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Abstract

Diffusion functions in term-structure models are measures of uncertainty about future price movements and are directly related to the risk associated with holding financial securities. Correct specification of diffusion functions is crucial in pricing options and other derivative securities. In contrast to the standard parametric two-factor models, we propose a non-parametric two-factor term-structure model that imposes no restrictions on the functional forms of the diffusion functions. Hence, this model allows for maximum flexibility when fitting diffusion functions into data. A non-parametric procedure is developed for estimating the diffusion functions, based on the discretely sampled observations. The convergence properties and the asymptotic distributions of the proposed non-parametric estimators of the diffusion functions with multivariate dimensions are also obtained. Based on U.S. data, the non-parametric prices of the bonds and bond options are computed and compared with those calculated under an alternative parametric model. The empirical results show that the non-parametric model generates significantly different prices for the derivative securities.

JEL classifications: C14, C22, G13.
Bank classification: Econometric and statistical methods; Market structure and pricing

Résumé

Les fonctions de diffusion utilisées dans les modèles relatifs à la structure des taux d'intérêt constituent des mesures de l'incertitude entourant l'évolution future des prix et sont directement liées au risque associé à la détention de titres financiers. La qualité de leur spécification est cruciale pour l'évaluation des options et des autres produits dérivés. Les auteurs recourent à un modèle non paramétrique à deux facteurs qui n'impose aucune restriction à la forme fonctionnelle de ces fonctions, contrairement aux modèles paramétriques à deux facteurs traditionnels. Ils disposent ainsi d'un maximum de souplesse aux fins de l'estimation des fonctions de diffusion à l'aide des données. La méthode non paramétrique que les auteurs ont mise au point pour estimer les fonctions de diffusion est fondée sur les observations tirées d'un échantillon discret. Les auteurs établissent aussi les propriétés de convergence et les distributions asymptotiques des estimateurs non paramétriques des fonctions de diffusion à plusieurs variables. Ils calculent les prix des obligations du Trésor américain et des options qui s'y rapportent au moyen de leur méthode non paramétrique et les comparent à ceux obtenus au moyen d'un modèle paramétrique. D'après les résultats empiriques qu'ils obtiennent, les prix des produits dérivés générés par les modèles paramétrique et non paramétrique diffèrent passablement.

Classifications JEL: C14, C22, G13.
Classification de la Banque: Méthodes économétriques et statistiques; Structure de marché et fixation des prix

Non-Technical Summary

Daily observations suggest that default-free bonds of different maturities have different prices. Economists and financial analysts have developed several theories to explain the relative prices. The pure expectations hypothesis asserts that forward rates are equal to expected future spot rates. Meanwhile, the liquidity premium hypothesis states that forward rates should exceed the corresponding expected future spot rates by a liquidity premium, which is required to compensate bondholders for greater capital risk inherent in long-term bonds.

On the basis of both the pure expectations hypothesis and the liquidity premium hypothesis, a two-factor term-structure model of interest rates is proposed by Brennan and Schwartz (1984). In that model, the current long-term rate of interest contains information about future dates of the spot rate. The short- and long-term interest rates are assumed to be exogeneous. The model attempts to explain only the intermediate portion of the yield curve in terms of its extremities (the short-end and long-end). The long rate and a spread (the difference of the short rate and long rate) are assumed to follow a joint stochastic process.

To describe the process, a drift function (conditional mean) and a diffusion function (conditional variance) need to be specified for the long rate and the spread. Empirical results suggest strong non-linearity in the diffusion terms. Thus, a variety of functional forms have been used in the literature, for example, linear and square root. It is important to note that the drift and diffusion functions cannot be assumed arbitrarily. In fact, the joint parameterizations of drift and diffusion functions imply specific forms for the marginal and transitional densities of the process that can be inferred from the data. (The marginal density provides the probability of observing a particular long rate [or spread] regardless of the value of the spread [or long rate]. The transitional density provides the probability of observing a particular combination of a long rate and a spread in a period given the combination observed in the previous period.)

Parameterizing with a particular functional form can lead to misspecification where the implied densities derived from the particular functional form do not match the observed distribution from the data. Recent research has employed the non-parametric approach to tackle the problem. A non-parametric approach starts with non-parametric estimates of the densities based on observed data and then constructs the drift and diffusion functions by matching the densities.

Until now, non-parametric estimation has focused on one-factor models. Our paper extends the non-parametric approach to two-factor models. We illustrate the approach by considering the particular two-factor model described above (a model with a long rate and a spread). Since prices of interest rate derivatives depend crucially on correct specification and estimation of diffusion functions, we compare call option prices on 5-year discount government bonds from the parametric model and the non-parametric model we develop here, and we find substantial differences.

The non-parametric two-factor model can be employed to analyze the effects of monetary policy actions on the term structure. The standard view of the transmission mechanism of monetary policy assigns a key role to medium- and long-term interest rates. According to this view, a monetary policy tightening pushes up both medium- and long-term interest rates, leading to less spending and slower growth. The empirical term-structure effects of monetary policy actions (control over the overnight rate and its volatility) can be estimated through the non-parametric technique developed in this paper.

1 Introduction

In the exchange-traded and over-the-counter markets, the volume of trading in the interest rate derivative securities has increased very quickly since the 1980s. Many new products were developed to meet particular needs of end users. To evaluate the securities, a variety of models and techniques have been developed. These models have characterized the term structure using either a single factor or multiple factors in a general-equilibrium or a partial-equilibrium framework. In a one-factor model, there is a single source of uncertainty driving the evolution of the yield curve. Examples of one-factor models include the models in Merton (1973), Vasicek (1977), Dothan (1978), Marsh and Rosenfeld (1983), Cox, Ingersoll, and Ross (1985) (CIR thereafter), Aït-Sahalia (1996), and Jiang and Knight (1997). However, a significant deficiency of one-factor models of the term structure is the unrealistic assumption about the stochastic process for the interest rate. A number of theoretical studies (for example, Stambaugh [1988], among others, argues that yields are driven by at least two risk factors[1]) and empirical evidence (notably Dybvig [1989] on the U.S. data and Steeley [1991] on the U.K. data) have concluded that the variability across rates of different maturities can better be explained by incorporating more than one source of uncertainty.

In view of the weakness of one-factor models, a number of authors have been proposing models of the term structure that incorporate two factors. Examples include the models in Richard (1978), Brennan and Schwartz (1979), Schaefer and Schwartz (1984), Longstaff and Schwartz (1992), Heath, Jarrow, and Morton (1992), Hsin (1995), and Duffie and Kan (1996). However, it should be noted that these models all rely on parametric specifications of

[1]Based on forward premiums with nonmatching maturities, the generalized methods latent-variable tests reject a CIR single-variable specification of the term structure for nominally default-free bonds and more than two risk factors are accepted.

the underlying processes and therefore may impose very strong restrictions on the diffusion process. There are two main reasons for these restrictions. First, in order to derive the exact pricing formula or the approximate analytical solution to the pricing equation, very specific assumptions about the nature of the continuous-time diffusion process must be presupposed. Second, from an econometric and statistical point of view, little progress has been made with the identification and estimation of a multivariate continuous-time diffusion process. Thus, most researchers have constrained their models to be simple in order to use the available estimation methods. It is worth noting that these restrictions on the diffusion functions can lead to serious misspecification even though they can constitute a convenient and simple formula to price financial assets.

In this paper, as in Schaefer and Schwartz (1984) and Bühler, Uhrig-Homburg, Walter, and Weber (1998), we express our model in terms of a long rate and the spread between the long rate and a short rate. Because the precise forms of the diffusion functions of the two factors are crucial to price derivative securities, and it is impossible to form a prior idea of the functional form of the diffusion functions, we impose no restrictions on the functional forms of the diffusion functions. Hence, the model allows for maximum flexibility in diffusion functions. To achieve identification and estimation, the drift term of spread is specified as a mean-reverting function while we leave the drift term of the long rate process unrestricted. Estimation of the diffusion functions is based on a non-parametric estimation procedure. Under regularity conditions, we prove that the estimators of diffusion functions have a standard asymptotic behaviour. The estimation of parameters in the drift term of the spread utilize a semi-parametric procedure to correct heteroskedasticity in the residuals from the regression. Prices of interest rate derivative securities based on the non-parametric model are investigated through comparison with the alternative parametric model in Schaefer and Schwartz (1984). The empirical results show that the non-parametric model generates significantly different prices of interest rate derivative securities.

This paper is organized as follows. In Section 2, we present a non-parametric two-factor term-structure model for pricing derivative securities. The non-parametric estimators of the diffusion functions are proposed in Section 3, without imposing any restriction on the drift functions. The asymptotic distributions of the non-parametric estimators of diffusion functions are also derived. In Section 4, based on the estimator of the diffusion function of the spread process, we construct the semi-parametric estimators of parameters in the drift term of the spread process. In Section 5, we suggest two approaches for computing the prices of derivative securities in our two-factor term-structure model, namely the partial-differential equation approach and the Monte Carlo simulation approach. In Section 6, an empirical examination of the model is carried out. Based on the U.S. data, the non-parametric prices of bonds and bond options are also computed and compared with those calculated under the alternative parametric model. Section 7 draws conclusions. Finally, the mathematical proofs of the theorems are given in the Appendix.

2 A non-parametric two-factor term-structure model

In parametric two-factor models, different functional forms of diffusion functions have been suggested. Table 1 lists several models that have factors either observable or estimable. Brennan and Schwartz (1979) consider a short rate and a long rate as the underlying variables. They specify a constant diffusion function for both the log of the short rate and the log of the long rate. Schaefer and Schwartz (1984) consider a consol rate and a spread between a short rate and the consol rate. They specify a constant diffusion function for the spread and a square root diffusion function for the consol rate. Longstaff and Schwartz (1992) extend the one-factor CIR (1985) model to a two-factor model, the factors being the short rate and its volatility. They assume that diffusion functions can be expressed as square root functions of the two factors. Balduzzi, Das, and Foresi (1998) consider a short rate and its central tendency and the diffusion functions are also square root functions of the corresponding factors.

In these models, the specifications of the diffusion functions are mainly for pure simplicity and tractability to price financial assets. For example, in the Longstaff and Schwartz (1992) model, they derive closed-form expressions for discount bonds based on the square root specifications of diffusion functions.

In this section, we develop a non-parametric two-factor term-structure model. Empirical findings are the guideline for the choice of the two factors in our two-factor model, which uses both a consol rate and the spread between the consol rate and the short rate as factors.[2] The non-parametric specifications of diffusion functions in our model are based on two important facts in the finance literature. First, it has long been recognized that one of the most important features for derivative security pricing is the specification of the diffusion function because (i) it is a measure of uncertainty about future price movements; (ii) it is directly related to the risk associated with holding financial securities and hence affects consumption/investment decisions and portfolio choice; and (iii) the diffusion function is the key component in the pricing of options and other derivative securities. Second,

[2]Principal component analyses in combination with the regression analyses reveal that the first component can be identified with the level of the yield curve, while the second is closely related to the spread between the long and the short rate. These empirical findings refer to Ayres and Barry (1979; 1980), Nelson and Schaefer (1983), and Litterman and Scheinkman (1991).

it is difficult to form a prior functional form of the diffusion function. Together, these two properties motivate us to leave the diffusion functions unrestricted and try to estimate them non-parametrically.

In our model, we assume the two factors, the spread s and the consol rate l, follow the system of stochastic differential equations:

$$ds_t = \beta(\alpha - s_t)dt + \sigma_1(s_t, l_t)dW_1, \tag{1}$$

$$dl_t = \mu_2(s_t, l_t)dt + \sigma_2(s_t, l_t)dW_2, \tag{2}$$

where t denotes the calendar time and β and α are constants; $\sigma_1(\cdot,\cdot), \sigma_2(\cdot,\cdot)$, and $\mu_2(\cdot,\cdot)$ are unknown functions; W_1 and W_2 are standard Wiener processes with $E(dW_1) = E(dW_2) = 0$, $dW_1^2 = dW_2^2 = dt$, $dW_1 \times dW_2 = \rho dt$; ρ is the instantaneous correlation between the processes.

The specification implies mean reversion of the spread level as in Schaefer and Schwartz (1984) and non-parametric specifications of the diffusion functions in the consol rate and the spread. In modelling term-structure dynamics in finance, it is common to specify interest rates as mean-reverting processes with levels that oscillate around a constant central value. Some of the most widely studied one-factor models - such as the model in Vasicek (1977) that is expressed by the Ornstein-Uhlenbeck process, the model in CIR (1985) that is expressed by a square root process, and the semi-parametric model in Aït-Sahalia (1996) - have been using a mean-reverting process to model the short-term rate. However, as mentioned in Schaefer and Schwartz (1984), it is more reasonable to assume that the spread rather than the short rate follows a process with mean-reversion because this kind of process may allow negative values. In addition, the mean-reverting specification of the drift term is also used to identify and estimate the spread process.

3 Non-parametric estimation of the diffusion functions

Given the specification of the term-structure model, the process has to be identified and estimated. The estimation of stochastic differential equations has been considered in the statistics and financial economics literature for many years. However, very few estimate techniques have been developed for a multivariate diffusion process because it is much more difficult to identify and estimate a multivariate diffusion process than to identify and estimate a univariate diffusion process. In this section, we propose non-parametric estimators for diffusion functions in our model. Under regularity conditions, we prove that the estimators have a standard asymptotic behaviour.

We define here non-parametric estimators of the diffusion functions $\sigma_1(s,l)$ and $\sigma_2(s,l)$ based on observing (s_t, l_t) at $(t = t_0, t_1, t_2, ..., t_n)$ of the finite time interval $[0, T]$.[3] Without loss of generality, we assume that $T = 1, t_i = i/n$.

We need to impose some regularity conditions on the drift and diffusion functions to guarantee the existence and uniqueness of a strong solution to our two-factor model, to guarantee the stochastic processes defined in (1) and (2) to be applied with Itô stochastic integration, and also to guarantee that its underlying process is a regular Markov process. Let (Ω, F, P) be a probability space, $(F_t, t \leq 0)$ a non-decreasing family of sub σ-algebra of F and $(W_1(\cdot), W_2(\cdot))$ is two-dimensional Brownian motion on (Ω, F, P). We assume that,

(A) $\mu_2(s,l), \sigma_1(s,l), \sigma_2(s,l)$: $R^2 \longrightarrow R$ are measurable functions and there exist constants C_1 and C_2 such that for any (s,l) and $(\overline{s}, \overline{l}) \in R^2$,

$$| \mu_2(s,l) - \mu_2(\overline{s}, \overline{l}) | + | \sigma_1(s,l) - \sigma_1(\overline{s}, \overline{l}) | + | \sigma_2(s,l) - \sigma_2(\overline{s}, \overline{l}) |$$

[3]Here we suppose that we cannot observe the continuous trajectory, only a discrete sample.

$$\leq C_1\sqrt{((s-\overline{s})^2+(l-\overline{l})^2)},$$

$$\mid \mu_2(s,l) \mid + \mid \sigma_1(s,l) \mid + \mid \sigma_2(s,l) \mid \leq C_2(1+\sqrt{(s^2+l^2)}).$$

(B) $\sigma_1(s,l)$ and $\sigma_2(s,l)$ are bounded by some positive constants. The initial random vector $\eta=(s_0,l_0)$ is F_0 measurable and satisfies $E[s_0^2+l_0^2]<\infty$.

(C) $\sigma_1(\cdot,\cdot)$ and $\sigma_2(\cdot,\cdot)$ are continuously differentiable, with bounded derivatives and the solution of stochastic differential equation is a stationary process.

Conditions (A) and (B) ensure that the stochastic process defined in (1) and (2) has a unique strong solution and a time-homogeneous transitional probability function.[4] Conditions (A), (B), and (C) guarantee the existence and uniqueness of a solution to the Kolmogorov backward equation with initial condition.

We estimate $\sigma_1^2(s,l)$ and $\sigma_2^2(s,l)$ by

$$\hat{\sigma}_1^2(s,l) = \frac{n\sum_{i=0}^{n-1} K[\frac{(s_{i/n},l_{i/n})-(s,l)}{h_n}].[s_{(i+1)/n}-s_{i/n}]^2}{\sum_{i=1}^{n-1} K[\frac{(s_{i/n},l_{i/n})-(s,l)}{h_n}]} \quad (3)$$

$$\hat{\sigma}_2^2(s,l) = \frac{n\sum_{i=0}^{n-1} K[\frac{(s_{i/n},l_{i/n})-(s,l)}{h_n}].[l_{(i+1)/n}-l_{i/n}]^2}{\sum_{i=0}^{n-1} K[\frac{(s_{i/n},l_{i/n})-(s,l)}{h_n}]} \quad (4)$$

where h_n is a positive sequence which converges to zero when n goes to infinity and $K(\cdot,\cdot)$ is a non-negative kernel function on R^2. It should be noted that the non-parametric estimators of diffusion functions are developed without imposing any restrictions on the drift terms so that it captures the true volatilities over different levels of the process. In the following theorem, we show that under conditions (A) and (B), the estimators $\hat{\sigma}_1^2(s,l)$ and $\hat{\sigma}_1^2(s,l)$ converge in probability to $\sigma_1^2(s,l)$ and $\sigma_2^2(s,l)$. Furthermore, under conditions (A), (B) and (C), the estimators are asymptotic normalities.

[4]Compared to the semi-parametric diffusion function estimator suggested by Aït-Sahalia (1996) in which the non-parametric estimator of the diffusion function is derived by a stationary-matching approach, we do not impose the stationary restriction on the density function. We only require the transition probability function to be time-homogeneous to consistently estimate parameters in the drift term of spread.

Theorem 1. *Denote the hitting time to (s,l) by $T_{(s,l)}$ and suppose $T_{(s,l)} < 1$, then we have*

(a) Under conditions (A) and (B), if $nh_n^4/(ln(h_n))^2$ goes to infinity when n goes to infinity, and the kernel function $K(\cdot,\cdot)$ satisfying (i)$K(\cdot,\cdot)$ is a nonnegative function on R^2 and (ii)$K(\cdot,\cdot) \geq \beta I_B$ for some $\beta > 0$ and some closed sphere B entered at the origin and having positive radius r, where I is the indicator function, then the estimators $\hat{\sigma}_1^2(s,l)$ and $\hat{\sigma}_2^2(s,l)$ converge to $\sigma_1^2(s,l)$ and $\sigma_2^2(s,l)$ in probability respectively.

(b) Under conditions (A), (B) and (C), if the kernel function $K(\cdot,\cdot)$ is a non-negative, bounded, and symmetric function on R^2, then as $nh_n^2 \longrightarrow \infty$ and $nh_n^4 \longrightarrow 0$, $\sqrt{nh_n^2}\,[(\hat{\sigma}_1^2(s,l)/\sigma_1^2(s,l)) - 1]$ and $\sqrt{nh_n^2}\,[(\hat{\sigma}_2^2(s,l)/\sigma_2^2(s,l)) - 1]$ converge respectively in distribution to $[\pi(s,l)]^{-1/2}\,[\int\int K^2(s,l)dsdl]\,N$ and $[\pi(s,l)]^{-1/2}\,[\int\int K^2(s,l)dsdl]^{1/2}\,Z$, where N and Z are two standard normal random variables, and $\pi(\cdot,\cdot)$ is the marginal density function of the solution process of (1) and (2).

Proof: See Appendix.

Remark 1. In the one-dimensional case, Jiang and Knight (1997) show that their estimators converge in distribution to the diffusion coefficient under certain conditions, but for this the local time is used. However, the corresponding result for the multivariate case of diffusion process is not available. Our result provides that the asymptotic distribution is available for our estimators in the multivariate case when the observations are from stationary processes (1) and (2). The asymptotic variance in our result depends not only on the marginal density function but also on the integration of the square of the kernel function.

Remark 2. The estimator developed in this paper is different from that in Brugiere (1991) in two aspects. First, the estimator in Brugiere (1991) is constructed from the indicator function rather than a general kernel function. According to Kumur and Markman's (1975)

Monte Carlo studies, the kernel estimator with a standard normal kernel or the optimal kernel of Epanechnikov performs better than the naive estimator of Rosenblatt (1975) and the orthogonal series estimator of Kronmal-Tartr. A well-known serious drawback of the naive method is that it is by definition not a continuous function, but has jumps at the endpoints of the window and zero derivative everywhere else. The discontinuity of the naive method could cause extreme difficulty when constructing the non-parametric drift function estimator of the diffusion process. Second, unfortunately, Brugiere (1991) did not obtain the asymptotic distributions of the estimators of diffusion functions because there does not exist the definition of local time for a multivariate diffusion process. However from Theorem 1, we know that, under the assumption of stationary process, the non-existence of the notion of local time in the multi-dimensional case is not an obstacle to derive the convergence property and asymptotic distribution of the estimator.

Remark 3. Part (a) of Theorem 1 can be extended to a more general specification of the stochastic differential equations. Suppose (s_t, l_t) satisfy the following stochastic differential equations,

$$\begin{aligned} ds_t &= \mu_1(s_t, l_t)dt + \sigma_1(s_t, l_t)dW_1(t) + \sigma_{12}(s_t, l_t)dW_2(t), \\ dl_t &= \mu_2(s_t, l_t)dt + \sigma_{21}(s_t, l_t)dW_1(t) + \sigma_2(s_t, l_t)dW_2(t). \end{aligned}$$

We define $\hat{\Sigma}_n(s,l) =$

$$\begin{pmatrix} \frac{n\sum_{i=0}^{n-1} K[\frac{(s_{i/n}, l_{i/n})-(s,l)}{h_n}].[s_{(i+1)/n}-s_{i/n}]^2}{\sum_{i=0}^{n-1} K[\frac{(s_{i/n}, l_{i/n})-(s,l)}{h_n}]} & \frac{n\sum_{i=0}^{n-1} K[\frac{(s_{i/n}, l_{i/n})-(s,l)}{h_n}].[s_{(i+1)/n}-s_{i/n}][l_{(i+1)/n}-l_{i/n}]}{\sum_{i=0}^{n-1} K[\frac{(s_{i/n}, l_{i/n})-(s,l)}{h_n}]} \\ \frac{n\sum_{i=0}^{n-1} K[\frac{(s_{i/n}, l_{i/n})-(s,l)}{h_n}].[s_{(i+1)/n}-s_{i/n}][l_{(i+1)/n}-l_{i/n}]}{\sum_{i=0}^{n-1} K[\frac{(s_{i/n}, l_{i/n})-(s,l)}{h_n}]} & \frac{\sum_{i=0}^{n-1} K[\frac{(s_{i/n}, l_{i/n})-(s,l)}{h_n}].[l_{(i+1)/n}-l_{i/n}]^2}{\sum_{i=0}^{n-1} K[\frac{(s_{i/n}, l_{i/n})-(s,l)}{h_n}]} \end{pmatrix}.$$

It can be verified that $\hat{\Sigma}_n(s,l)$ is a consistent estimator of

$$\Sigma(s,l) = \begin{pmatrix} \sigma_1(s,l) & \sigma_{12}(s,l) \\ \sigma_{21}(s,l) & \sigma_2(s,l) \end{pmatrix}' \begin{pmatrix} \sigma_1(s,l) & \sigma_{12}(s,l) \\ \sigma_{21}(s,l) & \sigma_2(s,l) \end{pmatrix}.$$

4 A semi-parametric estimator of the drift term

This section presents a semi-parametric estimation procedure for the parameters in the drift term of the spread. In Theorem 1, we proposed non-parametric estimators for the diffusion functions based on discrete sampling observations. In particular, the estimation of the diffusion functions places no restrictions on the functional form of the drift terms. In a single-factor diffusion model, Jiang and Knight (1997) identify the drift term by using information contained in the marginal density function of the single factor, along with the estimated diffusion function. However, without other constraints, the identification procedure in the single-factor diffusion model does not work in the multivariate diffusion model.[5] Therefore, to identify and estimate the drift terms, we have to impose restrictions on the form of the drift functions. An identifying restriction on the drift of the spread is the linear mean-reverting specification. The specification restriction is consistent with that used in Schaefer and Schwartz (1984) and this makes comparisons with their model possible.

Given the mean-reversion specification of the drift term of the spread process, we propose a semi-parametric weighted least-squares estimator of the drift term. The weighted least-squares approach is a natural extension of the conditional least-squares approach. The concept of conditional least squares, which is a general approach for estimation of the parameter involved in the conditional mean function of a stochastic process, was given a thorough treatment by Klimko and Nelson (1978).

The first step of estimating the parameters in the drift term of the spread process consists

[5]In the one-dimensional case, there exists a relationship between the drift, the diffusion, and the stationary density function such that given any two of these functions, we can determine the third by the simple relationship.

of deriving the conditional mean function for $s_{i/n}$

$$E[s_{t+\Delta} \mid s_t, l_t] = \gamma_0 + \gamma_1 s_t, \tag{5}$$

where $\gamma_0 = \alpha(1 - e^{-\beta\Delta}),\ \gamma_1 = e^{-\beta\Delta}$.

The proof of (5) as follow: let $v(s,l,t)$ be the solution of the Kolmogorov backward equation $\frac{\partial v(s,l,t)}{\partial t} = \frac{1}{2}\sigma_1^2(s,l)\frac{\partial^2 v(s,l,t)}{\partial^2 s} + \frac{1}{2}\sigma_2^2(s,l)\frac{\partial^2 v(s,l,t)}{\partial^2 l} + \mu_1(s,l)\frac{\partial v(s,l,t)}{\partial s} + \mu_2(s,l)\frac{\partial v(s,l,t)}{\partial l}$ with initial condition $v(s,l,0) = s$. Under conditions (A) and (B), the uniqueness solution of the partial-differential equation is $v(s,l,t) = E[s_t \mid s_0 = s, l_0 = l]$. Furthermore, by using the fact that the transition density function of the stochastic process has the property of being time-homogeneous, it is easy to verify directly that $E[s_{t+\Delta} \mid s_t = s, l_t = l] = E[s_\Delta \mid s_0 = s, l_0 = l]$. It is also easy to check that the function $g(s,t) = \alpha + e^{-\beta/n}(s - \alpha)$ satisfies the equation with the same initial condition. Therefore we have that $v(s,l,t) = g(s,t)$. By letting $t = \Delta$ in this equation we have that (5) holds. The estimator of $\sigma_1(s,l)$ is used to correct for heteroskedasticity in the residuals from the regression.

Theorem 2. *Under conditions (A), (B) and (C), the estimators:*

$$\hat{\alpha} = \hat{\gamma}_0/(1 - e^{-\hat{\beta}/n}), \tag{6}$$

$$\hat{\beta} = -n\log(\hat{\gamma}_1), \tag{7}$$

are consistent, where

$$\begin{aligned}
\hat{\gamma}_0 &= \frac{\sum_{i=0}^{n-1} s_{(i+1)/n} w_{i/n} \sum_{i=0}^{n-1} s_{i/n}^2 w_{i/n} - \sum_{i=0}^{n-1} s_{i/n} w_{i/n} . \sum_{i=0}^{n-1} s_{(i+1)/n} s_{i/n} w_{i/n}}{\sum_{i=0}^{n-1} w_{i/n} . \sum_{i=0}^{n-1} s_{i/n}^2 w_{i/n} - (\sum_{i=0}^{n-1} s_{i/n} w_{i/n})^2}, \\
\hat{\gamma}_1 &= \frac{\sum_{i=0}^{n-1} w_{i/n} . \sum_{i=0}^{n-1} s_{(i+1)/n} s_{i/n} w_{i/n} - \sum_{i=0}^{n-1} s_{(i+1)/n} w_{i/n} . \sum_{t=0}^{n-1} s_{i/n} w_{i/n}}{\sum_{i=0}^{n-1} w_{i/n} . \sum_{i=0}^{n-1} s_{i/n}^2 w_{i/n} - (\sum_{i=0}^{n-1} s_{i/n} w_{i/n})^2}, \\
w_{i/n} &= \frac{1}{\hat{\sigma}_1(s_{i/n}, l_{i/n})}.
\end{aligned}$$

Proof: See Appendix.

5 Approaches to price derivative securities

Given the non-parametric estimators of $\sigma_1(s, l)$ and $\sigma_2(s, l)$ in (1) and (2), and the semi-parametric estimator of α and β in (6) and (7), the prices of derivatives can be computed by numerical approaches. In this section, we suggest two numerical approaches for computing the prices of derivative securities in our two-factor term-structure model, namely the partial-differential equation approach and the Monte Carlo simulation approach. The partial-differential equation approach can handle American-style as well as European-style derivative securities. At the expense of a considerable increase in computer time, the partial-differential equation method can also be used when there are several state variables. Compared with the partial-differential equation approach, one limitation of the Monte Carlo simulation approach is that it can be used only for European-style derivative securities. However, the Monte Carlo simulation approach is relatively more efficient with respect to the partial-differential equation approach as the number of underlying variables increases. This is because the time taken to carry out a Monte Carlo simulation increases approximately linearly with the number of variables, whereas the time taken for the partial-differential equation approach increases exponentially with the number of variables.

5.1 The partial-differential equation approach

Given the underlying stochastic processes (1) and (2) for s_t and l_t, using the standard arbitrage arguments, we can derive the price $B(s, l, t)$ of any derivative security that depends on the spread s and consol rate l with time-to-maturity t. Let λ be the market price of spread risk, $c(s, l)$ the cash flow rate paid by the security per unit of time, and $f(s, l)$ the payoff of the derivative security at maturity. Using Itô's lemma on $B(s, l, t)$ and the absence of arbitrage opportunities, it is easy to show that $B(s,l,t)$ must satisfy the following partial-differential equation,

$$\frac{1}{2}\sigma_1^2(s,l)B_{ss} + \frac{1}{2}\sigma_2^2(s,l)B_{ll} + \rho\sigma_1(s,l)\sigma_2(s,l)B_{sl} + B_s(\mu_1(s,l,t) - \lambda(s,l)\sigma_1(s,l))$$
$$+B_l(\sigma_2^2/l - sl) - B_t - B(s+l) + c(s,l) = 0 \tag{8}$$

To derive equation (8), we have used the fact that the consol rate is inversely related to the price of the consol bond that must also satisfy the differential equation. It is interesting to note that the partial-differential equation is not only independent of the market price of the long-term interest rate risk, it is also independent of the drift function for the long-term interest rate, so that the solution is independent of the expected rate of return on the consol bond. This result is analogous to the finding within the simple Black-Scholes (1973) model for the pricing of stock options that the function expressing the equilibrium price of the option in terms of the price of the underlying stock is independent of the expected rate of return on the underlying stock. Actually, the reason for these two results is the same: There exists an asset for which the partial derivatives of its value with respect to all of the state variables is known, in this case the consol bond, and in the Black-Scholes case, the stock. In addition, we now take into account the empirical regularity that the spread is uncorrelated with the consol rate; therefore we can set $\rho = 0$.

Since the partial-differential equation (8) is valid for all types of default-free derivative securities, it may be applied to the corresponding pricing of these securities by the introduction of the appropriate initial conditions defining the payoffs on the securities and boundary conditions depending on the particular security considered.

Since there is no closed solution available to the differential equation (8), it has to be solved numerically. In one-factor models, Aït-Sahalia (1996) suggests the use of the Crank-

Nicolson scheme to solve numerically the partial-differential equation. However, the Crank-Nicolson scheme is inappropriate for the two-dimensional program (Smith 1985). In two-factor models, we suggest using the line-hopscotch method recommended by Gourlay and Mckee (1977). The advantage of implementating the line-hopscotch method is that it leads to a stable and consistent solution for the parabolic partial-differential equation. The idea of the line-hopscotch method is to solve alternative points explicitly and then employ an implicit scheme to solve for the remaining points. In practice, the block-wise bootstrap technique proposed in Künsch (1989) can be used to compute the standard errors. The bootstrap estimation procedure consists of the following three steps: (i) Redraw from the original data. The resampling procedure redraws from blocks of continuous observations to preserve the serial correlation existing in the original data. (ii) Estimate the diffusion functions, the drift function, and the market price of risk from resampled data. Then compute the bond prices $\hat{B}(s,l,t)$. (iii) The standard error of the bond price is the sample standard deviation of $\hat{B}(s,l,t)$.

5.2 The Monte Carlo simulation approach

The commonly used Monte Carlo simulation procedure for pricing derivative securities can be briefly described as follows. First, sample paths are simulated for the state variables. The paths of each state variable must be sampled on each simulation run. Second, the payoff of the derivative security is calculated on each simulation run from the sample paths. If the instantaneous risk-free interest rate, r, is a function of the state variables, the average value of r must also be calculated on each simulation run. The payoff is discounted at the average value of r before the next simulation run is begun. The price of derivative security can be obtained by averaging the simulated discount payoffs. However, for the purpose of the simulation, it should be noted that the diffusion processes for all state variables must be the processes that the variables would follow in a risk-neutral world. Particularly, in our two-factor term-structure model, the sample paths are simulated by the following risk-neutral dynamics:

$$ds_t = \beta(\alpha - \frac{\lambda\sigma_1}{\beta} - s_t)dt + \sigma_1(s_t, l_t)dW_1(t), \tag{9}$$

$$dl_t = (\sigma_2^2(s_t, l_t)/l_t - s_t l_t)dt + \sigma_2(s_t, l_t)dW_2(t). \tag{10}$$

The sample paths, all starting at $s_t = s, l_t = l$ at date t and finishing at date T, can be simulated with the risk-neutral drifts and diffusions replaced by their estimates. The conditional expectation under the risk-neutral dynamics gives the prices

$$\begin{aligned} &B(s, l, T-t) = E_t\{f(s_T, l_T)exp\{-\int_t^T (s_u + l_u)du\} \\ &+ \int_t^T exp\{-\int_t^\tau (s_u + l_u)du\}c(s_\tau, l_\tau, \tau)d\tau | s_t = s, l_t = l\}, \end{aligned} \tag{11}$$

where $f(\cdot, \cdot)$ is the payoff of the security at maturity time T. The price $B(s, l, T-t)$ can then be obtained by averaging the argument of the conditional expectation over the simulated sample paths. The standard deviations of the estimates can also be calculated by the simulation method. The simulations of the sample path can be performed using the following Euler scheme method (Talay [1996]) with a discretization step T/n over the time interval $[0, T]$,

$$\begin{aligned} s^j_{(k+1)T/n} &= s^j_{kT/n} + \beta(\alpha - \frac{\lambda\sigma_1}{\beta} - s^j_{kT/n})T/n \\ &+ \sigma_1(s^j_{kT/n}, l^j_{kT/n})(W^j_{1,(k+1)T/n} - W^j_{1,kT/n}), \end{aligned} \tag{12}$$

$$\begin{aligned} l^j_{(k+1)T/n} &= l^j_{kT/n} + (\sigma_2^2(s^j_{kT/n}, l^j_{kT/n})/l^j_{kT/n} - s^j_{kT/n} l^j_{kT/n})T/n \\ &+ \sigma_2(s^j_{kT/n}, l^j_{kT/n})(W^j_{2,(k+1)T/n} - W^j_{2,kT/n}), \end{aligned} \tag{13}$$

with $(s_0^j, l_0^j) = (s, l)$, where j is the index for the simulation run ($j = 1, 2, ..., m$) and k is the index for the step during each run ($k = 0, 1, ..., n-1$). Each simulation run involves obtaining a sample of $(W_{1,T/n}, W_{1,2T/n} - W_{1,T/n}, ..., W_{1,T} - W_{1,T})$ and of $(W_{2,T/n}, W_{2,2T/n} - W_{2,T/n}, ..., W_{2,T} - W_{2,T})$ from independent Gaussian random variables. These are substituted into (12) and (13) to produce simulated paths for the spread and the consol rate and enable a sample value for the derivative security to be calculated.

In financial applications of the Monte Carlo simulation methods, a number of variance reduction methods have been proposed, e.g., the control variate approach, the antithetic variate method, the moment matching method, the importance sampling method, the conditional Monte Carlo methods, and the quasi-random Monte Carlo methods (see, for example, Boyle, Broadie, and Glasserman [1997]). Also it should be mentioned that the Monte Carlo simulation approach is one of the approaches most often used to solve the partial-differential equation when the usual methods are relatively difficult to implement.

6 Empirical pricing of discount bonds and options

In this section, we carry out an empirical analysis of the non-parametric two-factor model and report the prices of bonds and bond options based on our non-parametric model and the Schaefer and Schwartz (1984) model. The time series used in this paper are daily yields of 30-day U.S. Treasury bills and 10-year-above bonds from January 1988 to August 1999. The consol rate is approximated by a long rate, the 10-year-and-above bond yield, and the spread is the yield difference between the 30-day U.S. Treasury bill and the 10-year-above bond. The time series daily data and its first difference are plotted in Figures 1 and 2. The summary statistics of the data and the stationary test results are given in Table 2.

The autocorrelations of the long rates and the spreads decay very slowly. The augmented Dickey-Fuller non-stationarity tests indicate that the null hypothesis of non-stationarity is rejected at a 10 per cent significance level for both the long rates and the spreads. The non-parametric kernel density estimation results are plotted in Figures 3 and 4.

6.1 Estimation results of diffusion functions

We apply the non-parametric estimation technique of diffusion functions in the two-factor model to the U.S. data. The non-parametric estimators of the diffusion functions are reported in Figures 5 to 8. Figure 5 and Figure 7 plot the three-dimensional graphs of the non-parametric estimators of the diffusion functions of the spread and long rate, while Figure 6 and Figure 8 plot the two-dimensional graphs. First, the diffusion function of the spread process exhibits noticeable variations from low to high values of both the long rate and spread. This may suggest the model with a constant diffusion function of the spread as in Schaefer and Schwartz (1984) is misspecified. Second, the non-parametric diffusion function of the long rate is obviously not an increasing function of the long rate, which also contradicts the model with a squared-root specification of the diffusion function for the long rate as in Schaefer and Schwartz (1984) in which the long rate at a high level is expected to vary more than at a low level, namely, the "level-effect" specification of the long rate. Third, both diffusion functions of the spread and the long rate depend on both state variables.

6.2 Estimation of drift functions and market price of spread risk

Given the non-parametric estimators of the diffusion function of the spread process, we apply the estimation method for the drift in Theorem 2 to get the semi-parametric weighted least-square estimate of the drift term of the spread process. The estimates of α and β are reported in Table 2.

In the same table, we also report the estimates from the Schaefer and Schwartz (1984) model. The estimates of α, β are obtained by the unweighted conditional least-squares method, i.e., simply taking the weight function as 1 in Theorem 2. We can estimate γ^2 by $\hat{\gamma}^2 = \sum_{i=0}^{n-1}(s_{(i+1)/n} - s_{i/n})^2/n$ (see e.g. Dohnal (1987)). As we have already mentioned that any drift term of the long-rate process would be compatible with equation (8), we specify the drift term of the long rate as $\beta_l(\alpha_l - l)$. To estimate σ^2, we first estimate β_l and α_l by following the same way as we estimate α and β for the spread equation. Then we obtain the conditional second moment function (see Overbeck and Ryden (1997)): $E[(l_{t+\Delta} - E[l_t|l_t])^2|l_t = l] = \sigma^2(\eta_0 + \eta_1 l)$ with $\eta_0 = \frac{\alpha_l}{2\beta_l}(e^{-\beta_l\Delta} - 1)$ and $\eta_1 = -\frac{1}{\beta_l}e^{-\beta_l\Delta}(e^{-\beta_l\Delta} - 1)$. By standard linear regression, we can estimate σ^2 by $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=0}^{n-1}\frac{[l_{(i+1)/n} - (\hat{\mu}_0 + \hat{\mu}_1 l_{i/n})]^2}{\hat{\eta}_0 + \hat{\eta}_1 l_{i/n}}$ where $\hat{\mu}_0 = -\hat{\alpha}_l(e^{-\hat{\beta}_l\Delta} - 1)$, $\hat{\mu}_1 = e^{-\hat{\beta}_l\Delta}$ and $\hat{\eta}_0$ and $\hat{\eta}_1$ are evaluated at $\hat{\alpha}_l$ and $\hat{\beta}_l$. The standard errors are calculated by the block-wise bootstrap method.

From Table 3, we see that the estimates of α are similar, with -0.0266 in the non-parametric model and -0.0227 from the parametric model. The estimates of β are not significant but with expected signs.

The market price of spread risk, λ, is estimated for both the parametric model and the non-parametric model by minimizing the sum of squared deviations across maturities between a given target yield curve and the yields produced by the respective models. The target yield curve is obtained by averaging the yield curves over the sample period, January 1988 to August 1999. The estimates of the market prices of spread risk are also reported in Table 3.

6.3 Pricing discount bonds

With the parameter estimates, we can price discount bonds by the two methods mentioned in Section 5. In this paper, we use the Monte Carlo simulation approach. In Table 3, the bond prices are computed under parametric and non-parametric models by the Monte Carlo simulation approach. All prices correspond to a face value of the bond equal to $100. The three elements of each cell from top to bottom in Table 3 are the bond prices for the non-parametrically specified model, the standard deviations, and bond prices for the parametric model. In performing the Monte Carlo simulations, 500 risk-neutral paths for the long rate and the spread are simulated based on the Euler scheme. The standard errors of the non-parametric prices are also calculated through the simulations. For the short-, mid-, and long-term bond prices, the parametric model generates significantly different bond prices from the non-paremetric model as most parametric prices fall outside of the two-standard-deviation ranges of the non-parametric prices. This implies that bond prices not only reflect the differences in the risk-neutral first-moment of the underlying processes (i.e., similar specifications for two models), but also reflect the differences in the second moments of the underlying processes. This suggests that the misspecification of volatilities of the underlying processes can lead to significantly different prices of interest rate derivative securities.

6.4 Pricing bond options

Tables 5, 6, and 7 report the prices for in-the-money, at-the-money, and out-of-the-money call options, computed under the non-parametric model and the alternative parametric model, on a 5-year discount bond with face value of $100. The exercise prices, 0.98, 1.00 and 1.02, are expressed as proportions of the corresponding bond price for non-parametric and parametric models respectively. The three elements of each cell from top to bottom are the non-parametric price, the standard deviation, and the parametric price. It should be noted that it is never optimal to exercise early an American call option because the underlying bond pays no coupon, so call option values, which are reported in Tables 4, 5, and 6, express both American and European call option values. The valuation of a call option on a pure discount bond based on the two-factor term-structure model is a two-step procedure. First, the equilibrium value of the underlying bond at the maturity data of the option is estimated by the Monte Carlo simulation method subject to certain initial conditions. Then the value of the bond is substituted into the payoff function for the option as specified in previous section and the simulations are performed for option prices. Since derivative prices rely mostly on the second moments of the underlying processes, different estimates of the diffusion functions will lead to different call option prices. The estimates suggest substantial and significant differences of the prices based on the non-parametric model and the alternative parametric model. The differences vary over different values of the spread and long rate, the maturities of the options, and the moneyness of the options.

7 Conclusion

In this paper, we have proposed a non-parametric two-factor term-structure model with non-parametric specifications of the diffusion functions. The consistent non-parametric estimators of the diffusion functions have been obtained based on the discrete sampling observations. The estimators have been developed without imposing any restriction on the functional form of the drift terms, so that it can capture the true volatilities over different levels of the process. Under the condition that the processes are stationary, we obtain the asymptotic distribution of the proposed non-parametric kernel estimators of the multi-dimensional diffusion process. In view of the fact that the multivariate diffusion process is much more desirable in some important practices, for instance, the theory of stochastic control and modelling the term-structure movements of interest rate, the theoretical result obtained in this paper is very important.

The implementation of the model, based on performing Monte Carlo simulations of the sample paths of the risk-neutral process that is developed in this paper, provides evidence for rejecting the parametric specification of diffusion functions. This implies that it will be worthwhile to use the non-parametric technique to estimate the underlying multi-dimensional diffusion process of asset prices or interest rates in order to price more precisely derivative securities, to evaluate the value of contingent claims and other financial instruments, or to design optimal hedging strategies.

Appendix

In this appendix, we present proofs of the main results of this chapter. For simplicity, we assume that the drift functions are bounded functions. However, by a usual localization argument, we can extend all the results from the hypothesis of boundedness of drift functions to the Lipschitzian character of the drift functions.

To prove Theorem 1, we first state the notations. For any $\delta \in \left(0, \frac{1}{e}\right]$, we define $\varphi(\delta) = \sqrt{2\delta \ln\left(\frac{1}{\delta}\right)}$. The inverse of φ is denoted by ψ. Therefore ψ can be defined by: for any $y \in \varphi\left(\left(0, \frac{1}{e}\right]\right)$, any $y' \in \left(0, \frac{1}{e}\right]$, $\varphi(y') = y \Longleftrightarrow y' = \psi(y)$. Let $X_t = (s_t, l_t)'$ and $x = (s, l)$. Then the Levy's Modulus of a diffusion is defined as $\triangle_\varepsilon = \sup_{t,s\in[0,1],|t-s|\leq\varepsilon} \parallel X_t - X_s \parallel$.

Proof of Theorem 1: Denote $\triangle X_i = X_{(i+1)/n} - X_{i/n}$,

$\hat{\Sigma}_n(x) =$

$$\begin{pmatrix} n\dfrac{\sum_{i=0}^{n-1} K[\frac{(s_{i/n},l_{i/n})-(s,l)}{h_n}].[s_{i+1/n}-s_{i/n}]^2}{\sum_{i=0}^{n-1} K[\frac{(s_{i/n},l_{i/n})-(s,l)}{h_n}]} & n\dfrac{\sum_{i=0}^{n-1} K[\frac{(s_{i/n},l_{i/n})-(s,l)}{h_n}].[s_{i+1/n}-s_{i/n}][l_{i+1/n}-l_{i/n}]}{\sum_{i=0}^{n-1} K[\frac{(s_{i/n},l_{i/n})-(s,l)}{h_n}]} \\ n\dfrac{\sum_{i=0}^{n-1} K[\frac{(s_{i/n},l_{i/n})-(s,l)}{h_n}].[s_{i+1/n}-s_{i/n}][l_{i+1/n}-l_{i/n}]}{\sum_{i=0}^{n-1} K[\frac{(s_{i/n},l_{i/n})-(s,l)}{h_n}]} & n\dfrac{\sum_{i=0}^{n-1} K[\frac{(s_{i/n},l_{i/n})-(s,l)}{h_n}].[l_{i+1/n}-l_{i/n}]^2}{\sum_{i=0}^{n-1} K[\frac{(s_{i/n},l_{i/n})-(s,l)}{h_n}]} \end{pmatrix}.$$

On $\{T_x < 1\}$ we write

$$\hat{\Sigma}_n(x) - \Sigma(x) = A_n(x) + B_n(x),$$

where

$$\Sigma(x) = \begin{pmatrix} \sigma_1(s,l) & \sigma_{12}(s,l) \\ \sigma_{21}(s,l) & \sigma_2(s,l) \end{pmatrix}' \begin{pmatrix} \sigma_1(s,l) & \sigma_{12}(s,l) \\ \sigma_{21}(s,l) & \sigma_2(s,l) \end{pmatrix},$$

and

$$\begin{aligned} A_n(x) &= \frac{n\sum_{i=0}^{n-1} K\left(\frac{X_{i/n}-x}{h_n}\right)\left(\triangle X_i \triangle X_i' - \int_{i/n}^{(i+1)/n} \Sigma(X_s)\,ds\right)}{D_n}, \\ B_n(x) &= \frac{n\sum_{i=0}^{n-1} K\left(\frac{X_{i/n}-x}{h_n}\right)\left(\int_{i/n}^{(i+1)/n} \Sigma(X_s)\,ds - (\Sigma(x)/n)\right)}{D_n}, \\ D_n &= \sum_{i=0}^{n-1} K\left(\frac{X_{i/n}-x}{h_n}\right). \end{aligned}$$

To prove (a) of Theorem 1, we only need to prove that $A_n(x)$ and $B_n(x)$ converge in probability to zero. This can be proved by the following three Lemmas. The structure of proofs follows Brugiere (1991).

Lemma 1. There exists constant $\varepsilon > 0$, such that

$$\underline{Lim} \frac{1}{n\psi(h_n)} \sum_{i=0}^{n-1} K\left(\frac{X_{i/n} - x}{h_n}\right) \geq \varepsilon.$$

Proof of Lemma 1:

We define

$$\Omega_0 = \left\{\omega \in \Omega \mid \overline{Lim} \frac{\triangle_\varepsilon}{\varphi(\varepsilon)} = \max_{0 \leq t \leq 1} \sqrt{\rho(X_t)}\right\}.$$

Following Brugiere (1991), we have $P(\Omega_0) = 1$. Let $\omega \in \Omega_0 \cap \{T_x < 1\}$, then there exists $s_\omega \in [0,1)$ such that $X_{s_\omega}(\omega) = x$ and by the definition of Levy's modulus of continuity of a diffusion, for any given positive constant r, there exists $N_\omega > 0$, when $n > N_\omega$ we have

$$\| X_{i/n}(\omega) - X_{s_\omega}(\omega) \| \leq (1 + \sqrt{\rho_2})\,\varphi\left[\psi\left(\frac{rh_n}{1+\sqrt{\rho_2}}\right)\right], \ \mid \frac{i}{n} - s_\omega \mid \leq \psi\left(\frac{rh_n}{1+\sqrt{\rho_2}}\right),$$

which means $\| \frac{X_{i/n}-x}{h_n} \| \leq r$. Thus, when $n > N_\omega$, we have $K\left(\frac{X_{i/n}-x}{h_n}\right) \geq \beta$ because $\frac{X_{i/n}-x}{h_n} \in B$. Using the similar discussion as in Brugiere (1991), we have at least $n\psi\left(rh_n/\left(1+\sqrt{\rho_2}\right)\right)$ observations available. Therefore, $\frac{D_n(\omega)}{n\psi\left(rh_n/\left(1+\sqrt{\rho_2}\right)\right)} \geq \beta$.

By the definition of ψ and the fact that $\psi\left(\frac{rh_n}{1+\sqrt{\rho_2}}\right) \geq \frac{1}{2\left(1+\sqrt{\rho_2}\right)^2}\psi(rh_n)$ when n is sufficiently large, we can select ε as $\varepsilon = \frac{\beta}{2\left(1+\sqrt{\rho_2}\right)^2}$ #

Lemma 2. On $\{T_x < 1\}$, $B_n(x)$ convergences in probability to zero.

Proof of Lemma 2:

The proof directly follows Lemma 2 of Brugiere (1991) by replacing the indicator function by the kernel function $K(\cdot)$. The proof simply makes use of the Levy's modulus of continuity of a diffusion process. #

Lemma 3. On $\{T_x < 1\}$, $A_n(x)$ convergences in probability to zero.

Proof of Lemma 3:

let u be a unit vector in R^2, we can write, as a consequence of the Itô formula,

$$u'\left[(\triangle X)_i'(\triangle X)_i - \int_{i/n}^{(i+1)/n} \Sigma(X_s)ds\right]u = 2\int_{i/n}^{(i+1)/n} u'(X_s - X_{i/n})u'dX_s.$$

We can decompose $u'A_n(x)u$ in two terms as $u'A_n(x)u = C_n^u + F_n^u$, where

$$\begin{aligned} C_n^u &= 2n\frac{\sum_{i=0}^{n-1} K\left(\frac{X_{i/n}-x}{h_n}\right)\int_{i/n}^{(i+1)/n} u'(X_s - X_{i/n})u'\mu(X_s)\,ds}{D_n}, \\ F_n^u &= 2n\frac{\sum_{i=0}^{n-1} K\left(\frac{X_{i/n}-x}{h_n}\right)\int_{i/n}^{(i+1)/n} u'(X_s - X_{i/n})u'\Sigma^{1/2}(X_s)\,dW_1(s)}{D_n}. \end{aligned}$$

Then the following results hold: $(i)C_n^u$ converges in probability to zero, $(ii)F_n^u$ converges in probability to zero. The two results can be shown by following Lemma 3 and Lemma 4 of Brugiere (1991) and noting that $\Pi_n = \sum_{i=0}^{n-1}\Psi_{i,n}$ is a martingale with quadratic variation of over n^{-2},where $\Psi_{i,n} = K\left(\frac{X_{i/n}-x}{h_n}\right)\int_{i/n}^{(i+1)/n} u'(X_s - X_{i/n})u'\sigma(X_s)\,dW_1(s)$. #

Denoting $E^{i,n}$ the conditional expectation with respective to $F_{i/n}^{s,l} = \sigma[(s_u, l_u); u \le i/n]$ and setting for any $0 < t \le 1$.

$$\begin{aligned} m_{i+1} &= \sqrt{n/h_n^2}K\left(\frac{(s_{i/n}, l_{i/n}) - (s,l)}{h_n}\right)\left[(s_{i+1/n} - s_{i/n})^2 - \frac{1}{n}\sigma_1^2(s,l)\right]. \\ M_t^n &= \sum_{i=0}^{[nt]-1} m_{i+1}. \end{aligned}$$

To prove part (b) of Theorem 1, we need to prove the following results:

(i) $\sum_{i=0}^{[nt]-1} E^{i,n}(m_{i+1})$ converges in probability to zero,

(ii) $\sum_{i=0}^{[nt]-1} E^{i,n}(m_{i+1}^2)$ converges in probability to $t\sigma_1^4(s,l)\pi(s,l)\int\int K^2(s,l)dsdl$,

(iii) $\sum_{i=0}^{[nt]-1} E^{i,n}|(m_{i+1})|^3$ converges in probability to zero.

Proof of (i) An application of Itô's forma gives

$$
\begin{aligned}
|E^{i,n}m_{i+1}| &= \sqrt{\frac{n}{h_n^2}}|E^{i,n}K(\frac{(s_{i/n},l_{i/n})-(s,l)}{h_n})(\int_{i/n}^{(i+1)/n} 2(s_u-s_{i/n})\mu_1(s_u,l_u)du \\
&+ \int_{i/n}^{(i+1)/n}(\sigma_1^2(s_u,l_u)-\sigma_1^2(s,l))du)|.
\end{aligned}
$$

By using the Burkholder-Davis-Gundy inequality, we can obtain that

$$
|E^{i,n}m_{i+1}| = O(1)\sqrt{\frac{n}{h_n^2}}(\frac{1}{n^{3/2}}+\frac{h_n}{n})K(\frac{(s_{i/n},l_{i/n})-(s,l)}{h_n}).
$$

Thus

$$
|\sum_{i=0}^{[nt]-1} E^{i,n}m_{i+1}| = O(1)(h_n+h_n^2 n^{\frac{1}{2}})\left[\frac{1}{nh_n^2}\sum_{i=0}^{[nt]-1} K(\frac{(s_{i/n},l_{i/n})-(s,l)}{h_n})\right],
$$

which approaches to zero in probability because $\frac{1}{nh_n^2}\sum_{i=0}^{[nt]-1}K(\frac{(s_{i/n},l_{i/n})-(s,l)}{h_n})$ converges to $t\pi(s,l)$ and nh_n^4 tends to zero.

Proof of (ii) Using Itô's formula and the Burkholder-Davis-Gundy inequality, we have

$$
\begin{aligned}
\sum_{i=0}^{[nt]-1} E^{i,n}m_{i+1}^2 &= \sum_{i=0}^{[nt]-1} E^{i,n}(\frac{n}{h_n^2})K^2(\frac{(s_{i/n},l_{i/n})-(s,l)}{h_n})\{4\int_{i/n}^{(i+1)/n}(s_u-s_{i/n})^3\mu_1(s_u,l_u)du \\
&+6\int_{i/n}^{(i+1)/n}(s_u-s_{i/n})^2(\sigma_1^2(s_u,l_u)-\sigma_1^2(s,l))du \\
&-4\frac{\sigma_1^2}{n}(s,l)\int_{i/n}^{(i+1)/n}(s_u-s_{i/n})\mu_1(s_u,l_u)du \\
&+12\int_{i/n}^{(i+1)/n}du\int_u^{i/n}(s_v-s_{i/n})\mu_1(s_v,l_v)dv \\
&+6\sigma_1^2(s,l)\int_{i/n}^{(i+1)/n}du\int_u^{i/n}(\sigma_1^2(s_v,l_v)-\sigma_1^2(s,l)dv \\
&-2\frac{\sigma_1^2}{n}\int_{i/n}^{(i=1)/n}(\sigma_1^2(s_v,l_v)-\sigma_1^2(s,l))dv\} \\
&= o_p(1)+(\frac{\sigma_1^4(s,l)}{nh^2})\sum_{i=0}^{[nt]-1}K^2(\frac{(s_{i/n},l_{i/n})-(s,l)}{h_n}). \qquad (14)
\end{aligned}
$$

However it can be shown that the last term of (14) approaches to $t\sigma_1^4(s,l)\pi(s,l)\int\int K^2(s,l)dsdl$.

Proof of (iii)

$$
\sum_{i=0}^{[nt]-1} E^{i,n}|m_{i+1}|^3 = O(\frac{1}{h_n n^{\frac{1}{2}}})\left[(\frac{1}{nh_n^2})\sum_{i=0}^{[nt]-1}K(\frac{(s_{i/n},l_{i/n})-(s,l)}{h_n})\right],
$$

which approaches to zero in probability because of $nh_n^2 \longrightarrow 0$ and $(\frac{1}{nh_n^2})\sum_{i=0}^{[nt]-1} K(\frac{(s_{i/n}, l_{i/n})-(s,l)}{h_n})$ approaches $\pi(s,l)$ in probability.

From **(i)**,**(ii)**, and **(iii)**, we know that M_t^n converges to the martingale M_t with increasing process $< M_t >= t\sigma_1^4(s,l)\pi(s,l)\int\int K^2(s,l)dsdl$. Then we can write $M_t = B_{t\sigma_1^4(s,l)\pi(s,l)\int\int K^2(s,l)dsdl}$ where B_t is Brownian motion. Thus we have that M_1^n converges in distribution to $\sqrt{\sigma_1^4(s,l)\pi(s,l)\int\int K^2(s,l)dsdl}N$, where N is a standard normal variable. Since we can write

$$\sqrt{nh_n^2}\left[(\frac{\hat{\sigma}_1^2(s,l)}{\sigma_1^2(s,l)}) - 1\right] = \frac{M_1^n}{\sigma_1^2(s,l)(\frac{1}{nh_n^2})\sum_{i=0}^{n-1} K(\frac{(s_{i/n}, l_{i/n})-(s,l)}{h_n})}$$

thus $\sqrt{nh_n^2}\left[(\frac{\hat{\sigma}_1^2(s,l)}{\sigma_1^2(s,l)}) - 1\right]$ approaches to $\sqrt{(\frac{\int\int K^2(s,l)dsdl}{\pi(s,l)})}N$. #

Proof of Theorem 2:

It is easily derived by using standard method of proving consistent about weighted conditional least-square estimate. Therefore we omit the detailed proof here. #

Table 1: Selected two-factor parametric models

Model	Factors	Drift term	Diffusion term
Brennan and Schwartz (1979)	lnr	$\alpha(lnl - lnr - lnp)$	σ_1
	lnl	$q - k_1 lnr + k_2 lnl$	σ_2
Schaefer and Schwartz (1984)	s	$\beta(\alpha - s)$	γ
	l	$\mu_2(s, l, t)$	$\sigma\sqrt{l}$
Longstaff and Schwartz (1992)	r	linear fn of (r, V)	sqrt of linear fn of r, V
	V	linear fn of (r, V)	sqrt of linear fn of r, V
Balduzzi, Das and Foresi (1998)	r	$\beta(\alpha - r)$	sqrt of linear fn of r
	θ	linear fn (θ)	sqrt of linear fn of θ

Table 2: Summary statistics of the data and stationary test

	s_t	$s_t - s_{t-1}$	l_t	$l_t - l_{t-1}$
N	1919	1918	1919	1918
Mean	-0.0250	1.19E-5	0.0669	7.24E-6
Standard deviation	0.0128	0.0012	0.0077	0.0005
Test statistics	-2.74	372.85	-5.86	-462.63
(H_0:Nonstationary)	(Reject)	(Reject)	(Reject)	(Reject)

Note: N is the number of observations. The stationary test is the augumented Dickey-Fuller test. The testing results are based on a 10 per cent signficance level.

Table 3: Parameter estimates of the two-factor models

	Non-parametric Model	Parametric Model
β	1.1711 (2.1469)	1.3029 (1.2657)
α	-0.0266 (0.0024)	-0.0227 (0.0027)
γ	Non-parametric diffusion (Figure 5)	0.0204 (0.0002)
σ	Non-parametric diffusion (Figure 7)	0.0316 (0.0007)
λ	-1.3982 (0.2575)	-0.7310 (0.0987)

Note: For the non-parametric model, diffusion functions are estimated by (3) and (4). The drift estimators are estimated by (6) and (7). The numbers in brackets are standard errors which are obtained by the blockwise bootstrap method. The market price of spread risk λ is estimated by minimizing the squared deviations between the respective model's bond yields and the average yield curve.

Table 4: Discount bond prices under alternative models

Maturity (Years)	Spread s	Long rate l			
		5.92	6.69	7.46	8.23
0.5	-3.79	98.4741	98.3064	97.7265	97.3530
		(0.0444)	(0.0207)	(0.0330)	(0.0267)
		98.6001	98.2184	97.8381	97.4593
	-2.50	97.8377	97.6178	97.2101	96.9224
		(0.0602)	(0.0279)	(0.0393)	(0.0265)
		98.1218	97.7427	97.3652	96.9891
	-1.21	97.4826	97.2228	96.8698	96.4644
		(0.0243)	(0.0168)	(0.0099)	(0.0149)
		97.6457	97.2694	96.8945	96.5211
1	-3.79	96.2272	96.0341	94.7446	94.0758
		(0.1025)	(0.0531)	(0.0571)	(0.0408)
		96.6640	95.9118	95.1656	94.4251
	-2.50	95.2516	94.8967	94.1482	93.5539
		(0.1024)	(0.0457)	(0.0528)	(0.0384)
		95.9749	95.2313	94.4934	93.7613
	-1.21	94.7887	94.4172	93.7714	92.9856
		(0.0401)	(0.0285)	(0.0159)	(0.0261)
		95.2906	94.5554	93.8258	93.1019
5	-3.79	76.3494	74.7017	71.8303	69.9700
		(0.1367)	(0.1267)	(0.1041)	(0.0821)
		78.2278	75.1381	72.1705	69.3201
	-2.50	75.3619	73.8589	71.4331	69.6461
		(0.1345)	(0.1453)	(0.0676)	(0.0828)
		77.6608	74.6187	71.6958	68.8874
	-1.21	74.9888	73.7051	71.2086	69.0096
		(0.0741)	(0.1389)	(0.0468)	(0.0739)
		77.0963	74.1011	71.2223	68.4553

Note: The face value of the underlying bond is \$100. The top element of each cell is the non-parametric price. The standard errors are in parentheses. The bottom element is the parametric price.

Table 5: In-the-money call option prices on a 5-year discount bond (X=0.98)

Maturity (Years)	Spread s	Long rate l			
		5.92	6.69	7.46	8.23
0.5	-3.79	3.0360	3.1132	3.0185	2.9516
		(0.0199)	(0.0088)	(0.0067)	(0.0085)
		3.0884	3.1105	3.1135	3.1001
	-2.50	2.9442	2.9979	2.9900	2.9293
		(0.0147)	(0.0092)	(0.0077)	(0.0073)
		3.0233	3.0459	3.0500	3.0382
	-1.21	2.8760	2.9443	2.9484	2.8966
		(0.0107)	(0.0108)	(0.0059)	(0.0081)
		2.9596	2.9827	2.9878	2.9773
1	-3.79	5.0312	5.2763	5.2077	5.1559
		(0.0277)	(0.0185)	(0.0126)	(0.0153)
		5.0808	5.2557	5.3855	5.4754
	-2.50	4.9086	5.0745	5.1460	5.0873
		(0.0267)	(0.0205)	(0.0125)	(0.0119)
		4.9639	5.1365	5.2655	5.3556
	-1.21	4.8140	4.9579	5.0783	5.0541
		(0.0197)	(0.0236)	(0.0098)	(0.0134)
		4.8495	5.0200	5.1480	5.2383

Note: The underlying 5-year discount bond has a face value of $100. The strike price X is a proportion of the current bond price for each model. The top element of each cell is the non-parametric price. The standard errors are in parentheses. The second element is the parametric price.

Table 6: At-the-money call option prices on a 5-year discount bond (X=1.00)

Maturity (Years)	Spread s	Long rate l			
		5.92	6.69	7.46	8.23
0.5	-3.79	1.8261	1.9605	1.9502	1.9384
		(0.0185)	(0.0096)	(0.0059)	(0.0083)
		1.8290	1.9437	2.0326	2.0988
	-2.50	1.7671	1.8701	1.9334	1.9178
		(0.0131)	(0.0110)	(0.0073)	(0.0069)
		1.7824	1.8956	1.9837	2.0497
	-1.21	1.7189	1.8123	1.8962	1.9057
		(0.0111)	(0.0130)	(0.0059)	(0.0078)
		1.7370	1.8487	1.9359	2.0017
1	-3.79	3.7853	4.0852	4.1006	4.1038
		(0.0260)	(0.0199)	(0.0120)	(0.0153)
		3.7853	4.0506	4.2645	4.4327
	-2.50	3.6961	3.9097	4.0513	4.0378
		(0.0252)	(0.0228)	(0.0123)	(0.0118)
		3.6879	3.9489	4.1600	4.3267
	-1.21	3.6221	3.7899	3.9883	4.0253
		(0.0204)	(0.0261)	(0.0098)	(0.0135)
		3.5928	3.8495	4.0579	4.2231

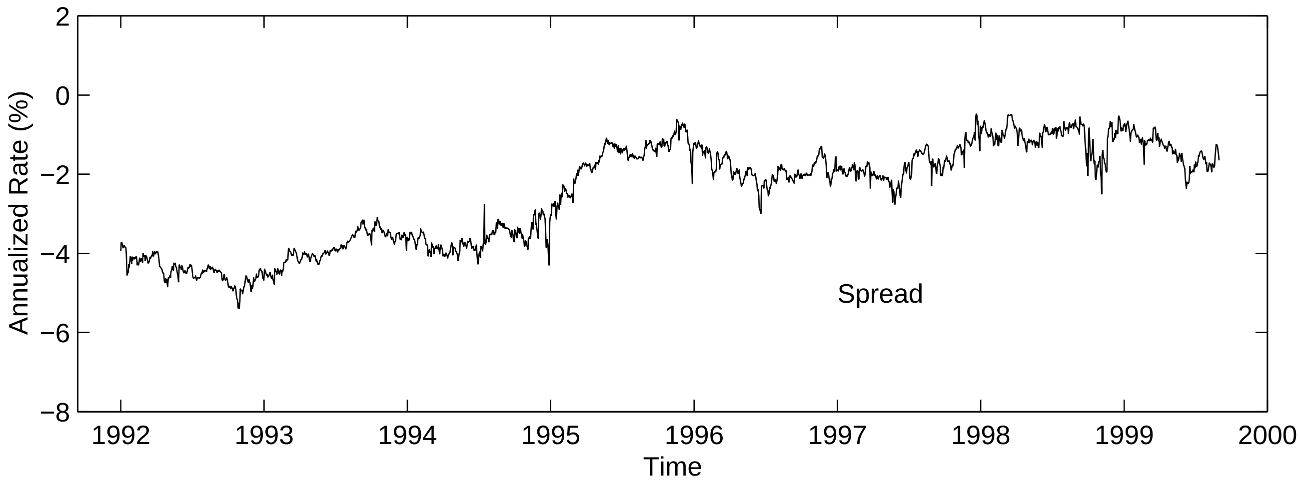
Note: The underlying 5-year discount bond has a face value of $100. The strike price X is a proportion of the current bond price for each model. The top element of each cell is the non-parametric price. The standard errors are in parentheses. The second element is the parametric price.

Table 7: Out-of-the-money call option prices on a 5-Year discount bond (X=1.02)

Maturity (Years)	Spread s	Long rate l			
		5.92	6.69	7.46	8.23
0.5	-3.79	0.6161	0.8079	0.8818	0.9252
		(0.0177)	(0.0116)	(0.0063)	(0.0087)
		0.5695	0.7769	0.9517	1.0974
	-2.50	0.5899	0.7424	0.8769	0.9064
		(0.0126)	(0.0138)	(0.0075)	(0.0074)
		0.5415	0.7453	0.9174	1.0613
	-1.21	0.5619	0.6803	0.8439	0.9148
		(0.0120)	(0.0157)	(0.0062)	(0.0079)
		0.5145	0.7147	0.8840	1.0260
1	-3.79	2.5395	2.8940	2.9935	3.0518
		(0.0247)	(0.0218)	(0.0120)	(0.0156)
		2.4899	2.8456	3.1435	3.3899
	-2.50	2.4836	2.7449	2.9566	2.9883
		(0.0243)	(0.0256)	(0.0124)	(0.0123)
		2.4119	2.7612	3.0546	3.2978
	-1.21	2.4301	2.6219	2.8984	2.9965
		(0.0214)	(0.0288)	(0.0100)	(0.0139)
		2.3361	2.6791	2.9678	3.2079

Note: The underlying 5-year discount bond has a face value of $100. The strike price X is a proportion of the current bond price for each model. The top element of each cell is the non-parametric price. The standard errors are in parentheses. The second element is the parametric price.

Figure 1. Spreads and Long Rates (1988:01:02 − 1999:08:31)



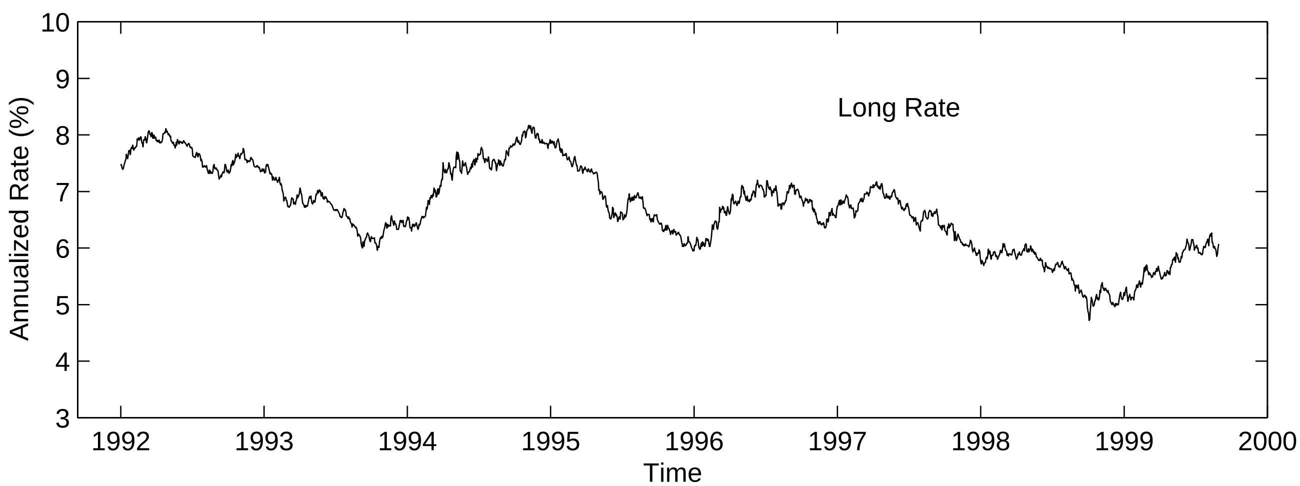


Figure 2. First Differences of Spreads and Long Rates

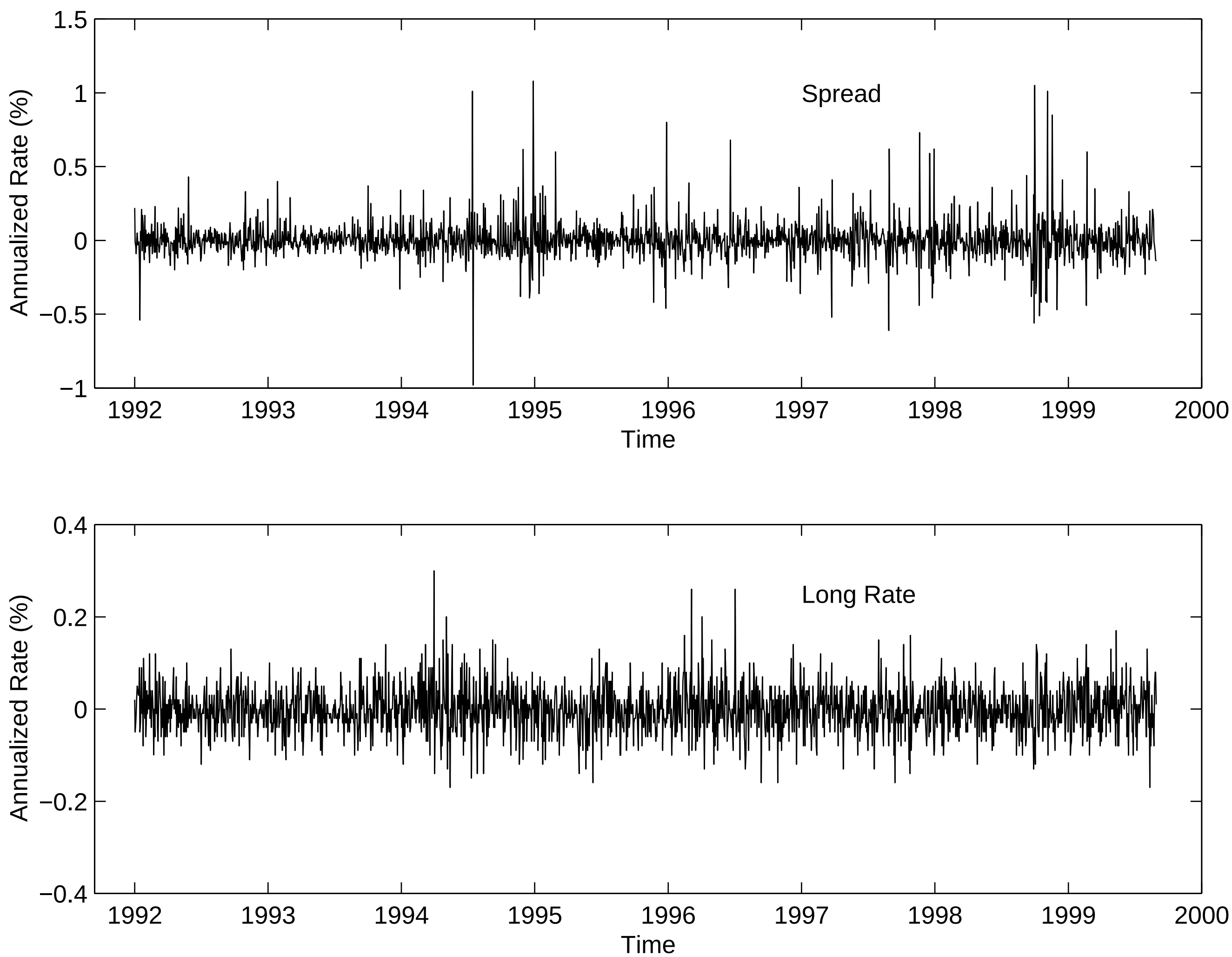


Figure 3. Two Dimensional Nonparametric Kernel Density

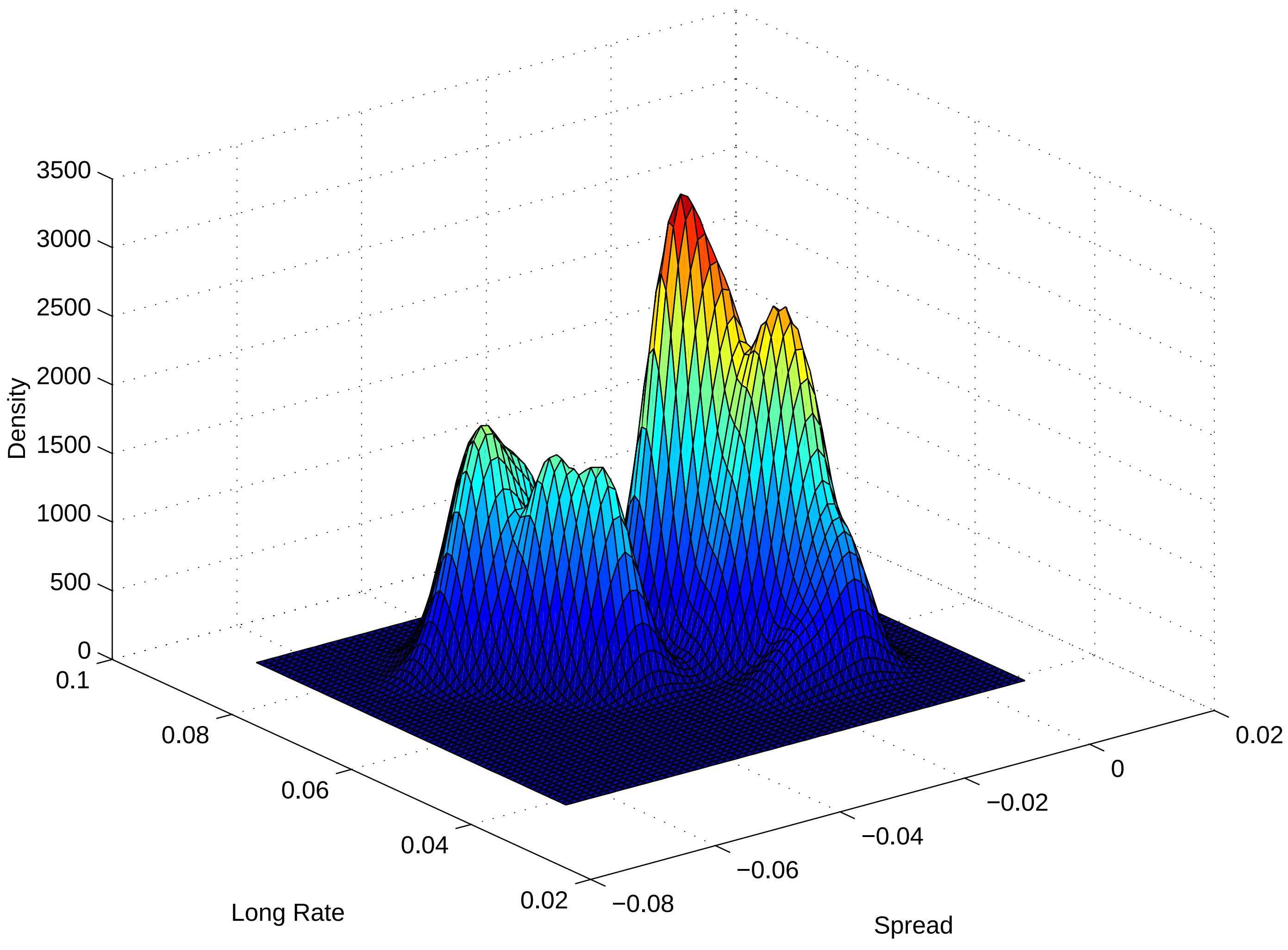


Figure 4. Selected Estimates of Nonparametric Kernel Density

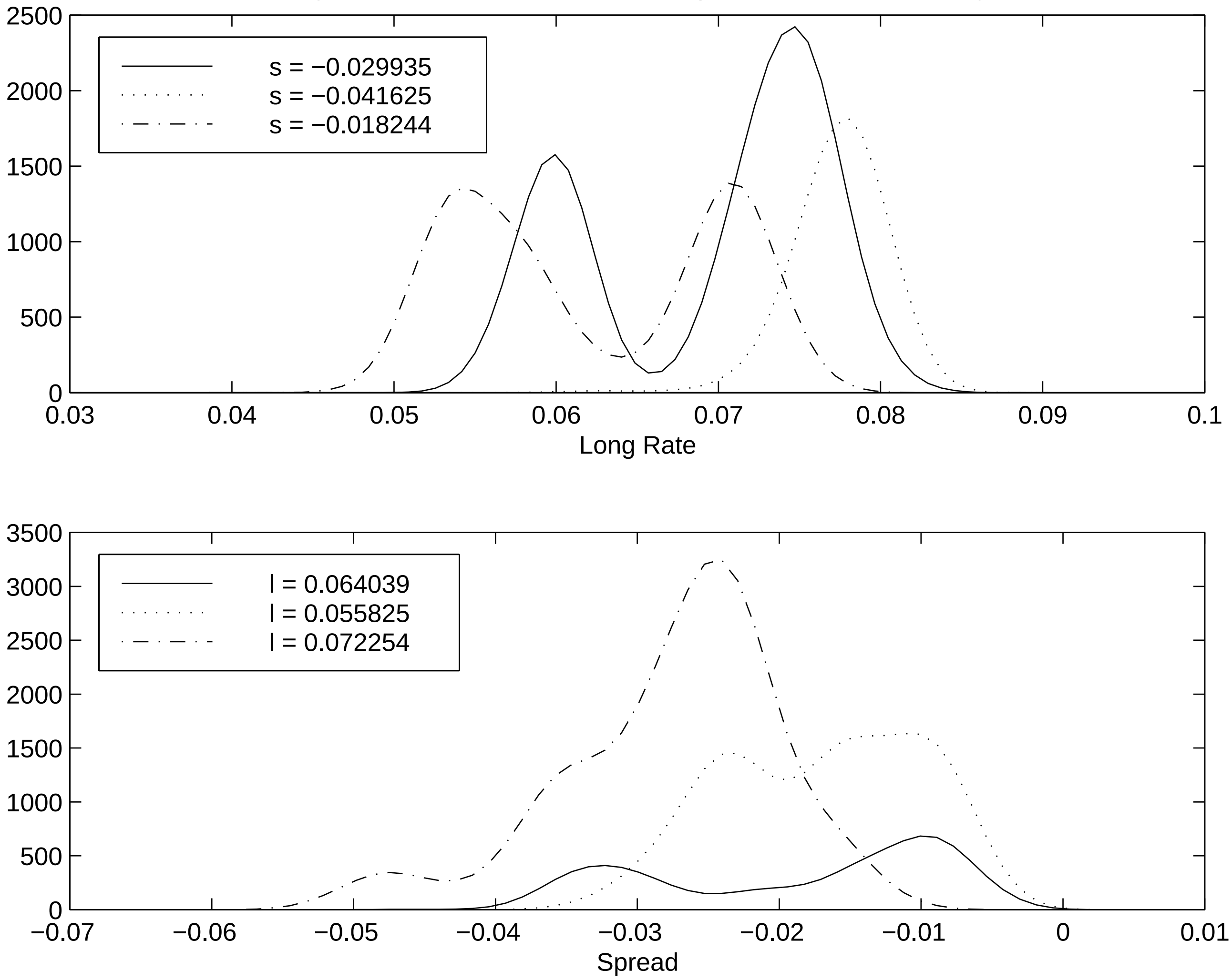


Figure 5. Nonparametric Estimation of Spread Diffusions ($\sigma_1^2(s,l)$)

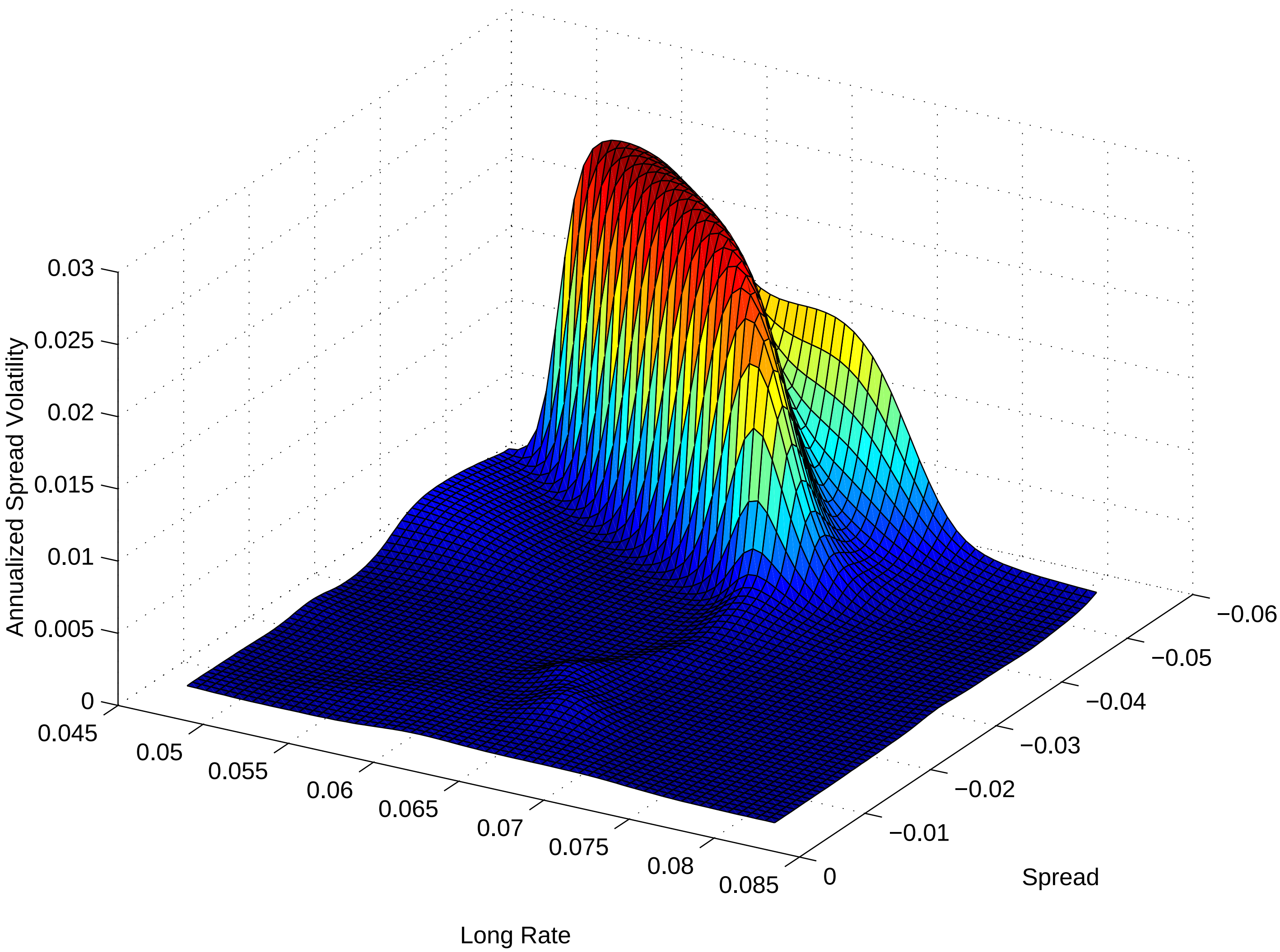


Figure 6. Selected Estimates of Spread Diffusions

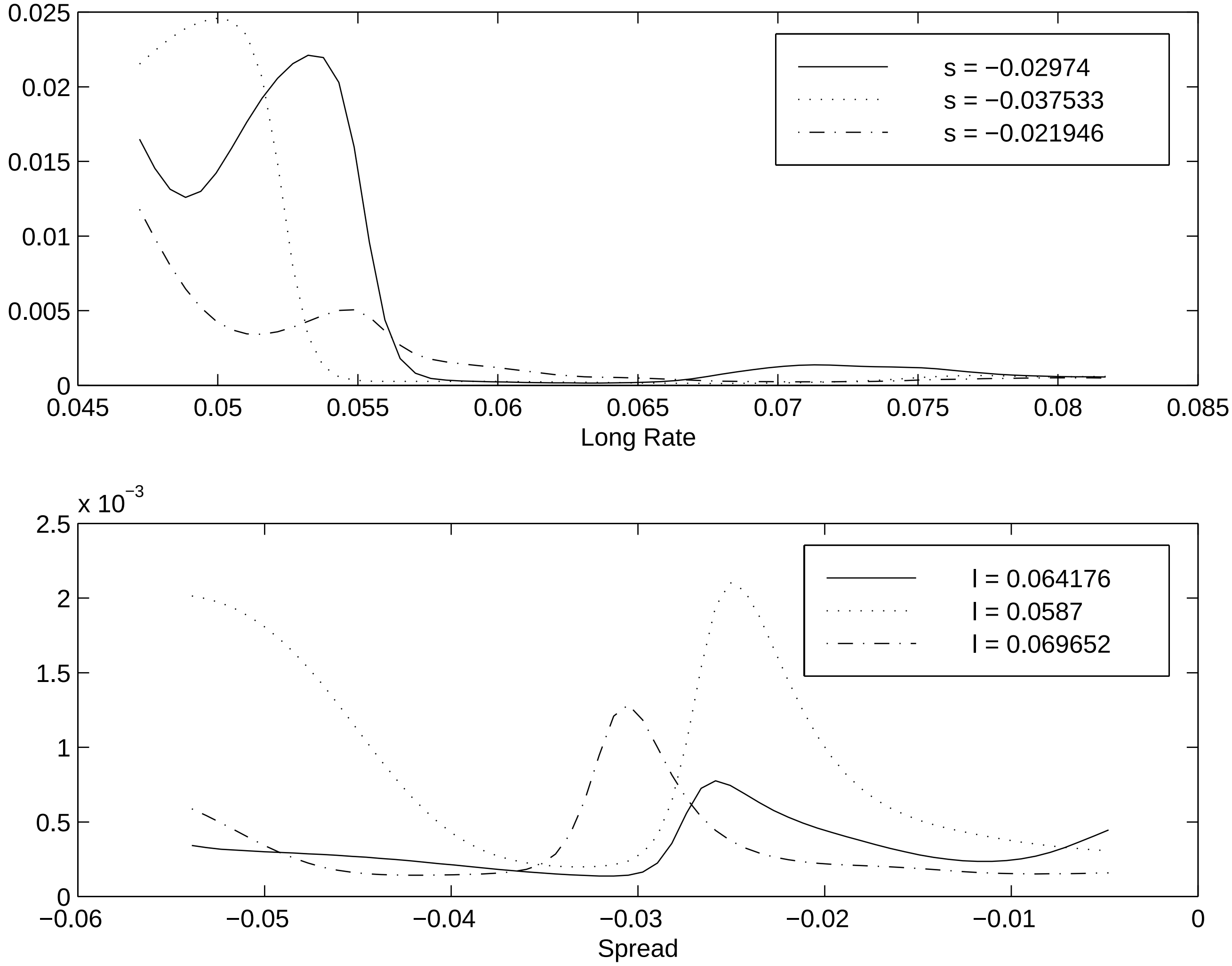


Figure 7. Nonparametric Estimation of Long Rate Diffusions ($\sigma_2^2(s,l)$)

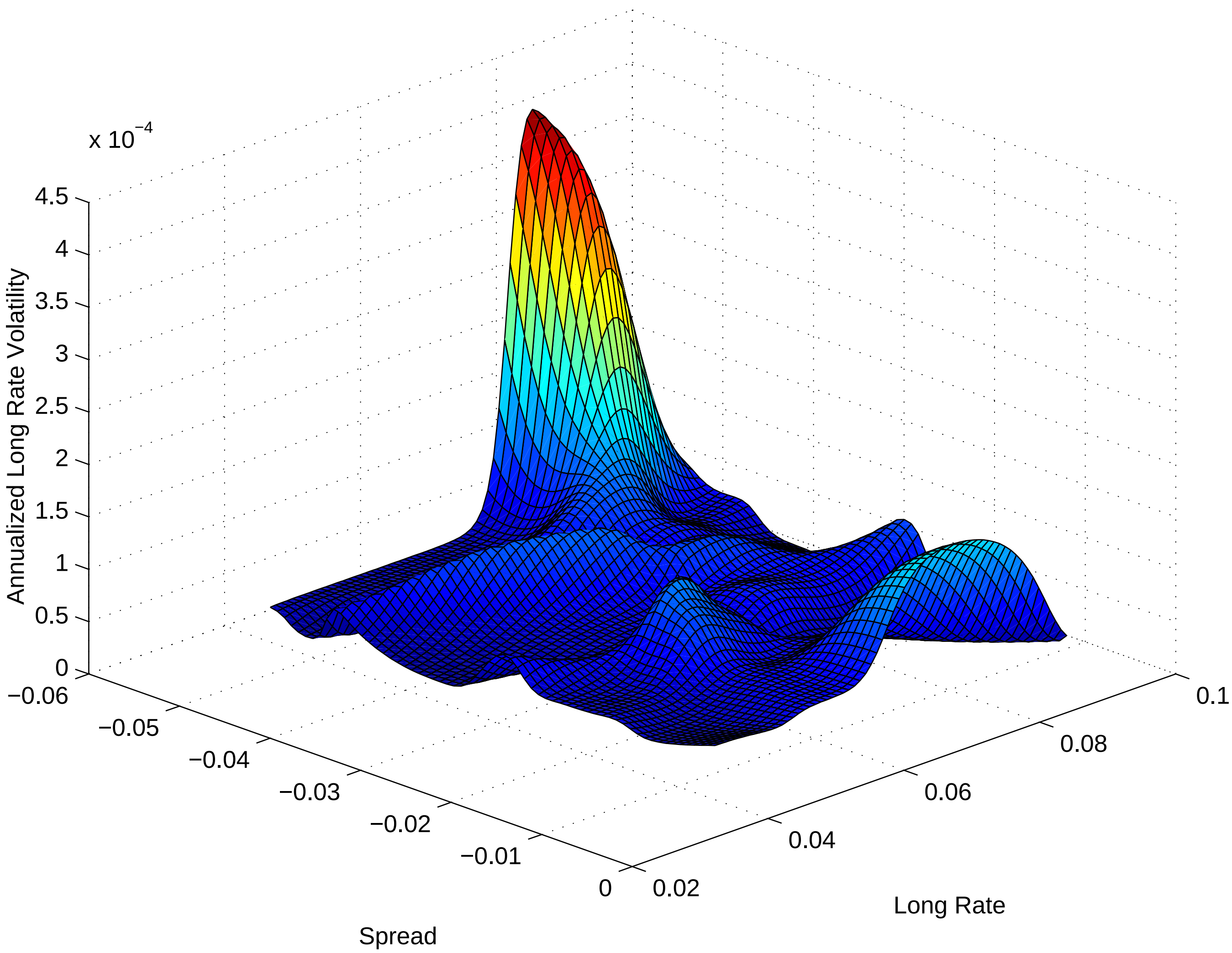
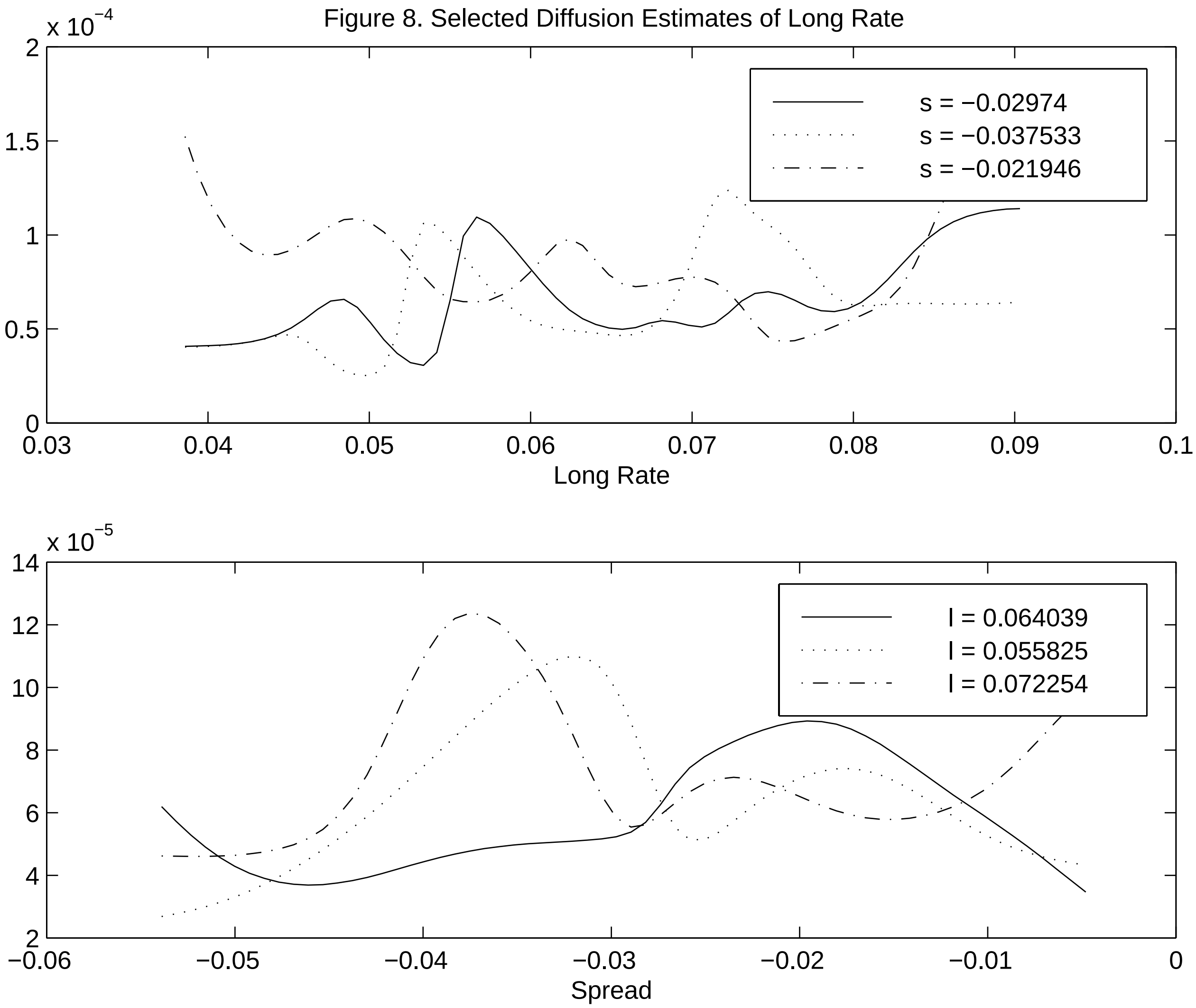


Figure 8. Selected Diffusion Estimates of Long Rate



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