Best Instruments for Market Discipline in Banking

by Greg Caldwell
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Abstract

The author develops a dynamic model of banking competition to determine which capital instrument is most effective in disciplining banks’ risk choice. Comparisons are conducted between equity, subordinated debentures (SD), and uninsured deposits (UD) as funding sources. The model, adapted from Repullo (2004), analyzes the effectiveness of regulatory capital when banks incorporate charter value and competition for depositors into their risk-taking decision. The paper’s main finding is that although all three instruments can induce market discipline on banks, equity weakly dominates SD and UD (with SD weakly dominating UD).

JEL classification: G21, G28
Bank classification: Financial institutions

Résumé

L’auteur construit un modèle dynamique de concurrence bancaire pour déterminer quelle source de capitaux constitue l’instrument de discipline le plus efficace pour limiter la prise de risques par les banques. Les sources de financement qu’il compare sont les capitaux propres, les titres de dette de rang inférieur et les dépôts non assurés. À partir d’une version adaptée du modèle de Repullo (2004), l’auteur évalue l’efficacité de la structure du capital réglementaire dans un cadre où les banques décident du niveau de risque assumé en tenant compte de la valeur de licence et de la concurrence dont les déposants sont l’objet. Sa principale conclusion est la suivante : bien que les trois sources de financement considérées puissent toutes être un vecteur de la discipline de marché, ce sont les capitaux propres qui dominent généralement, devant les titres de dette de rang inférieur, lesquels dépassent légèrement en efficacité les dépôts non assurés.

Classification JEL : G21, G28
Classification de la Banque : Institutions financières
1. Introduction

Given the increased complexity of banking operations and the corresponding supervisory requirements imposed on banks, more attention has been focussed on identifying sources of market discipline for banks. Market discipline rests on the notion that market prices and information signals to third parties provided by the market prices of bank liabilities can alter the risk-shifting incentives of banks. If effective, market discipline could either complement the current regulatory framework or, perhaps, act as a substitute for it. The Basel Committee has recognized the importance of market discipline in the supervisory environment, making it the third pillar of new capital Accord (BIS, 2004). The goal of this pillar is to ensure that banks provide clear and accurate disclosure of their financial status to the market. If this is done properly, the market’s judgment about a bank’s riskiness will be reflected in the prices of its liabilities. Fear of being punished for excessive risk-taking will align management’s decisions with the desires of supervisors.

Yet, even with better information being available, a question remains unaddressed. Namely, which instruments are most effective at providing market discipline for banks? Since capital is scarce and monitoring is a costly endeavour why not get the most out of the instrument of choice for disciplining banks? This paper examines three potential choices: Equity, subordinated debt (SD), and uninsured deposits (UD).

The theoretical literature analyzing the merits of SD versus equity is not as advanced as the empirical or policy literature in advocating SD. Although the literature has looked at general debt instruments as mechanisms for disciplining a bank (e.g., Calmoris and Kahn (1991) and Dewatripont and Tirole (1994)) it has not specifically identified attributes in SD that make it attractive relative to equity. If anything the literature has undermined this commonly-held notion. Levonian (2000) using a Merton-approach to conduct marginal analysis of equity and SD prices to changes in a single-bank’s risk profile. The policy variables are probability of failure and liability of the deposit insurer. There is no analysis of optimal capital structure and risk is not endogenously determined. Blum (2002) considers a stripped down model of banking where there is no equity, in order to show that adding SD can actually

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1Market discipline can affect the bank’s behaviour directly or through indirect channels. The model developed in this paper is concerned with direct sources. It determines which funding source most effectively induces the bank manager/owner to choose an efficient level of risk-taking. This represents a direct effect on bank behaviour since the cost of using these instruments to fund the bank’s operations increases with risk-taking. The alternative, indirect channel, which this paper abstracts from, relies on either supervisory intervention, or the market’s reaction, being sensitive to price movements within the liability instrument. For a discussion on the differences between direct and indirect market discipline, see either Bliss (2001) or Flannery (2001).
increase risk-taking if the bank cannot commit to a level risk before issuing.

In this paper we consider a richer regulatory and market structure. The methodology here is based on a theoretical model of dynamic banking developed by Repullo (2004). This framework analyzes the incentives for a bank to take on risk, in the presence of regulatory constraints, imperfect competition and endogenous charter value. Banks have a choice between investing in an inefficient gambling asset or an efficient prudent asset. This paper then extends the Repullo model by introducing the three liability instruments and ranking their effectiveness at inducing a prudent equilibrium (PE) under different parametric conditions. This determines the minimum necessary capital requirements for sustaining a PE, i.e., combinations of liabilities that form a PE frontier.

The main finding of this paper is that, in terms of discipline per unit of capital, equity weakly dominates both SD and UD (while SD weakly dominates UD). Although all three types of liabilities impose some degree of market discipline on bank lending, equity is the most efficient. The effectiveness of each instrument conforms to the absolute priority of claims in the event of insolvency, which is quite intuitive. Uninsured depositors have claims which are senior to both SD and equity in bankruptcy proceedings therefore, they have the highest probability of receiving funds and thus impose the least discipline on the bank. Subordinated debt holders receive the residual assets after senior claimants and general creditors. Finally, equity holders expect the greatest loss if the bank is unsuccessful. Therefore they impose the most discipline. The paper finds this ranking to be robust to all various parameterizations. Next, these results are shown to be robust to more realistic assumptions about the costs of information acquisition and monitoring by market participants. Evidence from other empirical studies is summarized which suggest secondary markets for equity are more liquid than SD markets for banks.

Section 2 reviews the literature and discusses the ‘merits’ of each of these three liability instruments. Section 3 develops the general model, while Section 4 considers the performance of SD relative to equity. Section 5 repeats this analysis for UD and provides a comparison between various combinations of all three liabilities. Section 6 reviews the results and discusses the implications of more realistic assumptions on monitoring costs. Section 7 gives the conclusions.

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2 A prudent equilibrium occurs when all banks invest in the prudent asset in each period. The equilibrium satisfies a Markov-perfect criterion in that it is robust to any single, individual bank, deviating from this strategy and investing in the gambling asset for one period.
2. A Review of the Options: Equity, SD and UD

Curtailing excessive risk-taking by banks, in a deposit insurance regime, is one of the main objectives of regulatory capital requirements. In order to mitigate the deposit insurance liability a regime of supervision is established so that banks remain sufficiently capitalized.\(^3\)

Equity has traditionally been the staple source of discipline for a bank from both the regulatory and economic perspective. To the regulator, equity represents a source of permanence within the bank, the potential owners have an interest in the bank’s long-term performance. Equity is also appealing since it acts as a buffer against unexpected losses in the loan portfolio. Due to its residual status, it performs much like a deductible on an insurance policy. If the bank is required, by regulation, to hold sufficient equity capital, in proportion to the risk-weighted assets it carries, then stock prices should be sensitive to changes in the bank’s loan portfolio. Concern about adverse stock price movements can positively affect the behaviour of the bank’s managers. If they wish to keep the bank’s financing costs down and minimize regulatory scrutiny, they will have incentives to behave more prudently than otherwise.

As a disciplinary tool, equity has its drawbacks. One concern is that the contingent payoff structure of equity is similar to a put option on the underlying cash flow of the bank with a strike price equal to its debt obligations (Merton, 1974). Under standard assumptions used for valuing derivative products, the value of equity can be shown to be positively and monotonically related to the volatility of the return on the bank’s assets.\(^4\) This is, however, the opposite of what is desired which raises concerns that the price of equity might not be sensitive to changes in the risk of the bank’s loan portfolio. Empirically, Krainer and Lopez (2004) show that equity prices are able to predict downgrades in supervisory ratings of banks, but this may reflect ex post performance as opposed to ex ante discipline.

Consider, alternatively, SD as a potential source of market discipline. The payoff structure of SD has limited upside potential as risk increases. Meanwhile the downside exposure increases with bank risk-taking, making potential holders of SD sensitive to the bank’s risk profile. Empirical evidence supports the argument that secondary market and issuance spreads on SD are sensitive to bank insolvency risk (Flannery and Sorescu, 1996; Covitz et al., 2003). SD also has many policy advocates for its use as a tool for market discipline.\(^5\)

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\(^3\)See Berger et al. (1993) for a review of the rationale behind regulatory capital.

\(^4\)This is easily shown if the option is European. However under an American option the relationship is non-monotonic. Geske and Shastri (1981) showed that when the exercise power is optional over the life of the bank, increased risk-taking can lower the value of the bank.

\(^5\)The Board of Governors (1999) and the Board and Treasury (2000) provide a complete documentation
On the theoretical side, Levonian (2000) questions whether SD brings additional stability beyond what can be found in equity. Blum (2002) points out that SD cannot mitigate post-contractual incentives to risk-shift, a time consistency problem.

Another potential source of discipline is UD. If a bank is near default, evidence suggests that uninsured depositors become unwilling to supply funding at any reasonable price. This ‘flight-to-quality’ effect has been documented empirically for deposit withdrawals (Calomiris and Powell, 2000) and for large certificates of deposit (James, 1988). Given the leveraged nature of banking, lacking capability to access the uninsured deposit market could create funding constraints on a bank at a critical moment.

Aside from Levonian (2000), there is little guidance in the literature on which of these instruments are most effective at disciplining a bank. Levonian argues SD does not discipline bank risk-taking any better than equity. He finds that replacing equity with SD one-for-one, does not change bank incentives. Holders of SD demand higher risk premia if the bank is risk-taking but this is offset by the increased leverage of the bank. The diluted equity that remains has a greater appetite for risk-shifting.

This paper shows that a lower risk premium on SD (and UD), not increased leverage, reduces the effectiveness of SD (and UD) as a tool for market discipline. In a dynamic setting, increased leverage might enhance the gains of risk-shifting in the current period but at the expense of risking future rents. If these rents are sufficiently large then a bank could be steered towards an efficient investment choice. Once charter value is incorporated, SD holders and UD holders suffer less than equity holders.


The Repullo (2004) model consists of a dynamic, universally risk neutral, banking economy with \( n \) banks located at equal distances around a circular city. There is a continuum of depositors, of measure one, that are uniformly dispersed around the city. In each period, a new set of depositors emerge, each endowed with one unit of insurable deposits that will be used for consumption at the end of the period. If their bank fails, an exogenous deposit insurer will reimburse principal and any accrued interest. Insured depositors face linear transaction costs, \( \mu \), per unit of distance, to travel to any bank to deposit their funds. Each depositor enjoys a non-pecuniary and homogeneous benefit, denoted \( v \), from their deposit

\footnote{of the various proposals over the prior 20 years. (Herring, 2004) provides a more up-to-date proposal for mandatory SD as an alternative to the new Basel Capital Accord, Basel II.}
account.\textsuperscript{6} This benefit will enable gross deposit rates to vary below one, depending on the level of banking competition and the inconvenience of travelling to another bank.

Define the utility of a generic depositor located $x$ units of distance away from bank $j$ as:

$$U(x, j) = v + r_j - \mu x.$$ 

Without loss of generality, it is assumed that $v$ is sufficiently large such that, regardless of the industry structure, equilibrium deposit rates will be high enough to attract all deposits.\textsuperscript{7}

The marginal depositor, denoted $\tilde{x}$, is indifferent between the two closest banks under the following identity:\textsuperscript{8}

$$v + r_j - \mu \tilde{x} = v + r_{j+1} - \mu \left( \frac{1}{n} - \tilde{x} \right). \quad (1)$$

Solving for $\tilde{x}$ gives the location of the marginal depositor

$$\tilde{x}(r_j, r_{j+1}) = \frac{1}{2n} + \frac{r_j - r_{j+1}}{2\mu}. \quad (2)$$

The measure $\tilde{x}(r_j, r_{j+1})$ represents bank $j$’s insured deposit base on the interval $[j, j + 1]$. Given the symmetrical features of this model, the total deposit base for bank $j$ is $2\tilde{x}$ or

$$D(r_j, r_{j+1}) = \frac{1}{n} + \frac{r_j - r_{j+1}}{\mu}. \quad (3)$$

The bank’s objective is to maximize the value of the bank to its shareholders by choosing, in each period, an insured deposit rate, $r_j$, and a choice between investing in a prudent or a gambling asset.\textsuperscript{9} The prudent asset is risk-free and pays $1 + \alpha > 1$ for each dollar invested. The gambling asset pays $1 + \gamma > 1$ with probability $1 - \pi$ (success) and $\beta \geq 0$ with probability

\textsuperscript{6}These benefits can be thought of as payment services or a comparative advantage that banks have over depositors in currency-storage technology.

\textsuperscript{7}This implies $v > \mu/2$ since if there were only one bank, located perhaps at $x = 0$ the greatest disutility of any depositor is incurred by the individual located at $x = 1/2$.

\textsuperscript{8}Banking at bank $j + 1$ will give $U(x, j + 1) = v + r_{j+1} (1/n - x)$ to depositor $x$.

\textsuperscript{9}This paper abstracts from agency problems between the management of the bank and the equity holders. Decisions of the bank management are assumed throughout to be perfectly aligned with the interests of equity holders.
π (default). The gambling asset is assumed to be stochastically dominated however it does offer a higher return in the successful state. The following condition summarizes these points:

\[ 1 + \gamma > 1 + \alpha > (1 - \pi)(1 + \gamma) + \pi \beta \equiv E_\gamma. \] (4)

Condition (4) deserves further comment. The reader might wonder why a banker would ever be interested in the gambling asset given it has greater risk and lower expected return. The problem is that, due to limited liability, if the banker gambles it only repays depositors if the gamble succeeds. Also if other banks are lending prudently, a gambling banker can raise its offered rate on insured deposits and attract a greater market share to gamble with.\(^{10}\) If it has incentives to do this then only a gambling equilibrium can be sustained, since a one-period race to the bottom ensues for all market participants.

On the liability side, the bank meets whatever capital requirements are imposed by supervisory authorities. The extension introduced by this paper to the Repullo model is three instruments available, denoted by a \(1 \times 3\) vector, \(\chi \equiv (k, s, u)\), representing minimum equity, (k), subordinated debt, (s), and uninsured deposit, (u), requirements.

Each capital instrument is modeled as being costly to hold in excess: The expected return demanded on capital, \(\rho\), is assumed to be greater than the return offered by the prudent asset \(\alpha\). This assumption implies that the owners of the bank will require a minimum amount of insured deposits in order to not exit. Insured depositors benefit since they get a higher return depositing their funds than in autarky. The remainder (i.e., \(1 + \alpha - r\)) compensates the bank owner for the negative return on its funds: \(k(\alpha - \rho) < 0\). This could be thought of as gains from intermediation resulting from banks' comparative advantage of being able to reallocate these funds towards investment projects with returns above what insured depositors can find.\(^{11}\)

The bank must choose its lending technology, deposit rates and the amount of capital it raises, subject to minimum capital requirements imposed by the supervisor. Repullo (2004) shows that the equity capital requirement is always binding for this environment, a feature

\(^{10}\)Loosely, if it gambles it expects \((1 - \pi)(1 + \gamma - r_{gp})D_{gp}\) while if it lends prudently it expects \((1 + \alpha - r_{p})D_{p}\), where \(r_{gp} > r_{p}\) and \(D_{gp} > D_{p}\) are the amount of insured deposits attracted. As \(r_{p} \to 1 + \alpha\), perhaps due to increased competition, the second term approaches zero. Yet there is still a possibility to gamble and raise the deposit rate further as a best response.

\(^{11}\)De Young and Rice (2004), point out that deposit accounts provide revenues for banks both from explicit fees but also foregone interest revenue by depositors.
which also holds for both minimum SD and UD requirements in this model extension.\footnote{This result arises from a bank’s access to insured deposits and an assumption of risk neutrality. In the absence of a regulatory capital requirement, a risk neutral bank would choose to use insured deposits exclusively and hold zero capital. The economic capital requirement in this model must be zero and non-binding in the presence of a positive regulatory capital requirement. This feature is important to understand since there is a debate about whether the regulatory or economic capital requirement binds for banks (Jones, 2000; Elizakle and Repullo, 2004). This model abstracts from that debate and assumes the binding capital requirement is regulatory. In reality, banks in Canada hold capital in excess of the minimum requirements stated by the Office of the Superintendent of Financial Institutions (OSFI). Risk aversion is a likely factor. Namely, a risk averse bank manager would maintain a buffer if dropping below the requirement implied regulatory intervention.} Consequently, this paper assumes throughout that the minimum regulatory capital requirement binds and focusses on the bank’s decisions on lending technology and deposit rates.

Each bank is assumed to be infinitely lived, provided it always invests in the prudent asset. If it gambles it will be shut down by the supervisor only if the gamble is unsuccessful. A bank shut down at time \( t \) will be replaced in the subsequent period by a new bank at the exact same location of the failed bank. This discussion has summarized the Repullo template. The next two sections describe in further detail extensions to the model of subordinated debt and uninsured deposits markets.

\textbf{3.1 Subordinated debt}

For the moment, consider the mechanics of SD in isolation. Define an all-SD regime as \( \chi^s \equiv (0, s, 0) \). For every $1 in insured deposits raised the bank is required to issue $s$ of SD at the market determined rate \( R^s_j \). After the next period, if the liabilities \( r_j + sR^s_j \) exceed the realized payoff from the investment by the bank then it will be closed by the supervisor. SD holders then have a claim on the residual, after insured deposits (or the deposit insurer) have been remunerated: max\( \{0, \beta - r_j\} \).

We abstract from any informational asymmetry between the SD holder and the bank concerning the latter’s choice of lending. Consequently, the equilibrium rate will reflect the inherent riskiness derived from the underlying loan. If the bank invests in the gambling asset, the equilibrium rate demanded by SD holders satisfies the following “no-arbitrage” argument:

\[
(1 - \pi)R^s_{g,j} + \pi \max\{0, \beta - r_{g,j}\} = (1 + \rho)
\]

or
$$R_{g,j}^s(r_{g,j}) = \frac{1 + \rho - \pi \max\{0, \beta - r_{g,j}\}}{1 - \pi}.$$  \hspace{1cm} (5)

If the bank invests in the prudent asset, there is no risk of loss for SD holders and
\begin{align*}
R_{p,j}^s(r_{g,j}) &= 1 + \rho. \hspace{1cm} (6)
\end{align*}

### 3.2 Uninsured deposits

Alternatively, banks could be ordered to exclusively raise funds in the form of UD. This regime is defined as $\chi^u \equiv (0, 0, u)$. For every dollar of insured deposits the bank brings in it must also raise $\$u$ of UD, paying a rate $R_j^u$. Again, it is assumed that there is no informational asymmetry between the UD holder and the bank.

In the event that the bank is closed, uninsured depositors have a prorated share of the banks recovery value with the deposit insurer. The amount of liabilities the bank owes is defined as $A_o \equiv r_j + uR_j^u$. The bank is closed only if its realized asset value is below its interest and principle owed from its liabilities (i.e., if $A < A_o$). In this event, the share of the realized asset value $\beta$ accruing to uninsured depositors is $R_{g,j}^u/A_o$. By similar no-arbitrage arguments, if the bank invests in the gambling asset then:

\begin{align*}
(1 - \pi)R_{g,j}^u + \pi\beta R_{g,j}^u/A_o &= (1 + \rho)
\end{align*}

This defines a quadratic function in $R_{g,j}^u$ with solution:

\begin{align*}
R_{g,j}^u &= \frac{-(1 - \pi)r_{g,j} + u(\pi \beta (1 + \rho)) + \sqrt{\Theta}}{2(1 - \pi)u}, \hspace{1cm} (7)
\end{align*}

where
\begin{align*}
\Theta &= ((1 - \pi)r_{g,j} + u(\pi \beta - (1 + \rho))^2 + 4(1 - \pi)(1 + \rho)ur_{g,j}.
\end{align*}

Note that this function, $R_{g,j}^u$, is increasing and concave in $r_{g,j}$. If instead the bank invests in the prudent asset, the return will again be
\begin{align*}
R_{p,j}^u &= 1 + \rho. \hspace{1cm} (8)
\end{align*}
As a point of comparison, both UD and SD offer the same return when the bank is choosing the prudent asset. The difference between these instruments’ prices arises when the bank gambles. The SD rate increases linearly in the insured deposit rate, $r_{g,j}$, before flattening out once the recovery threshold, $\beta$, is passed. Beyond this point holders of SD expect 100 percent losses if the bank fails. The relationship between the uninsured deposit rate and the insured deposit rate is increasing but at a diminishing rate. In fact, the uninsured deposit rate never exceeds $$(1 + \rho)/(1 - \pi),$$ the maximum SD equilibrium return which occurs when $r_{g,j} > \beta$. This feature reflects the different level of seniority of the two types of claims. When the insured deposit rate increases, this lowers the share of $\beta$ that an uninsured deposit claim receives relative to the deposit insurer. However the same insured deposit rate increase directly lowers the amount of the bank’s recovery value left to the SD holder one-for-one. Consequently, the effect on the SD rate versus the uninsured deposit rate is more pronounced when the bank moves from prudent behaviour to gambling.

4. Banking with subordinated debt

The purpose of this section is to derive conditions for sustaining a prudent equilibrium with only equity and SD combinations.$^{13}$ The regime is denoted $\chi^{k,s} = (k, s, 0)$. At that point, we know the amount of capital assets that the bank must hold in order to be deterred from gambling. To accomplish this, first, an all-prudent asset economy, where no gambling assets exist is analyzed. Next, an all-gambling asset economy is examined. This provides analytical information that is useful in describing conditions for sustaining the prudent equilibrium (PE).

The last subsection will allow the bank to choose between the prudent and gambling asset and minimum conditions are derived whereby all banks voluntarily choose the prudent asset in each period. Provided no single bank has an incentive to deviate and gamble, one-shot, then the specified environment constitutes a sustainable Markov-perfect prudent equilibrium.$^{14}$ The next two sections lay out some of the groundwork necessary for the construction of a Markov-perfect criterion.

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$^{13}$The next section will describe conditions whereby combinations of equity and UD can sustain a PE.

$^{14}$A Markov-perfect equilibrium (MPE), the criteria that Repullo (2004) used, is a dynamic strategy profile that is robust to one-shot deviations. In this case, a strategy profile where every bank is choosing to invest in the prudent asset in every period will be the conjectured MPE. Provided there is no incentive for a single period deviation by one bank towards investing in the gambling asset and then returning to investing in the prudent asset in subsequent periods, then the strategy profile conjectured as a MPE holds.
4.1 Prudent only asset regime

Suppose, the bank can only invest in the prudent asset. Since there is no risk of default the value equation for bank $j$ is

$$V_p = \frac{1}{1 + \rho} \left( (1 + \alpha)(1 + s + k) - r_{p,j} - (s + k)(1 + \rho) \right) D(r_{p,j}, r_{p,j+1}) + \frac{1}{1 + \rho} V_p.$$  \hfill (9)

The first term represents the spread (or margin) per unit of insured deposits times the amount of insured deposits the bank takes in. The last term represents the present (continuation) value of investing in the prudent asset each period forever, i.e., the charter value of the bank.

The value function is concave in $r_{p,j}$, so, taking first-order conditions, and imposing symmetry (implying $D(r, r) = 1/n$), results in the PE insured deposit rate for bank $j$\(^{15}\)

$$r_p = (1 + \alpha)(1 + s + k) - (s + k)(1 + \rho) - \frac{\mu}{n}. \hfill (10)$$

Given a steady state, we can solve for the value function:

$$V_p = \frac{1}{1 + \rho} \left[ \frac{\mu}{n^2} \right] + \frac{1}{1 + \rho} V_p = \frac{\mu}{\rho n^2}. \hfill (11)$$

Provided the bank is making positive returns (or, provided insured deposits are imperfect substitutes) the charter value is pivotal to the bank’s choice between the prudent and gambling asset in the current period. As competition increases, either through $n$ increasing or $\mu$ decreasing, the incentive to remain prudent diminishes, ceteris paribus.

**Result 1** Under a prudent equilibrium, only insured depositors are affected by changes to regulatory capital requirements $\chi^{ksd}$.

This result can be seen from three key observations: $V_p$, the bank’s future charter value, is invariant to either $k$ or $s$; the single period return to the shareholders of the bank is always $1 + \rho$, regardless of $k$ and $s$; and both $k$ and $s$ lower the insured deposit rate. The first observation follows from the definition of $V_p$. For the second observation, the competitive price for ownership of the bank, in a PE, would be $V_p + kD(r_p, r_p)$, given a regulatory requirement of $\chi^{ksd}$. If bought for this price, the realized residual value of the bank after one

\(^{15}\) Without loss of generality, we drop the subscript denoting bank $j$, which was arbitrarily chosen.
period is $V_p + (k(1 + \rho) + \mu/n)D(r_p, r_p)$. This leads to a gross return of $1 + \rho$.\footnote{Explicitly,}

Finally, the insured deposit interest rate $r_p$ is strictly decreasing in either $k$ or $s$, provided $\alpha < \rho$. In conclusion, as banks face intermediation cost increases through tougher capital requirements, this gets passed onto insured depositors through lower rates, regardless of the degree of market power banks have.

4.2 Gambling only asset regime

Assuming, for the moment, that the bank can only invest in the gambling asset in each period which introduces the possibility of bank default. Bank $j$ survives each period with probability $1 - \pi$ and, consequently, its value equation reflects this feature:

$$V_{g,j} = \frac{1 - \pi}{1 + \rho} \left[ (1 + \gamma)(1 + s + k) - r_{g,j} - sR_{g,j} - k \frac{1 + \rho}{1 - \pi} \right] D(r_{g,j}, r_{g,j+1}) + \frac{1 - \pi}{1 + \rho} V_{g,j}.$$  (12)

Following the same procedure, as in the previous section, the insured deposit rate under a symmetric gambling equilibrium is derived from the first-order conditions:

$$r_g = \left[ (1 + \gamma)(1 + s + k) - (k + s) \frac{1 + \rho}{1 - \pi} \right] - \frac{\mu}{n}, \quad \text{if } \beta \leq r_g,$$  (13)

whenever there is no residual recovery value for SD holders (i.e., $\beta \leq r_g$). Otherwise, the deposit rate is more involved:

$$r_g = \frac{1 - \pi}{(1 - \pi + \pi s)} \left[ (1 + \gamma)(1 + k + s) - (s + k) \left( \frac{1 + \rho}{1 - \pi} \right) + s \pi \beta \frac{1 - \pi}{1 - \pi} \right] - \frac{\mu}{n}, \quad \text{if } \beta > r_g.$$

Notice that the insured deposit rate is now affected by the level of subordinated debt in a non-linear fashion as $s$ appears in the denominator. This reflects a feedback effect between the insured deposit rate and the subordinated debt rate. As $s$ increases we already know this will lower the insured deposit rate. If $\beta > r_{g,j}$, the lowered insured deposit rate further lowers the cost of raising SD, which increases the desire of the bank to further lower $s$. Hence the first derivative of $r_{g,j}$, with respect to $s$, is decreasing, whereas before it was zero.

\footnote{Explicitly,} $$\frac{V_p + (k(1 + \rho) + \mu/n)D(r_p, r_p)}{V_p + kD(r_p, r_p)} = 1 + \rho,$$

Where $V_p$ is given in equation (11).
The gambling equilibrium charter value is derived from substituting \( r_g \) into the bank’s value function

\[
V_g = \frac{1 - \pi \mu}{\rho + \pi n^2}, \quad \text{if } \beta \leq r_j, \tag{14}
\]

and

\[
V_g = \frac{1 - \pi + \pi s \mu}{\rho + \pi n^2}, \quad \text{if } \beta > r_j, \tag{15}
\]

Next, we mimic the results of the previous subsection:

**Result 2** Under a gambling equilibrium, if \( \beta \leq r_j \), only insured depositors are affected by changes to regulatory capital requirements \( \chi^{kad} \).

This result comes from the exact same observations as in the previous section. The charter value \( V_g \) is still invariant to \( k \) and \( s \).\(^{17}\) The single-period return offered to shareholders of the bank is \((1 + \rho)/(1 - \pi)\), leading to an expected return of \( 1 + \rho \), regardless of \( k \) or \( s \).\(^{18}\) However the insured deposit rate is decreasing in \( s \) and \( k \), provided \( (1 - \pi)(1 + \gamma) < 1 + \rho \) which holds by condition (4). These three characteristics lead to the same conclusions as in the all-prudent asset equilibrium: The bank is not affected by increases in equity or SD above zero yet insured depositors are made worse off. There is another point to consider.

**Corollary 1** Provided \( \beta \leq r_j \), or \( s < (1 - \rho)/\rho \) then, \( V_p > V_g \).

What this corollary implies is that banks are worth more, in a charter value sense, under a prudent equilibrium. If, however, the recovery value of the bank exceeds the insured deposit rate then it is still likely that the charter value of the bank under a prudent equilibrium will exceed its value under gambling. The condition where this would not hold requires the discount rate to be sizeable, in order for subordinated debt to exceed \((1 - \rho)/\rho \). This raises the question of why, in the absence of capital requirements, would a gambling equilibrium ever arise? As the next section shows a prudent equilibrium is not always robust to single-bank deviations.

\(^{17}\) This is provided \( \beta \leq r_j \). If this condition fails, then charter value is increasing in \( s \) under a gambling equilibrium.

\(^{18}\) The competitive price for ownership of the bank, in a gambling equilibrium, would be \( V_g + kD(r_g, r_g) \), given a regulatory requirement of \( \chi^{kad} \). If bought for this price, the expected residual value of the bank, is \((1 + \pi)(V_g + (k(1 + \rho) + \mu/n)D(r_g, r_g))\), leading to an expected return of \( 1 + \rho \).
4.3 Prudent equilibrium stability under subordinated debt

We now have the necessary ingredients to determine conditions under which a PE is sustainable, under the Markov-perfection criterion, when each bank can choose between the prudent or gambling asset in each period. Given a conjectured regulatory regime, $\chi^{k_{sd}}$, if all $n$ banks are investing in the prudent asset in each period does any single bank have incentive to deviate for one period? If the answer is no then $\chi^{k_{sd}}$ can sustain the PE.

The sustainability condition is

$$\max_{r_j} \left[ \frac{1 - \pi}{1 + \rho} \left( (1 + \gamma)(1 + s + k) - r_j - sR_j^s(r_j) - k \frac{1 + \rho}{1 - \pi} \right) D(r_j, r_p) \right]$$

$$+ \frac{1 - \pi}{1 + \rho} V_p \equiv V_{gp} \leq V_p. \quad (16)$$

The left-hand side represents the expected value to an arbitrary bank $j$ defecting from prudent lending for one period, while all other $n - 1$ banks continue to lend prudently, and then reverting back to prudent lending in the future, provided it survives the period. The right-hand side represents the value to bank $j$ of continuing to lend prudently forever, given all other banks also continue to lend prudently forever. If the latter value is large enough, then there is no incentive for bank $j$ to unilaterally defect. Since bank $j$ was chosen arbitrarily, and given the symmetry of banks, this implies no bank would defect from lending prudently forever under regime $\chi^{k_{sd}}$, implying the PE is sustainable.

Inequality (16) highlights the trade-offs of deviating from a prudent equilibrium. To gamble now, the owner of bank stock risks losing $V_p$ in the future. Subordinated debt holders demand higher returns in order to be induced to accept the risk of principal loss. Where does the bank recoup the funds in the successful state in order to meet these obligations? As alluded to in the discussion around condition (4), the defecting bank must increase the insured deposit rate it offers in order to attract a larger deposit base. The amount that insured deposits must rise in order to be able to gamble is further constrained by charter value. If there is little competition or $\rho$ is small then the cost of unsuccessful gambling becomes prohibitive. On the opposite side, as competition increases and charter value is driven to zero, the risks of gambling diminish.

If the bank can raise its insured deposit rate, $r_{gp}$, above $r_p$ then it increases its share of the insured deposit market. From the first order conditions on $V_{gp}$ with respect to $r_j$ the
insured deposit rate bank $j$ offers when it defects is:

$$r_{gp} = \frac{r_p + r_g}{2} \tag{17}$$

Since both $r_p$ and $r_g$ are decreasing in $k$ and $s$ this is how capital requirements can stifle the incentive to defect if it causes $r_{gp}$ to decrease. What is important is the relative decrease. If an increase in $k$ or $s$ has a greater impact on $r_{gp}$ relative to $r_p$ then eventually $r_{gp}$ will be less than $r_p$. Intuitively, $r_{gp}$ drops faster than $r_p$ since gambling becomes more expensive as $k$ or $s$ increases since a higher proportion of funding sources (SD and equity versus insured deposits) demand greater compensation for the risk incurred. This makes defection infeasible for bank $j$ since it cannot increase this period’s revenue above what it gets from lending prudently. The following proposition summarizes the relationship between regime $\chi^{ksd}$ and $r_{gp}$.

**Proposition 1** Given $\beta \leq r_{gp}$, $r_{gp}$ exceeds $r_p$ whenever $r_g > r_p$, or equivalently

$$s + k < \frac{(\gamma - \alpha)(1 - \pi)}{(\pi(1 + \rho) - (1 - \pi)(\gamma - \alpha))}.$$ 

Proposition 1 shows that for low capital requirements, revenue can be increased by raising the insured deposit rate above $r_p$ and attracting more insured deposits for gambling this one period. Eventually, as the capital requirements increase this becomes too costly and there is no incentive to invest in the gambling asset. The best response deposit rate to all other banks paying $r_p$ would need to be below $r_p$ in order to maintain an adequate spread to account for costs of capital.

Substituting $r_{gp}$ into $V_{gp}$ one can construct a frontier of minimal combinations of equity and SD that are capable of sustaining a PE.

This equity and SD (KSD) frontier is implicitly defined as

$$k = f(s; \gamma, \beta, \alpha, \pi, \rho, \mu, n|V_{gp}(r_{gp}) = V_p) \tag{18}$$

The following proposition summarizes the relationship between equity and SD, as described by the KSD frontier.

**Proposition 2** When $\beta \leq r_j$, equity and SD are perfect substitutes for sustaining a Markov-
perfect PE. If $\beta > r_j$ then SD is inferior to equity as an instrument for sustaining a PE.

Proposition 2 argues that SD and equity are equally-capable of sustaining a PE only when they have the same contingent payment streams. When the recovery value of the bank is not sufficient to cover senior creditors’ claims then neither the equity holders nor the SD holders receive anything if the bank is unsuccessful. If instead, there is some residual recovery value for the SD holders then they will not demand as high a premium from the bank as the case where the loss under default is 100 percent. In this latter case if a regulator wants to rely on SD to induce a PE it must impose a larger requirement relative to equity.

Figure 1: The Equity Subordinated Debt (KSD) Frontier of Sustainable Prudent Equilibrium Options

![Graph showing the KSD frontier tradeoff between equity and SD.](image)

Figure 1 heuristically describes the KSD frontier tradeoff between equity and SD. The negative relationship between equity and SD suggests they are substitutes in terms of sustaining a PE. In the extreme, both instruments are capable of unilaterally sustaining a PE provided capital requirements are sufficiently high. When $\beta \leq r_j$, equity and SD are perfect substitutes implying a frontier slope of minus one. When $\beta > r_j$, SD is less effective at disciplining the bank since there is residual recovery value in the unsuccessful state. More SD is required to bring about the same impact that can be achieved with equity. This causes the KSD frontier to bow out as the ratio of $s$ to $k$ increases.
Comparative statics are performed to determine how the KSD frontier responds to exogenous shocks, *ceteris paribus.*\(^{19}\)

**Result 3**  
(a) The KSD frontier shifts outwards if either (i) competition increases ($n$ increases or $\mu$ decreases), or (ii) gambling becomes more attractive ($E\gamma$ increases or $\alpha$ decreases or $\pi$ decreases).

(b) The KSD frontier may shift inwards or outwards when the cost of capital, $\rho$, increases. As $n \to \infty$ the KSD will shift inwards when $\rho$ increases, while for low $n$ it will shift outwards as $\rho$ increases.

These results are quite intuitive. As competitive pressures (the number of banks or the homogeneity of their products as measured by $\mu$) increase it requires more equity and/or SD to sustain the PE since market forces have eroded charter value. When gambling becomes more attractive there is a greater need for increased equity or subordinated debt, and the KSD frontier shifts out. There is some ambiguity about the effect of the cost of capital on market discipline. When there is no charter value to the bank ($n$ is large) an increase in the cost of capital will make sustaining a PE easier since the spread must be even larger to make deviating to the gambling asset worthwhile. Meanwhile, if there is a lack of competition, and $\rho$ increases, this reduces the risk associated with losing the charter in the future, since its present value has diminished. Since the bank has greater market power it will have less difficulty increasing its spread. Hence there could be a need for greater capital requirements to sustain the PE. The main point, however, is that each of these comparative effects does not change the ranking between equity and SD.

5. Banking with Uninsured Deposits

This section replicates the analysis of the previous section but with uninsured deposits replacing SD as a capital instrument for the supervisor to use to sustain the PE. The regime is defined as $\chi^{kud} = (k, 0, u)$. Based on the formula for the equilibrium uninsured deposit rate, (7), when a bank gambles, $R^n$ is a function of both $r_j$ and $\beta$, regardless of the level of recovery value (provided $\beta > 0$). Uninsured depositors are sure to get at least a share of the recovery value since their claims that are senior to both equity and SD. Unfortunately, $R^n$,\(^{19}\) This analysis will assume that $\beta \leq r_j$. When $\beta > r_j$ this introduces a non-linearity into the KSD frontier, resulting in a less-tractable analytical solution. Instead this paper relied on numerical methods for this special case. Regardless of the parameterization, the directional effects on comparative statics were not found to change significantly.
in equation (7), is continuously and infinitely differentiable in \( r_j \) which makes an analytical solution to the insured deposit rate \( r_j \) intractable. Consequently, the derivation of the equity and UD (KUD) frontier relies on numerical methods.\(^{20}\) Again the implicit KUD frontier is defined by combinations of \( u \) and \( k \) capable of sustaining a prudent equilibrium:

\[
k = f(u; \gamma, \beta, \alpha, \pi, \rho, \mu, n|V_{sp}(r_{gp}) = V_{p}^u).
\]  

(19)

Figures 2 and 3 compare the KUD and KSD frontiers for various parameter specifications, and summarize the key comparative statics.

Figure 2: Sustainable Prudent Equilibria for the KSD and KUD frontiers. Base parameter values are: \( n = 20, \mu = 0.5, \rho = 0.1, \alpha = 0.05, \gamma = 0.12, \beta = 0.3, \) and \( \pi = 0.15. \)

\(^{20}\) An iterative two-stage process is used. In the first stage, the uninsured deposit rate, \( R^{u(1)} \), is initially set to one, then the optimal insured deposit rate is determined under a one-shot deviation toward gambling \( r_{sp}^{(1)} \). Then a new uninsured deposit rate is given \( R^{u(2)}(r_{gp}^{(1)}) \). A subsequent new insured deposit rate is determined \( r_{sp}^{(2)} \). This process continues until convergence is achieved. In all the cases examined, convergence has been achieved after one iteration.
Figure 3: Sustainable Prudent Equilibria for the KSD and KUD frontiers. Base parameter values are: $n = 20$, $\mu = 0.5$, $\rho = 0.1$, $\alpha = 0.05$, $\gamma = 0.12$, $\beta = 0.3$, and $\pi = 0.15$. 

![Graph showing sustainable prudent equilibria for the KSD and KUD frontiers.](image-url)
The main highlights of these Figures are:

(i) The KUD frontier is weakly dominated by the KSD frontier.

The KUD frontier is never below the KSD frontier. This suggests that SD, combined with equity is more effective than UD combined with equity. The careful reader might notice that direct results of minimum frontiers necessary to sustain a prudent equilibrium with all three capital instruments were never presented. The frontiers essentially mimic the results of Figures 2 and 3 in three dimensions. The 3-dimensional frontier bows out as the intensity of UD or SD increases relative to equity. All combinations of this three dimensional frontier that involve equity and SD require less capital instruments than any slice that involves some use of UD. Consequently, since closed form solutions are not tractable in the presence of UD, these results are represented indirectly using these two-dimensional figures.

Suppose that a regulator is deciding between an all-SD regime, defined as \( \chi^s \equiv (0, s, 0) \), and an all-UD regime, defined as \( \chi^u \equiv (0, 0, u) \). Each of which is capable of sustaining a PE. In each of the charts in Figures 2 and 3, sustaining a PE requires at least as much UD as SD when equity is set to zero, suggesting that \( \chi^u \) is weakly dominated by \( \chi^s \).

(ii) The KUD frontier exhibits the same directional responses to various parametric shocks as the KSD frontier.

Since both frontiers have similar directional responses, there are not any circumstances where the KUD frontier behaves differently from the KSD frontier.

(iii) Only when there is a very low recovery rate (i.e., as \( \beta \) approaches zero) does UD perform just as well as SD in sustaining a PE.

As \( \beta \to 0 \), the UD rate approaches the equilibrium return on SD, \( ((1 + \rho)/(1 - \pi)) \).\(^{21}\) In the limit UD has the same contingent payoffs as SD and, for that matter, equity. Otherwise, for \( \beta > 0 \), the uninsured depositor expects a greater recovery rate than a holder of SD since the former ranks at the same level of seniority as the deposit insurer.

The intuition behind the relationship between the interest paid to UD holders and SD holders with respect to the recovery rate is captured in Figure 4. The latter rate is always at least as great as the former. Below the gambling insured deposit rate, \( r_g \), the premium

\(^{21}\)This can be seen from the no-arbitrage condition, loosely stated as: \( (1 - \pi)R + \pi\beta R/A_0 = 1 + \rho \). When \( \beta = 0 \), solving for \( R \) we get \( R = (1 + \rho)/(1 - \pi) \).
demanded by SD holders is the same as equity holders, since there is no recovery value left after the deposit insurer is compensated. In contrast, uninsured depositors share any recovery value with the deposit insurer and are guaranteed something provided the recovery value is positive. As the recovery value increases, eventually there is enough recovery value in the unsuccessful state for both the deposit insurer, SD holders, and UD holders to be fully compensated. At that point, neither provides any disciplinary effectiveness.

Figure 4: The Subordinated Debt Rate and Uninsured Deposit Rate as a Function of the Recovery Rate $\beta$

6. Summary of sections 4 and 5

The two previous sections have provided some insights into the effectiveness of various forms of debt and equity on the liability side of the bank’s balance sheet. Briefly, the following results were found.

(i) Equity, SD and UD each have the capacity to behave as instruments for market discipline.

This suggests that all three types of instruments offer some degree of market discipline for banks, since their presence can limit incentives for excessive gambling. If one wanted to, one could induce market discipline with any one instrument.

(ii) As a source of market discipline, equity always performs at least as well as SD which performs at least as well as UD.
At best, SD is equivalent to equity in sustaining a PE, and that is only when SD looks a lot like equity (i.e., it pays off when the bank is successful and pays nothing if the bank fails). If instead, a SD holder expects to recoup some of its investment when the bank fails, it will demand a lower risk premium and not be as effective at disciplining bank gambling. Likewise, uninsured depositors always expect to recoup some of their investment when the bank is unsuccessful so discipline is even weaker with UD than with SD. An implication is that UD should not be relied upon in its current form since it requires more resources to discipline banks, than either equity or SD. This ranking was found to be robust to different parameterizations.

(ii) Comparative statics show that all three instruments respond directionally the same way to exogenous changes.

So if competition is intensifying (due to either more entrants or less heterogeneity between banks) increasing any of the three instruments would be required to sustain a PE. Competition erodes charter value, which increases the incentive for a bank to defect from choosing a prudent portfolio. Other parameters that tend to cause a change in the relative attractiveness of gambling versus prudence (for instance the likelihood of failure, return on gambling, recovery costs) were analyzed. If gambling becomes more attractive, more of these disciplinary instruments are going to be required to sustain a PE.

6.1 Discussion

The analysis in this paper has concluded that equity is the ‘best’ capital instrument for market discipline in an environment where there are no costs to holders of K, SE, UD for acquiring information about the bank’s lending decision. This is an unrealistic assumption. Generally, evaluating all the publicly available information for a fully transparent non-bank corporation is a costly exercise. For a bank, which is generally less transparent and more complex, the monitoring costs for creditors could be prohibitive.

One counter-argument, based on the principles of an efficient market, is that market prices for capital instruments already reflect all available information. Even this argument requires there be a critical mass of participants, monitoring the activities of the issuing institution (Grossman and Stiglitz, 1980). If not, then the secondary market will suffer from an illiquidity premium. If we assume that there is a liquidity premium in primary markets for capital instruments, the price of a bank’s equity is still likely to provide a more accurate indicator of risk-taking than its other debt obligations.
Anecdotal evidence supplied by market participants suggests that “off-the-run” SD prices tend to become stale over time, and when a new SD series is issued the secondary market prices of the former are sensitive to new information revealed in an issuance prospectus (Board of Governors, 1999). A bank’s equity price in contrast, is widely quoted and easily accessible relative to SD. As a consequence, when a bank is issuing SD, prospective holders that are aware of this will demand a higher liquidity premium to hold SD than equity. This is confirmed by a Board of Governors (1999) study which found that only the largest 15-20 banking organizations in the United States were active issuers of SD due to onerous liquidity premiums attached to smaller issues.  

Liquidity in equity markets for banking organizations in the United States go deeper than the largest 15 participants. There are over 1000 banks that are publicly listed and trade either on an exchange or over the counter (FDIC, 2005). Yet the fixed costs of issuing SD are lower than equity (Board and Treasury, 2000), which further suggests a liquidity premium is driving smaller banks towards raising capital in equity markets instead of issuing SD.

Another argument is that since SD holders do not gain on the upside from risk-taking by the bank while equity holders benefit when there is imperfect information, equity is less effective at disciplining the bank. This would not be the case, however. If SD holders do not know which strategy the bank will choose the SD price will be a weighted average of holders’ priors about the likelihood of gambling and prudent behaviour. The consequence is that the bank will pay a higher price for SD even when it does choose the prudent lending strategy. This will raise its cost of financing and lower the charter value of remaining in a prudent equilibrium, making gambling more attractive. This paper took the assumption that market discipline was perfect. As soon as the riskiness of the bank’s lending changed the SD holder could renegotiate a higher premium. If this is no longer the case, SD will become less effective and more prone to induce gambling behaviour.

The conclusion from this discussion is that the ranking of equity above SD is likely robust to more realistic assumptions about monitoring costs.

7. Conclusions

This paper used an established template for modelling a banking economy and an environment devoid of asymmetries of monitoring costs, to determine the most effective instruments

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22Sironi (2000) found the same characteristic to exist for European banks. Only the larger institutions were active issuers. Caldwell (2005) also confirmed this characteristic in the Canadian context.
for market discipline. Although results suggest that each instrument is capable of mitigating
risk-shifting incentives, equity was found to perform at least as well as either SD or UD. These latter instruments displayed some substitutability under special conditions (when SD or UD started to look like equity the disciplinary effect was identical).

While each instrument can be utilized by a supervisor to limit the incentive of a bank
to gamble, comparative statics show that each responds the same to exogenous change. As
the level of competition increased, or the heterogeneity of bank deposits diminished, more
equity, SD, or UD would be required to sustain a PE. Similar results hold for increases in the
expected value of gambling relative to prudent lending. Each instrument behaves the same
way at inducing a PE although not as effectively as equity.

Empirical evidence from other studies reaffirm that even in a costly-monitoring environ-
ment equity is the most effective capital instrument. This paper’s contribution is to show
that instead of being purely the result of lower monitoring costs, equity outperforms SD and
UD through another channel based on absolute priority of claims.

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Appendix A: Proofs of lemmata and propositions in the main text

Proof of Proposition 1

The SD price under gambling is \( R^* = (1 + \rho)/(1 - \pi) \). The condition for the PE is given by inequality (13). Taking first-order conditions with respect to \( r_j \):

\[
\frac{\partial V_{gp}}{\partial r_j} = -k \left( \frac{1}{\mu} \right) + \frac{1 - \pi}{1 + \rho} \left[(1 + \gamma)(1 + s + k) - (s + k) \left( \frac{1 + \rho}{1 - \pi} \right) \right] \frac{1}{\mu} \\
- \frac{1 - \pi}{1 + \rho} \left[ \frac{1}{n} + \frac{r_j - r_p}{\mu} \right] = 0.
\]

Solving

\( r_{gp} = \frac{1}{2} \left[(1 + \gamma)(1 + s + k) - (s + k) \left( \frac{1 + \rho}{1 - \pi} \right) - \frac{4}{n} + r_p \right] = \frac{r_g + r_p}{2} \), since in this scenario \( r_g = (1 + \gamma)(1 + s + k) - (k + s)(1 + \rho)/(1 - \pi) \). Hence \( r_{gp} > r_p \) is equivalent to the condition: \( r_g > r_p \). Hence algebra shows that \( r_g > r_p \) is equivalent to:

\[
s + k < \frac{(\gamma - \alpha)(1 - \pi)}{\pi(1 + \rho) - (1 - \pi)(\gamma - \alpha)}
\]

Proof of Proposition 2

If \( \beta \leq r_j \) then substituting \( r_{gp} \) and \( V_p \) into the sustainability condition inequality (13) gives:

\[
\frac{1 - \pi}{1 + \rho} \left[ (1 + \gamma)(1 + k + s) - \frac{r_g + r_p}{2} - (k + s) \left( \frac{1 + \rho}{1 - \pi} \right) \right] \times \left[ \frac{1}{n} - \frac{r_g + r_p - r_p}{\mu} \right] \leq \frac{\rho + \pi}{1 + \rho} \frac{\mu}{\rho n}
\]

or

\[
\left[ \frac{\mu}{n} + \frac{r_g - r_p}{2} \right]^2 \leq \frac{\rho + \pi}{(1 - \pi)\rho} \left( \frac{\mu}{n} \right)^2.
\]

Squaring both sides and substituting in \( r_g \) and \( r_p \) gives:

\[
\left[ \frac{\mu}{n} + \frac{(\gamma - \alpha)(1 + k + s)}{2} - \frac{(s + k)}{2} \left( \frac{(1 + \rho)\pi}{1 - \pi} \right) \right] \leq \sqrt{\frac{\rho + n}{(1 - \pi)\rho n}} \mu
\]

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or, isolating $k$:
\[
k \geq \frac{(\gamma - \alpha) - 2\omega \left(\frac{H}{n}\right)}{\lambda - (\gamma - \alpha)} - s \equiv m - s
\]
where $\omega \equiv \sqrt{(\rho + \pi)/(\rho(1 - \pi))} - 1$ and $\lambda \equiv (1 + \rho)\pi/(1 - \pi)$. Equality represents the KSD frontier. Since all combinations of $s$ and $k$ that on the KSD frontier sum to $m$, SD and equity are perfect substitutes.

If $\beta > r_j$ then, following the same steps the KSD frontier satisfies:
\[
\left(\frac{1 - \pi + s\pi}{1 - \pi}\right) \left[\frac{\mu}{n} + \frac{r_p - r_p}{2}\right]^2 \leq \frac{\rho + \pi}{(1 - \pi)\rho} \left(\frac{\mu}{n}\right)^2
\]
Solving for $k$:
\[
k = \frac{\left[\frac{(1-\pi)(1+\gamma)}{1-\pi+s\pi} - (1 + \alpha) - 2\omega' \left(\frac{H}{n}\right)\right]}{\frac{1+\rho-(1-\pi)(1+\gamma)}{1-\pi+s\pi} - (\rho - \alpha)} - \left[1 - \frac{\pi \beta/(1 - \pi + s\pi)}{1+\rho-(1-\pi)(1+\gamma)} - (\rho - \alpha)\right] s
\]
where
\[
\omega' = \sqrt{\frac{\rho + \pi}{(1 - \pi + s\pi)\rho}}.
\]

Now consider the end points. If $s = 0$ then
\[
k^* = \frac{\gamma - \alpha - 2\omega \left(\frac{H}{n}\right)}{\lambda - (\gamma - \alpha)}
\]
which is the same as when $\beta \leq r_j$. Letting $k = 0$ gives:
\[
s = \frac{\left[\frac{(1-\pi)(1+\gamma)}{1-\pi+s\pi} - (1 + \alpha) - 2\omega' \left(\frac{H}{n}\right)\right]}{\frac{1+\rho-(1-\pi)(1+\gamma)}{1-\pi+s\pi} - (\rho - \alpha) - \frac{\pi \beta}{1-\pi+s\pi}}
\]
Again consider $\pi s \to 0$.
\[
s_{(\pi s=0)} = \frac{\gamma - \alpha - 2\omega \left(\frac{H}{n}\right)}{\lambda - (\gamma - \alpha) - \frac{\pi \beta}{1-\pi}}
\]
which is greater than $s_{(\beta=0)}$ which is equivalent to $k^*$. Hence the $s$-intercept for the KSD frontier is greater whenever $\beta > r_j$.

**Proof of Result 3**

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(a) When $\beta \leq r_j$,

$$k^* = \frac{\gamma - \alpha - 2\omega \left( \frac{\mu}{n} \right)}{\lambda - (\gamma - \alpha)}.$$

Taking derivatives:

$$\frac{\partial k^*}{\partial n} = \frac{2\omega \left( \frac{\mu}{n} \right)}{\lambda - (\gamma - \alpha)} > 0, \quad \frac{\partial k^*}{\partial \mu} = \frac{-2\omega \left( \frac{1}{n} \right)}{\lambda - (\gamma - \alpha)} < 0,$$

and

$$\frac{\partial k^*}{\partial \gamma} = \frac{\lambda - 2\omega \left( \frac{\mu}{n} \right)}{[\lambda - (\gamma - \alpha)]^2} > 0, \quad \frac{\partial k^*}{\partial \alpha} = \frac{2\omega \left( \frac{\mu}{n} \right) - \lambda}{[\lambda - (\gamma - \alpha)]^2} < 0.$$

With respect to $\pi$, first there are some preliminary derivatives with respect to $\lambda$ and $\pi$ used in the final measure:

$$\frac{\partial \lambda}{\partial \pi} = \frac{1 + \rho}{(1 - \pi)^2} > 0, \quad \frac{\partial \omega}{\partial \pi} = \frac{1}{2} \left[ \frac{\rho + \pi}{\rho(1 - \pi)} \right]^{-\frac{1}{2}} \left[ \frac{\rho(1 + \rho)}{(\rho(1 - \pi))^2} \right] > 0.$$

So

$$\frac{\partial k^*}{\partial \pi} = -2 \left( \frac{\mu}{n} \right) \frac{\lambda - (\gamma - \alpha)}{[\lambda - (\gamma - \alpha)]^2} - \left( \gamma - \alpha - 2\omega \left( \frac{\mu}{n} \right) \right) \frac{\partial \lambda}{\partial \pi} < 0.$$

(b) First

$$\frac{\partial \lambda}{\partial \rho} = \frac{\pi}{1 - \pi} > 0, \quad \frac{\partial \omega}{\partial \rho} = \frac{1}{2} \left[ \frac{\rho + \pi}{\rho(1 - \pi)} \right]^{-\frac{1}{2}} \left[ \frac{-\pi(1 - \pi)}{(\rho(1 - \pi))^2} \right] < 0.$$

So

$$\frac{\partial k^*}{\partial \rho} = -2 \left( \frac{\mu}{n} \right) \frac{\lambda - (\gamma - \alpha)}{[\lambda - (\gamma - \alpha)]^2} - \left[ \gamma - \alpha - 2\omega \left( \frac{\mu}{n} \right) \right] \frac{\partial \lambda}{\partial \rho},$$

for which the sign is ambiguous. However when $n \rightarrow \infty$

$$k^* \rightarrow \frac{\gamma - \alpha}{\lambda - (\gamma - \alpha)}.$$

and

$$\frac{\partial k^*}{\partial \rho} = \frac{-(\gamma - \alpha) \frac{\partial \lambda}{\partial \rho}}{[\lambda - (\gamma - \alpha)]^2} < 0.$$