Discussion

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Classifying asset-pricing models is a treacherous venture, because many seemingly distinct modelling frameworks are intricately related. However, for this discussion, I will put all financial asset-pricing models into two broad categories: equilibrium models and arbitrage-based models.

Carmichael's paper focuses on the first group of models. Lucas's seminal 1978 article laid the foundation of consumption-based or dynamic-equilibrium models, and his 1982 article extended the model to a two-country setting. Other major contributors in this area include: Merton (1973); Breeden (1979); and Cox, Ingersoll, and Ross (1985).

A distinctive feature of consumption-based models is that the agent's risk-preference or utility functions must be specified. Once a utility function is in place, maximizing the intertemporal expected utility leads to the usual first-order conditions, the so-called Euler equations, which ultimately govern the prices of financial assets. Insofar as Euler equations are the source of asset price dynamics, specifying the agent's utility function is crucial. In contrast, arbitrage-based models are free of investors' risk preferences.

In the remainder of this discussion, I will mainly review arbitrage-based models to complement Carmichael's paper. The review will discuss the framework of arbitrage pricing as it is applied to derivative financial assets. In the spirit of the conference theme, I will try to relate the discussions to the market's information structures, and compare the two groups of models whenever possible and appropriate. Most of the materials are drawn from Dothan (1990), Huang and Litzenberger (1988), Ingersoll (1987), Duffie (1996), Neftci (1996), and Pliska (1997).

Background

The arbitrage-pricing theory (APT) was pioneered by Ross (1976). Unlike the capital-asset-pricing model (CAPM), credited to Sharpe (1964) and Lintner (1965), the APT requires no assumptions about investors' preferences or return distributions. It builds on the concepts of diversification and the absence of arbitrage in equilibrium. It is similar to CAPM, in that both models treat idiosyncratic risks as diversifiable and, hence, do not assign a risk premium to them.

However, the two models differ in the way they treat systematic risk. Under CAPM, the risk premium is quantified through two-fund separation and utility maximization, whereas under APT, it is based on the principle of no arbitrage. Since the two models were developed to value primary financial assets such as stocks, and since APT itself does not specify what the systematic risk factors are, CAPM has historically won the favour of practitioners and most academics.

APT is more powerful than CAPM for recovering information from securities returns, in that it allows an unlimited number of systematic risk factors to affect returns. But this also becomes its chief weakness: researchers must assign economic meanings to the systematic risk factors recovered from securities returns. As far as pricing primary securities are concerned, APT is handicapped by its own merits. However, as history has proven, the principle of no arbitrage, when applied to valuing derivative financial securities, is not only powerful but also readily feasible. The key to its success, of course, lies in relative pricing.

Before we explore the deeper world of arbitrage pricing, we must introduce another key element, namely, stochastic processes, the building block for financial modelling. The most widely used process is perhaps the "Brownian motion," named after the British botanist Robert Brown. But the actual development was due to Bachelier (1900) and Einstein (1905) and other mathematicians in the first half of the century. Although both Bachelier (1900) and Samuelson (1965) applied the tools to options pricing, with the former using Brownian motion and the latter geometric Brownian motion, the early work of Merton (1969 and 1971) planted the seeds to grow the modern applications. The apparatus in stochastic calculus and modern probability theory, together with the principle of no arbitrage, enable researchers to conveniently value derivative securities while maintaining a realistic information structure.

Black and Scholes (1973) derived their celebrated options-pricing formula by solving a partial differential equation that is independent of the on expected return on the underlying asset. Three years later, Cox and Ross (1976) derived an important implication from the Black–Scholes differential equation: Since the expected return of the underlying asset does not enter into the pricing, one can assume that the investors are risk-neutral, hence, the expected return is the risk-free rate. This observation was formalized by Harrison and Kreps (1979), who showed that risk-neutral valuation is equivalent to arbitrage-free pricing, and in this system all prices are Martingales upon proper normalization and change of probability measure.

In the next section, I will survey the key elements of arbitrage pricing for derivative securities. We will start with a discrete setting and then move on to a continuous setting.

Arbitrage Pricing in a Single-Period Model

The economic meaning of arbitrage-free

A single period model can be specified as:

- initial time $t = 0$
- terminal time $t = 1$
- trading is possible only at $t = 0, 1$
- sample space is finite: $\Omega = \{\omega_1, \omega_2, \omega_3, \ldots, \omega_K\}$ $(K < \infty)$
- there exists a probability measure P on Ω, with $P(\omega) > 0, \forall \omega \in \Omega$
- a money-market account $B = \{B_t : t = 0, 1\}$, with $B_0 = \$1$
- a price process $S = \{S_t : t = 0, 1\}$ for N risky securities where $S_t = (S_1(t), S_2(t), S_3(t), \ldots, S_N(t))$.

Definition

Trading strategy: $H = (H_0, H_1, H_2, \ldots, H_N)$ where H_0 is number of dollars in a money-market account and H_n is number of shares in security $n (n = 1, 2, 3, \ldots, N)$.

Value process: $V = \{V_t : t = 0, 1\}$ where

$$V_t = H_0 B_t + \sum_{n=1}^{N} H_n S_n(t), \quad t = 0, 1 .$$

Gains process:

$$G = H_0(B_t - 1) + \sum_{n=1}^{N} H_n \Delta S_n$$

where $(B_t - 1)$ is the interest rate, and $\Delta S_n = S_n(1) - S_n(0)$. Also, $V_1 = V_0 + G$.

Discounted price process:

$S^* = \{S_t^* : t = 0, 1\}$ where $S_t^* = (S_1^*(t), S_2^*(t), S_3^*(t), \ldots, S_N^*(t))$ and $S_n^*(t) \equiv S_n(t)/B_t$, $n = 1, 2, 3, \ldots, N$; $t = 0, 1$. B_t is the numeraire.

For any modelling to be sensible, the resulting price system must comply with economic rationale. For instance, if there are two trading strategies H^A and H^B such that $V_1^A(\omega) = V_1^B(\omega) \forall \omega \in \Omega$ and yet $V_0^A > V_0^B$, then the law of one price is violated. This is the least desirable system because there is ambiguity of prices—portfolios with identical payoffs can have differing current values, a result flying in the face of economic reality. When the law of one price does not hold, an investor will be able to obtain a sure amount of trading profits today with zero net investment.

Will the prevalence of the law of one price guarantee a sensible price system? Hardly. The law of one price merely ensures that trading strategies with the same terminal values have the same current values. However, two trading strategies with differing terminal values *may* have the same current values. For instance, if there exists H^A and H^B such that $V_0^A = V_0^B$, but $V_1^A(\omega) > V_1^B(\omega) \forall \omega \in \Omega$, then the law of one price is not violated, but the price system is not sensible. In this case, there are "dominant trading strategies" that allow investors to guarantee a strictly positive-dollar return with zero net investments (by shorting with strategy H^B and going long with strategy H^A). Equivalently, a dominant trading strategy can also be characterized as a strategy where $V_0 < 0$, and $V_1(\omega) \geq 0 \ \forall \omega \in \Omega$.

A securities-market model allowing a dominant trading strategy to exist will not be realistic, because it leads to illogical prices. When dominant trading strategies exist, an investor can guarantee a positive-dollar return with zero net investments; but the investor is no longer sure of the dollar amount of returns, unlike when the law of one price is broken. Depending on which state of the world is realized, the dollar return could be high or low, but it is always positive. So, the absence of dominant trading strategies is a more stringent condition than is the prevalence of the law of one price.

Perhaps the condition of no dominant trading strategy would lead to a sensible securities-market model. Unfortunately, this is not so. A tighter condition is needed to ensure stable, equilibrium securities prices, and this condition is the absence of arbitrage opportunities. Formally, an arbitrage opportunity is a trading strategy H, such that $V_0 = 0$, $V_1 \geq 0$ and $E(V_1) > 0$. Here, an investor can expect to achieve a positive-dollar return with zero investment, and he may end up with a zero return. An arbitrage opportunity is still riskless because it requires zero investment and cannot bring the investor into debt.

It becomes clear to this point that the three conditions are progressively more stringent. A violation of the law of one price implies the existence of dominant trading strategies, which in turn implies arbitrage opportunities. The reverse is not necessarily true. The set of securities prices that are compatible with the absence of arbitrage is the smallest. Only those prices are sensible prices. Therefore, any sensible pricing model must ensure the absence of arbitrage opportunities. How does one verify that a model is free of arbitrage opportunities?

Arbitrage-free versus risk-neutral probability measures

It turns out that there is a close link between the absence of arbitrage opportunities and the existence of a risk-neutral probability measure.

> Definition: Probability measure Q is a risk-neutral probability measure if:
>
> $Q(\omega) > 0, \forall \omega \in \Omega$ and $E^Q[S_n^*(t)] = 0 \ (n = 1, 2, 3, \ldots, N)$.

Obviously, a probability measure should satisfy $Q(\omega) > 0, \forall \omega \in \Omega$ and $\sum_{k=1}^{K} Q(\omega_k) = 1$. The term *risk-neutral* refers to $E^Q[S_n^*(t)] = 0$, which says the discounted prices are Martingales under Q or, equivalently, the current price of a security is the expected future price discounted at the risk-free rate. An important result:

> A pricing model is arbitrage-free only if there exists a risk-neutral probability measure, Q. (1)

Because the condition is both necessary and sufficient, the existence of a risk-neutral probability measure would fully guarantee the absence of arbitrage opportunities. To understand the above, let's look at an example. Suppose there are two risky securities besides the risk-free asset, which are priced at $S_1(0)$ and $S_2(0)$. There are three states, ω_1, ω_2, and ω_3. The payoffs are:

State	Money market account	$S_1(1)$	$S_2(1)$
ω_1	$1+r$	$S_1(1)(\omega_1)$	$S_2(1)(\omega_1)$
ω_2	$1+r$	$S_1(1)(\omega_2)$	$S_2(1)(\omega_2)$
ω_3	$1+r$	$S_1(1)(\omega_3)$	$S_2(1)(\omega_3)$

If the Arrow–Debreu prices are ψ_1, ψ_2, and ψ_3, then:

$$1 = (1+r)\psi_1 + (1+r)\psi_2 + (1+r)\psi_3$$

$$S_1(0) = S_1(1)(\omega_1)\psi_1 + S_1(1)(\omega_2)\psi_2 + S_1(1)(\omega_3)\psi_3$$

$$S_2(0) = S_2(1)(\omega_1)\psi_1 + S_2(1)(\omega_2)\psi_2 + S_2(1)(\omega_3)\psi_3 . \qquad (2)$$

Define $Q(\omega_k) \equiv (1+r)\psi_k \ (k = 1, 2, 3)$. Then the above equation system becomes:

$$1 = Q(\omega_1) + Q(\omega_2) + Q(\omega_3) .$$

$$S_1(0) = [S_1(1)(\omega_1)Q(\omega_1) + S_1(1)(\omega_2)Q(\omega_2)$$

$$+ S_3(1)(\omega_3)Q(\omega_3)]/1 + r . \qquad (3)$$

$$S_2(0) = [S_2(1)(\omega_1)Q(\omega_1) + S_2(1)(\omega_2)Q(\omega_2) \qquad (4)$$

$$+ S_2(1)(\omega_3)Q(\omega_3)]/(1+r) . \qquad (5)$$

Clearly, $Q(\omega)$ can be interpreted as probabilities, and the requirement that discounted prices are Martingales is evidently satisfied. When the number of states equals the number of securities (as in this example), the system always yields solutions. For example, for:

$r = 0.2, S_1(1) = \{2, 3, 6\}, S_2(1) = \{4, 9, 6\}, S_1(0) = 2.5$, and

$S_2(0) = 5.0$,

we have

$Q(\omega_1) = 1/2, Q(\omega_2) = 1/3$, and $Q(\omega_3) = 1/6$.

Unfortunately there is no guarantee that $Q(\omega) > 0, \forall \omega \in \Omega$. In this case a risk-neutral probability measure does not exist and arbitrage exists. For instance, for:

$r = 0.2, S_1(1) = \{2, 3, 8\}, S_2(1) = \{4, 8, 6\}, S_1(0) = 5$, and

$S_2(0) = 7.5$,

we have

$$Q(\omega_1) = -1/2,\ Q(\omega_2) = 1,\ \text{and}\ Q(\omega_3) = 1/2.$$

These solutions are not probabilities and arbitrage exists. (One arbitrage-trading strategy would be to short one unit of each risk security and invest the proceeds of $12.50 into the risk-free asset to yield $15. At time 1, the maximum short-position liability is $8 + $6 = $14, and a positive return of $1 is guaranteed without initial investment.) The negative probability corresponds to a negative Arrow–Debreu state price.

Note that the risk-neutral probability measure does not have to be unique to exclude arbitrage opportunities. As long as at least one viable measure can be identified, then arbitrage is excluded. To continue the above example, if we modify the payoff and price of the second risky security to

$$S_2(1) = \{4, 6, 16\}\ \text{and}\ S_2(0) = 10,$$

then we have a set of risk-neutral probability measures:

$$Q = \{5\lambda - 3, 4 - 6\lambda, 2/3 > \lambda > 3/5\}.$$

In this case, the second risky security is redundant and there is no arbitrage opportunity. Of course, if the number of securities is bigger than the number of states, a risk-neutral probability measure certainly does not exist, unless there are redundant securities.

The formal proof of the equivalence between the existence of risk-neutral probability measures and the absence of arbitrage is a bit complicated. But we could at least summarize matters so far by identifying three possible situations for any pricing model: (i) a risk-neutral probability measure does not exist (either because there are more independent securities than there are states or because there are negative Arrow–Debreu prices); (ii) there are more than one risk-neutral probability measures; and (iii) there is a unique risk-neutral probability measure.

According to the result in (1), arbitrage exists only under (i). Here is an intuitive explanation. When there are more independent securities than there are states, a set of Arrow–Debreu prices will fail to fit all securities prices—state prices can be security-specific. But this contradicts the very definition of state prices, which should be the same for all securities. The discrepancy in state price for a particular state between two securities is the source of arbitrage.

How about negative Arrow–Debreu prices? This is an easy one. Any possible positive payoff should command a positive price per unit of payoff. A negative price implies that an investor is paid up front, and at the same time is granted another possible payment in the future. This is of course a money machine, or a source of arbitrage.

Contingent-claims pricing and market completeness

We have established that as long as a risk-neutral probability measure exists, there will be no arbitrage opportunities. No arbitrage would be associated with “reasonable” prices. Does this mean that contingent-claims pricing can be accomplished once a risk-neutral probability measure is identified? Before we answer the question, let me clarify that the price of any contingent claim, if it exists at all, should simply be:

$$V = \frac{1}{1+r} \sum_{k=1}^{K} G(\omega_k) Q(\omega_k), \tag{6}$$

where $G(\omega_k)$ $(k = 1, 2, \ldots, K)$ are payoffs of the claim. In light of (2) and (3), the above amounts to weighing the payoffs by Arrow–Debreu prices, which is how any security should be priced. To continue the first example following (3), suppose a call option with a strike price of $5 is written on the second risky security. The payoff is $G(\omega_k) = \{0, 4, 1\}$, and by (4) the call price is $1.25.

Now I will return to the previous question. It turns out that once a risk-neutral probability measure is identified, some value can always be found for a contingent claim, but it is uncertain if this value is unique. To continue the previous example, in which infinitely many risk-neutral probability measures are found,

$$Q = \{5\lambda - 3, 4 - 6\lambda, 2/3 > \lambda > 3/5\},$$

if a call with a strike price of $6 is written on the second security with payoff $S_2(1) = \{4, 6, 16\}$, then the call option’s payoff is $G(\omega) = \{0, 0, 10\}$, and by (4) the price of the call is $10\lambda/1.2$. Since $2/3 > \lambda > 3/5$, all that is known about the call’s value is its lower and upper bounds (5, 50/9). The price is not unique.

It becomes evident that absence of arbitrage is not sufficient for us to obtain a unique value of a contingent claim. For a contingent claim to have a unique value, it must be “marketable” or “attainable.” A contingent claim is marketable if its payoff can be replicated by a trading strategy H, or a replicating portfolio. When marketable, the contingent claim has a value equal to that of the replicating portfolio, and this value is unique under many potentially risk-neutral probability measures. To understand this result, suppose a model consists of a money-market account and a risky security, and there are three possible states. Then there will be an infinite number of risk-neutral probability measures, as I showed before. Suppose a contingent claim has a payoff $G(\omega) = \{X_1, X_2, X_3\}$ and the underlying security's current price is $S(0)$ with payoffs $S(\omega) = \{S_1, S_2, S_3\}$. The replicating portfolio is $H = \{H_0, H_1\}$. Then by (4), the value of the claim is:

$$\begin{aligned}
&\frac{1}{1+r}(X_1Q_1 + X_2Q_2 + X_3Q_3) \\
&= \frac{1}{1+r}[(H_0(1+r) + S_1H_1)Q_1 + (H_0(1+r) + S_2H_1)Q_2 \\
&\quad + (H_0(1+r) + S_3H_1)Q_3] \\
&= H_0 + \frac{(S_1Q_1 + S_2Q_2 + S_3Q_3)}{1+r}H_1 \\
&= H_0 + S(0)H_1 .
\end{aligned}$$

Clearly, when the contingent claim is marketable, its value is unique and independent of the many risk-neutral probability measures. This is the heart of valuation by replication.

Within a particular model, some contingent claims are marketable and some are not. When every contingent claim is marketable, then the market is said to be "complete."

> The market is complete only if the number of possible states equals the number of independent securities. Alternatively, the market is complete only if the risk-neutral probability measure is unique. (7)

At this point, I will summarize for single-period, discrete-time pricing: (i) in a complete market, arbitrage does not exist and all contingent claims can be valued; and (ii) in an incomplete market where multiple risk-neutral probability measures prevail, arbitrage does not exist, and only marketable or attainable contingent claims can be valued; prices of unmarketable claims are bound within a known range.

Remarkably, the above results also hold for multi-period, discrete-time models and continuous-time models. The generalization to multiple-period models does not involve new concepts, and will be omitted for brevity. Generalization to the continuous case requires some additional apparatus. Owing to space limits, I will only highlight the essence of continuous-time arbitrage-free pricing.

Arbitrage Pricing in a Continuous-Time Model

In a continuous-time model, information is revealed through a "filtration," which is a growing area of sub-σ-algebra. If the stochastic processes modelling the economic variables are the sole source of information, then we say the processes are "adapted" to the filtration, meaning that investors always know the current and past prices of all securities concerned.

Many other new concepts are required to fully discuss continuous-time models, two of which are "self-financing" and "admissible" trading strategies. We define them only loosely.

> A trading strategy is self-financing if, after initialization, are neither additional funds invested into it nor are funds withdrawn from it.

Harrison and Kreps (1979) showed that if none of the self-financing trading strategies in a model leads to arbitrage opportunities, then there exists a probability measure Q equivalent to P such that all discounted price processes are Q-Martingales. This is the origin of risk-neutral valuation and valuation by replication in a continuous-time setting. But the picture is not complete yet.

> A self-financing trading strategy is admissible if the process for gains and losses is a Q-Martingale.

It can be shown that when all the trading strategies are admissible, there are no arbitrage opportunities. As in the discrete case, a contingent claim is said to be marketable or attainable if it can be dynamically replicated by an admissible trading strategy.

When all trading strategies are admissible, arbitrage opportunities are absent and an equivalent probability measure Q exists. A contingent claim, if marketable, can be valued by discounting its expected payoff under Q. This is valid because the value of a marketable contingent claim is also a Q-Martingale.

There is only one remaining question now: When is a contingent claim marketable? As in the discrete case, all contingent claims are marketable if the securities market is dynamically complete. Markets are dynamically complete if all Q-Martingales can be represented as stochastic integrals with respect to the Q-Martingale prices of the underlying risky securities. This is the "Martingale representation property." In most cases, diffusion processes, together with continuous trading, would ensure market completeness.

In summary, in a continuous-time model, a unique price for a contingent claim can be found if the model is dynamically complete. In that event, the unique price is the expected payoff of the claim under the equivalent Martingale measure Q, discounted at the risk-free rate.

Comparing Consumption-Based and Arbitrage-Based Pricing Models

It would be difficult to judge which type of model is superior. But I will make the following remarks.

First, as Carmichael pointed out, consumption-based models offer a unified framework; all securities are priced via a Euler equation, whereby all cash flows are discounted at the marginal rate of intertemporal substitution of consumption. As such, the models usually involve parameters of investors' consumption choices and risk preferences, which can be a source of difficulty. Most works in this area derive pricing results for a "representative agent." It is an open question as to how close investors' risk-preference profiles can be. If diverse risk preferences (or utility functions) are used, then obtaining a unique security price is nearly impossible. Again, as Carmichael shows, the commonly used isoelastic utility function and its variants perform poorly when fitting market data to the derived pricing models.

Second, it appears that consumption-based models are often used when the market is not yet complete. For example, most stochastic volatility models complete the market by specifying investor's risk preferences. Studies of this include Hull and White (1987), Johnson and Shannon (1987), Scott (1987), Wiggins (1987), Melino and Turnbull (1990), Stein and Stein (1991), Heston (1993), and Duan (1995). Duan's study is unique in that it combines the consumption-based approach with the Martingale-based approach in a GARCH[1] setting, and the final pricing equation is independent of the utility function parameter (but still dependent on the risk premium).[2] When the market is incomplete due to discontinuous information (or jumps), consumption-based models can also be used. The study by Naik and Lee (1990) is an example.

Third, and perhaps this is a biased personal observation, practitioners seem to prefer arbitrage-based models. This is especially true with term-structure modelling and the pricing of interest rate-derivative securities. For instance, the popular models by Heath, Jarrow, and Morton (1992) and Hull and White (1990) and their variants are all arbitrage-based. These models

1. Generalized autoregressive conditional heteroscedasticity.

2. Kallsen and Taqqu (1998) are able to complete the market for an autoregressive conditional heteroscedasticity (ARCH)-type setting by letting the process evolve continuously between two discrete ARCH times. Rubinstein (1976) and Brennan (1979) have shown that when particular combinations of consumption preference and distribution are assumed, risk-neutral pricing obtains.

are appealing partly because of the ease of estimation and the independence of risk-preference parameters.

It is remarkable that, by excluding arbitrage and ensuring market completeness, investors can agree on a unique price for a security. After all, different investors would value the same payoff contingency differently if their risk preferences differ. Of course, there is no contradiction. Investors' preferences are built into underlying securities prices already. Contingent claims in a complete market are redundant securities by definition, and they do not represent new consumption opportunities. To put it another way, contingent-claims pricing is relative pricing, which should be free of risk preferences.

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