 lag vol		-.18		-.19		.03
(s.e.)	(.13)	(.17)		(.11)		(.10)
E[Sharpe]	.40		1.02		1.12	
σ[Sharpe]	.10		.18		.20	
$\rho[Sharpe, \widetilde{my}]$	.26		.32		.50	

is positive in the data and equal to .25, .32, and .50 for 1, 5 and 10 year cumulative excess returns. The corresponding correlations in the model are large and positive (.50, .59 and .39). Figure 8 plots the Sharpe ratio in the model on 5-year and 10-year cumulative returns for the collateral model. Figure 9 plots the estimated Sharpe ratio for US stocks at a 5-year and a 10-year horizon against the collateral scarcity measure, $\widetilde{my}_t = \frac{max(my_t) - my_t}{max(my_t) - min(my_t)}$. The collateral scarcity measure $\widetilde{my}_t$ is constructed to lie between 0 and 1 for all t. We see a positive correlation between the Sharpe ratio and the collateral scarcity.

3.4. Long Horizon Predictability

One of the implications of the time-variation in the equity premium is that the housing collateral model should predict returns. We explore this predictability in depth in our empirical paper (Lustig and VanNieuwerburgh (2004b)). Panel 1 of table 3 summarizes the predictability results of the housing collateral ratio for 1 to 8 year ahead cumulative excess returns (data for 1927-2003 and 1945-2003). The housing collateral ratio we use in this table is based on the outstanding value

Figure 8. Housing Collateral Ratio and Long-Horizon Sharpe Ratio in Model.
The average collateral share is 5 percent, the discount factor is .95 and the coefficient of risk aversion is 8. This the Sharpe ratio on a 10 year and 5 year cumulative excess return on a non-levered consumption claim (dotted line), and the collateral ratio my is the ratio of housing wealth to total wealth (full line) for a one hundred period model simulation.

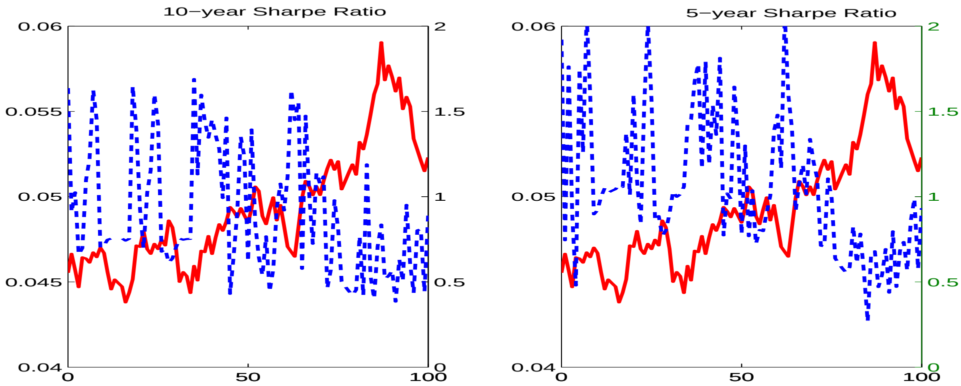
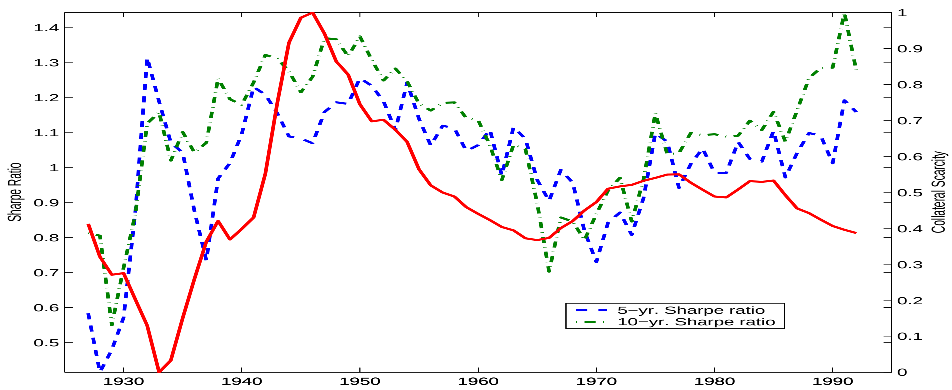


Figure 9. Housing Collateral Ratio and Long- Horizon Sharpe Ratio in Data.
This is the Sharpe ratio on 5-year and 10-year cumulative stock market returns in the data for 1928-1997. The housing collateral measure $\widetilde{my}$ measures the scarcity of collateral and is scaled to be between 0 and 1.



of mortgage debt (see appendix A.4 for a detailed description of its construction). The data are supportive of the collateral effect: excess returns are higher when collateral is scarce ($b_1 > 0$). The effect becomes larger and statistically more significant with the horizon. The R^2 increases.[16]

Our model replicates the pattern of predictability for the housing collateral ratio. The second panel of table 3 reports regression results *inside* the model of excess returns on our measure of housing collateral ratio scarcity. When housing collateral is scarce (my is low), the excess return

[16]Predictability results for the other two collateral measures we consider are reported in Lustig and VanNieuwerburgh (2004b).

Table 3

Predictability of K-Year Excess Returns.

Results of regressing log K-horizon excess returns on the housing collateral ratio. The intercept is b_0, the slope coefficient is b_1. The first panel reports the results in the data. The t-stats in brackets are computed using the Newey West covariance matrix with K lags. The returns are cum-dividend returns on the value-weighted CRSP index. The collateral scarcity measure $\widetilde{my}_t$ is based on the market value of outstanding mortgages. The long sample contains annual data from 1930-2003. The post-war sample is from 1945-2003. The second panel reports the same regressions inside the model. The regressions were obtained by simulating the model for 10,000 periods under the benchmark parametrization with risk aversion 5. The expenditure share process is an AR(1) with an aggregate consumption growth term: $\log r_t = \bar{r} + \rho_r \log r_{t-1} + b_r \Delta(\log(c_{t+1})) + \sigma_r \nu_t$.

	b_0	b_1	R^2	b_0	b_1	R^2
			Panel 1: Data			
Horizon	**Entire Sample**			**Post-War Sample**		
1	0.01	0.15	0.02	0.02	0.15	0.04
	[0.24]	[1.25]		[0.63]	[1.73]	
2	0.07	0.25	0.03	0.06	0.31	0.07
	[0.66]	[1.11]		[0.83]	[1.75]	
3	0.16	0.31	0.03	0.12	0.46	0.09
	[1.02]	[0.97]		[1.04]	[1.80]	
4	0.29	0.32	0.02	0.22	0.56	0.09
	[1.50]	[0.80]		[1.42]	[1.82]	
5	0.38	0.47	0.03	0.34	0.66	0.08
	[1.83]	[1.16]		[1.67]	[1.77]	
6	0.41	0.79	0.07	0.43	0.89	0.10
	[1.70]	[1.73]		[1.58]	[1.86]	
7	0.41	1.20	0.12	0.49	1.26	0.14
	[1.32]	[2.11]		[1.30]	[2.02]	
8	0.37	1.80	0.18	0.50	1.79	0.19
	[0.91]	[2.71]		[1.01]	[2.26]	
			Panel 2: Model			
Horizon	**Leverage =1**			**Leverage=4**		
1	−0.00	0.05	0.00	−0.00	0.09	0.00
2	−0.07	0.18	0.02	−0.06	0.23	0.01
3	−0.15	0.33	0.04	−0.14	0.41	0.02
4	−0.26	0.51	0.06	−0.24	0.62	0.03
5	−0.39	0.72	0.09	−0.37	0.85	0.04
6	−0.55	0.96	0.11	−0.51	1.10	0.06
7	−0.72	1.22	0.14	−0.69	1.40	0.07
8	−0.92	1.51	0.17	−0.88	1.73	0.09

is high. The magnitude of the slope coefficients is close to the one we find in the data. Moreover, the R^2 of the predictability regression increase with the predictability horizon, just as in the data. We find this negative relationship between my_t and the excess return for a non-levered claim, as well as for a levered claim to aggregate consumption. If we regress long-horizon returns on the housing expenditure share in the model, we get identical results because the expenditure share and the collateral ratio are perfectly correlated. The representative agent counterpart to our model does not generate any predictability. Piazzesi, Schneider, and Tuzel (2004) generate this pattern of predictability by the housing share in a representative agent model by choosing a heteroscedastic process for innovations to the expenditure share that depends on its level.

Other Predictors Other variables, such as the price-dividend ratio and the risk-free rate have also been shown to predict excess returns. In the data, a higher dividend yield and a lower risk-free rate forecast higher future excess returns. The former becomes more important at longer horizons, whereas the latter is more important at short horizons. We explore the predictability of the price-dividend ratio for returns inside the model. The theory predicts a negative sign for returns and a positive sign for *excess* returns, whereas the data show a negative sign in both regressions.[17] Lastly, the model predicts a negative relationship between current risk-free rate and future excess returns but a positive relationship with future returns. The data show a negative relationship in either case. This pattern is due to the risk-free rate dynamics. When collateral is scarce, the price of insurance increases, lowering the risk-free rate and pushing up the price-dividend ratio. The theory predicts that the equity premium is high (see figure 1). Because of the persistence in the housing collateral ratio, future equity premia are also high. This must mean that future realized *excess* returns are high on average. However, the high excess returns are the result of *lower* realized returns and even lower future risk-free rates.[18]

4. Cross-sectional Variation in Risk Premia

Value firms, with a high ratio of book equity to market equity, historically pay higher returns than growth firms, with a low book-to-market ratio. We use annual return data for 1927-2003 for the US for 10 value portfolios from Fama and French (1992). The decile portfolios are formed every year by sorting the universe of stocks on the ratio of book value to market value of equity. Table 4 reports sample means for the excess return, its unconditional volatility and the Sharpe ratio on the ten book-to-market deciles. The annual excess return on a zero-cost investment strategy that goes long in the highest book-to-market decile and short in the lowest decile is 5.5 percent for 1927-2003. The value premium is 5.7 percent for quintile portfolios. Similar value premia are found for monthly and quarterly returns. Using quarterly data for 1951-2002, Lettau and Wachter (2004) document that the unconditional Sharpe ratio for value stocks (.64) is twice as large as for growth stocks (.32). In table 4, we find a similar increase for annual data from 1945-2003 (from .37 to .56), but a smaller increase over the entire period 1927-2003 (.32 to .42).

[17] Because the risk-free rate is not very volatile in the data, the empirical results for regressions where the left-hand side variable is the real returns instead of the excess return are very similar.

[18] Results for the data and model are available upon request.

Table 4
Value Premium in Data and in the Collateral Model

The moments for the US data are excess returns in excess of a risk-free rate, the sample volatility and the Sharpe ratio. The risk-free rate is the average yield on a 3-month T-Bill. The value-weighted stock returns on the book-to-market deciles are from Kenneth French's web site, annual data for 1930-2003 and 1945-2003 (source CRSP). For the model, the table reports expected returns, standard deviation and Sharpe ratio on an artificial asset generated with a set of betas listed in Table 10, but with intercept β_0^j chosen so that the Euler equation is satisfied for this asset in the model. Each panel corresponds to a set of betas for a different collateral measure. The block uses a housing collateral ratio based on the market value of outstanding mortgages ($mymo$), the second block uses a measure based on residential real estate wealth ($myrw$) and the third panel employs a measure based on residential fixed asset values ($myfa$). The parametrization is the benchmark one with expenditure share process is an AR(1) without aggregate consumption growth term.

Decile	1	2	3	4	5	6	7	8	9	10
	Panel 1: US Data									
	Sample 1930-2003									
$E(R^e)$	0.071	0.084	0.080	0.081	0.100	0.099	0.108	0.127	0.131	0.139
$\sigma(R^e)$	0.222	0.194	0.196	0.229	0.228	0.238	0.250	0.274	0.291	0.332
$E(R^e)/\sigma(R^e)$	0.321	0.431	0.411	0.355	0.437	0.417	0.433	0.464	0.449	0.419
	Panel 2: Sample 1945-2003									
$E(R^e)$	0.078	0.087	0.086	0.084	0.106	0.107	0.110	0.130	0.126	0.143
$\sigma(R^e)$	0.209	0.175	0.175	0.178	0.182	0.178	0.194	0.214	0.212	0.257
$E(R^e)/\sigma(R^e)$	0.372	0.497	0.492	0.473	0.580	0.599	0.566	0.604	0.592	0.558
	Panel 2: Risk Aversion 5									
	Mortgage-Based Collateral Measure									
$E(R^e)$	0.01	0.01	0.01	0.01	0.02	0.03	0.03	0.03	0.02	0.03
$\sigma(R^e)$	0.06	0.04	0.05	0.07	0.07	0.10	0.10	0.10	0.09	0.12
$E(R^e)/\sigma(R^e)$	0.18	0.23	0.23	0.23	0.23	0.28	0.26	0.26	0.26	0.27
	Residential Wealth-Based Collateral Measure									
$E(R^e)$	0.01	0.01	0.01	0.02	0.02	0.03	0.03	0.03	0.03	0.04
$\sigma(R^e)$	0.07	0.04	0.05	0.07	0.07	0.11	0.12	0.11	0.11	0.14
$E(R^e)/\sigma(R^e)$	0.15	0.23	0.23	0.24	0.24	0.28	0.26	0.27	0.26	0.27
	Fixed Assets-Based Collateral Measure									
$E(R^e)$	0.01	0.02	0.02	0.02	0.02	0.04	0.04	0.04	0.04	0.05
$\sigma(R^e)$	0.07	0.06	0.07	0.09	0.10	0.13	0.17	0.15	0.16	0.19
$E(R^e)/\sigma(R^e)$	0.17	0.26	0.25	0.24	0.23	0.26	0.25	0.25	0.25	0.25
	Panel 3: Risk Aversion 8									
	Mortgage-Based Collateral Measure									
$E(R^e)$	0.01	0.01	0.02	0.02	0.02	0.04	0.04	0.03	0.03	0.05
$\sigma(R^e)$	0.06	0.04	0.05	0.06	0.07	0.10	0.10	0.10	0.09	0.12
$E(R^e)/\sigma(R^e)$	0.16	0.24	0.30	0.31	0.26	0.38	0.40	0.35	0.36	0.38
	Residential Wealth-Based Collateral Measure									
$E(R^e)$	0.01	0.01	0.02	0.02	0.02	0.04	0.05	0.04	0.04	0.05
$\sigma(R^e)$	0.07	0.04	0.05	0.07	0.07	0.11	0.12	0.11	0.11	0.14
$E(R^e)/\sigma(R^e)$	0.15	0.33	0.35	0.38	0.31	0.40	0.42	0.37	0.39	0.40
	Fixed Assets-Based Collateral Measure									
$E(R^e)$	0.01	0.03	0.03	0.04	0.03	0.05	0.07	0.06	0.07	0.07
$\sigma(R^e)$	0.07	0.06	0.07	0.10	0.10	0.14	0.17	0.15	0.16	0.19
$E(R^e)/\sigma(R^e)$	0.15	0.43	0.41	0.40	0.34	0.40	0.43	0.38	0.41	0.36

Lustig and VanNieuwerburgh (2004b) show that value stocks command higher expected returns because their returns co-vary more strongly with aggregate consumption growth when collateral is scarce. This mechanism accounts for more than 80 percent if the cross-sectional variation in the book-to-market portfolio returns in the data. The second main exercise of this paper is to show the collateral model can endogenously generate a value premium of the magnitude observed in the data. We do two exercises to substantiate this claim. In the first exercise we generate excess returns on value decile portfolios from an empirically plausible factor model (section 4.1). In a second exercise, we specify the dividend processes on the value portfolios and compute returns on these dividend streams (section 4.2).

4.1. Plugging the Empirical Betas into our Model

Decile Return Processes in Data In a first step, we use the data on the decile value portfolio returns to describe the return-generating process for each of the book-to-market decile portfolios. We specify return processes for value and growth stocks as linear functions of the state variables in our model. These are aggregate consumption growth, aggregate expenditure share growth, the housing collateral ratio and the interaction terms of the housing collateral ratio with aggregate consumption growth and expenditure share growth. The return in excess of a risk-free rate on the j^{th} book-to-market decile portfolio is:

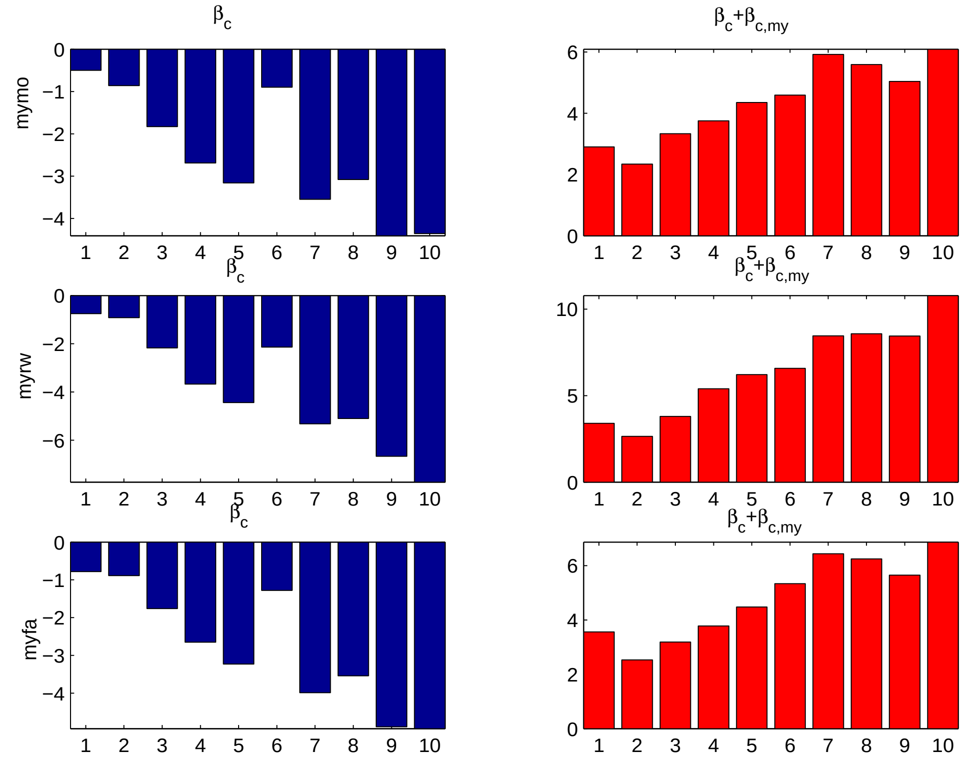
$$\begin{aligned} R^{e,j}_{t+1} &= \beta^j_0 + \beta^j_{my}\widetilde{my}_{t+1} + \beta^j_c \Delta \log c^a_{t+1} + \beta^j_{c,my}\widetilde{my}_{t+1}\Delta \log c^a_{t+1} + \\ & \quad \beta^j_\alpha \Delta \log \alpha^a_{t+1} + \beta^j_{\alpha,my}\widetilde{my}_{t+1}\Delta \log \alpha^a_{t+1} + \nu^j_{t+1}, \end{aligned} \tag{18}$$

where the collateral scarcity measure $\widetilde{my}_{t+1} = \frac{my^{max} - my_{t+1}}{my^{max} - my^{min}}$ is always between 0 and 1. The beta vector in equation (18) is estimated by ordinary least squares.[19]

Figure 10 plots the consumption beta β_c in the first column and the sum of the consumption beta and the beta on the interaction term with the collateral ratio $\beta_{c,my}$ in the second column. The returns on value stocks (decile 10) are high in recessions, i.e. they co-vary negatively with aggregate consumption growth, while growth stocks (decile 1) are much less sensitive to aggregate consumption growth; $|\beta_c|$ increases monotonically from decile 1 to decile 10. This pattern is robust across different collateral measures. The first column of figure 10 plots the total consumption beta, $\beta_c + \beta_{c,my}\widetilde{my}_t$, when collateral is very abundant ($\widetilde{my}_t = 0$). The second column shows the total consumption beta when collateral is very scarce ($\widetilde{my}_t = 1$). Value stocks are more sensitive to consumption growth shocks when collateral is scarce ($\beta_{c,my} > 0$). As a result, they have higher total consumption betas when collateral is scarce. That is why they command a risk premium. For growth stocks this effect is much less pronounced; $\beta_{c,my}$ increases monotonically from decile 1

[19] Results are available in table 10 in a separate appendix.

Figure 10. Beta Estimates for Book-to-Market Decile Returns in Data.
OLS regression of excess returns of the 10 book-to-market deciles on a constant, the collateral scarcity measure $\widetilde{my}_t$, the aggregate consumption growth rate $\Delta \log c^a_{t+1}$, the interaction term $\widetilde{my}_t \Delta \log c^a_{t+1}$, the aggregate expenditure share growth rate $\Delta \log \alpha^a_{t+1}$, and the interaction term $\widetilde{my}_t \Delta \log \alpha^a_{t+1}$. These are the five risk factors in the collateral model. The portfolios are organized from the lowest book-to-market decile (growth) on the left to the highest book-to-market decile (value) on the right of each horizontal axis. In the first panel the housing collateral ratio is based on the value of outstanding mortgages ($mymo$), in the second panel the housing collateral ratio is based on the value of residential real estate wealth ($myrw$), and in the third panel it is based on the value of residential fixed assets ($myfa$). The data are annual for the period 1930-2003.



to decile 10. This pattern is robust across different collateral measures. Lastly, value stocks are also more sensitive to aggregate expenditure share shocks; β_α increases monotonically for all three collateral measures.

Decile Return Processes in Model In a second step, we generate ten excess return processes as the product of the factor loadings $\left(\beta^j_{my}, \beta^j_c, \beta^j_{c,my}, \beta^j_\alpha, \beta^j_{\alpha,my}\right)$ displayed in figure 10 and the aggregate state variables (factors) generated in the model. For each excess return, the intercept β^j_0 is determined such that the Euler equation $E_t[m_{t+1} R^{j,e}_{t+1}] = 0$ is satisfied. This is a consistency requirement that the SDF of the model prices these assets correctly. We then simulate the model for 10,000 periods and compute unconditional mean and standard deviation of each of the decile portfolio returns.

Collateral Model The second and third panel of table 4 report the excess returns on the ten value portfolios predicted by the collateral model, ordered from growth (B1) to value (B10) for $\gamma = 5$ and $\gamma = 8$ respectively. In each panel we use three sets of empirical factor loadings, corresponding to the three housing collateral measures.

For the fixed asset measure, the value spread is 6 percent for $\gamma = 8$ and 4 percent for $\gamma = 5$. For the other two collateral measure, the value spread is 4 percent for $\gamma = 8$ and 3 percent for $\gamma = 5$. For $\gamma = 8$, the model is able to replicate the 5.2 percent value spread in the data. Furthermore, the model predicts a noticeable increase the Sharpe ratio between the first decile (growth) and the tenth decile (value). The Sharpe ratio doubles for all three collateral measures and $\gamma = 8$. In the data there is similar increase in the Sharpe ratio for post-war data. Figure 2 in the introduction plots the spreads against the lowest B/M portfolio for the model ($\gamma = 8$) and for the data, using the three different factor loadings from the three different collateral measures. The model does quite well in reproducing these spreads. The mean return levels are off by roughly 6 percent, but this could be due to the fact that the returns in the data depend on the history of consumption growth shocks in ways that we are not capturing here.

Representative Agent Model In contrast, the representative agent economy generates no value premium. We estimate the consumption betas from an equation like (18), but with aggregate consumption growth and aggregate expenditure share growth as the factors. These are the only two risk factors in the representative agent economy with non-separable preferences. There is much less of a pattern in the betas than what we found for the collateral model. The estimated betas are used together with aggregate consumption growth shocks in the model to generate 10 excess return processes. There is no pattern in the excess returns. The value premium is zero. The Sharpe ratios on growth stocks are higher than the ones on value stocks, the opposite pattern from the one found in the data.[20]

[20]Detailed results available upon request.

4.2. Pricing Stocks with Different Duration

Growth stocks (value stocks) can be thought of as a basket of consumption strips that is heavily weighted towards longer (shorter) maturities (Dechow, Sloan, and Soliman (2002) and Lettau and Wachter (2004)). Consumption strips are claims to period $t+k$ aggregate consumption (c_{t+k}), where k is the horizon in years.

Formally, the multiplicative (one year) equity premium on a non-levered claim to the stream of aggregate consumption $\{c_k\}_{k=1}^{\infty}$, $E_0 R_{0,1}^e[\{c_k\}]$, can be written as a weighted sum of expected excess returns on consumption strips:

$$1+\nu_0 = 1+E_0[R_{0,1}^e[\{c_k\}]] \quad = \quad \sum_{k=1}^{\infty} \omega_k \frac{E_0 R_{0,1}\left[c_k\right]}{R_{0,1}\left[1\right]},$$

with weights

$$\omega_k = \frac{E_0 M_k c_k}{\sum_{l=1}^{\infty} E_0 M_l c_l}.$$

The second term in the sum is the expected return on a period k consumption strip $E_0 R_{0,1}\left[c_k\right]$ in excess of the risk-free rate $R_{0,1}\left[1\right]$. The weights can be interpreted as the value of the period k consumption strip relative to the total value of all consumption strips. M_k is the pricing kernel in period k. It is linked to the stochastic discount factor m by $M_k = m_1 \times \cdots \times m_k$. Appendix A.2 shows the derivation of the value premium in more detail.

Value stocks can then be modelled as a claim to a weighted stream of aggregate consumption $\{f^v(k)c_k\}_{k=1}^{\infty}$, where the function $f(\cdot)$ puts more weight on the consumption realizations in the near future. For example, $f^v(k) = Ce^{ak}$, where a is a negative number and C is a normalization constant, $C = \frac{\sum_{k=1}^{\infty} c_k}{\sum_{k=1}^{\infty} e^{ak} c_k}$. Likewise, growth stocks can be thought of as a claim to a weighted stream of aggregate consumption $\{f^g(k)c_k\}_{k=1}^{\infty}$, where the function $f(\cdot)$ puts more weight on the consumption realizations in the far future. For example, $f^g(k) = Ce^{ak}$, where a is a positive number. The multiplicative equity premium on such a claim is

$$1+\tilde{\nu}_0 = \sum_{k=1}^{\infty} \tilde{\omega}_k \frac{E_0 R_{0,1}\left[c_k\right]}{R_{0,1}\left[1\right]},$$

with modified weights

$$\tilde{\omega}_k = \frac{f(k) E_0 M_k c_k}{\sum_{l=1}^{\infty} f(l) E_0 M_l c_l}.$$

The following proposition shows that the properties of the pricing kernel determine the sign of the value spread. In particular, if the pricing kernel has no permanent component, then the model generates a *growth premium.*

Proposition 4. *If* $\gamma > 1$ *and* $f(k) = Ce^{ak}, a > 0$ *then*

$$\lim_{k\to\infty} \frac{E_{t+1}M_{t+k}c_{t+k}}{E_t M_{t+k}c_{t+k}} = 1 \Rightarrow \lim_{a\to\infty} 1 + \widetilde{\nu}_0 = \lim_{k\to\infty} R^c_{t+1,k} \geq 1 + \nu_0,$$

for any other sequence of weights $\{\omega_k\}$

Proof: see appendix. The proof relies on insights in Alvarez and Jermann (2001b). The proposition implies that the highest equity premium is the one on the farthest out consumption strip. In the absence of a permanent component in the pricing kernel, there is a growth premium. The pricing kernel in our model contains a permanent component through the multiplicative component stemming from the risk of binding solvency constraints: The aggregate weight shock $(\xi^a)^\gamma$ is a non-decreasing stochastic process. This is a necessary condition for generating a value premium.

Consumption-CAPM In the representative agent economy, the equity premia on consumption strips do not change with the horizon. This is easy to show for additive preferences that are separable in both commodities and aggregate endowment growth that is i.i.d with mean $\bar{\lambda}$. The pricing kernel is simply a function of the aggregate consumption growth rate between period 1 and period k: $M_k = \lambda_k^{-\gamma}\lambda_{k-1}^{-\gamma}\cdots\lambda_1^{-\gamma}$. Because the aggregate endowment grows every period at the rate λ, $M_1 c_1 = \lambda_1^{-\gamma}\lambda_1 c_0$. For the period k strip, $M_k c_k = \lambda_k^{1-\gamma}\lambda_{k-1}^{1-\gamma}\cdots\lambda_1^{1-\gamma}c_0$. Hence, the expected return on a period k strip is:

$$E_0 R_{0,1}[c_k] = \frac{E_1(M_k c_k)}{E_0(M_k c_k)} = \frac{E_1\left(\lambda_k^{1-\gamma}\lambda_{k-1}^{1-\gamma}\cdots\lambda_1^{1-\gamma}c_0\right)}{E_0\left(\lambda_k^{1-\gamma}\lambda_{k-1}^{1-\gamma}\cdots\lambda_1^{1-\gamma}c_0\right)} = \left(\frac{\lambda_1}{\bar{\lambda}}\right)^{1-\gamma}$$

The expression does not depend on the horizon k. This shows that equity premia are constant across strips of different horizons in the representative agent economy. The term structure of consumption strips is flat. A similar result obtains if preferences are non-separable and aggregate expenditure share growth is i.i.d., even when aggregate expenditure share growth is correlated with aggregate consumption growth.

Collateral-Consumption-CAPM Our model endogenously generates a downward sloping term structure of consumption strip risk premia. The building blocks of the equity premia on value and growth stocks are equity premia on consumption strips. Ultimately, the collateral model generates a higher expected return and a higher Sharpe ratio for value stocks than growth stocks because short term assets are more risky than long term assets.

Figure 11. Term Structure of Sharpe Ratios on Consumption Strips.
The average collateral share is 5 percent, the discount factor is .95 and the coefficient of risk aversion γ is between 3 and 8. The figure plots the conditional Sharpe ratio on a claim to aggregate consumption k periods from now, $k = 2, 3, ..., 30$.

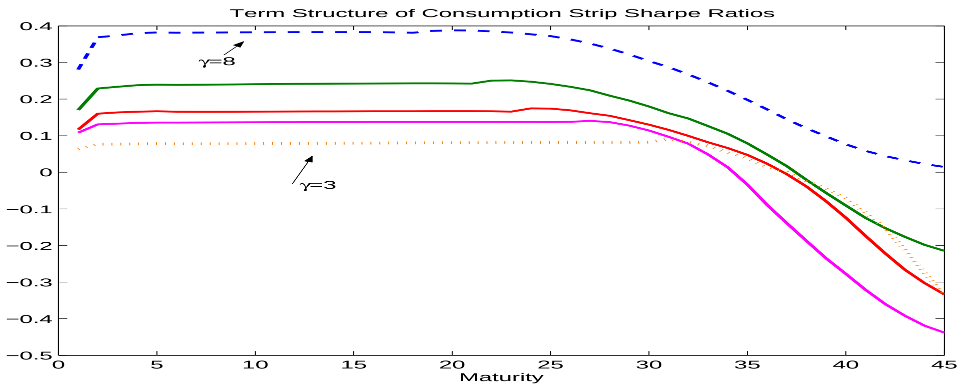


Figure 11 plots the Sharpe ratios on consumption strips of horizons 2 to 45 years for the benchmark model with additive utility ($\gamma = 3 - 8$). Risk premia and Sharpe ratios on consumption strips are lower for long horizon strips. The duration effect is even stronger in the collateral model with recursive utility.[21] When a low aggregate consumption growth shock arrives, the conditional market price of risk decreases, but the state prices decreases for all states for the next couple of periods, or, interest rates rise. This lowers the price of consumption strips with short maturities, but it does not affect the price of strips at longer maturities. This explains why $cov\left(m, R_{0,1}\left[c_k\right]\right)$ decreases in absolute value in the maturity k. Of course, this effect increases in size the lower the collateral ratio and this steepens the slope of the term structure of strip risk premia when the collateral ratio is low. [22] Value stocks are baskets of long maturity strips and as a result their expected return increases when the collateral ratio is low. This is the effect picked up by the interaction of consumption growth and the collateral ratio in the factor loadings.

[21]Results available in the separate appendix.

[22]When the collateral ratio is high, the term structure is close to flat, as in the representative agent model (see above).

Table 5
Risk Premia on Portfolios of Consumption Strips.

This table reports the expected excess return, the conditional standard deviation and the Sharpe ratio on baskets of consumption strips. The consumption strips are computed for the baseline model with additive utility and $\gamma = 5, 8$. The first row denotes the duration of each basket in years. The baskets are weighted combinations of consumption strips of different horizons k, with weights governed by Ce^{ak}. Each column reports the basket for a value of parameter a ranging from -.5 to .5.

Panel 1: Risk Aversion 5											
Duration	2.4	3.0	4.2	5.6	8.2	14.3	25.6	35.2	39.5	41.4	43.4
$E[R^e]$	0.031	0.032	0.034	0.035	0.036	0.035	0.025	0.008	-0.004	-0.011	-0.020
$\sigma[R^e]$	0.106	0.112	0.119	0.123	0.128	0.132	0.131	0.125	0.120	0.117	0.113
$E[R^e]/\sigma[R^e]$	0.290	0.288	0.285	0.285	0.283	0.266	0.190	0.065	-0.033	-0.095	-0.180
Panel 1: Risk Aversion 8											
Duration	2.3	2.9	4.1	5.4	7.9	13.9	25.7	35.7	40.0	41.7	43.4
$E[R^e]$	0.059	0.063	0.068	0.071	0.074	0.072	0.052	0.027	0.014	0.007	0.002
$\sigma[R^e]$	0.131	0.141	0.154	0.161	0.168	0.174	0.172	0.166	0.161	0.159	0.157
$E[R^e]/\sigma[R^e]$	0.451	0.447	0.443	0.442	0.438	0.411	0.304	0.164	0.084	0.047	0.013

Table 5 reports equity premia on claims to $\{Ce^{ak}c_k\}$. These are baskets of consumption strips of different maturities, where the constant a governs the duration of the basket. We vary a from -.5 to .5. The corresponding baskets have a duration between 2.3 years and 43 years. We think of the basket with duration of 5 years as the value stock and the basket with duration of 40 years as the growth stock (Dechow, Sloan, and Soliman (2002)). The value spread is 3.9 percent for $\gamma = 5$ and 5.7 percent for $\gamma = 8$. The latter matches the value spread in the data of 5.7 percent. In addition, Sharpe ratios on value portfolios are much higher than on growth portfolios. For $\gamma = 8$, the Sharpe ratio on the 5-year duration portfolio is .44 and the Sharpe ratio on the 40-year duration portfolio is .08.

5. Unconditional Asset Pricing Moments

Before we conclude, we compare the *unconditional* moments against the data. This reveals (1) how we chose the preference parameters governing risk aversion and the intratemporal elasticity of substitution, (2) that the model matches key unconditional moments such as the equity premium, its unconditional volatility, the Sharpe ratio, and the risk-free rate, and (3) that the model overstates the volatility of the risk-free rate and the correlation between housing and stock returns. Throughout, we contrast the results with those of a representative agent economy with non-separable preferences.

5.1. Collateral Model

Table 6 summarizes the unconditional first and second moments of asset returns for the collateral model with non-separable preferences, and it compares these to those for the representative agent model with non-separable preferences, and those for the US economy.

Table 6
Unconditional Asset Pricing Moments for Collateral Model.

Averages from a simulation of the model for 5,000 agents and 10,000 periods. In the first column, $R^{l,e}$ denotes the excess return on a levered claim to aggregate consumption growth, with leverage parameter $\kappa = 3$. $R^{c,e}$ denotes the excess return on a non-levered claim to aggregate consumption growth. The third column reports the unconditional mean of the risk-free rate. Columns four to six report unconditional standard deviations of levered and non-levered consumption claims and risk-free rate. The last two columns report Sharpe ratios on levered and non-levered consumption claims. The expenditure share process is an AR(1) with an aggregate consumption growth term: $\log r_t = \bar{r} + \rho_r \log r_{t-1} + b_r \Delta(\log(c_{t+1})) + \sigma_r \nu_t$. ε is fixed at .05. We vary the coefficient of relative risk aversion γ. All other parameters are held constant at their benchmark level and there is 5 percent collateral. Panel 1 reports historical averages for 1927-2001 (data from Kenneth French) and for 1889-1979 (data from Shiller). Panel 2 reports the results for the benchmark collateral economy with 5 percent collateral. Panel 3 reports results for the representative agent economy. Panel 4 reports the moments for the collateral economy with 10 percent collateral.

(γ)	$E(R^{l,e})$	$E(R^{c,e})$	$E(r^f)$	$\sigma(R^{l,e})$	$\sigma(R^{c,e})$	$\sigma(r^f)$	$\frac{E(R^{l,e})}{\sigma(R^{l,e})}$	$\frac{E(R^{c,e})}{\sigma(R^{c,e})}$
	Panel 1: Data							
1927-2002	0.075		0.039	0.198		0.032	0.419	
1889-1979	0.060		0.014	0.192		0.065	0.313	
	Panel 2: Benchmark 5 percent Collateral Model							
(5)	0.051	0.030	0.057	0.213	0.129	0.073	0.238	0.230
(8)	0.108	0.071	0.029	0.257	0.179	0.125	0.421	0.398
	Panel 3: Representative Agent Model							
(5)	0.024	0.010	0.129	0.127	0.057	0.031	0.186	0.177
(8)	0.045	0.023	0.158	0.146	0.074	0.055	0.309	0.310
	Panel 4: 10 percent Collateral Model							
(5)	0.033	0.028	0.101	0.159	0.090	0.051	0.207	0.318
(8)	0.078	0.056	0.075	0.207	0.150	0.107	0.376	0.373

Risk Premium Because consumption growth is less volatile in the data than dividend growth, the relevant comparison of the excess stock market return in the data is with a levered claim to aggregate consumption in the model ($\kappa = 3$). In the data (panel 1 of table 6), the excess return on the market portfolio is 7.5 percent with a volatility of 19.8 percent. The benchmark model with five percent collateral is able to generate a high and volatile levered equity risk premium. The calibration with $\gamma = 5$ and $\varepsilon = .05$ in panel 2 of table 6 generates a 5.1 percent equity premium. The standard deviation is 21.3 percent. To understand the effect of the leverage, we also price a non-levered consumption claim. Its equity premium is 3.0 percent for $\gamma = 5$ and 7.1 percent for $\gamma = 8$. The Sharpe ratio in the model with $\gamma = 8$ is 0.42, equal to the Sharpe ratio observed for 1927-2002.

The model with $\gamma = 8$ ($\gamma = 5$) generates an average risk free rate of 2.9 (5.7) percent, close to the 3.9 percent in the data. For a higher γ, a given size aggregate weight shocks has a much larger effect on the state price of consumption, and this liquidity effect increases the conditional expectation of the stochastic discount factor, and pushes down the risk-free rate.

No Conditional Heteroscedasticity in Income If we shut down the Constantinides and Duffie (1996) mechanism, the housing collateral mechanism still generates a sizeable equity premium of 7.4 percent ($\gamma = 8$, 5 percent collateral). This amounts to 70 percent of the magnitudes we found for our benchmark model in table 7. The mean risk-free rate is slightly higher, but less volatile.

Representative Agent We contrast the results from the collateral model with those same moments in the representative agent economy. Preferences are non-separable between non-durable and housing services consumption, but the collateral effect is shut down. The equity premium is compensation for aggregate consumption growth risk (as in Lucas (1978)) and aggregate composition risk (as in Piazzesi, Schneider, and Tuzel (2004)). Panel 3 of table 6 also shows that the equity premium in the representative agent economy is substantially smaller than in the collateral model. The levered consumption claim has an expected excess return of 2.4 percent for ($\gamma = 5, \varepsilon = .05$) and 4.5 percent for ($\gamma = 8, \varepsilon = .05$). This is less than half as big as for the collateral model with five percent collateral. The equity premium on a non-levered claim is one-third the magnitude of the collateral model. In addition, the levels of the risk-free rate and the stock return are much too high in the representative agent economy. For $\gamma = 8$, the risk-free rate is 15.8 percent and the stock return is 20.3 percent on average. This compares to 3.9 percent and 11.4 percent in the data for 1927-2002. Moreover, the risk-free rate *increases* with γ. A more risk-averse representative agent is less willing to substitute intertemporally and wants to borrow more against her growing labor income; this drives up the risk-free rate.

Risk-Free Rate Volatility The biggest failure of the collateral model is the unconditional volatility of the risk-free rate. It is 7.3 percent in the economy with $\gamma = 5$, 12.5 percent in the economy with $\gamma = 8$, but only 3.2 percent in the data. There are two forces driving the volatility of the risk-free rate and both work though changes in the demand for insurance. At high frequencies, the variation in the expected fraction of households facing binding constraints tomorrow (due to shocks to the wealth distribution) moves the risk-free rate, while at lower frequencies the shocks to the risk-sharing technology due to the destruction of housing collateral drive the variations in the risk-free rate. It is important to note that risk-free rates were more volatile prior to 1927. The unconditional standard deviation of the real risk-free rate is 6.5 percent for the 1889-1979 period (data from Shiller's web site). A model with recursive preferences, described in a separate technical appendix, generates a volatility of the risk-free rate of 6.5 percent. It still generates a large equity premium (6 percent), a low risk-free rate, and volatile stock market returns.

Collateral Finally, the economy with ten percent collateral (panel 4 of Table 6) is closer to the representative agent economy, because the collateral constraints are not as tight. The expected excess return on a levered consumption claim is still high. For $(\gamma = 8, \varepsilon = .05)$, the equity premium is 7.8 percent (panel 4). With 10 percent collateral, the risk-free rate is higher on average, but less volatile.

Table 7
Unconditional Asset Pricing Moments for Collateral Model.

Averages from a simulation of the model for 5,000 agents and 10,000 periods. In the first column, $R^{l,e}$ denotes the excess return on a levered claim to aggregate consumption growth, with leverage parameter $\kappa = 3$. $R^{c,e}$ denotes the excess return on a non-levered claim to aggregate consumption growth. The third column reports the unconditional mean of the risk-free rate. Columns four to six report unconditional standard deviations of levered and non-levered consumption claims and risk-free rate. The last two columns report Sharpe ratios on levered and non-levered consumption claims. Panel 1 is the benchmark calibration with 5 percent collateral. The expenditure share process is an AR(1) with an aggregate consumption growth term: $\log r_t = \bar{r} + \rho_r \log r_{t-1} + b_r \Delta(\log(c_{t+1})) + \sigma_r \nu_t$. Panel 1 reports results for different parameters of intratemporal elasticity of substitution between non-durable and housing services consumption ε. All other parameters are held constant at their benchmark level and there is 5 percent collateral. Panel 2 varies the coefficient of relative risk aversion γ (5 percent collateral). Panel 3 is the benchmark calibration with 5 percent collateral but with an expenditure share process is an AR(1) without consumption growth term: $\log r_t = \bar{r} + \rho_r \log r_{t-1} + \sigma_r \nu_t$.

(γ, ε)	$E(R^{l,e})$	$E(R^{c,e})$	$E(r^f)$	$\sigma(R^{l,e})$	$\sigma(R^{c,e})$	$\sigma(r^f)$	$\frac{E(R^{l,e})}{\sigma(R^{l,e})}$	$\frac{E(R^{c,e})}{\sigma(R^{c,e})}$
	Panel 1: Varying ε and $\gamma = 8$							
$(8, 0.15)$	0.110	0.091	0.024	0.258	0.175	0.128	0.425	0.516
$(8, 0.75)$	0.153	0.123	−0.015	0.274	0.188	0.136	0.558	0.657
	Panel 2: Varying γ and $\epsilon = .05$							
$(2, 0.05)$	0.013	0.005	0.069	0.162	0.067	0.022	0.081	0.079
$(10, 0.05)$	0.151	0.110	−0.003	0.285	0.213	0.159	0.530	0.514
	Panel 3: AR(1) process for $\log(r)$							
$(8, 0.05)$	0.091	0.066	0.012	0.207	0.154	0.119	0.440	0.427
$(8, 0.75)$	0.104	0.080	−0.001	0.220	0.176	0.124	0.475	0.456

5.2. Exploring the Parameter Space

Composition Effect Table 7 explores the effect of varying the value of the intratemporal elasticity parameter ε. The effect of a higher intratemporal elasticity of substitution is to increase the equity premium and the market price of risk and to lower the risk-free rate. In the case of $\gamma = 8$, the equity premium for $\varepsilon = .75$ is 4.5 percent higher than for $\varepsilon = .05$.

This effect also shows up in the representative agent economy; Piazzesi, Schneider, and Tuzel (2004) refer to it as a *negative composition* effect. Agents want to hedge by investing in assets that pay off in low non-housing expenditure share growth states, i.e. recessions (see equation 13). An increase in the intratemporal elasticity of substitution ε from .05 to .75 increase the equity premium by 4 percent (1.7) percent in a representative agent economy with $\gamma = 8$ ($\gamma = 5$). The risk-free rate goes down by 6 (1.9) percent. The average Sharpe ratio for the representative agent economy with $\varepsilon = .75$ is .47 (.27), in line with the historical average. However, the empirically plausible equity premium, risk-free rate and Sharpe ratio for $\varepsilon = .75$ come at the expense of an implausibly high rental price growth volatility. For $\varepsilon = .75$, the unconditional standard deviation of rental price growth is 19 percent per annum (see equation 14). In the data, rental price growth volatility is below 5 percent. Driving ε even closer to one leads to exponentially increasing rental price growth volatility.[23] This is the reason we choose $\varepsilon = .05$ as our benchmark economy. In addition, the volatility of the risk-free rate increases sharply with ε.[24]

Risk Aversion Increasing the coefficient of relative risk aversion γ from 2 to 10 in panel 2 increases the equity premium on a levered (non-levered) consumption claim from 1.3 to 15.1 (.5 to 11) percent. The unconditional Sharpe ratio increases from .08 to 0.53. The increase in γ *decreases* the risk-free rate from 7 to 0 percent. The first reason for this risk-free rate effect is that households cannot borrow as much as they would like because of binding collateral constraints. Second, the risk-free asset provides insurance against binding constraints, and this lowers the risk-free rate even further. This is a precautionary savings effect coming from the collateral constraints. If households are more risk averse, they want better insurance against the risk of binding constraints. The main effect of a higher coefficient of relative risk aversion is to amplify the collateral effect, coming through the second part of the stochastic discount factor, g_{t+1}^{γ}.

[23]For $\varepsilon > 1$ the representative agent model generates a negative equity premium.
[24]Results for the representative agent economy are available upon request.

Expenditure Share Process Lastly, panel 3 shows that the results are not very sensitive to a change in the expenditure share process in equation (13). This table displays the results for an AR(1) specification for $\log r_t$ without consumption growth term on the right-hand side ($b_r = 0$). Under this specification, the housing collateral ratio is less volatile. The stock return and the risk-free rate are one percent lower on average. The volatility of $R^{l,e}$ is 5 percent lower than in the benchmark economy. Changes in ε have a smaller effect on the unconditional asset pricing moments.

5.3. Housing Market Statistics

The model also has predictions for the return on housing in ownership. In the collateral model, the expected excess return on a claim to the aggregate housing dividend stream is similar to the return on the aggregate consumption stream. Panel 1 of table 8 shows that the housing equity premium is 6.8 (2.6) percent for $\gamma = 8$ ($\gamma = 5$). The standard deviation of this return is 19.2 (12.7) percent, leading to a Sharpe ratio of .35 (.20). The Sharpe ratio on a non-levered consumption claim in table 7 was .40 (.23). Using household-level data for 1968-1992 from the PSID, Flavin and Yamashita (2002) estimate the expected excess return and its standard deviation on home ownership to be 6.6 percent and 14.2 percent. This implies a Sharpe ratio for housing of .46. Our benchmark model with $\gamma = 8$ generates expected returns and Sharpe ratios for the housing market that are broadly consistent with the Flavin and Yamashita (2002) numbers. In the representative agent economy (panel 3 of 8), the expected excess return on home ownership is too low, the expected return too high and not sufficiently volatile to match the data.

Correlation of Housing and Stock Markets Finally, the implied correlation between housing returns and stock returns in our model is close to one, regardless of the specific calibration. This is much too high: In the household-level data of Flavin and Yamashita (2002), the correlation is close to zero. The same SDF prices all payoffs in this economy, including flows of rental services. We conjecture that a model where financial assets and housing are collateralizeable to a different extent may lower this correlation, because it would imply a different SDF for stocks and for real estate.

Table 8
Unconditional Asset Pricing Moments for Housing Market.

In the left block the coefficient of relative risk aversion γ is 5; in the right block it is 8. The rows are for different intratemporal elasticities of substitution between non-durable and housing services consumption ε. All other parameters are held constant at their benchmark level. In the first column, $\sigma(\Delta\log(\rho_t))$ denotes the volatility of rental price growth. In the second column $R^{h,e}$ denotes the return on a claim to the aggregate housing endowment in excess of the risk-free rate. The third column denotes the unconditional standard deviation of $R^{h,e}$. The fourth column gives the average housing collateral ratio, the ratio of housing wealth to total wealth my. Panel 1 is the collateral model with 5 percent housing collateral, panel 2 is the collateral model with 10 percent housing collateral and panel 3 is the representative agent economy.

ϵ	$\sigma(\Delta\log(\rho_t))$	$E(R^{h,e})$	$\sigma(R^{h,e})$	my_t	$\sigma(\Delta\log(\rho_t))$	$E(R^{h,e})$	$\sigma(R^{h,e})$	my_t
	Panel 1: Benchmark 5 percent Collateral Model							
	Risk Aversion 5				*Risk Aversion 8*			
.05	0.050	0.026	0.127	0.056	0.050	0.068	0.192	0.057
.75	0.190	0.044	0.151	0.058	0.189	0.103	0.221	0.062
	Panel 2: 10 percent Collateral Model							
	Risk Aversion 5				*Risk Aversion 8*			
.05	0.050	0.016	0.088	0.105	0.050	0.048	0.156	0.106
.75	0.189	0.028	0.112	0.108	0.189	0.088	0.201	0.114
	Panel 3: Representative Agent Economy							
	Risk Aversion 5				*Risk Aversion 8*			
0.05	0.050	0.006	0.053	0.055	0.050	0.015	0.079	0.056
0.75	0.188	0.016	0.078	0.058	0.189	0.041	0.122	0.059

6. Conclusion

This paper specifies, calibrates and solves a general equilibrium asset pricing model with housing collateral. Agents write state-contingent promises backed by the value of the housing stock. Time variation in the price of housing induces time variation in the economy's ability to share labor income risk. When the ratio of housing wealth to total wealth is low, households with binding collateral constraints experience a larger change in the consumption share. In such periods of limited risk-sharing possibilities, agents demand a higher risk premium on financial assets. The housing collateral mechanism endogenously generates time-varying volatility in the Sharpe ratio on equity. This is a novel feature of the model. It quantitatively matches the dynamics of the Sharpe ratio on equity in US data: It is high in periods of collateral scarcity and volatile. In contrast, the representative agent model delivers virtually no variation in the Sharpe ratio. Other equilibrium models have similar difficulties generating enough volatility in the Sharpe ratio. The model also explains cross-sectional variation in excess returns along the value dimension. Excess returns on short duration assets (such as value stocks) are higher than returns on long duration assets (such as growth stocks). The reason for this duration effect is that a negative consumption growth shock leads to lower future Sharpe ratios. Lastly, the model quantitatively matches key unconditional asset pricing moments. To the best of our knowledge, this is one of the few models that starts from first principles and quantitatively matches the unconditional and conditional moments of aggregate asset prices, and generates a meaningful variation in the cross-section of returns.

Why does the collateral model work better than the standard consumption CAPM? The model suggest that the answer lies in allowing for time-variation in risk-sharing among heterogeneous agents. The standard CCAPM implies that risk-sharing is always perfect. In Lustig and Van-Nieuwerburgh (2004a), we provide direct empirical support for the underlying time-variation in risk-sharing. Using US metropolitan area data, we find that the degree of insurance between regions decreases when the housing collateral ratio is low. This is consistent with evidence from Blundell, Pistaferri, and Preston (2002), who find evidence for time-variation the economy's risk sharing capacity. Regional income and consumption data provide direct support for the existence and importance of the collateral mechanism, whose asset pricing implications were the focus of this paper.

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A. Appendix

A.1. Arrow-Debreu Equilibrium

This appendix spells out the household problem in an economy where all trade takes place at time zero.

Household Problem A household of type (θ_0, s_0) purchases a complete contingent consumption plan $\{c(\theta_0, s_0), h(\theta_0, s_0)\}$ at time-zero market state prices $\{p, p\rho\}$. The household solves:

$$\sup_{\{c,h\}} U(c(\theta_0, s_0), h(\theta_0, s_0))$$

subject to the time-zero budget constraint

$$\Pi_{s_0}\left[\{c(\theta_0, s_0) + \rho h(\theta_0, s_0)\}\right] \leqslant \theta_0 + \Pi_{s_0}\left[\{\eta\}\right],$$

and an infinite sequence of collateral constraints for each t and s^t

$$\Pi_{s^t}\left[\{c(\theta_0, s_0) + \rho h(\theta_0, s_0)\}\right] \geq \Pi_{s^t}\left[\{\eta\}\right], \forall s^t.$$

Dual Problem Given Arrow-Debreu prices $\{p, \rho\}$ the household with label (θ_0, s_0) minimizes the cost $C(\cdot)$ of delivering initial utility w_0 to itself:

$$\begin{aligned} C(w_0, s_0) \quad = \quad & \min_{\{c,h\}} \left(c_0(w_0 s_0) + h_0(w_0, s_0)\rho_0(s_0)\right) \\ & + \sum_{s^t} p(s^t|s_0)\left(c_t(w_0, s^t|s_0) + h_t(w_0, s^t|s_0)\rho_t(s^t|s_0)\right) \end{aligned}$$

subject to the promise-keeping constraint

$$U_0(\{c\}, \{h\}; w_0, s_0) \geq w_0$$

and the collateral constraints

$$\Pi_{s^t}\left[\{c(w_0, s_0) + \rho h(w_0, s_0)\}\right] \geq \Pi_{s^t}\left[\{\eta\}\right], \forall s^t.$$

The initial promised value w_0 is determined such that the household spends its entire initial wealth:

$$C(w_0, s_0) = \theta_0 + \Pi\left[\{\eta\}\right].$$

There is a monotone relationship between θ_0 and w_0.

The above problem is a convex programming problem. We set up the saddle point problem and then make it recursive by defining cumulative multipliers (Marcet and Marimon (1999)). Let ℓ be the Lagrange multiplier on the promise keeping constraint and $\gamma_t(w_0, s^t)$ be the Lagrange multiplier on the collateral constraint in history s^t. Define a cumulative multiplier at each node: $\zeta_t(w_0, s^t) = 1 - \sum_{s^t} \gamma_t(w_0, s^t)$. Finally, we rescale the market state price $\hat{p}_t(s^t) = p_t(z^t)/\delta^t \pi_t(s^t|s_0)$. By using Abel's partial summation formula and the law of iterated expectations to the

Lagrangian, we obtain an objective function that is a function of the cumulative multiplier process ζ^i :

$$D(c,h,\zeta;w_0,s_0)=\sum_{t\geq 0}\sum_{s^t}\left\{\delta^t\pi(s^t|s_0)\left[\begin{array}{c}\zeta_t(w_0,s^t|s_0)\hat{p}_t(s^t)\left(c_t(w_0,s^t)+\rho_t(s^t)h_t(w_0,s^t)\right)\\+\gamma_t(w_0,s^t)\Pi_{s^t}\left[\{\eta\}\right]\end{array}\right]\right\}$$

such that

$$\zeta_t(w_0,s^t)=\zeta_{t-1}(w_0,s^{t-1})-\gamma_t(w_0,s^t),\ \zeta_0(w_0,s_0)=1$$

Then the **recursive dual** saddle point problem is given by:

$$\inf_{\{c,h\}}\sup_{\{\zeta\}}D(c,h,\zeta;w_0,s_0)$$

such that

$$\sum_{t\geq 0}\sum_{s^t}\delta^t\pi(s^t|s_0)u(c_t(w_0,s^t),h_t(w_0,s^t))\geq w_0$$

To keep the mechanics of the model in line with standard practice, we re-scale the multipliers. Let

$$\xi_t(\ell,s^t)=\frac{\ell}{\zeta_t(w_0,s^t)},$$

The cumulative multiplier $\xi(\ell,s^t)$ is a non-decreasing stochastic sequence (sub-martingale). If the constraint for household (ℓ,s_0) binds, it goes up, else it stays put.

First Order Necessary Conditions The f.o.c. for $c(\ell,s^t)$ is :

$$\hat{p}(s^t)=\xi_t(\ell,s^t)u_c(c_t(\ell,s^t),h_t(\ell,s^t)).$$

Upon division of the first order conditions for any two households ℓ' and ℓ'', the following restriction on the joint evolution of marginal utilities over time and across states must hold:

$$\frac{u_c(c_t(\ell',s^t),h_t(\ell',s^t))}{u_c(c_t(\ell'',s^t),h_t(\ell'',s^t))}=\frac{\xi_t(\ell'',s^t)}{\xi_t(\ell',s^t)}. \tag{19}$$

Growth rates of marginal utility of non-durable consumption, weighted by the multipliers, are equalized across agents:

$$\frac{\xi_{t+1}(\ell',s^{t+1})}{\xi_t(\ell',s^t)}\frac{u_c(c_{t+1}(\ell',s^{t+1}),h_{t+1}(\ell',s^{t+1}))}{u_c(c_t(\ell',s^t),h_t(\ell',s^t))}=\frac{\hat{p}_{t+1}(s^{t+1})}{\hat{p}_t(s^t)}=\frac{\xi_{t+1}(\ell'',s^{t+1})}{\xi_t(\ell'',s^t)}\frac{u_c(c_{t+1}(\ell'',s^{t+1}),h_{t+1}(\ell'',s^{t+1}))}{u_c(c_t(\ell'',s^t),h_t(\ell'',s^t))}.$$

There is a mapping from the multipliers at s^t to the equilibrium allocations of both commodities. We refer to this mapping as the risk-sharing rule.

$$c_t(\ell,s^t)=\frac{\xi_t(\ell,s^t)^{\frac{1}{\gamma}}}{\xi_t^a(z^t)}c_t^a(z^t)\text{ and }h_t(\ell,s^t)=\frac{\xi_t(\ell,s^t)^{\frac{1}{\gamma}}}{\xi_t^a(z^t)}h_t^a(z^t). \tag{20}$$

It is easy to verify that this rule satisfies the optimality condition and the market clearing conditions.

The time zero ratio of marginal utilities is pinned down by the ratio of multipliers on the promise-keeping constraints. For $t>0$, it tracks the stochastic weights ξ. From the first order condition w.r.t. $\xi_t(\ell,s^t)$ we obtain a reservation weight policy:

$$\xi_t = \xi_{t-1} \text{ if } \xi_{t-1} > \ell^c(y_t, z^t), \tag{21}$$

$$\xi_t = \ell^c(y_t, z^t) \text{ otherwise.} \tag{22}$$

and the collateral constraints hold with equality at the bounds:

$$\Pi_{s^t}\left[\left\{c_t(\ell, s^t; \underline{\xi}_t(\ell, s^t)) + \rho h^i(\ell, s^t; \underline{\xi}(\ell, s^t))\right\}\right] = \Pi_{s^t}\left[\{\eta\}\right].$$

A.2. Collateral Effect

Proof of Proposition 1 Denote the price of a claim under perfect risk-sharing by $\Pi^*[\{\cdot\}]$. Perfect risk sharing can be sustained if and only if

$$\Pi^*_z\left[\left\{c^a\left(1+\frac{1}{r}\right)\right\}\right] \geq \Pi^*_{z,y}\left[\{\eta(y,z)\}\right] \text{ for all } (y,z,r)$$

If this condition is satisfied, each household can get a constant and equal share of the aggregate non-durable and housing endowment at all future nodes. Perfect risk-sharing is possible. q.e.d.

Proof of Proposition 2 Assume utility is separable. Let $C(\ell, y_t, z^t)$ denote the cost of claim to consumption in state (y_t, z^t) for a household who enters the period with weight ξ . The cutoff rule $\ell^c(y_t, z^t)$ is determined such that the solvency constrain binds exactly: $\Pi_{y,z^t}\left[\{\eta\}\right] = C(\xi, y_t, z^t)$, where $C(\xi, y_t, z^t)$ is defined recursively as:

$$C(\xi, y_t, z^t) = \frac{\ell^c(y_t, z^t)}{\xi^a_t(z^t)}\left(1+\frac{1}{r_t}\right) + \delta\sum_{z_{t+1}}\pi(z_{t+1}|z_t)\sum_{y'}\frac{\pi(y_{t+1}, z_{t+1}|y_t, z_t)}{\pi(z_{t+1}|z_t)}m_{t+1}(z^{t+1})C(\xi', y_{t+1}, z^{t+1}),$$

and ξ' is determined by the cutoff rule. Note that the stochastic discount factor $m_{t+1}(z^{t+1})$ does not depend on $r_t(z^t)$ because we assumed that utility is separable. This also implies that the cost of a claim to labor income $\Pi_{y,z^t}\left[\{\eta\}\right]$ does not depend on r.

We prove the result for a finite horizon version of this economy. We first assume some arbitrary state prices $\{p_t(s^t|s_0)\}$ for both of these economies. $\{m_t(z^t)\}$ denotes the SDF process implied by these state prices. Finally, we use T^i to denote the operator that maps the aggregate weight functions $\{\xi^a_t(z^t)\}$ we start with into a new aggregate function $\{\xi'^{,a}_t(z^t)\}$.

In the last period T, the cutoff rule is determined such that:

$$\eta(y_{T-1}, z^{T-1}) = \frac{\ell^c(y_{T-1}, z^{T-1})}{\xi^a_T(z^{T-1})}\left(1+\frac{1}{r_{T-1}(z^{T-1})}\right) + \delta\sum_{z_{t+1}}\pi(z_T|z_{T-1})$$
$$\sum_{y'}\frac{\pi(y_T, z_T|y_{T-1}, z_{T-1})}{\pi(z_T|z_{T-1})}m_T(z^T|z_{T-1})\left(\frac{\xi'^{1/\gamma}}{\xi^a_T(z^T)}\left(1+\frac{1}{r_T(z^T)}\right) - \eta(y_T, z^T\right),$$

where $\frac{\xi'^{1/\gamma}}{\xi^a_T(z^T)}\left(1+\frac{1}{r_T}\right) \geq \eta\left(y_T, z^T\right)$. Given $r^1_T < r^2_T$ and $r^1_{T-1}(z^{T-1}) < r^2_{T-1}(z^{T-1})$, this implies that $\ell^{1,c}(y_{T-1}, z^{T-1}) <$

$\ell^{2,c}(y_{T-1}, z^{T-1})$ for all (y_{T-1}, z^{T-1}). By backward induction we get that, for a given sequence of $\{\xi_t^a(z^t)\}$, $\ell^{1,c}(y_t, z^t) < \ell^{2,c}(y_t, z^t)$ for all nodes (y_t, z^t) in the finite horizon economy. This in turn implies that $T^1(\{\xi_t^a(z^t)\}) \leq T^2(\{\xi_t^a(z^t)\})$ for all z^t, with strict inequality if at least one of the constraints binds. This follows directly from the definition of

$$\xi_t^a(z^t) = \sum_{y^t} \int \xi_t(\ell, y^t, z^t) \frac{\pi(z^t, y^t|z_0, y_0)}{\pi(z^t|z_0)} d\Phi_0 \tag{23}$$

$$= \sum_{y^t} \int_{\ell^c(y_t, z^t)} \xi_{t-1}(\ell, y^t, z^t) \frac{\pi(z^t, y^t|z_0, y_0)}{\pi(z^t|z_0)} d\Phi_0 \tag{24}$$

$$+ \sum_{y^t} \int^{\ell^c(y_t, z^t)} \ell^c(y_t, z^t) \frac{\pi(z^t, y^t|z_0, y_0)}{\pi(z^t|z_0)} d\Phi_0 \tag{25}$$

$\xi_t^a(z^t)$ is non-decreasing in $\ell^c(y_t, z^t)$.

The proof extends to the infinite horizon economy if the transition matrix has no absorbing states. The reason is that $lim_{T\to\infty} E_t\left[\beta^{T-t} m_T(z^T|z_t)\pi_{z^T,y_T}\right]$ does not depend on the current state (y_t, z_t). Now, this also implies a new state price function, for each s^t:

$$p_t'(s^t|s_0) = p_t^a(s^t|s_0) \frac{\xi_t^a(z^t)}{\xi_0^a(z^0)}^{\gamma},$$

where $p_t^a(s^t|s_0)$ is the representative agent state price. So if we start with the equilibrium state prices for the second economy $\{p_t^2(s^t|s_0)\}$, the implied aggregate weights for the first economy will be smaller:

$$T^1\left(\{\xi_t^{2,a}(z^t)^\star\}\right) < \{\xi_t^{2,a}(z^t)^\star\} = T^2\left(\{\xi_t^{2,a}(z^t)^\star\}\right),$$

where the last equality follows because we started with the equilibrium prices for the second economy, and similarly,

$$T^1(\{\xi_t^{1,a}(z^t)^\star\}) = \{\xi_t^{1,a}(z^t)^\star\} < T^2(\{\xi_t^{1,a}(z^t)^\star\}),$$

if we start with the equilibrium prices in the first economy. Now, it can be shown that $T^i(\{\xi_t'(z^t)\}) \leq T^i(\{\xi_t(z^t)\})$ if $\{\xi_t'(z^t)\} > \{\xi_t(z^t)\}$ (Lustig (2003). Finally, using the previous results:

$$T^1\left(\{\xi_t^{2,a}(z^t)^\star\}\right) < \{\xi_t^{2,a}(z^t)^\star\} \; and \; T^1(\{\xi_t^{1,a}(z^t)^\star\}) = \{\xi_t^{1,a}(z^t)^\star\},$$

we obtain that $\{\xi_t^{1,a}\left(z^t\right)^\star\} < \{\xi_t^{2,a}\left(z^t\right)^\star\}$

q.e.d.

Proof of Corollary 1 Follows from the definition of the cutoff level in the previous proof. For a given sequence of $\{\xi_t^a(z^t)\}$, it is obvious that $\ell^{1,c}(y_t, z^t) < \ell^{2,c}(y_t, z^t)$ for all nodes (y_t, z^t). This in turn implies that $\{\xi_t^{a,1}(z^t)\} \leq \{\xi_t^{a,2}(z^t)\}$. This follows directly from the definition of the aggregate weight shock (25). As a result, $\xi_t^a(z^t)$ is non-decreasing in $\ell^c(y_t, z^t)$. This implies the state prices at time 0 for consumption to be delivered in s^t are higher, and this is true for all nodes s^t.

Interest rates between time zero and time t are given by $R_{0,t}^f = E_0[M_{0,t}]^{-1}$, where the pricing kernel between time 0 and time t is $M_{0,t} = m_0 \cdot m_1 \cdots m_t$. A lower aggregate weight shock $\frac{\xi_t^a}{\xi_0^a}$ at time t *in all nodes* s^t implies a lower pricing kernel on average and higher interest rates on average. q.e.d.

Proof of Proposition 3 The proof follows Alvarez and Jermann (2000). Appendix A.3 explains the relationship between the static and sequential budget constraints and solvency constraints.

Derivation of Value Premium

$$\begin{aligned} 1+\nu_0 = 1+E_0[R^e_{0,1}[\{c_k\}]] &= E_0 M_1 E_0 \left(\frac{\sum_{k=1}^{\infty} E_1 M_k c_k}{\sum_{k=1}^{\infty} E_0 M_k c_k} \right) = \sum_{k=1}^{\infty} \frac{E_0 M_k c_k}{\sum_{k=1}^{\infty} E_0 M_k c_k} \frac{\frac{E_1 M_k c_k}{E_0 M_k c_k}}{\frac{1}{E_0 M_1}} \\ &= \sum_{k=1}^{\infty} \omega_k \frac{E_0 R_{0,1}\left[c_k\right]}{R_{0,1}\left[1\right]} = \sum_{k=1}^{\infty} \omega_k E_0 R^e_{0,1}\left[c_k\right], \end{aligned}$$

with weights

$$\omega_k = \frac{E_0 M_k c_k}{\sum_{k=1}^{\infty} E_0 M_k c_k}.$$

Proof of Proposition 4 Following the definition of Alvarez and Jermann (2001b), the pricing kernel M has no permanent component if

$$\lim_{k\to\infty} \frac{E_{t+1} M_{t+k}}{E_t M_{t+k}} = 1.$$

We focus on a slightly different condition:

$$\lim_{k\to\infty} \frac{E_{t+1} M_{t+k} c_{t+k}}{E_t M_{t+k} c_{t+k}} = 1.$$

Let the one period holding return on a period-k consumption strip be given by:

$$R^c_{t+1,k} = \frac{M_t}{M_{t+1}} \frac{E_{t+1} M_{t+k} c_{t+k}}{E_t M_{t+k} c_{t+k}},$$

then we know, from the derivation above, that

$$\lim_{k\to\infty} R^c_{t+1,k} = \frac{M_t}{M_{t+1}}.$$

Furthermore, for any return $E_t[\frac{M_{t+1}}{M_t} R_{t+1}] = 1$, we know that $E_t[\log(\frac{M_{t+1}}{M_t} R_{t+1})] \leq \log E_t[\frac{M_{t+1}}{M_t} R_{t+1}] = 0$ by Jensen's inequality. This implies that $E_t \log(\frac{M_t}{M_{t+1}}) \geq E_t \log(R_{t+1})$ or

$$E_t \log \lim_{k\to\infty} R^c_{t+1,k} = \log \frac{M_t}{M_{t+1}} \geq E_t \log(R_{t+1}) \text{ for any asset return } R_{t+1}$$

This implies that the expected log excess return exceeds that any other asset:

$$E_t \log \lim_{k\to\infty} \frac{R^c_{t+1,k}}{R_{t+1,1}} \geq E_t \log \frac{(R_{t+1})}{R_{t+1,1}}$$

Let $f(k) = Ce^{ak}$ with $a > 0$ for growth stocks. In the absence of a permanent component in the pricing kernel:

$$\lim_{a\to\infty} 1 + \widetilde{\nu}_0 = \lim_{k\to\infty} R^c_{t+1,k} \geq 1 + \nu_0 \text{ for any other sequence of weights } \{\omega_k\}$$

This implies that the highest equity premium is the one on the farthest out consumption strip. In the absence of a permanent component in the pricing kernel, there is a growth premium. q.e.d.

A.3. Sequential versus Time-Zero Constraints

We show under which conditions the sequence of budget constraints and collateral constraints in the sequential market setup can be rewritten as one time-zero budget constraint and a collection of solvency constraints, one for each node s^t. The proof strategy follows Sargent (1984) (Chapter 8).

Budget Constraint First, we show how the Arrow-Debreu budget constraint obtains from aggregating successive sequential budget constraints.

Let Π_{s^t} be the value of a dividend stream $\{d\}$ starting in history s^t priced using the market state prices $\{p\}$:

$$\Pi_{s^t}\left[\{d\}\right] = \sum_{j\geq 0}\sum_{s^{t+j}|s^t} p_{t+j}(s^{t+j})d_{t+j}(s^{t+j}),$$

where for a given path s^{t+j} following history s^t, p is defined as

$$p_{t+j}(s^{t+j}|s^t) = q_{t+j}\left(s^{t+j}|s^{t+j-1}\right)q_{t+2}(s^{t+2}|s^{t+1})...q_{t+1}(s^{t+1}|s^t).$$

Let $\{\widetilde{\eta}\}$ be the largest possible labor income stream.

Assumption 1. *Interest rates are sufficiently high: The value of a claim to the largest possible labor income stream at time 0 is finite:* $\Pi_{s^0}\left[\{\widetilde{\eta}\}\right] < \infty$*,*

The sequential budget constraint is:

$$c_t(\ell, s^t) + \rho_t(z^t)h^r_t(\ell, s^t) + \sum_{s'} q_t(s^t, s')a_t(\ell, s^t, s') + p^h_t(s^t)h^o_{t+1}(\ell, s^t) \leq W_t(\ell, s^t).$$

Next period wealth is:

$$W_{t+1}(\ell, s^t, s') = \eta_{t+1}(s^t, s') + a_t(\ell, s^t, s') + h^o_{t+1}(\ell, s^t)\left[p^h_{t+1}(s^t, s') + \rho_{t+1}(s^t, s')\right].$$

Multiply the second equation by $q_{t+1}(s')$ and sum over states. Then substitute the expression for $\sum q_{t+1}(s')a_{t+1}(s')$ into the first equation.

$$c_t + \rho_t h^r_t + \sum_{s'} q_{t+1}(s')W_{t+1}(s') \leqslant W_t + \sum_{s'} q_{t+1}(s')\eta_{t+1}(s') + h^o_{t+1}\left(\sum_{s'} q_{t+1}(s')\left[p^h_{t+1}(s') + \rho_{t+1}(s')\right] - p^h_t\right).$$

Similarly, for period $t+1$:

$$c_{t+1} + \rho_{t+1} h^r_{t+1} + \sum_{s''} q_{t+2}(s'') W_{t+2}(s'') \leqslant W_{t+1} + \sum_{s''} q_{t+2}(s'')\eta_{t+2}(s'') + h^o_{t+2} \left(\sum_{s''} q_{t+2}(s'') \left[p^h_{t+2}(s'') + \rho_{t+2}(s'') \right] - p^h_{t+1} \right).$$

Substituting the expression for $t+1$ into the expression for t by substituting out W_{t+1}, we get:

$$c_t + \rho_t h^r_t + \sum_{s'} q_{t+1}(s') \left[c_{t+1} + \rho_{t+1} h^r_{t+1} \right] + \sum_{s'} \sum_{s''} q_{t+1}(s') q_{t+2}(s'') W_{t+2}(s'') \leqslant$$

$$W_t + \sum_{s'} q_{t+1}(s')\eta_{t+1}(s') + \sum_{s'} \sum_{s''} q_{t+1}(s') q_{t+2}(s'')\eta_{t+2}(s'') + h^o_{t+1} \left(\sum_{s'} q_{t+1}(s') \left[p^h_{t+1}(s') + \rho_{t+1}(s') \right] - p^h_t \right) +$$

$$\sum_{s'} q_{t+1}(s') h^o_{t+2}(s') \left(\sum_{s''} q_{t+2}(s'') \left[p^h_{t+2}(s'') + \rho_{t+2}(s'') \right] - p^h_{t+1} \right).$$

Repeating these substitutions, we obtain the following inequality at time t:

$$\Pi_{s^t} \left[\{ c + \rho h^r \} \right] \leqslant W_t - \eta_t + \Pi_{s^t} \left[\{ \eta \} \right], \tag{26}$$

where we have used: (1) the transversality condition

$$\lim_{j \to \infty} \sum_{s^{t+j}} p_{t+j}(s^{t+j}) W_{t+j}(s^{t+j}) = 0, \tag{27}$$

and (2) a no-arbitrage condition:

$$p^h_{t+j-1}(s^{t+j-1}) = \sum_{s^{t+j} | s^{t+j-1}} q_{t+j}(s^{t+j}) \left[p^h_{t+j}(s^{t+j}) + \rho_{t+j}(s^{t+j}) \right], \qquad \forall j \geq 0, \forall s^{t+j} \tag{28}$$

If the latter condition were not satisfied, a household could achieve unbounded consumption by investing sufficiently high amounts in housing shares h^o and financing this by borrowing. This is a feasible strategy because ownership shares in the housing tree are collateralizable.

Because $W_0 = \eta_0 + \theta_0$, and relabelling $h^r_t = h_t$, we recover from equation (26) the Arrow-Debreu budget constraint at time 0:

$$\Pi_{s^0} \left[\{ c + \rho h \} \right] \leqslant \theta_0 + \Pi_{s^0} \left[\{ \eta \} \right],$$

where we have used the assumption that interest rates are sufficiently high (see Assumption 1). This implies that the AD budget constraint is satisfied, if the sequential budget constraints are satisfied.

Collateral Constraints Second, we show the equivalence between the collateral constraints of the sequential markets setup and the solvency constraint in the static economy. The sequential collateral constraints are:

$$\left[p^h_t(z^t) + \rho_t(z^t) \right] h^o_{t-1}(s^{t-1}) + a_{t-1}(s^{t-1}, s_t) \geq 0,$$

and the collateral constraints in a history s^t:

$$\Pi_{s^t}\left[\{c+\rho h\}\right] \geq \Pi_{s^t}\left[\{\eta\}\right]. \tag{29}$$

The equivalence follows if and only if

$$a_{t-1}(s^{t-1}, s_t) + h^o_{t-1}(s^{t-1})\left[p^h_t(z^t) + \rho_t(z^t)\right] = \Pi_{s^t}\left[\{c+\rho h-\eta\}\right].$$

But this follows immediately from the budget constraint (26) holding with equality and the definition of W:

$$W_t(s^t) - \eta_t(s) = a_{t-1}(s^{t-1}, s_t) + h^o_{t-1}(s^{t-1})\left[p^h_t(z^t) + \rho_t(z^t)\right].$$

Under conditions (27) and (28) an allocation that is feasible and immune to the threat of default in sequential markets is feasible and immune to the threat of default in time-zero markets.

The equivalence implies that the allocation of home-ownership h^o is indeterminate in the sequential economy.

A.4. Data Appendix

We use two sets of variables: financial variables and aggregate macroeconomic variables. All variables are annual and for the United States.

Financial Data

Aggregate Dividends Aggregate dividends are the dividends on the Standard and Poor's composite stock price index. The data are available for the period 1889-2003 from Robert Shiller's web site. The standard deviation of real dividend growth is .14 for 1930-2003.

Market Return We also include the market return R^{vw}, the value-weighted return on all NYSE, AMEX and NASDAQ stocks.

Size and Book-to-Market Portfolios We use ten portfolios of NYSE, NASDAQ and AMEX stocks, grouped each year into ten value (book-to-market ratio) bins. Book-to-market is book equity at the end of the prior fiscal year divided by the market value of equity in December of the prior year. Portfolio returns are value-weighted. All returns are expressed in excess of an annual return on a one-month Treasury bill rate (from Ibbotson Associates). The returns are available for the period 1926-2002 from Kenneth French's web site and are described in more detail in Fama and French (1992).

Aggregate Macroeconomic Data

Consumption and Income Consumption is non-durable consumption C, measured by total expenditures minus apparel and minus rent and imputed rent. The housing expenditure ratio, r, is the ratio of non-durable expenditures to rent expenditures.

The income endowment in the model corresponds to an after-government income concept; it includes net transfer income. Aggregate income Y is labor income plus net transfer income. Nominal data are from the National Income and Product Accounts for 1930-2002. Consumption and income are deflated by the consumer price index and divided by the number of households N.

Price Indices Aggregate rental prices ρ_t are constructed as the ratio of the CPI rent component p_t^h and the CPI food component p_t^c. Data are for urban consumers from the Bureau of Labor Statistics for 1926-2001. The price of rent is a proxy for the price of shelter and the price of food is a proxy for the price of non-durables. We use the rent and food components because the shelter and non-durables components are only available from 1967 onwards. Two-thirds of consumer expenditures on shelter consists of owner-occupied housing. The BLS uses a rental equivalence approach to impute the price of owner-occupied housing. Because ρ_t is a relative *rental* price, our theory is conceptually consistent with the BLS approach. We also use the all items CPI, p_t^a, which goes back to 1889. All indices are normalized to 100 for the period 1982-84.

Housing Collateral We use three distinct measures of the housing collateral stock HV: the value of outstanding home mortgages (mo), the market value of residential real estate wealth (rw) and the net stock current cost value of owner-occupied and tenant occupied residential fixed assets (fa). The first two time series are from the Historical Statistics for the US (Bureau of the Census) for the period 1889-1945 and from the Flow of Funds data (Federal Board of Governors) for 1945-2001. The last series is from the Fixed Asset Tables (Bureau of Economic Analysis) for 1925-2001.

We use both the value of mortgages HV^{mo} and the total value of residential fixed assets HV^{rw} to be robust to changes in the extent to which housing can be used as a collateral asset. We use both HV^{rw}, which is a measure of the value of housing owned by households, and HV^{fa} which is a measure of the value of housing households live in, to be robust to changes in the home-ownership rate over time. Real per household variables are denoted by lower case letters. The real, per household housing collateral series $hv^{mo}, hv^{rw}, hv^{fa}$ are constructed using the all items CPI from the BLS, p^a, and the total number of households, N, from the Bureau of the Census.

Housing Collateral Ratio Log, real, per household real estate wealth ($\log hv$) and labor income plus transfers ($\log y$) are non-stationary. According to an augmented Dickey-Fuller test, the null hypothesis of a unit root cannot be rejected at the 1 percent level. This is true for all three measures of housing wealth ($hv = mo, rw, fa$).

If a linear combination of $\log hv$ and $\log y$, $\log(hv_t) + \varpi \log(y_t) + \chi$, is trend stationary, the components $\log hv$ and $\log y$ are said to be stochastically cointegrated with cointegrating vector $[1, \varpi, \chi]$. We additionally impose the restriction that the cointegrating vector eliminates the deterministic trends, so that $\log(hv_t) + \varpi \log(y_t) + \vartheta t + \chi$ is stationary. A likelihood-ratio test (Johansen and Juselius (1990)) shows that the null hypothesis of no cointegration relationship can be rejected, whereas the null hypothesis of one cointegration relationship cannot. This is evidence for one cointegration relationship between housing collateral and labor income plus transfers. We estimate the cointegration coefficients from vector error correction model:

$$\begin{bmatrix} \Delta \log (hv_t) \\ \Delta \log (y_t) \end{bmatrix} = \alpha \left[\log (hv_t) + \varpi \log (y_t) + \vartheta t + \chi\right] + \sum_{k=1}^{K} D_k \begin{bmatrix} \Delta \log (hv_{t-k}) \\ \Delta \log (y_{t-k}) \end{bmatrix} + \varepsilon_t. \tag{30}$$

The K error correction terms are included to eliminate the effect of regressor endogeneity on the distribution of the least squares estimators of $[1, \varpi, \vartheta, \chi]$. The housing collateral ratio my is measured as the cointegration relationship:

$$my_t = \log (hv_t) + \hat{\varpi} \log (y_t) + \hat{\vartheta} t + \hat{\chi}.$$

The OLS estimators of the cointegration parameters are superconsistent: They converge to their true value at rate $1/T$ (rather than $1/\sqrt{T}$). The superconsistency allows us to use the housing collateral ratio my as a regressor without need for an errors-in-variables standard error correction. For more details, see Lustig and VanNieuwerburgh (2004b).